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► **To cite this version:**

James Ledoux, Gerardo Rubino. Simple Formulas for Counting Processes in Reliability Models. *Advances in Applied Probability*, 1997, 29 (4), pp.1018-1038. hal-00852084

**HAL Id: hal-00852084**

**<https://hal.science/hal-00852084v1>**

Submitted on 20 Aug 2013

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# Simple Formulas for Counting Processes in Reliability Models

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## Abstract

*Dependability evaluation is a basic component in the assessment of the quality of repairable systems. We develop a model taking simultaneously into account the occurrence of failures and repairs, together with the observation of user-defined success events. The model is built from a Markovian description of the behavior of the system. We obtain the distribution function of the joint number of observed failures and of delivered services on a fixed mission period of the system. In particular, the marginal distribution of the number of failures can be directly related to the distribution of the Markovian Arrival Process extensively used in queueing theory. We give both the analytical expressions of the considered distributions and the algorithmic solutions for their evaluation. Asymptotical analysis is also provided.*

**Keywords:** *Counting Processes, Markov Chains, Uniformization.*

**AMS 1991 Subject Classification:** *Primary 60J10, 90B25*

*Secondary 60J27.*

## 1 Introduction

In the last years, special attention has been devoted to the quantitative analysis of queueing models with a Markovian Arrival Process (MAP) [11] (or a versatile Markovian point process according to [13]). The interest of such a point process is to keep the tractability of the Poisson arrivals but significantly generalize it allowing the inclusion of dependent interarrival times, non-exponential interarrival distributions, etc. Lucantoni's tutorial [11] reviews this class of stochastic process. We refer for instance to [14] for a discussion on qualitative features in traffic streams which can be captured by such a process. Extensive work has been performed about the stationary characteristics of MAPs and, in the last years, on the transient analysis of these processes as well, mainly on problems coming from communications systems (see [1] and references in [11], for instance). Here, we point out that MAP-type processes are also suitable for modeling some failure occurrence phenomenon in repairable systems. Moreover, from the viewpoint of reliability (or dependability [8]) theory, we are mainly interested in

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evaluating measures on transient features of the system. The aim of the paper is precisely to present a general dependability model and to give analytic and algorithmic results for evaluating it on an finite observation period.

In Section 2, the system is modeled by a discrete time Markov process. One of its aims is to handle the concept of *delivered service* with the help of a particular class of transitions. We also give a constructive definition of a model of failure occurrence and recovery times. We allow, for instance, to consider phase (PH) distributed recovery times. The joint distribution of the number of observed events (including failures) and of delivered services on an finite interval is then derived and we propose an efficient algorithm for the numerical computation of this distribution. In Section 3, we discuss the counterpart of the model in continuous time, using the uniformization technique. Finally, we report in Section 4 the extensions to our context of some well-known formulas on expectation measures associated with MAP processes. Section 5 consists of concluding remarks.

### Main notation

- $\mathcal{U} = \{1, 2, \dots, M\}$ ,  $\mathcal{R}$  : the sets of up and recovery states;
- $\alpha = (\alpha_1, \dots, \alpha_M)$  where  $\alpha_i = \mathbb{P}(\text{system starts at state } i)$ ;
- $Q = (Q(i, j))_{i, j \in \mathcal{U}}$  (resp.  $P = (P(i, j))_{i, j \in \mathcal{U}}$ ) where  $Q(i, j)$  (resp.  $P(i, j)$ ) is the transition rate (resp. probability) from state  $i$  to state  $j$  for a *failure-free* system;
- $X_t$ : the state occupied at time  $t$  for a *failure-free* system;
- $X_t^*$ : the up or recovery state occupied at time  $t$ , *when failures are taken into account*;
- $N_t$ : the number of failures up to time  $t$ ;
- $C_t$ : the number of delivered services up to time  $t$ ;
- $\lambda_i$  (resp.  $p_i$ ): *primary* failure rate (resp. probability) when the occupied state is  $i$ ;
- $\lambda(i, j)$ : occurrence probability of a *primary* failure during a control transfer from state  $i$  to state  $j$ ;
- $\mu_i$  (resp.  $\mu(i, j)$ ) has the same meaning as  $\lambda_i$  (resp.  $\lambda(i, j)$ ) for the *secondary* failure process;
- $\alpha(i, j)$ : probability that the first recovery state entered is  $j$  given that a failure occurs in the state  $i$ .

## 2 Discrete time model

### 2.1 Description

In this section, we are concerned with models evolving in discrete time. Such a model can be useful when a discrete temporal grain is more significant than a continuous one to represent system behavior. We consider that we have a model of the system in operation, which is assumed to be a discrete time homogeneous Markov chain  $X$  on the finite state space  $\mathcal{U} = \{1, \dots, M\}$ . The different states can represent the load on the system as it is usually done in performance evaluation, or some measure of performance level, or the active module of a software system, etc. This Markov chain is given by its transition probability matrix  $P = (P(i, j))_{i, j \in \mathcal{U}}$ , where  $P(i, j)$  denotes the transition probability from state  $i$  to state  $j$ , and by its initial distribution  $\alpha = (\alpha_1, \dots, \alpha_M)$ . By convention, vectors are row vectors. Column vectors are denoted by means of the transpose operator  $(\cdot)^\top$ .

Suppose that state  $i$  represents a computer system working to satisfy some request, and that after visiting state  $i$ , the next visited state is  $j$  (this happens with probability  $P(i, j)$ ). Given this, there are two possibilities: either the service request is satisfied at this point in time or it is not. In the first case, we say that an *execution cycle* ends successfully. The probability of this event is denoted by  $p_c(i, j)$  and the probability of its complementary event is  $p_f(i, j)$ . So, we have the decomposition  $P(i, j) = p_c(i, j) + p_f(i, j)$ . Of course, in a model we will usually have, for instance,  $p_f(i, j) = 0$  for many pairs  $i, j$ . Observe that if a cycle ends when the model jumps from state  $i$  to state  $j$ , we are assuming that the next cycle starts from state  $j$ .

Assume now that the system is subject to failures and that it includes repair facilities (that is, it is a *repairable* system). Let us describe the failure process. We distinguish two types of failure; the first one is associated with states, the second one with transitions. When the model is in state  $i$ , a failure occurs with probability  $p_i$ , thus depending on the identity of the state. This means that, at the next jump, a cycle ends unsuccessfully and a repair phase will start. We discuss above the representation of the repair time. In some applications, it can be useful to associate failures directly with transitions. To do this, suppose that a failure does not occur during a sojourn in state  $i$  (this event has probability  $1 - p_i$ ). If the next state to be visited is state  $j$  (which happens with probability  $P(i, j)$ ), suppose that a second class of failure can occur, called a *transfer failure*, and that this happens with probability  $\lambda(i, j)$ , thus depending on both the original and the next state. Such a failure causes an execution break.

Retrieving a safe state involves a general random delay which is assumed to have a phase type (PH) distribution; in other words, this delay can be seen as the time up to absorption in a finite discrete time Markov chain (for PH distributions, see [13]). The set of non-absorbing states associated with this PH random variable is denoted by

$\mathcal{R}$  and it represents the set of *recovery states* in our model. The sub-stochastic matrix of transition probabilities between recovery states is denoted by  $\widehat{R}$ <sup>1</sup>. The phase type assumption allows us to represent times to recover an operational state which depend on the identity of the state in which the failure has occurred. Indeed, the sequence of successive visited states in  $\mathcal{R}$  can be related to the first recovery state selected after a failure. We assume that the recovery state  $j$  is entered after a failure occurrence in state  $i$  (or at a transition instant from  $i$ ) with constant probability  $\alpha(i, j)$  (with  $\sum_{j \in \mathcal{R}} \alpha(i, j) = 1$ ). These delays can be interpreted, for instance, as either the time needed to restart the system or as the period of time used by a fault tolerant mechanism to recover a safe state. After a recovery period ended by state  $i \in \mathcal{R}$ , state  $j \in \mathcal{U}$  is entered with constant probability  $\widehat{S}(i, j)$ . Therefore, matrix  $\widehat{S} = \left( \widehat{S}(i, j) \right)_{(i, j) \in \mathcal{R} \times \mathcal{U}}$  is composed by the transition probabilities from the “down” states to the “up” states and we have  $(\widehat{S} + \widehat{R})\mathbf{1}^\top = \mathbf{1}^\top$ , where  $\mathbf{1}^\top$  represents a column vector with all its entries equal to one.

Let us define now the process  $X^* = (X_h^*)_{h \geq 0}$  where  $X_h^*$  is the occupied up-state at time  $h$  if the system is up, or the recovery state reached at time  $h$ . Its state space is the set  $\mathcal{E} = \mathcal{U} \cup \mathcal{R}$ . Given a sequence of states visited by  $X^*$  (a trajectory of the process), all the failure processes are assumed to be independent of each other and independent of the recovery process. We define three matrices  $\widehat{A}, \widehat{C}, \widehat{D}$  by their respective entries:

$$\begin{aligned} \widehat{A}(i, j) &= p_c(i, j)(1 - p_i)(1 - \lambda(i, j)) & i, j \in \mathcal{U}, \\ \widehat{C}(i, j) &= p_f(i, j)(1 - p_i)(1 - \lambda(i, j)) & i, j \in \mathcal{U}, \\ \widehat{D}(i, j) &= [p_i + (1 - p_i) \sum_k P(i, k) \lambda(i, k)] \alpha(i, j) & i \in \mathcal{U}, j \in \mathcal{R}. \end{aligned}$$

The nonnegative number  $\widehat{A}(i, j)$  (respectively  $\widehat{C}(i, j)$ ) represents the probability that  $X^*$  jumps from state  $i$  to  $j$  without any occurrence of a failure or success event (respectively with no failure event and a delivered service). The entry  $\widehat{D}(i, j)$  is the probability that the  $X^*$  jumps from up-state  $i$  to repair-state  $j$ , that is after the occurrence of a failure event. It is immediate to check that  $X^*$  is a finite homogeneous Markov chain with transition probability matrix

$$P^* = \begin{pmatrix} \widehat{A} + \widehat{C} & \widehat{D} \\ \widehat{S} & \widehat{R} \end{pmatrix}$$

which is assumed to be irreducible. Therefore, the alternation of execution-recovery periods is infinite.

## 2.2 Example

Let us consider the Markov reliability-model for a modular software developed by Cheung [3]. The control structure is represented by a directed graph where every node is a program module. Each directed arc  $(i, j)$

<sup>1</sup>The reasons explaining some of the notation are better understood in the discussion of the continuous time model (Section 3).

represents a possible transfer of control from  $i$  to  $j$ . A probability  $P(i, j)$  that the associated transfer will take place when control is at module  $i$ , is attached to every arc  $(i, j)$ . The set of modules is assumed to be the state space of a finite Markov chain with transition probability matrix  $P$ . Suppose that we have 5 modules and that the transition probability matrix is

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 3/10 & 0 & 2/5 & 3/10 & 0 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 & 0 \end{pmatrix}.$$

In Cheung's formulation, there exists a single input module in the program. Here we can consider any module as an input state. For our example, module 1 and 3 are such input points. We suppose that the selection of the input module is done according to the probability distribution  $\alpha = (1/2, 0, 1/2, 0, 0)$ . Finally, an absorbing state (called a terminal node) is used in [3] to represent mission success of the software, that is, the fact that the software completes its task successfully. Here, we allow any module to be a terminal node for a class of tasks. In our example, we consider that 50% of the execution of module 2 corresponds to a completion of a task. If such a task is completed, in 40% (resp. 60%) of the cases the input module 3 (resp. module 1) is then executed. So we can write

$$\begin{aligned} p_f(2, 3) &= \frac{2}{5} \frac{1}{2} = \frac{1}{5} & \text{and} & \quad p_c(2, 3) = \frac{1}{5}, \\ p_f(2, 1) &= \frac{1}{2} \frac{3}{5} = \frac{3}{10} & \text{and} & \quad p_c(2, 1) = 0. \end{aligned}$$

Since  $p_c(2, 1) = 0$ , all the transitions from module 2 to module 1 correspond to the success of a task and the beginning of a new one. Since modules 1 and 3 are the only input modules, we have  $P(2, 4) = p_c(2, 4)$ , that is any transition from module 2 to module 4 means that module 4 must be executed after module 2 for the completion of some task. A similar assumption is made for modules 1, 3 and 4, that is  $[P(1, 2) = p_c(1, 2), P(1, 3) = p_c(1, 3)]$ ,  $[P(3, 4) = p_c(3, 4), P(3, 5) = p_c(3, 5)]$  and  $P(4, 5) = p_c(4, 5)$ . Finally, module 5 is always a terminal module, that is each execution of module 5 corresponds to the completion of a task. After such a mission success, the input modules 1 and 3 are executed according to the selection probability distribution  $\alpha$ .

Now let us describe the failure parameters. For each module  $i$ , we have a constant probability  $p_i$  that the module fails. For our numerical evaluations, these probabilities are

$$p_1 = p_2 = p_5 = 0; \quad p_3 = \frac{1}{100}, p_4 = \frac{3}{100}.$$

In [3], when such a failure event occurs, the programs are definitively stopped. We suppose here that each failure event is followed by a recovery period. There is no failure associated with transitions in the example for the sake of simplicity.

In our numerical example, we assume that there are two recovery states denoted by  $1_R$  and  $2_R$ . After a failure in module 3 the system recovers during a geometrically distributed period with parameter  $R(1_R, 1_R) = 1/1000$  (and thus,  $\alpha(3, 1_R) = 1$ ). In the same way, after a failure in module 4, the recovering period has a geometric distribution with parameter  $R(2_R, 2_R) = 3/1000$  (and thus,  $\alpha(4, 2_R) = 1$ ). After a recovering period, the input modules 1 and 3 are entered according to the initial distribution  $\alpha$ . The five associated matrices are then

$$\widehat{A} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{3}{10} & 0 \\ 0 & 0 & 0 & \frac{2}{5} \frac{99}{100} & \frac{3}{5} \frac{99}{100} \\ 0 & 0 & 0 & 0 & \frac{97}{100} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{3}{10} & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad \widehat{D} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{100} & 0 \\ 0 & \frac{3}{100} \\ 0 & 0 \end{pmatrix},$$

$$\widehat{R} = \begin{pmatrix} \frac{1}{1000} & 0 \\ 0 & \frac{3}{1000} \end{pmatrix}, \quad \widehat{S} = \begin{pmatrix} \frac{999}{1000} \frac{1}{2} & 0 & \frac{999}{1000} \frac{1}{2} & 0 & 0 \\ \frac{997}{1000} \frac{1}{2} & 0 & \frac{997}{1000} \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

### 2.3 Main joint distribution

We are interested in computing the distribution of the discrete time bi-dimensional process  $(N_h, C_h)_{h \geq 0}$  where  $N_h$  (respectively  $C_h$ ) is the cumulative number of failures (respectively of delivered services) observed at time  $h$ . To do this, let us consider the tri-dimensional process  $(N, C, X^*) = (N_h, C_h, X_h^*)_{h \geq 0}$  over the state space  $\mathbb{N} \times \mathbb{N} \times \mathcal{E}$ . It follows from the independence assumptions on failure and recovery processes and from the phase type assumption for recovery delays that  $(N, C, X^*)$  is a homogeneous Markovian process with initial distribution:

$$\begin{aligned} \mathbb{P}(N_0 = 0, C_0 = 0, X_0^* = i) &= \mathbb{P}(X_0^* = i) = \alpha_i, \\ \mathbb{P}(N_0 = k, C_0 = n, X_0^* = i) &= 0 \quad \text{for } k, n \geq 1 \text{ and any } i \in \mathcal{U}. \end{aligned}$$

The transition probabilities associated with  $(N, C, X^*)$  are given, for all  $k \geq 0$  and  $h \geq 1$ , by

$$\mathbb{P} \left( (N_h, C_h, X_h^*) = (k, n, j) \mid (N_{h-1}, C_{h-1}, X_{h-1}^*) = (k, n, i) \right) = \begin{cases} \widehat{A}(i, j) & (i, j) \in \mathcal{U} \times \mathcal{U} \\ \widehat{R}(i, j) & (i, j) \in \mathcal{R} \times \mathcal{R} \\ \widehat{S}(i, j) & (i, j) \in \mathcal{R} \times \mathcal{U} \\ 0 & (i, j) \in \mathcal{U} \times \mathcal{R} \end{cases}$$

$$\mathbb{P} \left( (N_h, C_h, X_h^*) = (k+1, n, j) \mid (N_{h-1}, C_{h-1}, X_{h-1}^*) = (k, n, i) \right) = \begin{cases} \widehat{D}(i, j) & (i, j) \in \mathcal{U} \times \mathcal{R} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbb{P} \left( (N_h, C_h, X_h^*) = (k, n+1, j) \mid (N_{h-1}, C_{h-1}, X_{h-1}^*) = (k, n, i) \right) = \begin{cases} \widehat{C}(i, j) & (i, j) \in \mathcal{U} \times \mathcal{U} \\ 0 & \text{otherwise.} \end{cases}$$

All other transition probabilities for  $(N, C, X^*)$  are null. The above theorem gives the distribution function of the counting process  $(N_h, C_h)_{h \geq 0}$ . In the sequel, for any  $n \in \mathbb{N}$  the expression  $(0)^n$  will denote a sequence of  $n$  values 0. We denote by  $\text{diag}(M_i)$  a diagonal matrix (respectively a block diagonal matrix), with the real number  $M_i$  (respectively matrix  $M_i$ ) as diagonal  $(i, i)$ -entry (respectively  $(i, i)$ -block entry).

**Theorem 2.1** *For all time  $h \geq 0$ , we have:*

$$\mathbb{P}(N_h \leq k, C_h \leq n) = (\alpha, (0)^n, (0)^{2k(n+1)}) P_{k+1, n+1}^h \mathbf{1}^\top \quad \forall k, n \geq 0,$$

where

$$P_{k+1, n+1} = \begin{pmatrix} B & \widehat{D}' & 0 & \cdots & \cdots & 0 \\ 0 & \widehat{R}' & \widehat{S}' & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & B & \widehat{D}' & 0 \\ \vdots & & & \ddots & \widehat{R}' & \widehat{S}' \\ 0 & \cdots & \cdots & \cdots & 0 & B \end{pmatrix}$$

is a  $(2k+1) \times (2k+1)$  block matrix with the  $(n+1) \times (n+1)$  block matrices  $B$  defined by

$$B = \begin{pmatrix} \widehat{A} & \widehat{C} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \widehat{C} \\ 0 & \cdots & \cdots & 0 & \widehat{A} \end{pmatrix},$$

and  $\widehat{R}' = \text{diag}(\widehat{R})$ ,  $\widehat{S}' = \text{diag}(\widehat{S})$ ,  $\widehat{D}' = \text{diag}(\widehat{D})$ .

**Proof.** Let us denote by  $\mathbb{P}_i(\cdot)$  the probability measure conditional to the event  $(X_0^* = i)$ , for any  $i \in \mathcal{U}$ . We use



the following renewal equations for each  $h \geq 1$ :

$$\begin{aligned} \forall i \in \mathcal{U} : \mathbb{P}_i(N_h \leq k, C_h \leq n) &= \sum_{j \in \mathcal{U}} \widehat{A}(i, j) \mathbb{P}_j(N_{h-1} \leq k, C_{h-1} \leq n) \\ &+ \sum_{j \in \mathcal{U}} \widehat{C}(i, j) \mathbb{P}_j(N_{h-1} \leq k, C_{h-1} \leq n-1) \\ &+ \sum_{j \in \mathcal{R}} \widehat{D}(i, j) \mathbb{P}_j(N_{h-1} \leq k-1, C_{h-1} \leq n) \quad \text{for } k, n \geq 1. \end{aligned}$$

$$\begin{aligned} \forall i \in \mathcal{U} : \mathbb{P}_i(N_h \leq k, C_h = 0) &= \sum_{j \in \mathcal{U}} \widehat{A}(i, j) \mathbb{P}_j(N_{h-1} \leq k, C_{h-1} = 0) \\ &+ \sum_{j \in \mathcal{R}} \widehat{D}(i, j) \mathbb{P}_j(N_{h-1} \leq k-1, C_{h-1} = 0) \quad \text{for } k \geq 1. \end{aligned}$$

$$\begin{aligned} \forall i \in \mathcal{U} : \mathbb{P}_i(N_h = 0, C_h \leq n) &= \sum_{j \in \mathcal{U}} \widehat{A}(i, j) \mathbb{P}_j(N_{h-1} = 0, C_{h-1} \leq n) \\ &+ \sum_{j \in \mathcal{U}} \widehat{C}(i, j) \mathbb{P}_j(N_{h-1} = 0, C_{h-1} \leq n-1) \quad \text{for } n \geq 1. \end{aligned}$$

$$\begin{aligned} \forall i \in \mathcal{R} : \mathbb{P}_i(N_h \leq k-1, C_h \leq n) &= \sum_{j \in \mathcal{R}} \widehat{R}(i, j) \mathbb{P}_j(N_{h-1} \leq k-1, C_{h-1} \leq n) \\ &+ \sum_{j \in \mathcal{U}} \widehat{S}(i, j) \mathbb{P}_j(N_{h-1} \leq k-1, C_{h-1} \leq n) \quad \text{for } k \geq 1, \forall n \geq 0. \end{aligned}$$

Let us define the row vectors

$$\begin{aligned} x_{\mathcal{U}}(k, n, h) &= (\mathbb{P}_i(N_h \leq k, C_h \leq n))_{i \in \mathcal{U}}, \\ x_{\mathcal{R}}(k, n, h) &= (\mathbb{P}_i(N_h \leq k-1, C_h \leq n))_{i \in \mathcal{R}} \end{aligned}$$

and denote the column vectors  $(x_{\mathcal{U}}(k, n, h))^{\top}$  and  $(x_{\mathcal{R}}(k, n, h))^{\top}$  respectively by  $x_{\mathcal{U}}^{\top}(k, n, h)$  and  $x_{\mathcal{R}}^{\top}(k, n, h)$ .

The previous relations can be rewritten with this notation: for  $h \geq 1$ ,

$$\left. \begin{aligned} x_{\mathcal{U}}^{\top}(k, n, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(k, n, h-1) + \widehat{C}x_{\mathcal{U}}^{\top}(k, n-1, h-1) + \widehat{D}x_{\mathcal{R}}^{\top}(k, n, h-1) & k, n \geq 1, \\ x_{\mathcal{U}}^{\top}(k, 0, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(k, 0, h-1) + \widehat{D}x_{\mathcal{R}}^{\top}(k, 0, h-1) & k \geq 1, \\ x_{\mathcal{U}}^{\top}(0, n, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(0, n, h-1) + \widehat{C}x_{\mathcal{U}}^{\top}(0, n-1, h-1) & n \geq 1, \\ x_{\mathcal{R}}^{\top}(k, n, h) &= \widehat{R}x_{\mathcal{R}}^{\top}(k, n, h-1) + \widehat{S}x_{\mathcal{U}}^{\top}(k-1, n, h-1), \quad x_{\mathcal{R}}^{\top}(k, n, 0) = 1^{\top} & k \geq 1, n \geq 0 \\ x_{\mathcal{U}}^{\top}(0, 0, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(0, 0, h-1), \quad x_{\mathcal{U}}^{\top}(k, n, 0) = 1^{\top}, & k, n \geq 0. \end{aligned} \right\} \quad (1)$$

If we define the column vector

$$x^\top(h) = \begin{bmatrix} x_{\mathcal{U}}(k, n, h), \dots, x_{\mathcal{U}}(k, 0, h), & x_{\mathcal{R}}(k, n, h), \dots, x_{\mathcal{R}}(k, 0, h), \\ x_{\mathcal{U}}(k-1, n, h), \dots, x_{\mathcal{U}}(k-1, 0, h), & x_{\mathcal{R}}(k-1, n, h), \dots, x_{\mathcal{R}}(k-1, 0, h), \\ \dots & \dots \\ x_{\mathcal{U}}(0, n, h), \dots, x_{\mathcal{U}}(0, 0, h) \end{bmatrix}^\top,$$

then we can verify that for  $h \geq 1$

$$x^\top(h) = P_{k+1, n+1} x^\top(h-1)$$

with  $x^\top(0) = 1^\top$ . Therefore, the proposed representation of the distribution function of  $(N_h, C_h)$  follows immediately.

System (1) can be used for the numerical evaluation of the distribution function  $\mathbb{P}(N_{h_0} \leq k_0, C_{h_0} \leq n_0)$ ,  $k_0, n_0 \geq 0$ . At step  $h \leq h_0$ , the value of the couple of vectors  $(x_{\mathcal{U}}^\top(k, n, h), x_{\mathcal{R}}^\top(k, n, h))$  is obtained from the computation performed at step  $h-1$ , as illustrated in Figure 1. Of course, a compact representation of the involved matrices can be used having regard to their probable sparsity.

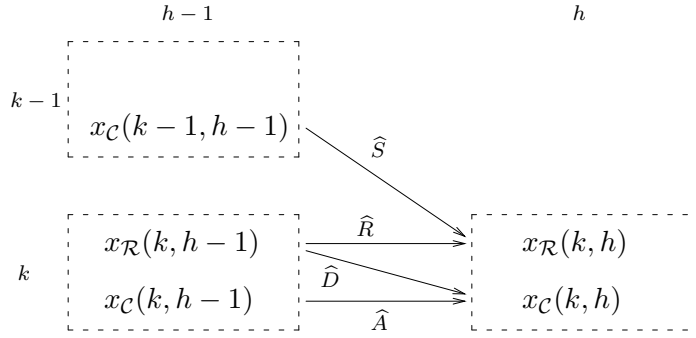


Figure 1: The contents of the cell  $(k, n, h)$  is the couple of vectors  $(x_{\mathcal{U}}^\top(k, n, h), x_{\mathcal{R}}^\top(k, n, h))$ .

## 2.4 Marginal distributions

From the distribution of the random variable  $(N_h, C_h)$ , we can derive the marginal distribution functions of the variables  $N_h$  and  $C_h$ :

$$\begin{aligned} \forall k \geq 0, \quad \mathbb{P}(N_h \leq k) &= \mathbb{P}(N_h \leq k, C_h \leq h) = \alpha x_{\mathcal{U}}^\top(k, h, h); \\ \forall n \geq 0, \quad \mathbb{P}(C_h \leq n) &= \mathbb{P}(N_h \leq h, C_h \leq n) = \alpha x_{\mathcal{U}}^\top(h, n, h). \end{aligned}$$

Since  $x_{\mathcal{U}}^{\top}(k, h, h-1) = x_{\mathcal{U}}^{\top}(k, h-1, h-1)$  for  $0 \leq k \leq h-1$  and  $x_{\mathcal{R}}^{\top}(k, h, h-1) = x_{\mathcal{R}}^{\top}(k, h-1, h-1)$  for  $1 \leq k \leq h-1$ , we can reduce system (1) for evaluating the marginal distribution  $\mathbb{P}(N_h \leq k)$ , for instance, to

$$\begin{aligned} x_{\mathcal{U}}^{\top}(k, h, h) &= (\widehat{A} + \widehat{C}) x_{\mathcal{U}}^{\top}(k, h-1, h-1) + \widehat{D} x_{\mathcal{R}}^{\top}(k, h-1, h-1), \quad k \geq 1; \\ x_{\mathcal{U}}^{\top}(0, h, h) &= (\widehat{A} + \widehat{C}) x_{\mathcal{U}}^{\top}(0, h-1, h-1); \\ x_{\mathcal{R}}^{\top}(k, h, h) &= \widehat{R} x_{\mathcal{R}}^{\top}(k, h-1, h-1) + \widehat{S} x_{\mathcal{U}}^{\top}(k-1, h-1, h-1), \quad k \geq 1; \\ x_{\mathcal{R}}^{\top}(k, 0, 0) &= \mathbf{1}^{\top}, \quad k \geq 1 \quad \text{and} \quad x_{\mathcal{U}}^{\top}(k, 0, 0) = \mathbf{1}^{\top}, \quad k \geq 0. \end{aligned}$$

In that case, it is clear that we only need a double index for the vectors.

## 2.5 Expected values

Let us briefly analyze now the mean number of failures up to time  $h$ . A first expression of this expectation can be easily derived from Theorem 2.1:

$$\begin{aligned} \mathbb{E}[N_h] &= \sum_{k=0}^{+\infty} \mathbb{P}(N_h > k) = \sum_{k=0}^{+\infty} (1 - \alpha x_{\mathcal{U}}^{\top}(k, h, h)) \\ &= \sum_{k=0}^{h-1} (1 - \alpha x_{\mathcal{U}}^{\top}(k, h, h)) \quad \text{since } x_{\mathcal{U}}^{\top}(k, h, h) = \mathbf{1}^{\top} \text{ for } k \geq h \\ &= h - \sum_{k=0}^{h-1} \alpha x_{\mathcal{U}}^{\top}(k, h, h). \end{aligned} \tag{2}$$

We obtain in the same way for the random variable  $C_h$ :

$$\mathbb{E}[C_h] = h - \sum_{n=0}^{h-1} \alpha x_{\mathcal{U}}^{\top}(h, n, h).$$

Therefore, for  $h_0$  fixed, the computation of  $\mathbb{E}[N_{h_0}]$  consists in summing the outputs of the previous algorithm with the successive values  $k_0 = 1, \dots, h_0 - 1$ . We will give other representation of these expectation measures in Section 4. They will be directly related to the classical relations in [12], [13] for a MAP in the continuous time context.

## 2.6 Example (continued)

The reliability measure adopted by Cheung [3] is the probability of reaching the terminal node from the input module, that is, the probability that the software completes successfully a task. It corresponds to the probability of absorption in the terminal state given that the Markov model starts in the input state. In [15], Siegrist modifies Cheung's model by eliminating the assumption of a terminal state and by considering the mean number of

transitions up to a failure as the measure of reliability. Our model and the results of previous subsections allows us to perform a much more deep analysis of such a system. Since we obtain the joint distribution function of the two main processes in the model, virtually any measure of interest can be numerically derived. The designer can also use this distribution in order to tune a control variable or to verify some dependability constraint. For instance, it is important to see if given  $\varepsilon > 0$ , we have  $\mathbb{P}(N_H \leq k, C_H \geq n) > 1 - \varepsilon$  for some  $H, k, n$ , or  $\mathbb{P}(N_H \leq k \mid C_H \geq n) > 1 - \varepsilon$ , or whatever. More simply, we can analyze the behavior of the joint distribution as a function of one or two variables. Let us illustrate this with some numerical values. In Figure 2 we plot the number  $\mathbb{P}(N_H = 0, C_H > n)$  as a function of  $n$ , for different values of the total interval length  $H$ . As one can

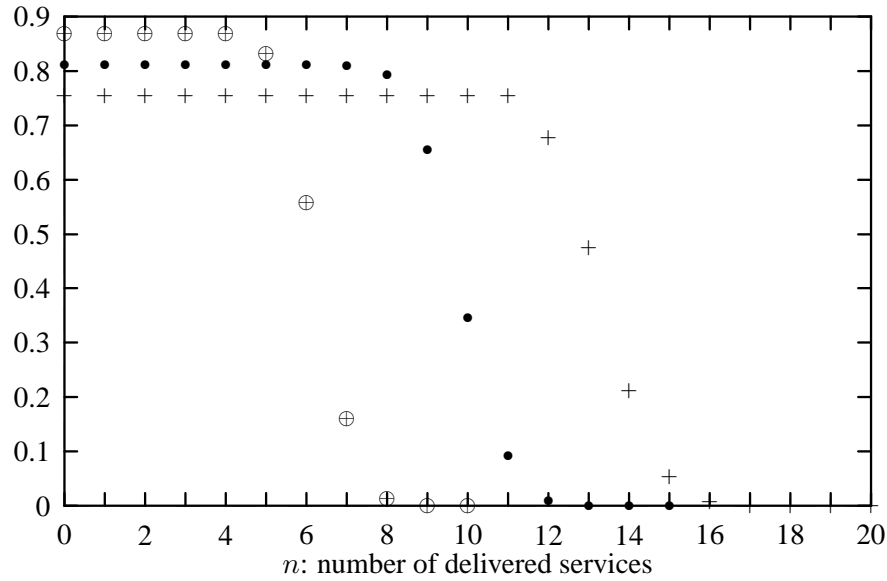


Figure 2:  $\mathbb{P}(N_H = 0, C_H > n)$  as a function of  $n$ , for different values of the total interval length  $H$ :  $\mathbb{P}(N_{40} = 0, C_{40} > n)$  + ,  $\mathbb{P}(N_{30} = 0, C_{30} > n)$  • ,  $\mathbb{P}(N_{20} = 0, C_{20} > n)$  ⊕

expect, this probability is not very dependent on  $n$  for small values of  $n$ , and gets quickly very small values when this dependency begins to increase. In the second example, we keep the total interval length constant and we look at  $\mathbb{P}(N_H \leq k, C_H > n)$  as a function of  $n$ , for different values of  $k$ . The behavior of these joint probabilities is similar as the behavior of the joint distributions illustrated in Figure 2.

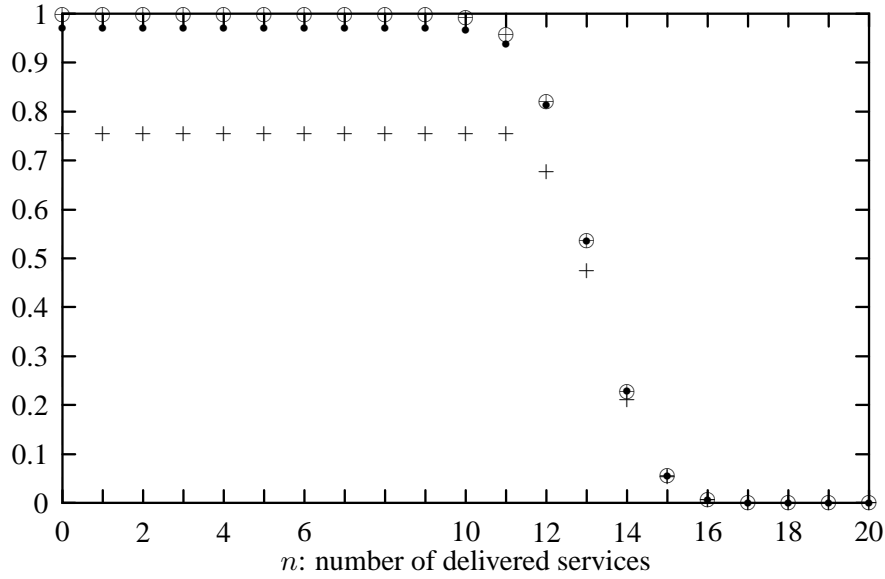


Figure 3:  $\mathbb{P}(N_{40} \leq k, C_{40} > n)$  as a function of  $n$ , with  $k = 0, 1, 2$ :  $\mathbb{P}(N_{40} = 0, C_{40} > n)$  + ,  $\mathbb{P}(N_{40} \leq 1, C_{40} > n)$  • ,  $\mathbb{P}(N_{40} \leq 2, C_{40} > n)$  ⊕

### 3 Continuous time model

#### 3.1 Description

In the continuous counterpart of the previous model, when the system is up, its behavior is described by a continuous time Markov chain  $X$  over the state space  $\mathcal{U} = \{1, \dots, M\}$ . The infinitesimal generator and the initial distribution of  $X$  are respectively denoted by  $Q = (Q(i, j))_{i, j \in \mathcal{U}}$  and  $\alpha = (\alpha_1, \dots, \alpha_M)$ . The entry  $Q(i, j)$  ( $i \neq j$ ) is the transition rate from state  $i$  to state  $j$  in absence of any disruption phenomenon.

We will count two classes of events. First, we consider what we call *secondary events*. When the occupied state is  $i$ , they occur according to a Poisson process having rate  $\mu_i$ . A secondary event can also take place with constant probability  $\mu(i, j)$  simultaneously with a transition from state  $i$  to state  $j$ . These events are just counted and they do not affect the behavior of the Markov chain  $X$ ; they are of interest for instance in reliability models as we will see in Remark 2 (see also [10]).

The other class of events is composed of the failures; they lead to an execution break and the execution is restarted (after some delay) for instance from a checkpoint or perhaps from the beginning. Their occurrence is a *primary event*. When the occupied state is  $i$ , a failure arrives with constant rate  $\lambda_i$ . Failures may also happen at a transition instant: when there is a switch from state  $i$  to state  $j$ , the probability of having a primary event is assumed to be constant and is denoted by  $\lambda(i, j)$ . In order to simplify the technical evaluation of the model,

we accept the simultaneous occurrence of the two types of events when a transition occurs from  $i$  to  $j$  (with probability  $\lambda(i, j)\mu(i, j)$ ), with the result that only the primary one will be taken into account. The point process constituted by the occurrence times of successive events is closely related to MAPs (see Remark 1).

As in Section 2, we distinguish the transitions between two states  $i$  and  $j$  where an execution cycle ends in state  $i$  and is followed by a jump to state  $j$ , from those occurring within those cycles. To do this, a probability  $p_f(i)$  that  $i$  is the last occupied state in an execution cycle is associated with each state  $i$ .

As in discrete time, we suppose that the delay following an execution break is a random variable with a phase type distribution. We denote the set of transient states associated with this PH-distribution by  $\mathcal{R}$ . The sub-generator composed by the transition rates between elements of  $\mathcal{R}$  is denoted by  $R$ . As argued in the discrete time context, we can then model a delay which depends on the state in which the failure has occurred. We assume that the first recovery state entered after a failure in state  $i \in \mathcal{U}$  (or at a transition time from state  $i$ ) is  $j \in \mathcal{R}$  with constant probability  $\alpha(i, j)$ . Matrix  $S = (S(i, j))_{(i, j) \in \mathcal{R} \times \mathcal{U}}$  is composed of the transition rates from a recovery state to an up-state of  $\mathcal{U}$ . We have  $(R + S)1^\top = 0$ .

Let us define  $X^* = (X_t^*)_{t \geq 0}$  as the process which gives either the up-state or the recovery state occupied at time  $t$ . Its state space is then the set  $\mathcal{E} = \mathcal{U} \cup \mathcal{R}$ . Given a sequence of visited states, the occurrence processes of events associated with each state are independent of each other. The occurrence processes of events during a transition are also independent of each other and of the occurrence processes of events in the up-states. Given a sequence of states of  $\mathcal{E}$ , the occurrence process of events and the recovery process are assumed to be independent too. It follows that  $X^*$  is a finite homogeneous Markov chain with initial distribution  $\alpha$  and generator  $Q^*$  given by

$$\begin{aligned} Q^*(i, j) &= Q(i, j)(1 - \lambda(i, j)) && \text{if } i \neq j \text{ and } i, j \in \mathcal{U}, \\ Q^*(i, i) &= Q(i, i) - \lambda_i && \text{if } i \in \mathcal{U}, \\ Q^*(i, j) &= [\lambda_i + \sum_{k \neq i, k \in \mathcal{U}} Q(i, k)\lambda(i, k)] \alpha(i, j) && \text{if } (i, j) \in \mathcal{U} \times \mathcal{R}, \\ Q^*(i, j) &= R(i, j) && \text{if } i, j \in \mathcal{R}, \\ Q^*(i, j) &= S(i, j) && \text{if } (i, j) \in \mathcal{R} \times \mathcal{U}. \end{aligned}$$

As in discrete time, we assume that  $X^*$  is irreducible, which ensures that the alternation execution-recovery is infinite.

### 3.2 Analysis

We will analyze the process  $(N, C) = (N_t, C_t)_{t \geq 0}$  where  $N_t$  (respectively  $C_t$ ) is the cumulative number of primary or secondary events (respectively of delivered services) in the interval  $]0, t]$ . To do this, let us consider

the tri-dimensional time-continuous process  $(N, C, X^*) = (N_t, C_t, X_t^*)_{t \geq 0}$ . It follows from the independence assumptions that this is a homogeneous Markovian process over the state space  $\mathbb{N} \times \mathbb{N} \times \mathcal{E}$ . We denote by  $N_t^{(p)}$  (resp.  $N_t^{(s)}$ ) the cumulative number of primary (resp. of secondary) events observed over  $]0, t]$ . Therefore, for all  $t \geq 0$ , we have by definition  $N_t = N_t^{(p)} + N_t^{(s)}$ . Formally, the transition rates of  $(N, C, X^*)$  are for all  $k, n \geq 0$ :

$$a(i, j) = \begin{cases} \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{P}((N_{t+dt} - N_t, C_{t+dt} - C_t, X_{t+dt}^*) = (0, 0, j) \mid (N_t, C_t, X_t^*) = (k, n, i)) & \text{if } i \neq j, \\ 0, & \text{if } i = j \end{cases}$$

$$i, j \in \mathcal{U}$$

$$c(i, j) = \begin{cases} \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{P}((N_{t+dt} - N_t, C_{t+dt} - C_t, X_{t+dt}^*) = (0, 1, j) \mid (N_t, C_t, X_t^*) = (k, n, i)) & \text{if } i \neq j, \\ 0, & \text{if } i = j \end{cases}$$

$$i, j \in \mathcal{U}$$

$$d^{(s)}(i, j) = \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{P}((N_{t+dt} - N_t = N_{t+dt}^{(s)} - N_t^{(s)}, C_{t+dt} - C_t, X_{t+dt}^*) = (1, 0, j) \mid (N_t, C_t, X_t^*) = (k, n, i))$$

$$i, j \in \mathcal{U}$$

$$d^{(p)}(i, j) = \lim_{dt \rightarrow 0} \frac{1}{dt} \mathbb{P}((N_{t+dt} - N_t = N_{t+dt}^{(p)} - N_t^{(p)}, C_{t+dt} - C_t, X_{t+dt}^*) = (1, 0, j) \mid (N_t, C_t, X_t^*) = (k, n, i)).$$

$$i \in \mathcal{U}, j \in \mathcal{R}$$

The following expressions for these rates can be derived by listing the different primary, secondary or recovery involved events:

$$\begin{aligned} a(i, j) &= (1 - p_f(i))Q(i, j)(1 - \lambda(i, j))(1 - \mu(i, j)) && \text{if } i \neq j \text{ and } i, j \in \mathcal{U}, \\ a(i, i) &= 0 && \text{if } i \in \mathcal{U}, \\ c(i, j) &= p_f(i)Q(i, j)(1 - \lambda(i, j))(1 - \mu(i, j)) && \text{if } i \neq j \text{ and } i, j \in \mathcal{U}, \\ c(i, i) &= 0 && \text{if } i \in \mathcal{U}, \\ d^{(s)}(i, i) &= \mu_i && \text{if } i \in \mathcal{U}, \\ d^{(s)}(i, j) &= Q(i, j)[1 - \lambda(i, j)]\mu(i, j) && \text{if } i \neq j \text{ and } i, j \in \mathcal{U}, \\ d^{(p)}(i, j) &= [\lambda_i + \sum_{k \neq i, k \in \mathcal{U}} Q(i, k)\lambda(i, k)]\alpha(i, j) && \text{if } (i, j) \in \mathcal{U} \times \mathcal{R}. \end{aligned}$$

After checking that, for any  $i$ , we have

$$\sum_{j \in \mathcal{U}} [a(i, j) + c(i, j) + d^{(s)}(i, j)] + \sum_{j \in \mathcal{R}} d^{(p)}(i, j) = -Q(i, i) + \lambda_i + \mu_i$$

and denoting by  $\delta_i$  this value, let us define the four matrices  $A$ ,  $D^{(p)}$ ,  $D^{(s)}$  and  $C$  by

$$A = (a(i, j))_{i, j \in \mathcal{U}} - \text{diag}(\delta_i)_{i \in \mathcal{U}}, \quad D^{(p)} = (d^{(p)}(i, j))_{(i, j) \in \mathcal{U} \times \mathcal{R}},$$

$$D^{(s)} = (d^{(s)}(i, j))_{i, j \in \mathcal{U}}, \quad C = (c(i, j))_{i, j \in \mathcal{U}}.$$

With the previous notation, the generator  $Q^*$  of  $X^*$  can be rewritten as

$$Q^* = \begin{pmatrix} A + C + D^{(s)} & D^{(p)} \\ S & R \end{pmatrix}.$$

We use the well-known uniformization technique [5] to compute the distribution function of  $(N_t, D_t)$ . Let us denote by  $(N_h, C_h, X_h^*)_{h \geq 0}$  the uniformized discrete time Markov chain with respect to the uniformization rate  $u$  and by  $\text{Pois}(t)$  the Poisson distributed (with parameter  $u$ ) random variable which gives the number of transitions over  $]0, t]$  for this uniformized chain. The quantity  $\text{Pois}(k, t)$  represents the sum

$$\text{Pois}(k, t) = \sum_{h=0}^k e^{-ut} \frac{(ut)^h}{h!}.$$

We can now state a PH-representation for the distribution function of the variable  $(N_t, C_t)$  at a fixed point.

**Theorem 3.1** *For all  $t \geq 0$ , we have*

$$\begin{aligned} \mathbb{P}(N_t \leq k, C_t \leq n) &= (\alpha, (0)^n, (0)^{2k(n+1)}) e^{A_{k+1, n+1} t} \mathbf{1}^\top \quad \forall k, n \geq 0 \\ &= \text{Pois}(\min(k, n), t) + \sum_{h=\min(k, n)+1}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} \alpha x_{\mathcal{U}}^\top(k, n, h) \end{aligned} \quad (3)$$

where  $u = \sup\{ |A(i, i)|, |R(j, j)| \}$  and

$$A_{k+1, n+1} = \begin{pmatrix} B & D^{(p)} & D^{(s)} & 0 & \dots & \dots & 0 \\ 0 & R' & S' & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & & \ddots & B & D^{(p)} & D^{(s)} \\ \vdots & & & & \ddots & R' & S' \\ 0 & \dots & \dots & \dots & \dots & 0 & B \end{pmatrix}$$

is a  $(2k + 1) \times (2k + 1)$  block matrix built with the following  $(n + 1) \times (n + 1)$  block matrices:  $B$  has the same structure as in Theorem 2.1 with matrices  $A$  and  $C$ ;  $R' = \text{diag}(R)$ ,  $S' = \text{diag}(S)$ ,  $D^{(p)} = \text{diag}(D^{(p)})$  and  $D^{(s)} = \text{diag}(D^{(s)})$ . Now, if  $\widehat{A}, \widehat{R}, \widehat{C}, \widehat{D}^{(p)}, \widehat{D}^{(s)}, \widehat{S}$  result from the uniformization transformation of matrices  $A, R, C, D^{(s)}, D^{(p)}, S$ , that is  $\widehat{A} = I + A/u$ ,  $\widehat{R} = I + R/u$ ,  $\widehat{C} = C/u$ ,  $\widehat{D}^{(p)} = D^{(p)}/u$ ,  $\widehat{D}^{(s)} = D^{(s)}/u$ ,  $\widehat{S} = S/u$ , then the column vectors  $x_{\mathcal{U}}^\top(k, n, h)$  in (3) are computed with system (4) (given in the proof of the theorem) analogous to system (1).



**Proof.** For all  $k, n \geq 0$  and  $t > 0$ , we can write

$$\begin{aligned}
\mathbb{P}(N_t \leq k, C_t \leq n) &= \sum_{h=0}^{+\infty} \mathbb{P}(\text{Poiss}(t) = h) \mathbb{P}(N_t \leq k, C_t \leq n \mid \text{Poiss}(t) = h) \\
&= \sum_{h=0}^{\min(k,n)} e^{-ut} \frac{(ut)^h}{h!} \mathbb{P}(N_t \leq k, C_t \leq n \mid \text{Poiss}(t) = h) \\
&\quad + \sum_{h=\min(k,n)+1}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} \mathbb{P}(N_t \leq k, C_t \leq n \mid \text{Poiss}(t) = h) \\
&= \text{Poiss}(\min(k, n), t) \quad (\text{since } \mathbb{P}(N_t \leq k, C_t \leq n \mid \text{Poiss}(t) = h) = 1 \text{ if } h \leq \min(k, n)) \\
&\quad + \sum_{h=\min(k,n)+1}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} \mathbb{P}(N_t \leq k, C_t \leq n \mid \text{Poiss}(t) = h).
\end{aligned}$$

Consequently, relation (3) follows from the equality  $\mathbb{P}(N_t \leq k, C_t \leq n \mid \text{Poiss}(t) = h) = \mathbb{P}(N_h \leq k, C_h \leq n)$ . Our problem reduces then to the computation of the distribution of  $(N_h, C_h)$  on the uniformized Markov chain  $(N_h, C_h, X_h^*)$ . This can be done with similar arguments as in Theorem 2.1. Indeed, we have a discrete time model as in Section 2, except that we allow two types of event: the first one leads to a restart in the state within the event has occurred; concerning the second one, we necessary have a recovery action to do. We can exhibit the same backward renewal equations than in Theorem 2.1, except that matrix  $\widehat{D}$  is replaced by matrix  $\widehat{D}^{(p)}$  and that we have an additional term in the two first equations which is respectively  $\sum_{j \in \mathcal{U}} \widehat{D}^{(s)}(i, j) \mathbb{P}_j(N_{h-1} \leq k-1, C_{h-1} \leq n)$  and  $\sum_{j \in \mathcal{U}} \widehat{D}^{(s)}(i, j) \mathbb{P}_j(N_{h-1} \leq k-1, C_{h-1} = 0)$  (for any  $i \in \mathcal{U}$ ). If we take up again the notation  $x_{\mathcal{U}}^{\top}(k, n, h) = (\mathbb{P}_i(N_h \leq k, C_h \leq n))_{i \in \mathcal{U}}^{\top}$ ,  $x_{\mathcal{R}}^{\top}(k, n, h) = (\mathbb{P}_i(N_h \leq k-1, C_h \leq n))_{i \in \mathcal{R}}^{\top}$  then we obtain, for  $h \geq 1$ ,

$$\left. \begin{aligned}
x_{\mathcal{U}}^{\top}(k, n, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(k, n, h-1) + \widehat{C}x_{\mathcal{U}}^{\top}(k, n-1, h-1) + \widehat{D}^{(s)}x_{\mathcal{U}}^{\top}(k-1, n, h-1) \\
&\quad + \widehat{D}^{(p)}x_{\mathcal{R}}^{\top}(k, n, h-1) && k, n \geq 1 \\
x_{\mathcal{U}}^{\top}(0, n, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(0, n, h-1) + \widehat{C}x_{\mathcal{U}}^{\top}(0, n-1, h-1) && n \geq 1 \\
x_{\mathcal{U}}^{\top}(k, 0, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(k, 0, h-1) + \widehat{D}^{(s)}x_{\mathcal{U}}^{\top}(k-1, 0, h-1) + \widehat{D}^{(p)}x_{\mathcal{R}}^{\top}(k, 0, h-1) && k \geq 1 \\
x_{\mathcal{R}}^{\top}(k, n, h) &= \widehat{R}x_{\mathcal{R}}^{\top}(k, n, h-1) + \widehat{S}x_{\mathcal{U}}^{\top}(k-1, n, h-1) \quad \text{with } x_{\mathcal{R}}^{\top}(k, n, 0) = \mathbf{1}^{\top} && k \geq 1, n \geq 0 \\
x_{\mathcal{U}}^{\top}(0, 0, h) &= \widehat{A}x_{\mathcal{U}}^{\top}(0, 0, h-1) \quad \text{and } x_{\mathcal{U}}^{\top}(k, n, 0) = \mathbf{1}^{\top} && k, n \geq 0.
\end{aligned} \right\} \quad (4)$$

Therefore, the remainder of the proof consists in writing  $\mathbb{P}(N_h \leq k, C_h \leq n)$  as  $\alpha x_{\mathcal{U}}^{\top}(k, n, h)$ . The exponential form of  $\mathbb{P}(N_t \leq k, C_t \leq n)$  follows from the ‘‘geometric’’ representation of  $\mathbb{P}(N_h \leq k, C_h \leq n)$  given in Theorem 2.1, that is

$$(\alpha, (0)^n, (0)^{2k(n+1)}) P_{k+1, n+1}^h \mathbf{1}^{\top}$$

with  $P_{k+1,n+1} = I + A_{k+1,n+1}/u$ . Therefore, relation (3) can be rewritten as

$$\mathbb{P}(N_t \leq k, C_t \leq n) = (\alpha, (0)^n, (0)^{2k(n+1)}) \sum_{h=0}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} P_{k+1,n+1}^h \mathbf{1}^\top.$$

The last term in the right hand side is merely the vector  $e^{A_{k+1,n+1}t} \mathbf{1}^\top$ .

Theorem 3.1 leads to a simple algorithm to evaluate the probability  $\mathbb{P}(N_{t_0} \leq k_0, C_{t_0} \leq n_0)$ , for  $k_0, n_0$  and  $t_0$  fixed. It can be resumed as follows:

- (a) choose the tolerance error  $\varepsilon$ ;
- (b) compute  $H$  such that  $\sum_{h=H+1}^{+\infty} e^{-ut_0} (ut_0)^h/h! < \varepsilon$ ; this can be done very efficiently (see [2] for instance) and will lead to a total absolute error on  $\mathbb{P}(N_{t_0} \leq k_0, C_{t_0} \leq n_0)$  bounded by  $\varepsilon$ ;
- (c) compute the vectors  $x_{\mathcal{U}}^\top(k, n, h)$  for  $k = 0, \dots, k_0$  and  $n = 0, \dots, n_0$  with system (4).

Note that we can get, with this algorithm, all the probabilities  $\mathbb{P}(N_{t_0} \leq k, C_{t_0} \leq n)$ ,  $k \leq k_0$ ;  $n \leq n_0$ .

**Remark 1: relationships with MAPs, MMPPs and PH-renewal processes.** If we drop the recovery assumptions and the counter of delivered services then we have a MAP (or a versatile point process). Let us adopt Lucantoni's formalism [11] for pointing out this fact. According to the notation in [11], the matrices  $D_0$  and  $D_1$  (on  $\mathcal{U}$ ) to identify are respectively matrix  $A$  defined in this section and matrix  $D^{(s)} + D^{(p)}$  where  $\alpha(i, j)$  ( $i, j \in \mathcal{U}$ ) becomes the constant probability for jumping to state  $j$  after a failure during the sojourn in state  $i$ . The generator of the Markov process  $X^*$  is  $Q^* = D_0 + D_1$ . It is clear from the MAP assumption on the primary process that we have as particular primary point processes the Markov Modulated Poisson Process (MMPP, with  $\lambda_i = 0, \mu(i, j) = 0, i, j \in \mathcal{U}$ , see [4]) and the PH-renewal process (with  $\mu_i = 0, \lambda(i, j) = \mu(i, j) = 0, i, j \in \mathcal{U}$  and  $\alpha(i, j) = \alpha_i, i, j \in \mathcal{U}$ ). The above discussion holds also for Section 2, with the discrete time versions of the MAP, MMPP and PH-renewal processes. Consequently, the distribution function of the counting variable  $N_t$  (respectively  $N_h$ ) at  $k$ , can be viewed as the absorption probability of the finite absorbing Markov chain with  $(k+1)M \times (k+1)M$  sub-generator (respectively sub-stochastic) matrix of transition rates (respectively transition probabilities) between transient states given by

$$A_{k+1} = \begin{pmatrix} D_0 & D_1 & 0 & \cdots & & 0 \\ 0 & D_0 & D_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & D_0 & D_1 \\ 0 & \cdots & \cdots & \cdots & 0 & D_0 \end{pmatrix}$$

(respectively by  $P_{k+1}$  which has the same structure as  $A_{k+1}$  with transition probabilities replacing rates).  $\triangleleft$

**Remark 2: counting only primary or secondary failures.** If we are not interested in counting the secondary events, that is, if we want to evaluate the distribution of  $N_t^{(p)}$  only, it is sufficient to let  $D^{(s)} \equiv 0$ . System (4) becomes then equivalent to system (1) given in Theorem 2.1. On the contrary, assume that we do not care about the primary counter. After examining the renewal equations given in the proof of Theorem 3.1, we note that to compute the distribution function of the secondary events counter  $N^{(s)}$  we can use the same system as (4) except that  $x_{\mathcal{R}}^{\top}(k, n, h)$  represents now the column vector  $(\mathbb{P}_i(N_h \leq k))_{i \in \mathcal{R}}^{\top}$ . Such a counter has been used in the continuous-time counterpart of the Cheung's reliability-model (see Example 2.2) developed by Littlewood [10]. Indeed, a software system is often decomposed into a set of  $M$  components; the execution control is in one and only one of them at each instant and the evolution of this control process, conditioned on the fact that there are no failures, is Markovian (in [9] we discuss on the relevance of these Markovian assumptions). When a failure occurs, the system is assumed to be restarted instantaneously where the failure appears. In other words, the occurrence of a failure does not affect the behavior of the model. Finally our results help us to describe the transient characteristics of Littlewood's model and in particular of the failure counting process  $N_t^{(s)}$ .  $\triangleleft$

### 3.3 Derivation of expected values

As in discrete time, we can exhibit different expressions of the expectation of the random variable  $N_t$ . The first one is obtained by conditioning with respect to the number of transitions of the uniformized chain in  $]0, t]$ :

$$\begin{aligned}
\mathbb{E}[N_t] &= \sum_{h=1}^{+\infty} \mathbb{P}(\text{Poiss}(t) = h) \mathbb{E}[N_t \mid \text{Poiss}(t) = h] \\
&= \sum_{h=1}^{+\infty} \mathbb{P}(\text{Poiss}(t) = h) \mathbb{E}[N_h] \\
&= \sum_{h=1}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} \left[ h - \sum_{k=0}^{h-1} \alpha x_{\mathcal{U}}^{\top}(k, h, h) \right] \quad (\text{with (2)}) \\
&= ut - \sum_{h=1}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} \left[ \sum_{k=0}^{h-1} \alpha x_{\mathcal{U}}^{\top}(k, h, h) \right]. \tag{5}
\end{aligned}$$

The uniformization of the continuous time Markov chain  $(X_t^*)_{t \geq 0}$  with respect to the rate  $u$  gives the discrete time Markov chain  $(X_h^*)_{h \geq 0}$ . The transition probability matrix of  $(X_h^*)_{h \geq 0}$  is given by  $\widehat{Q}^* = I + Q^*/u$ .

Last, we can, in the same way, derive expressions for  $\mathbb{E}[N_t^{(p)}]$  and  $\mathbb{E}[N_t^{(s)}]$ , simply from the relations  $\mathbb{E}[N_t^{(p)}] = \mathbb{E}[N_t] \Big|_{D^{(s)}=0}$  and  $\mathbb{E}[N_t^{(s)}] = \mathbb{E}[N_t] - \mathbb{E}[N_t^{(p)}]$ .

### 3.4 Example

Kao proposes a procedure for computing the renewal function of a PH-renewal process in [7]. As noted in Remark 1, the presented results can be used to compute the distribution and the renewal functions for such a process. We just give the numerical values obtained for a Erlang interrenewal distribution with order 2 and mean 2.0, that is, for a PH-distribution with representation (see [13])

$$\alpha = (1, 0), \quad T = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

In Figure 4, we first plot some values of the distribution function of  $N_t$ , that is, the probabilities  $\mathbb{P}(N_t \leq k)$  for  $k = 0, 1, 2$ . In [7], the author computes the renewal function using an error bound  $\varepsilon$  set to  $10^{-6}$ . We set the error bound to the same value in the example. The reader can verify in the table that the values given by our algorithm have an accuracy of six decimal digits (the ‘‘exact’’ values of the renewal function up to 8 decimal digits are obtained using Cox formulas). Note that 75% of the values given by our algorithm are closer to the exact values than those obtained in [7]. For completeness, note that the renewal density is computed in [1] for such an Erlang interrenewal distribution.

## 4 Asymptotic analysis

Let us establish another expression for the discrete time expectation  $\mathbb{E}[N_h]$  of the cumulative number of events of Section 2. It appears to be useful in studying the asymptotic behavior of  $\mathbb{E}[N_h]$ . Summing the relations satisfied by the family of vectors  $\{x_{\mathcal{U}}^{\top}(k, h, h), k = 0, \dots, h-1\}$  and  $\{x_{\mathcal{R}}^{\top}(k, h, h), k = 1, \dots, h\}$ , given in (1), we obtain

$$\begin{aligned} u_{\mathcal{U}}^{\top}(h) &\stackrel{\text{def}}{=} \sum_{k=0}^{h-1} x_{\mathcal{U}}^{\top}(k, h, h) = (\widehat{A} + \widehat{C}) u_{\mathcal{U}}^{\top}(h-1) + \widehat{D} u_{\mathcal{R}}^{\top}(h-1) + (\widehat{A} + \widehat{C}) \mathbf{1}^{\top} \quad \forall h \geq 2; \\ u_{\mathcal{R}}^{\top}(h) &\stackrel{\text{def}}{=} \sum_{k=1}^h x_{\mathcal{R}}^{\top}(k, h, h) = \widehat{R} u_{\mathcal{R}}^{\top}(h-1) + \widehat{S} u_{\mathcal{U}}^{\top}(h-1) + \mathbf{1}^{\top} \quad \forall h \geq 2. \end{aligned}$$

If  $U^{\top}(h)$  denotes the column vector  $\begin{pmatrix} u_{\mathcal{U}}^{\top}(h) \\ u_{\mathcal{R}}^{\top}(h) \end{pmatrix}$  and  $K_0 = \begin{pmatrix} (\widehat{A} + \widehat{C}) \mathbf{1}^{\top} \\ \mathbf{1}^{\top} \end{pmatrix}$  then it follows immediately that, for  $h \geq 1$ ,

$$U^{\top}(h) = \left( \sum_{k=0}^{h-1} P^{*k} \right) K_0.$$

We deduce from  $K_0 = 1^\top - \begin{pmatrix} (\widehat{D}^{(s)} + \widehat{D}^{(p)})1^\top \\ 0 \end{pmatrix} = 1^\top - L_0$ , that, for  $h \geq 1$ :

$$\mathbb{E}[N_h] = h - (\alpha, 0)U^\top(h) = (\alpha, 0) \left( \sum_{k=0}^{h-1} P^{*k} \right) L_0. \quad (6)$$

If we denote the stationary distribution of the irreducible matrix  $P^*$  by  $\pi$ , we can derive after some algebra that

$$\sum_{k=0}^{h-1} P^{*k} = h 1^\top \pi + (I - P^{*h})(I - P^* + 1^\top \pi)^{-1}. \quad (7)$$

Matrix  $(I - P^* + 1^\top \pi)^{-1}$  is the fundamental matrix associated with an irreducible Markov chain having transition probability matrix  $P^*$  (see [6, Chap. 4] for additional details). This last relation allows us to write  $\mathbb{E}[N_h]$  as

$$\mathbb{E}[N_h] = h\pi L_0 + (\alpha, 0) \left[ I - P^{*h} \right] (I - P^* + 1^\top \pi)^{-1} L_0 \quad (8)$$

and it yields another way to compute the function  $\mathbb{E}[N_h]$ . It is sufficient to evaluate the stationary distribution  $\pi$ , the inverse matrix  $(I - P^* + 1^\top \pi)^{-1}$  and the constant  $L_0$  from the data. From this point of view, the computation of  $\mathbb{E}[N_h]$  reduces to the evaluation of the state probabilities  $(\alpha, 0)P^{*h}$  of the Markov chain  $X^*$ . Note that relation (8) is the discrete time version of formula (10) given in [13] for the versatile point process. Finally, if  $P^*$  is an aperiodic matrix then formula (8) gives the linear asymptote of  $\mathbb{E}[N_h]$  as  $h$  tends to infinity:

$$\mathbb{E}[N_h] = h\pi L_0 + [(\alpha, 0) - \pi](I - P^* + 1^\top \pi)^{-1} L_0 + o(1).$$

In particular, we have

$$\lim_{h \rightarrow \infty} \frac{\mathbb{E}[N_h]}{h} = \sum_{i \in \mathcal{U}} \pi(i) \left[ \sum_{j \in \mathcal{R}} \widehat{D}(i, j) \right].$$

However, from (6), this last limit is clearly independent of the period of matrix  $P^*$ . The previous discussion can also be done in the same way for the random variable  $C_h$ . In particular, we have

$$\forall h \geq 1, \quad \mathbb{E}[C_h] = \alpha \left( \sum_{k=0}^{h-1} P^{*k} \right) \begin{pmatrix} \widehat{C}1^\top \\ 0 \end{pmatrix}.$$

Relation (6) holds for the counting variable  $N_h$  associated with the uniformized chain  $(N_h, C_h, X_h^*)_{h \geq 0}$  of Section 3 (matrix  $P^*$  is replaced by matrix  $\widehat{Q}^*$ ). We can thus derive the following expression of the continuous time expectation  $\mathbb{E}[N_t]$  of Section 3:

$$\mathbb{E}[N_t] = \sum_{h=1}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} (\alpha, 0) \left( \sum_{k=0}^{h-1} \widehat{Q}^{*k} \right) \widehat{L}_0. \quad (9)$$

Relations (5) or (9) can be exploited to evaluate numerically  $\mathbb{E}[N_t]$ . For a fixed tolerance error  $\varepsilon$ , it suffices to choose  $H$  such that  $ut \sum_{h=H}^{+\infty} e^{-ut} \frac{(ut)^h}{h!} < \varepsilon$ , in order to obtain a total error on  $\mathbb{E}[N_t]$  bounded by  $\varepsilon$ .

A third representation of  $\mathbb{E}[N_t]$  can be deduced from (7) and (9) ( $\widehat{Q}^*$  and  $Q^*$  have the same stationary distribution). It is the continuous time counterpart of relation (8):

$$\mathbb{E}[N_t] = (\pi L_0)t + (\alpha, 0)(I - e^{Q^*t})(1^\top \pi - Q^*)^{-1}L_0, \quad (10)$$

with  $L_0 = u\widehat{L}_0$ . Matrix  $(1^\top \pi - Q^*)^{-1}$  is the fundamental matrix associated with the irreducible continuous time Markov chain  $X_t^*$  [6]. This last formula can be viewed as the adaptation to our context of an analogous result published in [13] for the so-called versatile point process. It can be used for the computation of  $\mathbb{E}[N_t]$  as relation (8) for the discrete time expectation  $\mathbb{E}[N_h]$ . Finally, when  $t$  tends to infinity, we have

$$\begin{aligned} \mathbb{E}[N_t] &= (\pi L_0)t + [(\alpha, 0) - \pi](1^\top \pi - Q^*)^{-1}L_0 + o(1), \\ \text{and } \lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_t]}{t} &= \pi L_0 = \sum_{i \in \mathcal{U}} \left[ \sum_{j \in \mathcal{R}} D^{(p)}(i, j) + \sum_{j \in \mathcal{U}} D^{(s)}(i, j) \right]. \end{aligned}$$

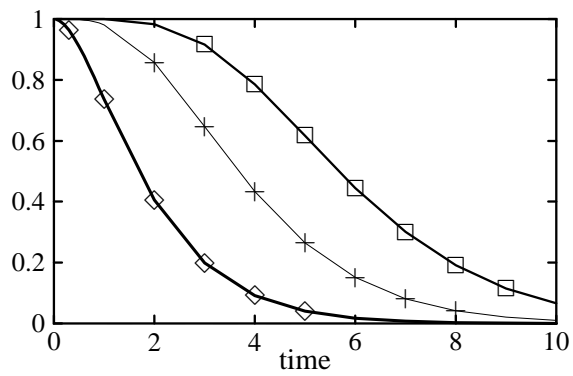
## 5 Conclusion

We investigate in this paper a general dependability model (in discrete or continuous time) based on a structural view of the given system. Mainly, we give the distribution function of the joint number of observed events (including failures) and delivered services on a fixed interval. We also discuss the computational issues associated with the derived formulas. The single failure process is closely related to the MAP or versatile point process used in queuing theory. A natural extension to our model is to allow for instance the occurrence of grouped failures. In that case, the failure point process has to be related to the BMAP and the distribution function (at a fixed point) of the corresponding counting random variable can also be viewed as an absorption probability in a finite absorbing Markov chain.

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$t$	Exact	Algo. in [7]	Proposed
.05	.00120935	.00120911	.00120935
.10	.00468269	.00468265	.00468253
.15	.01020456	.01020427	.01020451
.20	.01758001	.01757994	.01757979
.25	.02663266	.02663240	.02663264
.30	.03720291	.03720214	.03720281
.35	.04914633	.04914628	.04914604
.40	.06233224	.06233212	.06233220
.45	.07664241	.07664215	.07664230
.50	.09196986	.09196980	.09196961
.55	.10821777	.10821765	.10821774
.60	.12529855	.12529832	.12529849
.65	.14313295	.14313252	.14313282
.70	.16164924	.16164850	.16164900
.75	.18078254	.18078248	.18078211
.80	.20047413	.20047403	.20047406
.85	.22067088	.22067072	.22067076
.90	.24132472	.24132446	.24132451
.95	.26239215	.26239175	.26239181
1.00	.28383382	.28383371	.28383377

Figure 4: Distribution and renewal functions for Erlang interrenewal distribution with order 2 and mean 2.0.