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# Error estimates in periodic homogenization with a non-homogeneous Dirichlet condition.

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## Abstract

In this paper we investigate the homogenization problem with a non-homogeneous Dirichlet condition. Our aim is to give error estimates with boundary data in  $H^{1/2}(\partial\Omega)$ . The tools used are those of the unfolding method in periodic homogenization.

## 1 Introduction

We consider the following homogenization problem:

$$\phi^\varepsilon \in H^1(\Omega), \quad -\operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) = f \quad \text{in } \Omega, \quad \phi^\varepsilon = g \quad \text{on } \partial\Omega \quad (1.1)$$

where  $A_\varepsilon$  is a periodic matrix satisfying the usual condition of uniform ellipticity and where  $f \in L^2(\Omega)$  and  $g \in H^{1/2}(\partial\Omega)$ <sup>1</sup>. We know (see e.g. [4], [10], [13]) that the function  $\phi^\varepsilon$  weakly converges in  $H^1(\Omega)$  towards the solution  $\Phi$  of the homogenized problem

$$\Phi \in H^1(\Omega), \quad -\operatorname{div}(\mathcal{A} \nabla \Phi) = f \quad \text{in } \Omega, \quad \Phi = g \quad \text{on } \partial\Omega \quad (1.2)$$

where  $\mathcal{A}$  is the homogenized matrix (see (4.4) and (4.5)). Using the results in [10] we can give an approximation of  $\phi^\varepsilon$  belonging to  $H^1(\Omega)$  and we easily obtain

$$\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in } H^1(\Omega)$$

where  $\mathcal{Q}_\varepsilon$  is the *scale-splitting operator* (see [10] or Subsection 2.4) and where the  $\chi_i$  are the correctors (see (4.2)).

One of the aim of this paper is to give error estimates for this homogenization problem. Obviously, if we have  $g \in H^{3/2}(\partial\Omega)$  and the appropriate assumptions on

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<sup>1</sup>The homogenization problem with  $L^p$  boundary data is investigated in [3].

the boundary of the domain then we can apply the results in [4], [13], [14], [15], [16] and [22] to deduce error estimates. All of them require that the function  $\Phi$  belongs at least to  $H^2(\Omega)$ . Here, the solution  $\Phi$  of the homogenized problem (1.2) is only in  $H^1(\Omega) \cap H_{loc}^2(\Omega)$ . In this paper we have to deal with this lack of regularity; this is the main difficulty.

The tools of the unfolding method in periodic homogenization to obtain error estimates (see [14], [15] and [16]) are the projection theorems. This is why we prove two new projection theorems; the Theorems 3.1 and 3.2. Here, both theorems concern the functions  $\phi \in H_0^1(\Omega)$  satisfying  $\nabla\phi/\rho \in L^2(\Omega; \mathbb{R}^n)$  where  $\rho(x)$  is the distance between  $x$  and the boundary of  $\Omega$ . In the first one we give the distance between  $\mathcal{T}_\varepsilon(\phi)$  (see [10] or Subsection 2.4.1 for the definition of the unfolding operator  $\mathcal{T}_\varepsilon$ ) and the space  $L^2(\Omega; H_{per}^1(Y))$  in terms of the  $L^2$  norms of  $\phi/\rho$  and  $\nabla\phi/\rho$  and obviously  $\varepsilon$ . In the second one we prove an upper bound for the distance between  $\mathcal{T}_\varepsilon(\nabla\phi)$  and the space  $\nabla H^1(\Omega) \oplus \nabla_y L^2(\Omega; H_{per}^1(Y))$ ; again in terms of the  $L^2$  norms of  $\phi/\rho$  and  $\nabla\phi/\rho$  and  $\varepsilon$  (see Section 3). This last theorem is partially a consequence of the first one. In this paper we derive the new error estimates from the second projection theorem and those obtained in [16].

Different results are known about the global  $H^1$  error estimate regarding the classical homogenization problem (1.1) (see e.g. [4], [13]). Those with the minimal assumptions are given in [15]; if the solution of the homogenized problem (1.2) belongs to  $H^2(\Omega)$  -see Proposition 4.3 in [15]- (respectively  $H^{3/2}(\Omega)$ ; see Theorem 3.3 in [16]) then the  $H^1$  global error is of order  $\varepsilon^{1/2}$  (resp.  $\varepsilon^{1/4}$ ) while if this solution belongs to  $H_{loc}^2(\Omega) \cap W^{1,p}(\Omega)$  ( $p > 2$ ) the obtained  $H^1$  global error is smaller and depends on  $p$  (see Proposition 4.4 in [15])<sup>2</sup>. Here, with a non-homogeneous Dirichlet condition belonging only to  $H^{1/2}(\partial\Omega)$  we do not obtain a global  $H^1$  error estimate. The  $L^2$  global error estimate only requires a boundary of  $\Omega$  sufficiently smooth (of class  $C^{1,1}$ ) or a convex open set. Obviously if it is possible to make use of a global  $H^1$  error estimate, the  $L^2$  global error will be better (the reader will be able to compare the Theorem 3.2 in [16] with the Theorem 6.3). The  $H^1$  local error estimate is always linked to the  $L^2$  global error and never needs more assumption (see Theorem 3.2 in [16] or the proof of Theorem 6.1).

The paper is organized as follows. In Section 2 we introduce a few general notations, then we give some reminds<sup>3</sup> on lemmas, definitions and results about the unfolding method in periodic homogenization (see [10]), then we prove some new results involving the main operators of this method. Section 3 is devoted to the new projection theorems. In Section 4, we recall the main results on the classical homogenization problem. In Section 5 we introduce an operator which allows to lift the distributions belonging to  $H^{-1/2}(\partial\Omega)$  in functions belonging to  $L^2(\Omega)$ ; this lifting operator will play an important role in the case of strongly oscillating boundary data. In Section 6 we derive the error estimates results (Theorems 6.1 and 6.3) with a non-homogenous Dirichlet condition. We

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<sup>2</sup>These propositions or theorem are proved with a Dirichlet condition, with a non-homogenous Dirichlet condition belonging to  $H^{3/2}(\partial\Omega)$  the results are obviously the same.

<sup>3</sup>We want to simplify the reading to a non-familiar reader with the unfolding method

end the paper by investigating a case where the boundary data are strongly oscillating (see Theorem 7.1 in Section 7). A forthcoming paper will be devoted to homogenization problems with other strongly oscillating boundary data.

As general references on the homogenization theory we refer to [1], [4] and [13]. The reader is referred to [10], [12] and [13] for an introduction of the unfolding method in periodic homogenization. The following papers [5], [6], [7], [8], [11], [19], [24] give various applications of the unfolding method in periodic homogenization. As far as the error estimates are concerned, we refer to [2], [4], [14], [15], [16], [20], [22] and [23].

**Keywords:** periodic homogenization, error estimate, non-homogeneous Dirichlet condition, periodic unfolding method.

**Mathematics Subject Classification (2000):** 35B27, 65M15, 74Q15.

## 2 Preliminaries

### 2.1 Notations

- The space  $\mathbb{R}^k$  ( $k \geq 1$ ) is endowed with the standard basis  $(\mathbf{e}_1, \dots, \mathbf{e}_k)$ ; the euclidian norm is denoted  $|\cdot|$ .
- We denote by  $\Omega$  a bounded domain in  $\mathbb{R}^n$  with a Lipschitz boundary.<sup>4</sup> Let  $\rho(x)$  be the distance between  $x \in \mathbb{R}^n$  and the boundary of  $\Omega$ , we set

$$\tilde{\Omega}_\gamma = \{x \in \Omega \mid \rho(x) < \gamma\} \quad \tilde{\tilde{\Omega}}_\gamma = \{x \in \mathbb{R}^n \mid \rho(x) < \gamma\} \quad \gamma \in \mathbb{R}^{*+}.$$

- There exist constants  $a$ ,  $A$  and  $\gamma_0$  strictly positive and  $M \geq 1$ , a finite number  $N$  of local euclidian coordinate systems  $(O_r; \mathbf{e}_{1r}, \dots, \mathbf{e}_{nr})$  and mappings  $f_r : [-a, a]^{n-1} \rightarrow \mathbb{R}$ , Lipschitz continuous with ratio  $M$ ,  $1 \leq r \leq N$ , such that (see e.g. [17] or [18])

$$\partial\Omega = \bigcup_{r=1}^N \left\{ x = x'_r + x_{nr} \mathbf{e}_{nr} \in \mathbb{R}^n \mid x'_r \in \Delta_a \text{ and } x_{nr} = f_r(x'_r) \right\},$$

$$\text{where } x'_r = x_{1r} \mathbf{e}_{1r} + \dots + x_{n-1r} \mathbf{e}_{n-1r}, \quad \Delta_a = \left\{ x'_r \mid x_{ir} \in ]-a, a[, i \in \{1, \dots, n-1\} \right\}$$

$$\tilde{\tilde{\Omega}}_{\gamma_0} \subset \bigcup_{r=1}^N \Omega_r \subset \Omega, \quad \Omega_r = \left\{ x \in \mathbb{R}^n \mid x'_r \in \Delta_a \text{ and } f_r(x'_r) < x_{nr} < f_r(x'_r) + A \right\}$$

$$\tilde{\tilde{\tilde{\Omega}}}_{\gamma_0} \subset \bigcup_{r=1}^N \left\{ x \in \mathbb{R}^n \mid x'_r \in \Delta_a \text{ and } f_r(x'_r) - A < x_{nr} < f_r(x'_r) + A \right\}$$

$$\forall r \in \{1, \dots, N\}, \quad \forall x \in \Omega_r \quad \text{we have} \quad \frac{1}{2M}(x_{nr} - f_r(x'_r)) \leq \rho(x) \leq x_{nr} - f_r(x'_r). \quad (2.1)$$

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<sup>4</sup>In Section 5 and those which follow, we will assume that  $\Omega$  is a bounded domain of class  $\mathcal{C}^{1,1}$  or an open bounded convex set.

- We set

$$Y = ]0, 1[^n, \quad \Xi_\varepsilon = \{\xi \in \mathbb{Z}^n \mid \varepsilon(\xi + Y) \subset \Omega\},$$

$$\widehat{\Omega}_\varepsilon = \text{interior}\left(\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\right), \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$

where  $\varepsilon$  is a strictly positive real.

- We define

$$\star H_\rho^1(\Omega) = \left\{ \phi \in L^2(\Omega) \mid \rho \nabla \phi \in L^2(\Omega; \mathbb{R}^n) \right\},$$

$$\star L_{1/\rho}^2(\Omega) = \left\{ \phi \in L^2(\Omega) \mid \phi/\rho \in L^2(\Omega) \right\},$$

$$\star H_{1/\rho}^1(\Omega) = \left\{ \phi \in H_0^1(\Omega) \mid \nabla \phi/\rho \in L^2(\Omega; \mathbb{R}^n) \right\}.$$

We endow  $H_\rho^1(\Omega)$  (resp.  $H_{1/\rho}^1(\Omega)$ ) with the norm

$$\forall \phi \in H_\rho^1(\Omega), \quad \|\phi\|_\rho = \|\phi\|_{L^2(\Omega)} + \|\rho \nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}$$

$$(\text{ resp. } \forall \phi \in H_{1/\rho}^1(\Omega), \quad \|\phi\|_{1/\rho} = \|\nabla \phi/\rho\|_{L^2(\Omega; \mathbb{R}^n)}).$$

Note that if  $\phi$  belongs to  $H_\rho^1(\Omega)$  then the function  $\psi = \rho\phi$  is in  $H_0^1(\Omega)$  and vice versa if a function  $\psi$  belongs to  $H_0^1(\Omega)$  then  $\phi = \psi/\rho$  is in  $H_\rho^1(\Omega)$  since we have (see [9] or [21])

$$\forall \psi \in H_0^1(\Omega), \quad \|\psi/\rho\|_{L^2(\Omega)} \leq C \|\nabla \psi\|_{L^2(\Omega; \mathbb{R}^n)}. \quad (2.2)$$

Below we recall a classical extension lemma which is proved for example in [15] or which can be proved using the local charts (2.1).

**Lemma 2.1.** *Let  $\Omega$  be a bounded domain with a Lipschitz boundary, there exist  $c_0 \geq 1$  (which depends only on the boundary of  $\Omega$ ) and a linear and continuous extension operator  $\mathcal{P}$  from  $L^2(\Omega)$  into  $L^2(\mathbb{R}^n)$  which also maps  $H^1(\Omega)$  into  $H^1(\mathbb{R}^n)$  such that*

$$\forall \phi \in L^2(\Omega), \quad \mathcal{P}(\phi)|_\Omega = \phi, \quad \|\mathcal{P}(\phi)\|_{L^2(\mathbb{R}^n)} \leq C \|\phi\|_{L^2(\Omega)},$$

$$\|\mathcal{P}(\phi)\|_{L^2(\widetilde{\Omega}_{c_0\gamma})} \leq C \|\phi\|_{L^2(\widetilde{\Omega}_{c_0\gamma})}. \quad (2.3)$$

Moreover we have

$$\forall \phi \in H^1(\Omega), \quad \|\nabla \mathcal{P}(\phi)\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} \leq C \|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}.$$

From now on, if need be, a function  $\phi$  belonging to  $L^2(\Omega)$  (resp.  $H^1(\Omega)$ ) will be extended to a function belonging to  $L^2(\mathbb{R}^n)$  (resp.  $H^1(\mathbb{R}^n)$ ) using the above lemma. The extension will be still denoted  $\phi$ .

## 2.2 A characterization of the functions belonging to $H_{1/\rho}^1(\Omega)$

The two first projection theorems (see [15]) regarded the functions belonging to  $H_0^1(\Omega)$  while those in [16] regarded the functions in  $H^1(\Omega)$ . In this paper we prove two

new projection theorems which involve the functions in  $H_{1/\rho}^1(\Omega)$ ; this is why we first give a simple characterization of these functions in the Lemma 2.2 below.

Observe first that if a function  $\phi$  satisfies  $\phi/\rho \in H_0^1(\Omega)$  then  $\phi$  belongs to  $H_{1/\rho}^1(\Omega)$ . The reverse is true.

**Lemma 2.2.** *Let  $\Omega$  be a bounded domain with a Lipschitz boundary, we have*

$$\phi \in H_{1/\rho}^1(\Omega) \iff \phi/\rho \in H_0^1(\Omega).$$

Furthermore there exists a constant which depends only on  $\partial\Omega$  such that

$$\forall \phi \in H_{1/\rho}^1(\Omega) \quad \|\phi/\rho^2\|_{L^2(\Omega)} + \|\phi/\rho\|_{H^1(\Omega)} \leq C\|\phi\|_{1/\rho}. \quad (2.4)$$

*Proof. Step 1.* Let  $\phi$  be in  $H^1(]-a, a[^{n-1} \times ]0, A[)$  ( $a, A > 0$ ) satisfying  $\frac{1}{x_n} \nabla \phi(x) \in L^2(]-a, a[^{n-1} \times ]0, A[)$  and  $\phi(x) = 0$  for a.e.  $x$  in  $]-a, a[^{n-1} \times \{0\} \cup ]-a, a[^{n-1} \times \{A\}$ .

We have

$$\int_{]-a, a[^{n-1} \times ]0, A[} \frac{|\phi(x)|^2}{x_n^4} dx \leq \frac{1}{2} \int_{]-a, a[^{n-1} \times ]0, A[} \frac{|\nabla \phi(x)|^2}{x_n^2} dx. \quad (2.5)$$

To prove (2.5), we choose  $\eta > 0$  and we integrate by parts  $\int_{]-a, a[^{n-1} \times ]0, A[} \frac{|\phi(x)|^2}{(\eta + x_n)^4} dx$ , then thanks to the identity relation  $2bc \leq b^2 + c^2$  we obtain

$$\begin{aligned} \int_{]-a, a[^{n-1} \times ]0, A[} \frac{|\phi(x)|^2}{(\eta + x_n)^4} dx &\leq \frac{1}{2} \int_{]-a, a[^{n-1} \times ]0, A[} \frac{1}{(\eta + x_n)^2} \left| \frac{\partial \phi}{\partial x_n}(x) \right|^2 dx \\ &\leq \frac{1}{2} \int_{]-a, a[^{n-1} \times ]0, A[} \frac{|\nabla \phi(x)|^2}{x_n^2} dx. \end{aligned}$$

Passing to the limit ( $\eta \rightarrow 0$ ) it leads to (2.5).

*Step 2.* Let  $h$  be in  $W^{1,\infty}(\Omega)$  such that

$$\begin{aligned} h(x) &\in [0, 1], \\ \forall x \in \Omega, \quad h(x) &= 1 \quad \text{if } \rho(x) \geq \gamma_0, \\ h(x) &= 0 \quad \text{if } \rho(x) \leq \gamma_0/2. \end{aligned}$$

Let  $\phi$  be in  $H_{1/\rho}^1(\Omega)$ . The function  $\phi h/\rho^4$  belongs to  $H_0^1(\Omega)$ , therefore as a consequence of the Poincaré's inequality we obtain

$$\begin{aligned} \int_{\Omega} \frac{|\phi(x)h(x)|^2}{\rho(x)^4} dx &\leq C \int_{\Omega} \left| \nabla \left( \frac{\phi(x)h(x)}{\rho(x)^4} \right) \right|^2 dx \leq C \int_{\Omega} (|\nabla \phi(x)|^2 + |\phi(x)|^2) dx \\ &\leq C \int_{\Omega} |\nabla \phi(x)|^2 dx \leq C \int_{\Omega} \frac{|\nabla \phi(x)|^2}{\rho(x)^2} dx. \end{aligned} \quad (2.6)$$

Then using the local chart of  $\Omega_r$  given by (2.1), the inequality (2.5) and thanks to a simple change of variables we get

$$\int_{\Omega_r} \frac{|\phi(x)(1-h(x))|^2}{\rho(x)^4} dx \leq C \int_{\Omega_r} \frac{|\nabla(\phi(x)(1-h(x)))|^2}{\rho(x)^2} dx \leq C \int_{\Omega_r} \frac{|\nabla\phi(x)|^2 + |\phi(x)|^2}{\rho(x)^2} dx.$$

Since  $\phi \in H_0^1(\Omega)$  the function  $\phi/\rho$  belongs to  $L^2(\Omega)$  and we have (2.2). Hence, adding these inequalities ( $r = 1, \dots, N$ ) we obtain

$$\int_{\Omega} \frac{|\phi(x)(1-h(x))|^2}{\rho(x)^4} dx \leq C \int_{\Omega} \frac{|\nabla\phi(x)|^2}{\rho(x)^2} dx. \quad (2.7)$$

Finally  $\phi/\rho^2 \in L^2(\Omega)$  and (2.6)-(2.7) lead to  $\|\phi/\rho^2\|_{L^2(\Omega)} \leq C\|\phi\|_{1/\rho}$  and then (2.4).  $\square$

### 2.3 Two lemmas

In the Lemma 2.3 we give sharp estimates of a function on the boundary and in a neighborhood of the boundary of  $\Omega$ . The second estimate in (2.8) is used to obtain the  $L^2$  global error.

**Lemma 2.3.** *Let  $\Omega$  be a bounded domain with a Lipschitz boundary, there exists  $\gamma_0 > 0$  (see Subsection 2.2) such that for any  $\gamma \in ]0, \gamma_0]$  and for any  $\phi \in H^1(\Omega)$  we have*

$$\begin{aligned} \|\phi\|_{L^2(\partial\Omega)} &\leq \frac{C}{\gamma^{1/2}} (\|\phi\|_{L^2(\tilde{\Omega}_\gamma)} + \gamma\|\nabla\phi\|_{L^2(\tilde{\Omega}_\gamma; \mathbb{R}^n)}), \\ \|\phi\|_{L^2(\tilde{\Omega}_\gamma)} &\leq C(\gamma^{1/2}\|\phi\|_{L^2(\partial\Omega)} + \gamma\|\nabla\phi\|_{L^2(\tilde{\Omega}_\gamma; \mathbb{R}^n)}). \end{aligned} \quad (2.8)$$

The constants do not depend on  $\gamma$ .

*Proof.* Let  $\psi$  be in  $H^1(]-a, a[^{n-1} \times ]0, A[)$ . For  $\eta \in ]0, A[$  we have

$$\begin{aligned} \|\psi\|_{L^2(]-a, a[^{n-1} \times \{0\})}^2 &\leq \frac{C}{\eta} \|\psi\|_{L^2(]-a, a[^{n-1} \times ]0, \eta])}^2 + C\eta \|\nabla\psi\|_{L^2(]-a, a[^{n-1} \times ]0, \eta[; \mathbb{R}^n)}^2, \\ \|\psi\|_{L^2(]-a, a[^{n-1} \times ]0, \eta])}^2 &\leq C\eta \|\psi\|_{L^2(]-a, a[^{n-1} \times \{0\})}^2 + C\eta^2 \|\nabla\psi\|_{L^2(]-a, a[^{n-1} \times ]0, \eta[; \mathbb{R}^n)}^2. \end{aligned}$$

The constants do not depend on  $\eta$ . Now, let  $\phi$  be in  $H^1(\Omega)$ . We use the above estimates, the local charts of  $\tilde{\Omega}_{\gamma_0}$  given by (2.1) and a simple change of variables to obtain (2.8).  $\square$

In this second lemma we show that a function in  $H_0^1(\Omega)$  can be approached by functions vanishing close to the boundary of  $\Omega$ . Among other things this lemma is used to give an approximation of  $\phi$  via the scale-splitting operator  $\mathcal{Q}_\varepsilon$  (see Lemma 2.6) and it is also used in the main projection theorem (Theorem 3.2).

**Lemma 2.4.** *Let  $\phi$  be in  $H_0^1(\Omega)$ , there exists  $\phi_\varepsilon \in H^1(\mathbb{R}^n)$  satisfying*

$$\begin{aligned} \phi_\varepsilon(x) &= 0 && \text{for a.e. } x \notin \tilde{\Omega}_{6\sqrt{n\varepsilon}}, \\ \|\phi - \phi_\varepsilon\|_{L^2(\Omega)} &\leq C\varepsilon \|\nabla\phi\|_{L^2(\Omega; \mathbb{R}^n)}, && \|\phi_\varepsilon\|_{H^1(\Omega)} \leq C\|\phi\|_{H^1(\Omega)}. \end{aligned} \quad (2.9)$$

Moreover, if  $\phi \in H_{1/\rho}^1(\Omega)$  then we have

$$\|(\phi - \phi_\varepsilon)/\rho\|_{L^2(\Omega)} \leq C\varepsilon \|\nabla\phi\|_{1/\rho}, \quad \|\phi_\varepsilon\|_{1/\rho} \leq C\|\phi\|_{1/\rho}. \quad (2.10)$$

The constant  $C$  is independent of  $\varepsilon$ .

*Proof.* Let  $\phi$  be in  $H_0^1(\Omega)$ . We define  $\phi_\varepsilon$  by

$$\phi_\varepsilon(x) = \begin{cases} \frac{(\rho(x) - 6\sqrt{n\varepsilon})^+}{\rho(x)}\phi(x) & \text{for a. e. } x \in \Omega, \\ 0 & \text{for a. e. } x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

where  $\delta^+ = \max\{0, \delta\}$ . The above function  $\phi_\varepsilon$  belongs to  $H^1(\mathbb{R}^n)$  and satisfies  $\phi_\varepsilon = 0$  outside  $\tilde{\Omega}_{6\sqrt{n\varepsilon}}$ . Then due to the fact that  $\phi/\rho$  belongs to  $L^2(\Omega)$  and verifies  $\|\phi/\rho\|_{L^2(\Omega)} \leq C\|\nabla\phi\|_{L^2(\Omega; \mathbb{R}^n)}$  we obtain the estimates in (2.9). If  $\phi \in H_{1/\rho}^1(\Omega)$  we use the estimate (2.4) to obtain (2.10).  $\square$

## 2.4 Reminds and complements on the unfolding operators

In the sequel, we will make use of some definitions and results from [10] concerning the periodic unfolding method. Below we remind them briefly.

### 2.4.1 Some reminds

For almost every  $x \in \mathbb{R}^n$ , there exists an unique element in  $\mathbb{Z}^n$  denoted  $[x]$  such that

$$x = [x] + \{x\}, \quad \{x\} \in Y.$$

- *The unfolding operator  $\mathcal{T}_\varepsilon$ .*

For any  $\phi \in L^1(\Omega)$ , the function  $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega \times Y)$  is given by

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x, y) \in \hat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \quad (2.11)$$

Since  $\Lambda_\varepsilon \subset \tilde{\Omega}_{\sqrt{n\varepsilon}}$ , using Proposition 2.5 in [10] we get

$$\left| \int_{\Omega} \phi(x) dx - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi)(x, y) dx dy \right| \leq \int_{\Lambda_\varepsilon} |\phi(x)| dx \leq \|\phi\|_{L^1(\tilde{\Omega}_{\sqrt{n\varepsilon}})} \quad (2.12)$$



For  $\phi \in L^2(\Omega)$  we have

$$\|\mathcal{T}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Omega)}. \quad (2.13)$$

We also have (see Proposition 2.5 in [10]) for  $\phi \in H^1(\Omega)$  (resp.  $\psi \in H_0^1(\Omega)$ )

$$\begin{aligned} \|\mathcal{T}_\varepsilon(\phi) - \phi\|_{L^2(\widehat{\Omega}_\varepsilon \times Y)} &\leq C\varepsilon \|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)} \\ (\text{resp. } \|\mathcal{T}_\varepsilon(\psi) - \psi\|_{L^2(\Omega \times Y)} &\leq C\varepsilon \|\nabla \psi\|_{L^2(\Omega; \mathbb{R}^n)}). \end{aligned} \quad (2.14)$$

• *The local average operator  $\mathcal{M}_\varepsilon$*

For  $\phi \in L^1(\mathbb{R}^n)$ , the function  $\mathcal{M}_\varepsilon(\phi) \in L^\infty(\mathbb{R}^n)$  is defined by

$$\mathcal{M}_\varepsilon(\phi)(x) = \int_Y \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) dy \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (2.15)$$

The value of  $\mathcal{M}_\varepsilon(\phi)$  in the cell  $\varepsilon(\xi + Y)$  ( $\xi \in \mathbb{Z}^n$ ) will be denoted  $\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)$ . In [10] we proved the following results:

For  $\phi \in L^2(\Omega)$  we have

$$\|\mathcal{M}_\varepsilon(\phi)\|_{L^2(\Omega)} \leq C\|\phi\|_{L^2(\Omega)}, \quad \|\mathcal{M}_\varepsilon(\phi) - \phi\|_{H^{-1}(\Omega)} \leq C\varepsilon\|\phi\|_{L^2(\Omega)} \quad (2.16)$$

and for  $\psi \in H_0^1(\Omega)$  (resp.  $\phi \in H^1(\Omega)$ ) we have

$$\begin{aligned} \|\mathcal{M}_\varepsilon(\psi) - \psi\|_{L^2(\Omega)} &\leq C\varepsilon \|\nabla \psi\|_{L^2(\Omega; \mathbb{R}^n)} \\ (\text{resp. } \|\mathcal{M}_\varepsilon(\phi) - \phi\|_{L^2(\widehat{\Omega}_\varepsilon)} &\leq C\varepsilon \|\nabla \phi\|_{L^2(\Omega; \mathbb{R}^n)}). \end{aligned} \quad (2.17)$$

• *The scale-splitting operator  $\mathcal{Q}_\varepsilon$ .*

★ For  $\phi \in L^1(\mathbb{R}^n)$ , the function  $\mathcal{Q}_\varepsilon(\phi) \in W^{1,\infty}(\mathbb{R}^n)$  is given by

$$\mathcal{Q}_\varepsilon(\phi)(x) = \sum_{\xi \in \mathbb{Z}^n} \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) H_{\varepsilon,\xi}(x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where

$$\begin{aligned} H_{\varepsilon,\xi}(x) &= H\left(\frac{x - \varepsilon\xi}{\varepsilon}\right) \quad \text{with} \\ H(z) &= \begin{cases} (1 - |z_1|)(1 - |z_2|) \dots (1 - |z_n|) & \text{if } z \in [-1, 1]^n, \\ 0 & \text{if } z \in \mathbb{R}^n \setminus [-1, 1]^n. \end{cases} \end{aligned}$$

Below, we remind some results about  $\mathcal{Q}_\varepsilon$  proved in [10] and [16].

★ For  $\phi \in L^2(\mathbb{R}^n)$  we have

$$\|\mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n)} \leq C\|\phi\|_{L^2(\mathbb{R}^n)}, \quad \|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} \leq \frac{C}{\varepsilon} \|\phi\|_{L^2(\mathbb{R}^n)} \quad (2.18)$$

and

$$\mathcal{Q}_\varepsilon(\phi) \longrightarrow \phi \quad \text{strongly in } L^2(\mathbb{R}^n).$$

★ For  $\phi \in H^1(\mathbb{R}^n)$  we have

$$\begin{aligned} \|\nabla \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} &\leq C \|\nabla \phi\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)}, \\ \|\phi - \mathcal{Q}_\varepsilon(\phi)\|_{L^2(\mathbb{R}^n)} &\leq C\varepsilon \|\nabla \phi\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} \end{aligned} \quad (2.19)$$

and

$$\mathcal{Q}_\varepsilon(\phi) \longrightarrow \phi \quad \text{strongly in } H^1(\mathbb{R}^n). \quad (2.20)$$

★ For  $\phi \in L^2(\mathbb{R}^n)$  and  $\chi \in L^2(Y)$  we have  $\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \in L^2(\mathbb{R}^n)$ ,  $\nabla \mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right) \in L^2(\mathbb{R}^n)$  and

$$\begin{aligned} \|\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n)} &\leq C \|\phi\|_{L^2(\mathbb{R}^n)} \|\chi\|_{L^2(Y)}, \\ \|\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}})} &\leq C \|\phi\|_{L^2(\tilde{\Omega}_{3\sqrt{n\varepsilon}})} \|\chi\|_{L^2(Y)}. \end{aligned} \quad (2.21)$$

Moreover, if  $\phi \in H^1(\mathbb{R}^n)$  then we have

$$\begin{aligned} \|(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi))\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n)} &\leq C\varepsilon \|\nabla \phi\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} \|\chi\|_{L^2(Y)}, \\ \|\nabla \mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} &\leq C \|\nabla \phi\|_{L^2(\mathbb{R}^n; \mathbb{R}^n)} \|\chi\|_{L^2(Y)}, \\ \|\nabla \mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}}; \mathbb{R}^n)} &\leq C \|\nabla \phi\|_{L^2(\tilde{\Omega}_{3\sqrt{n\varepsilon}}; \mathbb{R}^n)} \|\chi\|_{L^2(Y)}. \end{aligned} \quad (2.22)$$

## 2.4.2 Some complements

In this subsection, we extend some results given above to functions belonging to  $H_\rho^1(\Omega)$ . These technical complements intervene in the proofs of the projection theorems and in the Theorem 6.1.

**Lemma 2.5.** *For  $\phi \in H_\rho^1(\Omega)$  we have*

$$\begin{aligned} \forall i \in \{1, \dots, n\}, \quad \|\rho(\mathcal{M}_\varepsilon(\phi) - \phi)\|_{L^2(\Omega)} &\leq C\varepsilon \|\phi\|_\rho, \\ \|\rho(\phi(\cdot + \varepsilon \mathbf{e}_i) - \phi)\|_{L^2(\Omega)} &\leq C\varepsilon \|\phi\|_\rho, \\ \|\rho(\mathcal{M}_\varepsilon(\phi)(\cdot + \varepsilon \mathbf{e}_i) - \mathcal{M}_\varepsilon(\phi))\|_{L^2(\Omega)} &\leq C\varepsilon \|\phi\|_\rho. \end{aligned} \quad (2.23)$$

For  $\phi \in L_{1/\rho}^2(\Omega)$  we have

$$\|\mathcal{M}_\varepsilon(\phi) - \phi\|_{(H_\rho^1(\Omega))'} \leq C\varepsilon \|\phi/\rho\|_{L^2(\Omega)}. \quad (2.24)$$

The constants do not depend on  $\varepsilon$ .

*Proof. Step 1.* We prove (2.23)<sub>1</sub>.

Let  $\phi$  be in  $H_\rho^1(\Omega)$  and let  $\varepsilon(\xi + Y)$  be a cell included in  $\Omega$ .

*Case 1:*  $\rho(\varepsilon\xi) \geq 2\sqrt{n\varepsilon}$ .

In this case, observing that

$$1 \leq \frac{\max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\}}{\min_{z \in \varepsilon(\xi+Y)} \{\rho(z)\}} \leq 3$$

and thanks to the Poincaré-Wirtinger's inequality we obtain

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx &\leq \left[ \max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 \int_{\varepsilon(\xi+Y)} |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx \\ &\leq \left[ \max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\nabla\phi(x)|^2 dx \\ &\leq C\varepsilon^2 \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\nabla\phi(x)|^2 dx. \end{aligned}$$

*Case 2:*  $\rho(\varepsilon\xi) \leq 2\sqrt{n}\varepsilon$ .

In this case we have

$$\int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\phi(x)|^2 dx.$$

The cases 1 and 2 lead to

$$\int_{\widehat{\Omega}_\varepsilon} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(x) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\widehat{\Omega}_\varepsilon} ([\rho(x)]^2 |\nabla\phi(x)|^2 + |\phi(x)|^2) dx. \quad (2.25)$$

Since  $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$  and due to Lemma 2.1 we get

$$\int_{\Lambda_\varepsilon} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(x) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\widetilde{\Omega}_{c_0\sqrt{n}\varepsilon}} |\phi(x)|^2 dx$$

which in turn with (2.25) gives (2.23)<sub>1</sub>. Proceeding in the same way we obtain (2.23)<sub>2</sub> and (2.23)<sub>3</sub>.

*Step 2.* We prove (2.24).

Let  $\phi$  be in  $L^2_{1/\rho}(\Omega)$  and  $\psi \in H^1_\rho(\Omega)$ . We have

$$\int_{\widehat{\Omega}_\varepsilon} (\mathcal{M}_\varepsilon(\phi) - \phi)\psi = \int_{\widehat{\Omega}_\varepsilon} (\mathcal{M}_\varepsilon(\psi) - \psi)\phi.$$

Consequently we obtain

$$\begin{aligned} \left| \int_{\Omega} (\mathcal{M}_\varepsilon(\phi) - \phi)\psi - \int_{\Omega} (\mathcal{M}_\varepsilon(\psi) - \psi)\phi \right| &\leq \int_{\Lambda_\varepsilon} |(\mathcal{M}_\varepsilon(\phi) - \phi)\psi| + \int_{\Lambda_\varepsilon} |(\mathcal{M}_\varepsilon(\psi) - \psi)\phi| \\ &\leq C(\|\phi\|_{L^2(\Lambda_\varepsilon)} + \|\mathcal{M}_\varepsilon(\phi)\|_{L^2(\Lambda_\varepsilon)})\|\psi\|_{L^2(\Omega)}. \end{aligned}$$

The inclusion  $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$ , the fact that  $\phi \in L^2_{1/\rho}(\Omega)$  and the estimates (2.3)<sub>1</sub>-(2.23)<sub>1</sub> lead to

$$\int_{\Omega} (\mathcal{M}_\varepsilon(\phi) - \phi)\psi \leq C\varepsilon\|\phi/\rho\|_{L^2(\Omega)}\|\psi\|_\rho.$$

Hence (2.24) is proved.  $\square$

**Lemma 2.6.** For  $\phi \in H_\rho^1(\Omega)$  we have

$$\|\rho(\mathcal{Q}_\varepsilon(\phi) - \phi)\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho \quad (2.26)$$

For  $\phi \in H_{1/\rho}^1(\Omega)$  and  $\phi_\varepsilon$  given by Lemma 2.4 we have

$$\begin{aligned} \|\mathcal{Q}_\varepsilon(\phi_\varepsilon)\|_{1/\rho} &\leq C\|\phi\|_{1/\rho}, & \|(\phi - \mathcal{Q}_\varepsilon(\phi_\varepsilon))/\rho\|_{L^2(\Omega)} &\leq C\varepsilon\|\phi\|_{1/\rho}, \\ \forall \mathbf{i} = i_1\mathbf{e}_1 + \dots + i_n\mathbf{e}_n, & (i_1, \dots, i_n) \in \{0, 1\}^n \\ \|(\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon))/\rho\|_{L^2(\Omega)} &\leq C\varepsilon\|\phi\|_{1/\rho}. \end{aligned} \quad (2.27)$$

For  $\phi \in L^2(\mathbb{R}^n)$  and  $\chi \in L^2(Y)$

$$\|(\mathcal{M}_\varepsilon(\rho\phi) - \rho\mathcal{M}_\varepsilon(\phi))\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon\|\phi\|_{L^2(\mathbb{R}^n)}\|\chi\|_{L^2(Y)}. \quad (2.28)$$

For  $\phi \in H_\rho^1(\Omega)$  and  $\chi \in L^2(Y)$

$$\begin{aligned} \|\rho(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi))\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\Omega)} &\leq C\varepsilon\|\phi\|_\rho\|\chi\|_{L^2(Y)}, \\ \|\rho\nabla\mathcal{Q}_\varepsilon(\phi)\chi\left(\left\{\frac{\cdot}{\varepsilon}\right\}\right)\|_{L^2(\Omega)} &\leq C\|\phi\|_\rho\|\chi\|_{L^2(Y)}. \end{aligned} \quad (2.29)$$

The constants do not depend on  $\varepsilon$ .

*Proof. Step 1.* Let  $\phi$  be in  $H_\rho^1(\Omega)$ . We first prove

$$\|\rho(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi))\|_{L^2(\Omega)} \leq C\varepsilon\|\phi\|_\rho. \quad (2.30)$$

To do that, we proceed as in the proof of (2.23)<sub>1</sub>. Let  $\varepsilon(\xi + Y)$  be a cell included in  $\Omega$ .

*Case 1:*  $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$ .

In this case we have

$$1 \leq \frac{\max_{z \in \varepsilon(\xi+Y)}\{\rho(z)\}}{\min_{z \in \varepsilon(\xi+2Y)}\{\rho(z)\}} \leq 4 \quad \text{and} \quad 1 \leq \frac{\max_{z \in \varepsilon(\xi+2Y)}\{\rho(z)\}}{\min_{z \in \varepsilon(\xi+Y)}\{\rho(z)\}} \leq \frac{5}{2}.$$

By definition of  $\mathcal{Q}_\varepsilon(\phi)$  we deduce that

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2 dx &\leq \left[ \max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 \int_{\varepsilon(\xi+Y)} |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2 dx \\ &\leq \left[ \max_{z \in \varepsilon(\xi+Y)} \{\rho(z)\} \right]^2 C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} |\nabla\phi(x)|^2 dx \\ &\leq C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} [\rho(x)]^2 |\nabla\phi(x)|^2 dx. \end{aligned}$$

*Case 2:*  $\rho(\varepsilon\xi) \leq 3\sqrt{n}\varepsilon$ . Then again by definition of  $\mathcal{Q}_\varepsilon(\phi)$  we get

$$\int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} |\phi(x)|^2 dx.$$

As a consequence of both cases we get

$$\int_{\widehat{\Omega}_\varepsilon} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(x)|^2 dx \leq C\varepsilon^2 \int_{\Omega} ([\rho(x)]^2 |\nabla\phi(x)|^2 + |\phi(x)|^2) dx. \quad (2.31)$$

Furthermore we have

$$\int_{\Lambda_\varepsilon} [\rho(x)]^2 |\mathcal{Q}_\varepsilon(\phi)(x)|^2 dx \leq C\varepsilon^2 \int_{\Lambda_\varepsilon} |\mathcal{Q}_\varepsilon(\phi)(x)|^2 dx \leq C\varepsilon^2 \int_{\Omega} |\phi(x)|^2 dx$$

which with (2.31) lead to (2.30). Then as a consequence of (2.23)<sub>1</sub> and (2.30) we get (2.26).

*Step 2.* We prove (2.27)<sub>1</sub>.

Let  $\phi$  be in  $H_{1/\rho}^1(\Omega)$  and  $\phi_\varepsilon$  given by Lemma 2.4. Due to the fact that  $\phi_\varepsilon(x) = 0$  for a.e.  $x \in \mathbb{R}^n \setminus \widetilde{\Omega}_{6\sqrt{n}\varepsilon}$ , hence  $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) = 0$  for every  $x \in \Omega$  such that  $\rho(x) \leq 4\sqrt{n}\varepsilon$ . Again we take a cell  $\varepsilon(\xi + Y)$  included in  $\Omega$  such that  $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$ . The values taken by  $\mathcal{Q}_\varepsilon(\phi_\varepsilon)$  in the cell  $\varepsilon(\xi + Y)$  depend only on the values of  $\phi_\varepsilon$  in  $\varepsilon(\xi + 2Y)$ . Then we have

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} \frac{1}{[\rho(x)]^2} |\nabla \mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)|^2 dx &\leq \frac{C}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \int_{\varepsilon(\xi+2Y)} |\nabla \phi_\varepsilon(x)|^2 dx \\ &\leq C \frac{[\max_{x \in \varepsilon(\xi+2Y)} \{\rho(x)\}]^2}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx \leq C \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx. \end{aligned}$$

Adding all these inequalities gives

$$\int_{\widetilde{\Omega}_{4\sqrt{n}\varepsilon}} \frac{1}{[\rho(x)]^2} |\nabla \mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)|^2 dx \leq C \int_{\Omega} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx$$

Since  $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) = 0$  for every  $x \in \Omega$  such that  $\rho(x) \leq 4\sqrt{n}\varepsilon$ , we get  $\|\mathcal{Q}_\varepsilon(\phi_\varepsilon)\|_{1/\rho} \leq C\|\phi_\varepsilon\|_{1/\rho}$ . We conclude using (2.10)<sub>2</sub>.

*Step 3.* Now we prove (2.27)<sub>2</sub>. Again we consider a cell  $\varepsilon(\xi + Y)$  included in  $\Omega$  such that  $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$ . We have

$$\begin{aligned} \int_{\varepsilon(\xi+Y)} \frac{1}{[\rho(x)]^2} |\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x)|^2 dx &\leq \frac{C}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \int_{\varepsilon(\xi+Y)} |\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x)|^2 dx \\ &\leq \frac{C}{[\min_{x \in \varepsilon(\xi+Y)} \{\rho(x)\}]^2} \sum_{\mathbf{i} \in \{0,1\}^n} \int_{\varepsilon(\xi+\mathbf{i}+Y)} |\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi + \varepsilon\mathbf{i}) - \phi_\varepsilon(x)|^2 dx \\ &\leq C\varepsilon^2 \frac{[\max_{z \in \varepsilon(\xi+2Y)} \{\rho(z)\}]^2}{[\min_{z \in \varepsilon(\xi+Y)} \{\rho(z)\}]^2} \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+2Y)} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx. \end{aligned}$$

Hence we get

$$\int_{\widetilde{\Omega}_{4\sqrt{n}\varepsilon}} \frac{1}{[\rho(x)]^2} |\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x)|^2 dx \leq C\varepsilon^2 \int_{\Omega} \frac{1}{[\rho(x)]^2} |\nabla \phi_\varepsilon(x)|^2 dx$$

The above estimate and the fact that  $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x) - \phi_\varepsilon(x) = 0$  for a.e.  $x \in \Omega$  such that  $\rho(x) \leq 4\sqrt{n}\varepsilon$  yield  $\|(\phi_\varepsilon - \mathcal{Q}_\varepsilon(\phi_\varepsilon))/\rho\|_{L^2(\Omega)} \leq C\varepsilon\|\phi_\varepsilon\|_{1/\rho}$ . We conclude using both estimates in (2.10).

Proceeding as in the Steps 2 and 3 we obtain (2.27)<sub>3</sub>, (2.28) and (2.29).  $\square$

### 3 Two new projection theorems

**Theorem 3.1.** *Let  $\phi$  be in  $H_{1/\rho}^1(\Omega)$ . There exists  $\widehat{\phi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that*

$$\begin{cases} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} \leq C\{\|\phi\|_{L^2(\Omega)} + \varepsilon\|\nabla\phi\|_{[L^2(\Omega)]^n}\} \\ \|\mathcal{T}_\varepsilon(\phi) - \widehat{\phi}_\varepsilon\|_{H^1(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon(\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon\|\phi\|_{1/\rho}). \end{cases} \quad (3.1)$$

The constants depend only on  $n$  and  $\partial\Omega$ .

*Proof.* Here, we proceed as in the proof of Proposition 3.3 in [15]. We first reintroduce the open sets  $\widehat{\Omega}_{\varepsilon,i}$  and the "double" unfolding operators  $\mathcal{T}_{\varepsilon,i}$ . We set

$$\widehat{\Omega}_{\varepsilon,i} = \widehat{\Omega}_\varepsilon \cap (\widehat{\Omega}_\varepsilon - \varepsilon\mathbf{e}_i), \quad K_i = \text{interior}(\overline{Y} \cup (\mathbf{e}_i + \overline{Y})), \quad i \in \{1, \dots, n\}.$$

The unfolding operator  $\mathcal{T}_{\varepsilon,i}$  from  $L^2(\Omega)$  into  $L^2(\Omega \times K_i)$  is defined by

$$\forall \psi \in L^2(\Omega), \quad \mathcal{T}_{\varepsilon,i}(\psi)(x, y) = \begin{cases} \psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for } x \in \widehat{\Omega}_{\varepsilon,i} \text{ and for a.e. } y \in K_i, \\ 0 & \text{for } x \in \Omega \setminus \widehat{\Omega}_{\varepsilon,i} \text{ and for a.e. } y \in K_i. \end{cases}$$

The restriction of  $\mathcal{T}_{\varepsilon,i}(\psi)$  to  $\widehat{\Omega}_{\varepsilon,i} \times Y$  is equal to  $\mathcal{T}_\varepsilon(\psi)$ .

*Step 1.* Let us first take  $\phi \in L_{1/\rho}^2(\Omega)$ . We set  $\psi = \frac{1}{\rho}\phi$  and we evaluate the difference  $\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)$  in  $L^2(Y; (H_\rho^1(\Omega))')$ . For any  $\Psi \in H_\rho^1(\Omega)$  a change of variables gives for a. e.  $y \in Y$

$$\begin{aligned} \int_\Omega \mathcal{T}_{\varepsilon,i}(\phi)(x, y + \mathbf{e}_i)\Psi(x)dx &= \int_{\widehat{\Omega}_{\varepsilon,i}} \mathcal{T}_\varepsilon(\phi)(x + \varepsilon\mathbf{e}_i, y)\Psi(x)dx \\ &= \int_{\widehat{\Omega}_{\varepsilon,i} + \varepsilon\mathbf{e}_i} \mathcal{T}_\varepsilon(\phi)(x, y)\Psi(x - \varepsilon\mathbf{e}_i)dx. \end{aligned}$$

Then we obtain for a. e.  $y \in Y$

$$\begin{aligned} &\left| \int_\Omega \{\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y)\}\Psi - \int_{\widehat{\Omega}_{\varepsilon,i}} \mathcal{T}_\varepsilon(\psi)(\cdot, y)\rho\{\Psi(\cdot - \varepsilon\mathbf{e}_i) - \Psi\} \right| \\ &\leq \left| \int_{\widehat{\Omega}_{\varepsilon,i}} \mathcal{T}_\varepsilon(\psi)(\cdot, y)(\mathcal{T}_\varepsilon(\rho) - \rho)\{\Psi(\cdot - \varepsilon\mathbf{e}_i) - \Psi\} \right| + C\|\mathcal{T}_\varepsilon(\phi)(\cdot, y)\|_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}\|\Psi\|_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}. \end{aligned}$$

Estimate (2.23)<sub>2</sub> leads to

$$\|\rho(\Psi(\cdot - \varepsilon \mathbf{e}_i) - \Psi)\|_{L^2(\tilde{\Omega}_{\varepsilon,i})} \leq C\varepsilon \|\Psi\|_{\rho} \quad \forall i \in \{1, \dots, n\}.$$

We have

$$\|\mathcal{T}_{\varepsilon}(\rho) - \rho\|_{L^{\infty}(\Omega)} \leq C\varepsilon. \quad (3.2)$$

The above inequalities imply

$$\begin{aligned} & \langle \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y), \Psi \rangle_{(H_{\rho}^1(\Omega))', H_{\rho}^1(\Omega)} \\ &= \int_{\Omega} \{ \mathcal{T}_{\varepsilon,i}(\phi)(x, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(x, y) \} \Psi(x) dx \\ &\leq C\varepsilon \|\Psi\|_{\rho} \|\mathcal{T}_{\varepsilon}(\psi)(\cdot, y)\|_{L^2(\Omega)} + C\varepsilon \|\Psi\|_{L^2(\Omega)} \|\mathcal{T}_{\varepsilon}(\psi)(\cdot, y)\|_{L^2(\Omega)} \\ &\quad + C \|\mathcal{T}_{\varepsilon}(\phi)(\cdot, y)\|_{L^2(\tilde{\Omega}_{2\sqrt{n}\varepsilon})} \|\Psi\|_{L^2(\tilde{\Omega}_{2\sqrt{n}\varepsilon})}. \end{aligned}$$

Therefore, for a.e.  $y \in Y$  we have

$$\|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(\cdot, y)\|_{(H_{\rho}^1(\Omega))'} \leq C\varepsilon \|\mathcal{T}_{\varepsilon}(\psi)(\cdot, y)\|_{L^2(\Omega)} + C \|\mathcal{T}_{\varepsilon}(\phi)(\cdot, y)\|_{L^2(\tilde{\Omega}_{2\sqrt{n}\varepsilon})}$$

which leads to the following estimate of the difference between  $\mathcal{T}_{\varepsilon,i}(\phi)|_{\Omega \times Y}$  and one of its translated :

$$\begin{aligned} \|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{L^2(Y; (H_{\rho}^1(\Omega))')} &\leq C\varepsilon \|\phi/\rho\|_{L^2(\Omega)} + C \|\phi\|_{L^2(\tilde{\Omega}_{2\sqrt{n}\varepsilon})} \\ &\leq C\varepsilon \|\phi/\rho\|_{L^2(\Omega)}. \end{aligned} \quad (3.3)$$

The constant depends only on the boundary of  $\Omega$ .

*Step 2.* Let  $\phi \in H_{1/\rho}^1(\Omega)$ . The above estimate (3.3) applied to  $\phi$  and its partial derivatives give

$$\begin{aligned} \|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{L^2(Y; (H_{\rho}^1(\Omega))')} &\leq C\varepsilon \|\phi/\rho\|_{L^2(\Omega)} \\ \|\mathcal{T}_{\varepsilon,i}(\nabla \phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\nabla \phi)\|_{[L^2(Y; (H_{\rho}^1(\Omega))')^n]} &\leq C\varepsilon \|\phi\|_{1/\rho}. \end{aligned}$$

which in turn lead to (we recall that  $\nabla_y(\mathcal{T}_{\varepsilon,i}(\phi)) = \varepsilon \mathcal{T}_{\varepsilon,i}(\nabla \phi)$ ).

$$\|\mathcal{T}_{\varepsilon,i}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)\|_{H^1(Y; (H_{\rho}^1(\Omega))')} \leq C\varepsilon (\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon \|\phi\|_{1/\rho}).$$

From these inequalities for  $i \in \{1, \dots, n\}$  we deduce the estimate of the difference of the traces of the function  $y \rightarrow \mathcal{T}_{\varepsilon}(\phi)(\cdot, y)$  on the faces  $Y_i \doteq \{y \in \bar{Y} \mid y_i = 0\}$  and  $\mathbf{e}_i + Y_i$

$$\|\mathcal{T}_{\varepsilon}(\phi)(\cdot, \cdot + \mathbf{e}_i) - \mathcal{T}_{\varepsilon}(\phi)\|_{H^{1/2}(Y_i; (H_{\rho}^1(\Omega))')} \leq C\varepsilon (\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon \|\phi\|_{1/\rho}). \quad (3.4)$$

These estimates ( $i \in \{1, \dots, n\}$ ) give a measure of the periodic defect of the function  $y \rightarrow \mathcal{T}_{\varepsilon}(\phi)(\cdot, y)$  (see [15]).

Then we decompose  $\mathcal{T}_\varepsilon(\phi)$  into the sum of an element belonging to  $H_{per}^1(Y; L^2(\Omega))$  and one to  $(H^1(Y; L^2(\Omega)))^\perp$  (the orthogonal of  $H_{per}^1(Y; L^2(\Omega))$  in  $H^1(Y; L^2(\Omega))$ ), see [15]

$$\mathcal{T}_\varepsilon(\phi) = \widehat{\phi}_\varepsilon + \overline{\phi}_\varepsilon, \quad \widehat{\phi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega)), \quad \overline{\phi}_\varepsilon \in (H^1(Y; L^2(\Omega)))^\perp. \quad (3.5)$$

The function  $y \rightarrow \mathcal{T}_\varepsilon(\phi)(\cdot, y)$  takes its values in a finite dimensional space,

$$\overline{\phi}_\varepsilon(\cdot, \cdot) = \sum_{\xi \in \Xi_\varepsilon} \overline{\phi}_{\varepsilon, \xi}(\cdot) \chi_{\varepsilon, \xi}(\cdot)$$

where  $\chi_{\varepsilon, \xi}(\cdot)$  is the characteristic function of the cell  $\varepsilon(\xi + Y)$  and where  $\overline{\phi}_{\varepsilon, \xi}(\cdot) \in (H^1(Y))^\perp$  (the orthogonal of  $H_{per}^1(Y)$  in  $H^1(Y)$ ), see [15]. The decomposition (3.5) is the same in  $H^1(Y; (H_\rho^1(\Omega))')$  and we have

$$\|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))}^2 + \|\overline{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))}^2 = \|\mathcal{T}_\varepsilon(\phi)\|_{H^1(Y; L^2(\Omega))}^2 \leq C \{ \|\phi\|_{L^2(\Omega)} + \varepsilon \|\nabla \phi\|_{[L^2(\Omega)]^n} \}^2.$$

It gives the first inequality in (3.1) and the estimate of  $\overline{\phi}_\varepsilon$  in  $H^1(Y; L^2(\Omega))$ . From Theorem 2.2 in [15] and (3.4) we obtain a finer estimate of  $\overline{\phi}_\varepsilon$  in  $H^1(Y; (H_\rho^1(\Omega))')$

$$\|\overline{\phi}_\varepsilon\|_{H^1(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon (\|\phi/\rho\|_{L^2(\Omega)} + \varepsilon \|\phi\|_{1/\rho}).$$

It is the second inequality in (3.1). □

**Theorem 3.2.** *For  $\phi \in H_{1/\rho}^1(\Omega)$ , there exists  $\widehat{\phi}_\varepsilon \in H_{per}^1(Y; L^2(\Omega))$  such that*

$$\begin{aligned} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} &\leq C \|\nabla \phi\|_{[L^2(\Omega)]^n}, \\ \|\mathcal{T}_\varepsilon(\nabla \phi) - \nabla \phi - \nabla_y \widehat{\phi}_\varepsilon\|_{[L^2(Y; (H_\rho^1(\Omega))')]^n} &\leq C\varepsilon \|\phi\|_{1/\rho}. \end{aligned} \quad (3.6)$$

*The constants depend only on  $\partial\Omega$ .*

*Proof.* Let  $\phi$  be in  $H_{1/\rho}^1(\Omega)$  and  $\psi = \phi/\rho \in H_0^1(\Omega)$ . The function  $\phi$  is extended by 0 outside of  $\Omega$ . We decompose  $\phi$  as

$$\phi = \Phi + \varepsilon \underline{\phi}, \quad \text{where } \Phi = \mathcal{Q}_\varepsilon(\phi_\varepsilon) \quad \text{and} \quad \underline{\phi} = \frac{1}{\varepsilon} \left( \phi - \mathcal{Q}_\varepsilon(\phi_\varepsilon) \right)$$

where  $\phi_\varepsilon$  is given by Lemma 2.4. We have  $\Phi$  and  $\underline{\phi} \in H_0^1(\Omega)$  and due to (2.27) we get the following estimates:

$$\|\Phi\|_{1/\rho} + \varepsilon \|\underline{\phi}\|_{1/\rho} + \|\underline{\phi}/\rho\|_{L^2(\Omega)} \leq C \|\phi\|_{1/\rho}. \quad (3.7)$$

The projection Theorem 3.1 applied to  $\underline{\phi} \in H_{1/\rho}^1(\Omega)$  gives an element  $\widehat{\phi}_\varepsilon$  in  $H_{per}^1(Y; L^2(\Omega))$  such that

$$\begin{aligned} \|\widehat{\phi}_\varepsilon\|_{H^1(Y; L^2(\Omega))} &\leq C \|\phi\|_{1/\rho}, \\ \|\mathcal{T}_\varepsilon(\underline{\phi}) - \widehat{\phi}_\varepsilon\|_{H^1(Y; (H_\rho^1(\Omega))')} &\leq C\varepsilon \|\phi\|_{1/\rho}. \end{aligned} \quad (3.8)$$



Now we evaluate  $\|\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\Phi\|_{[L^2(Y; (H_\rho^1(\Omega))') ]^n}$ .

From (2.24), (2.27)<sub>1</sub> and (3.7) we get

$$\|\nabla\Phi - \mathcal{M}_\varepsilon(\nabla\Phi)\|_{(H_\rho^1(\Omega; \mathbb{R}^n))'} \leq C\varepsilon\|\phi\|_{1/\rho}. \quad (3.9)$$

We set

$$H^{(1)}(z) = \begin{cases} (1 - |z_2|)(1 - |z_3|) \dots (1 - |z_n|) & \text{if } z = (z_1, z_2, \dots, z_n) \in [-1, 1]^n, \\ 0 & \text{if } z \in \mathbb{R}^n \setminus [-1, 1]^n. \end{cases}$$

$$\mathbf{I} = \left\{ \mathbf{i} \mid \mathbf{i} = i_2 \mathbf{e}_2 + \dots + i_n \mathbf{e}_n, \quad (i_2, \dots, i_n) \in \{0, 1\}^{n-1} \right\}$$

For  $\xi \in \mathbb{Z}^n$  and for every  $(x, y) \in \varepsilon(\xi + Y) \times Y$  we have

$$\mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x, y) = \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon} H^{(1)}(y - \mathbf{i})$$

$$\mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\varepsilon\xi) = \frac{1}{2^{n-1}} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon}.$$

Now, let us take  $\psi \in H_\rho^1(\Omega)$ . We recall that  $\phi_\varepsilon(x) = 0$  for a.e.  $x \in \mathbb{R}^n \setminus \widetilde{\Omega}_{6\sqrt{n}\varepsilon}$ , hence  $\Phi(x) = 0$  for  $x \in \mathbb{R}^n \setminus \widetilde{\Omega}_{3\sqrt{n}\varepsilon}$ ; as a first consequence  $\mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) = 0$  in  $\Lambda_\varepsilon$ .

For  $y \in Y$  we have

$$\begin{aligned} \langle \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi \rangle_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} &= \int_{\Omega} \left\{ \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x) \right\} \psi(x) dx \\ &= \int_{\widehat{\Omega}_\varepsilon} \left\{ \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x) \right\} \mathcal{M}_\varepsilon(\psi)(x) dx. \end{aligned}$$

Besides we have

$$\begin{aligned} \int_{\widehat{\Omega}_\varepsilon} \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x) \mathcal{M}_\varepsilon(\psi)(x) dx &= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\varepsilon\xi) \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi) \\ &= \frac{\varepsilon^n}{2^{n-1}} \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon} \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi) \\ &= \frac{\varepsilon^n}{2^{n-1}} \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) \end{aligned}$$

and

$$\begin{aligned}
& \int_{\widetilde{\Omega}_\varepsilon} \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(x, y) \mathcal{M}_\varepsilon(\psi)(x) dx \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \left[ \frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{e}_1 + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))}{\varepsilon} \right] H^{(1)}(y - \mathbf{i}) \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi) \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} H^{(1)}(y - \mathbf{i}) \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i}))
\end{aligned}$$

Due to the fact that  $\phi_\varepsilon(x) = 0$  for a.e.  $x \in \mathbb{R}^n \setminus \widetilde{\Omega}_{6\sqrt{n}\varepsilon}$ , in the above summations we only take the  $\xi$ 's belonging to  $\Xi_\varepsilon$  and satisfying  $\rho(\varepsilon\xi) \geq 3\sqrt{n}\varepsilon$ . Hence

$$\begin{aligned}
& \langle \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi \rangle_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} \sum_{\mathbf{i} \in \mathbf{I}} \left[ H^{(1)}(y - \mathbf{i}) - \frac{1}{2^{n-1}} \right] \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})).
\end{aligned}$$

Thanks to the identity relation  $\sum_{\mathbf{i} \in \mathbf{I}} \left[ H^{(1)}(y - \mathbf{i}) - \frac{1}{2^{n-1}} \right] = 0$  we obtain that

$$\left| \sum_{\mathbf{i} \in \mathbf{I}} \left[ H^{(1)}(y - \mathbf{i}) - \frac{1}{2^{n-1}} \right] \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) \right| \leq \sum_{\mathbf{i} \in \mathbf{I}} \left| \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi) \right|.$$

Taking into account the last equality and inequality above we deduce that

$$\begin{aligned}
& \langle \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi \rangle_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} \\
&= \varepsilon^n \sum_{\xi \in \mathbb{Z}^n} \sum_{\mathbf{i} \in \mathbf{I}} \left| \frac{\mathcal{M}_\varepsilon(\psi)(\varepsilon(\xi - \mathbf{e}_1)) - \mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)}{\varepsilon} \right| \left| \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi + \mathbf{i})) - \mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi) \right| \\
&= \frac{1}{\varepsilon} \sum_{\mathbf{i} \in \mathbf{I}} \int_{\Omega} \left| \mathcal{M}_\varepsilon(\psi)(\cdot - \varepsilon\mathbf{e}_1) - \mathcal{M}_\varepsilon(\psi) \right| \left| \mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon) \right| \\
&\leq \frac{C}{\varepsilon} \sum_{\mathbf{i} \in \mathbf{I}} \left\| \rho(\mathcal{M}_\varepsilon(\psi)(\cdot - \varepsilon\mathbf{e}_1) - \mathcal{M}_\varepsilon(\psi)) \right\|_{L^2(\Omega)} \left\| \frac{1}{\rho} (\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon)) \right\|_{L^2(\Omega)}.
\end{aligned}$$

Due to (2.23)<sub>3</sub> and (2.27)<sub>3</sub> we finally get

$$\langle \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right)(\cdot, y) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right), \psi \rangle_{(H_\rho^1(\Omega))', H_\rho^1(\Omega)} \leq C\varepsilon \|\phi_\varepsilon\|_{1/\rho} \|\psi\|_\rho.$$

It leads to

$$\left\| \mathcal{T}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) - \mathcal{M}_\varepsilon\left(\frac{\partial\Phi}{\partial x_1}\right) \right\|_{L^\infty(Y; (H_\rho^1(\Omega))')} \leq C\varepsilon \|\phi_\varepsilon\|_{1/\rho}. \quad (3.10)$$

Besides we have

$$\int_{\Omega} \frac{\partial \phi}{\partial x_1}(x) \psi(x) dx = - \int_{\Omega} \underline{\phi}(x) \frac{\partial \psi}{\partial x_1}(x) dx \leq C \|\underline{\phi}/\rho\|_{L^2(\Omega)} \|\psi\|_{\rho} \leq C \|\phi\|_{1/\rho} \|\psi\|_{\rho}.$$

Hence  $\left\| \varepsilon \frac{\partial \phi}{\partial x_1} \right\|_{(H^1_{\rho}(\Omega; \mathbb{R}^n))'} \leq C\varepsilon \|\phi\|_{1/\rho}$ . This last estimate with (2.10)<sub>2</sub>, (3.9) and (3.10) yield

$$\left\| \mathcal{T}_{\varepsilon} \left( \frac{\partial \Phi}{\partial x_1} \right) - \frac{\partial \phi}{\partial x_1} \right\|_{L^{\infty}(Y; (H^1_{\rho}(\Omega))')} \leq C\varepsilon \|\phi_{\varepsilon}\|_{1/\rho}.$$

In the same way we prove the estimates for the partial derivatives of  $\Phi$  with respect to  $x_i$ ,  $i \in \{2, \dots, n\}$ . Hence we get  $\|\mathcal{T}_{\varepsilon}(\nabla \Phi) - \nabla \phi\|_{[L^{\infty}(Y; (H^1_{\rho}(\Omega))')]^n} \leq C\varepsilon \|\phi_{\varepsilon}\|_{1/\rho}$ . Then thanks to (3.8) the second estimate in (3.6) is proved.  $\square$

## 4 Reminds about the classical periodic homogenization problem

We consider the homogenization problem

$$\phi^{\varepsilon} \in H^1_0(\Omega), \quad \int_{\Omega} A_{\varepsilon}(x) \nabla \phi^{\varepsilon}(x) \nabla \psi(x) dx = \int_{\Omega} f(x) \psi(x) dx, \quad \forall \psi \in H^1_0(\Omega), \quad (4.1)$$

where

- $A_{\varepsilon}(x) = A\left(\left\{\frac{x}{\varepsilon}\right\}\right)$  for a.e.  $x \in \Omega$ , where  $A$  is a square matrix belonging to  $L^{\infty}(Y; \mathbb{R}^{n \times n})$  and satisfying the condition of uniform ellipticity  $c|\xi|^2 \leq A(y)\xi \cdot \xi$  for a.e.  $y \in Y$ , with  $c$  a strictly positive constant,
- $f \in L^2(\Omega)$ .

We showed in [10] that

$$\mathcal{T}_{\varepsilon}(\nabla \phi^{\varepsilon}) \longrightarrow \nabla \Phi + \nabla_y \widehat{\phi} \quad \text{strongly in } L^2(\Omega \times Y; \mathbb{R}^n)$$

where  $(\Phi, \widehat{\phi}) \in H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y))$  is the solution of the problem of unfolding homogenization

$$\begin{aligned} \forall (\Psi, \widehat{\psi}) \in H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y)) \\ \int_{\Omega} \int_Y A(y) \{ \nabla \Phi(x) + \nabla_y \widehat{\phi}(x, y) \} \{ \nabla \Psi(x) + \nabla_y \widehat{\psi}(x, y) \} dx dy = \int_{\Omega} f(x) \Psi(x) dx. \end{aligned}$$

The correctors  $\chi_i$ ,  $i \in \{1, \dots, n\}$ , are the solutions of the variational problems

$$\begin{aligned} \chi_i \in H^1_{per}(Y), \quad \int_Y \chi_i = 0, \\ \int_Y A(y) \nabla_y (\chi_i(y) + y_i) \nabla_y \psi(y) dy = 0, \quad \forall \psi \in H^1_{per}(Y). \end{aligned} \quad (4.2)$$

They allow to express  $\widehat{\phi}$  in terms of the partial derivatives of  $\Phi$

$$\widehat{\phi} = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} \chi_i \quad (4.3)$$

and to give the homogenized problem satisfied by  $\Phi$

$$\Phi \in H_0^1(\Omega), \quad \int_{\Omega} \mathcal{A} \nabla \Phi(x) \nabla \Psi(x) dx = \int_{\Omega} f(x) \Psi(x) dx, \quad \forall \Psi \in H_0^1(\Omega) \quad (4.4)$$

where (see [10])

$$\mathcal{A}_{ij} = \sum_{k,l=1}^n \int_Y a_{kl}(y) \frac{\partial(y_j + \chi_j(y))}{\partial y_l} \frac{\partial(y_i + \chi_i(y))}{\partial y_k} dy. \quad (4.5)$$

## 5 An operator from $H^{-1/2}(\partial\Omega)$ into $L^2(\Omega)$

From now on,  $\Omega$  is a bounded domain with a  $\mathcal{C}^{1,1}$  boundary or an open bounded convex set.

In this section we first introduce a lifting operator  $\mathbf{T}$  (defined by (5.1)) from  $H^{1/2}(\partial\Omega)$  into  $H^1(\Omega)$ . This operator and the estimate (5.2) are in fact sufficient to obtain the error estimates with a non-homogeneous Dirichlet condition (Theorem 6.3); one of the aim of this paper. Then we extend this operator. The extension of  $\mathbf{T}$  from  $H^{-1/2}(\partial\Omega)$  into  $H^1(\Omega)$  is essential in order to get a sharper estimate (6.3) than (6.2)<sub>1</sub>. In Theorem 7.1 we give an application based on (6.3), in this theorem we investigate a first case of strongly oscillating boundary data.

Let  $g$  be in  $H^{1/2}(\partial\Omega)$ , there exists one  $\phi_g \in H^1(\Omega)$  such that

$$\operatorname{div}(\mathcal{A} \nabla \phi_g) = 0 \quad \text{in } \Omega, \quad \phi_g = g \quad \text{on } \partial\Omega \quad (5.1)$$

where  $\mathcal{A}$  is the matrix given by (4.5). We have

$$\|\phi_g\|_{H^1(\Omega)} \leq C \|g\|_{H^{1/2}(\partial\Omega)}. \quad (5.2)$$

We denote by  $\mathbf{T}$  the operator from  $H^{1/2}(\partial\Omega)$  into  $H^1(\Omega)$  which associates to  $g \in H^{1/2}(\partial\Omega)$  the function  $\phi_g \in H^1(\Omega)$ .

Now, let  $(\psi, \Psi)$  be a couple in  $[\mathcal{C}^\infty(\overline{\Omega})]^2$ , integrating by parts over  $\Omega$  gives

$$\int_{\Omega} \mathcal{A} \nabla \psi(x) \nabla \Psi(x) dx = - \int_{\Omega} \psi(x) \operatorname{div}(\mathcal{A}^T \nabla \Psi)(x) dx + \int_{\partial\Omega} \psi(x) (\mathcal{A}^T \nabla \Psi)(x) dx \cdot \nu(x) d\sigma.$$

The space  $\mathcal{C}^\infty(\overline{\Omega})$  being dense in  $H^1(\Omega)$  and  $H^2(\Omega)$ , hence the above equality holds true for any  $\psi \in H^1(\Omega)$  and any  $\Psi \in H^2(\Omega)$ . Hence, for  $\Psi \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $\phi_g$  defined by (5.1) we get

$$\int_{\Omega} \phi_g(x) \operatorname{div}(\mathcal{A}^T \nabla \Psi)(x) dx = \int_{\partial\Omega} g(x) (\mathcal{A}^T \nabla \Psi)(x) \cdot \nu(x) d\sigma. \quad (5.3)$$

Under the assumption on  $\Omega$  the function  $\Psi(g)$  defined by

$$\Psi(g) \in H_0^1(\Omega), \quad \operatorname{div}(\mathcal{A}^T \nabla \Psi(g)) = \phi_g \quad \text{in } \Omega$$

belongs to  $H_0^1(\Omega) \cap H^2(\Omega)$  and satisfies

$$\|\Psi(g)\|_{H^2(\Omega)} \leq C \|\phi_g\|_{L^2(\Omega)}.$$

Taking  $\Psi = \Psi(g)$  in the above equality (5.3) we obtain

$$\begin{aligned} \int_{\Omega} |\phi_g(x)|^2 dx &= \int_{\partial\Omega} g(x) (\mathcal{A}^T \nabla \Psi(g)(x)) \cdot \nu(x) d\sigma \leq \|g\|_{H^{-1/2}(\partial\Omega)} \|(\mathcal{A}^T \nabla \Psi(g)) \cdot \nu\|_{H^{1/2}(\partial\Omega)} \\ &\leq C \|g\|_{H^{-1/2}(\partial\Omega)} \|\Psi(g)\|_{H^2(\Omega)}. \end{aligned}$$

This leads to

$$\|\phi_g\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}. \quad (5.4)$$

Due to (5.4), the operator  $\mathbf{T}$  admits an extension (still denoted  $\mathbf{T}$ ) from  $H^{-1/2}(\partial\Omega)$  into  $L^2(\Omega)$  and we have

$$\forall g \in H^{-1/2}(\partial\Omega), \quad \|\mathbf{T}(g)\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}.$$

For  $g \in H^{-1/2}(\partial\Omega)$ , we also denote  $\phi_g = \mathbf{T}(g)$ . This function is the "very weak" solution of the problem

$$\phi_g \in L^2(\Omega), \quad \operatorname{div}(\mathcal{A} \nabla \phi_g) = 0 \quad \text{in } \Omega, \quad \phi_g = g \quad \text{on } \partial\Omega$$

or the solution of the following:

$$\begin{aligned} \phi_g &\in L^2(\Omega), \\ \int_{\Omega} \phi_g(x) \operatorname{div}(\mathcal{A}^T \nabla \psi(x)) dx &= \langle g, (\mathcal{A}^T \nabla \psi) \cdot \nu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \\ \forall \psi &\in H_0^1(\Omega) \cap H^2(\Omega). \end{aligned} \quad (5.5)$$

**Lemma 5.1.** *The operator  $\mathbf{T}$  is a bicontinuous linear operator from  $H^{-1/2}(\partial\Omega)$  onto*

$$\mathbf{H} = \left\{ \phi \in L^2(\Omega) \mid \operatorname{div}(\mathcal{A} \nabla \phi) = 0 \quad \text{in } \Omega \right\}.$$

*There exists a constant  $C \geq 1$  such that*

$$\forall g \in H^{-1/2}(\partial\Omega), \quad \frac{1}{C} \|g\|_{H^{-1/2}(\partial\Omega)} \leq \|\mathbf{T}(g)\|_{L^2(\Omega)} \leq C \|g\|_{H^{-1/2}(\partial\Omega)}. \quad (5.6)$$

*Proof.* Let  $\phi$  be in  $\mathbf{H}$  we are going to prove that there exists an element  $g \in H^{-1/2}(\partial\Omega)$  such that  $\mathbf{T}(g) = \phi$ . To do that, we consider a continuous linear lifting operator  $\mathbf{R}$  from  $H^{1/2}(\partial\Omega)$  into  $H_0^1(\Omega) \cap H^2(\Omega)$  satisfying for any  $h \in H^{1/2}(\partial\Omega)$

$$\begin{aligned} \mathbf{R}(h) &\in H_0^1(\Omega) \cap H^2(\Omega), \\ \mathcal{A}^T \nabla \mathbf{R}(h)|_{\partial\Omega} \cdot \nu &= h \quad \text{on } \partial\Omega, \\ \|\mathbf{R}(h)\|_{H^2(\Omega)} &\leq C \|h\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

The map  $h \mapsto \int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \mathbf{R}(h))$  is a continuous linear form defined over  $H^{1/2}(\partial\Omega)$ . Thus, there exists  $g \in H^{-1/2}(\partial\Omega)$  such that

$$\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \mathbf{R}(h)) = \langle g, h \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \quad (5.7)$$

Since  $\phi \in \mathbf{H}$ , we deduce that for any  $\psi \in \mathcal{C}_0^\infty(\Omega)$  we have  $\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \psi) = 0$ . Therefore, for any  $\psi \in H_0^2(\Omega)$  we have  $\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \psi) = 0$ . Taking into account (5.7) we get

$$\int_{\Omega} \phi \operatorname{div}(\mathcal{A}^T \nabla \psi) = \langle g, (\mathcal{A}^T \nabla \psi) \cdot \nu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \quad \forall \psi \in H_0^1(\Omega) \cap H^2(\Omega).$$

It yields  $\phi = \phi_g$  and then (5.6).  $\square$

**Remark 5.2.** *It is well known (see e.g. [18]) that every function  $\phi \in \mathbf{H}$  also belongs to  $H_\rho^1(\Omega)$  and verifies*

$$\|\phi\|_\rho \leq C \|\phi\|_{L^2(\Omega)}. \quad (5.8)$$

## 6 Error estimates with a non-homogeneous Dirichlet condition

**Theorem 6.1.** *Let  $(\phi^\varepsilon)_{\varepsilon>0}$  be a sequence of functions belonging to  $H^1(\Omega)$  such that*

$$\operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) = 0 \quad \text{in } \Omega. \quad (6.1)$$

*Setting  $g_\varepsilon = \phi^\varepsilon|_{\partial\Omega}$  and  $\phi_{g_\varepsilon} = \mathbf{T}(g_\varepsilon) \in H^1(\Omega)$ , there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \leq \varepsilon_0$  we have*

$$\begin{aligned} \|\phi^\varepsilon\|_{H^1(\Omega)} &\leq C \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}, & \|\phi^\varepsilon - \phi_{g_\varepsilon}\|_{L^2(\Omega)} &\leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}, \\ \left\| \rho \left( \nabla \phi^\varepsilon - \nabla \phi_{g_\varepsilon} - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} &\leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \end{aligned} \quad (6.2)$$

*Moreover we have*

$$\|\phi^\varepsilon\|_\rho \leq C (\varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} + \|g_\varepsilon\|_{H^{-1/2}(\partial\Omega)}). \quad (6.3)$$

*The  $\chi_i$ 's are the correctors introduced in Section 4 and  $\mathbf{T}$  is the operator defined in Section 5.*

*Proof. Step 1.* We prove the first estimate in (6.2). From Section 5 we get

$$\|\phi_{g_\varepsilon}\|_{H^1(\Omega)} \leq C\|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \quad \|\phi_{g_\varepsilon}\|_\rho \leq C\|g_\varepsilon\|_{H^{-1/2}(\partial\Omega)}. \quad (6.4)$$

We write (6.1) in the following weak form:

$$\begin{aligned} \phi^\varepsilon &= \check{\phi}_\varepsilon + \phi_{g_\varepsilon}, \quad \check{\phi}_\varepsilon \in H_0^1(\Omega) \\ \int_\Omega A_\varepsilon \nabla \check{\phi}_\varepsilon \nabla v &= - \int_\Omega A_\varepsilon \nabla \phi_{g_\varepsilon} \nabla v \quad \forall v \in H_0^1(\Omega). \end{aligned} \quad (6.5)$$

The solution  $\check{\phi}_\varepsilon$  of the above variational problem satisfies

$$\|\check{\phi}_\varepsilon\|_{H^1(\Omega)} \leq C\|\nabla \phi_{g_\varepsilon}\|_{L^2(\Omega; \mathbb{R}^n)}.$$

Hence, from (6.4)<sub>1</sub> and the above estimate we get the first inequality in (6.2).

*Step 2.* We prove the second estimate in (6.2).

For every test function  $v \in H_0^1(\Omega)$  we have

$$\int_\Omega A_\varepsilon \nabla \phi^\varepsilon \nabla v = 0. \quad (6.6)$$

Now, in order to obtain the  $L^2$  error estimate we proceed as in the proof of the Theorem 3.2 in [16]. We first recall that for any  $\phi \in H^1(\Omega)$  we have (see Lemma 2.3) for every  $\varepsilon \leq \varepsilon_0 \doteq \gamma_0/3\sqrt{n}$

$$\|\phi\|_{L^2(\tilde{\Omega}_{3c_0\sqrt{n\varepsilon}})} \leq C\varepsilon^{1/2}\|\phi\|_{H^1(\Omega)}.$$

Let  $U$  be a test function belonging to  $H_0^1(\Omega) \cap H^2(\Omega)$ . The above estimate yields

$$\|\nabla U\|_{L^2(\tilde{\Omega}_{3c_0\sqrt{n\varepsilon}; \mathbb{R}^n})} \leq C\varepsilon^{1/2}\|U\|_{H^2(\Omega)} \quad (6.7)$$

which in turn with (2.12)-(2.13)-(2.14)<sub>1</sub> and (6.2)<sub>1</sub>-(6.6) lead to

$$\left| \int_{\Omega \times Y} A(y) \mathcal{T}_\varepsilon(\nabla \phi^\varepsilon)(x, y) \nabla U(x) dx dy \right| \leq C\varepsilon^{1/2}\|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}\|U\|_{H^2(\Omega)}. \quad (6.8)$$

The Theorem 2.3 in [16] gives an element  $\widehat{\phi}_\varepsilon \in L^2(\Omega; H_{per}^1(Y))$  such that

$$\begin{aligned} \|\mathcal{T}(\nabla \phi^\varepsilon) - \nabla \phi^\varepsilon - \nabla_y \widehat{\phi}_\varepsilon\|_{[L^2(Y; (H^1(\Omega))')]^n} &\leq C\varepsilon^{1/2}\|\nabla \phi^\varepsilon\|_{L^2(\Omega; \mathbb{R}^n)} \\ &\leq C\varepsilon^{1/2}\|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \end{aligned} \quad (6.9)$$

The above inequalities (6.8) and (6.9) yield

$$\left| \int_{\Omega \times Y} A(\nabla \phi^\varepsilon + \nabla_y \widehat{\phi}_\varepsilon) \nabla U \right| \leq C\varepsilon^{1/2}\|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}\|U\|_{H^2(\Omega)}. \quad (6.10)$$

We set

$$\forall x \in \mathbb{R}^n, \quad \rho_\varepsilon(x) = \inf \left\{ 1, \frac{\rho(x)}{\varepsilon} \right\}.$$

Now, we take  $\bar{\chi} \in H_{per}^1(Y)$  and we consider the test function  $u_\varepsilon \in H_0^1(\Omega)$  defined for a.e.  $x \in \Omega$  by

$$u_\varepsilon(x) = \varepsilon \rho_\varepsilon(x) \mathcal{Q}_\varepsilon \left( \frac{\partial U}{\partial x_i} \right) (x) \bar{\chi} \left( \frac{x}{\varepsilon} \right).$$

Due to (2.21)<sub>2</sub> and (6.7) we get

$$\left\| \mathcal{Q}_\varepsilon \left( \frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left( \frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}; \mathbb{R}^n})} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)} \quad (6.11)$$

Then by a straightforward calculation and thanks to (2.21)<sub>2</sub>-(2.22)<sub>2</sub> and (6.7)-(6.11) we obtain

$$\left\| \nabla u_\varepsilon - \mathcal{Q}_\varepsilon \left( \frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left( \frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}$$

which in turn with again (6.11) give

$$\|\nabla u_\varepsilon\|_{L^2(\tilde{\Omega}_{\sqrt{n\varepsilon}; \mathbb{R}^n})} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)} \quad (6.12)$$

and then with (2.22)<sub>1</sub> they lead to

$$\left\| \nabla u_\varepsilon - \mathcal{M}_\varepsilon \left( \frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left( \frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \varepsilon^{1/2} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}.$$

In (6.6) we replace  $\nabla u_\varepsilon$  with  $\mathcal{M}_\varepsilon \left( \frac{\partial U}{\partial x_i} \right) \nabla_y \bar{\chi} \left( \frac{\cdot}{\varepsilon} \right)$ ; we continue using (2.12)-(2.13) and (6.2)<sub>1</sub>-(6.12) to obtain

$$\left| \int_{\Omega \times Y} A(y) \mathcal{T}_\varepsilon(\nabla \phi^\varepsilon)(x, y) \mathcal{M}_\varepsilon \left( \frac{\partial U}{\partial x_i} \right) (x) \nabla_y \bar{\chi}(y) dx dy \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}$$

which with (2.17)<sub>2</sub> and then (6.9) give

$$\left| \int_{\Omega \times Y} A(y) (\nabla \phi^\varepsilon(x) + \nabla_y \hat{\phi}_\varepsilon(x, y)) \frac{\partial U}{\partial x_i}(x) \nabla_y \bar{\chi}(y) dx dy \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)} \|\bar{\chi}\|_{H^1(Y)}. \quad (6.13)$$

As in [16] we introduce the adjoint correctors  $\bar{\chi}_i \in H_{per}^1(Y)$ ,  $i \in \{1, \dots, n\}$ , defined by

$$\int_Y A(y) \nabla_y \psi(y) \nabla_y (\bar{\chi}_i(y) + y_i) dy = 0 \quad \forall \psi \in H_{per}^1(Y). \quad (6.14)$$

From (6.13) we get

$$\left| \int_{\Omega \times Y} A(\nabla \phi^\varepsilon + \nabla_y \hat{\phi}_\varepsilon) \nabla_y \left( \sum_{i=1}^n \frac{\partial U}{\partial x_i} \bar{\chi}_i \right) \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}$$



and from the definition (4.2) of the correctors  $\chi_i$  we have

$$\int_{\Omega \times Y} A \left( \nabla \phi^\varepsilon + \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \nabla_y \chi_i \right) \nabla_y \left( \sum_{j=1}^n \frac{\partial U}{\partial x_j} \bar{\chi}_j \right) = 0.$$

Thus

$$\left| \int_{\Omega \times Y} A \nabla_y \left( \widehat{\phi}_\varepsilon - \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \chi_i \right) \nabla_y \left( \sum_{j=1}^n \frac{\partial U}{\partial x_j} \bar{\chi}_j \right) \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}$$

and thanks to (6.14) we obtain

$$\left| \int_{\Omega \times Y} A \nabla_y \left( \widehat{\phi}_\varepsilon - \sum_{i=1}^n \frac{\partial \phi^\varepsilon}{\partial x_i} \chi_i \right) \nabla U \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}.$$

The above estimate, (6.10) and the expression (4.5) of the matrix  $\mathcal{A}$  yield

$$\left| \int_{\Omega} \mathcal{A} \nabla \phi^\varepsilon \nabla U \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}.$$

Finally, since we have  $\int_{\Omega} \mathcal{A} \nabla \phi_{g_\varepsilon} \nabla v = 0$  for any  $v \in H_0^1(\Omega)$ , we deduce that

$$\forall U \in H_0^1(\Omega) \cap H^2(\Omega), \quad \left| \int_{\Omega} \mathcal{A} \nabla (\phi^\varepsilon - \phi_{g_\varepsilon}) \nabla U \right| \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \|U\|_{H^2(\Omega)}.$$

Now, let  $U_\varepsilon \in H_0^1(\Omega)$  be the solution of the following variational problem:

$$\int_{\Omega} \mathcal{A} \nabla v \nabla U_\varepsilon = \int_{\Omega} v (\phi^\varepsilon - \phi_{g_\varepsilon}), \quad \forall v \in H_0^1(\Omega).$$

Under the assumption on the boundary of  $\Omega$ , we know that  $U_\varepsilon$  belongs to  $H_0^1(\Omega) \cap H^2(\Omega)$  and satisfies  $\|U_\varepsilon\|_{H^2(\Omega)} \leq C \|\phi^\varepsilon - \phi_{g_\varepsilon}\|_{L^2(\Omega)}$  (the constant do not depend on  $\varepsilon$ ). Therefore, the second estimate in (6.2) is proved.

*Step 3.* We prove the third estimate in (6.2) and (6.3). The partial derivative  $\frac{\partial \phi_{g_\varepsilon}}{\partial x_i}$  satisfies

$$\operatorname{div} \left( \mathcal{A} \nabla \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \in L^2(\Omega).$$

Thus, from Remark 5.8 and estimate (6.4)<sub>2</sub> we get

$$\left\| \rho \nabla \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \left\| \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right\|_{L^2(\Omega)} \leq C \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \quad (6.15)$$

Now, let  $U$  be in  $H_0^1(\Omega)$ , the function  $\rho U$  belongs to  $H_{1/\rho}^1(\Omega)$ . Applying the Theorem 3.2 with the function  $\rho U$ , there exists  $\widehat{u}_\varepsilon \in L^2(\Omega; H_{per}^1(Y))$  such that

$$\|\mathcal{T}_\varepsilon(\nabla(\rho U)) - \nabla(\rho U) - \nabla_y \widehat{u}_\varepsilon\|_{L^2(Y; (H_{1/\rho}^1(\Omega; \mathbb{R}^n))')} \leq C \varepsilon \|\rho U\|_{H_{1/\rho}^1(\Omega)} \leq C \varepsilon \|U\|_{H^1(\Omega)}. \quad (6.16)$$

The above estimates (6.15) and (6.16) lead to

$$\left| \int_{\Omega \times Y} A \left( \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \left( \mathcal{T}_\varepsilon(\nabla(\rho U)) - \nabla(\rho U) - \nabla_y \widehat{u}_\varepsilon \right) \right| \leq C\varepsilon \|U\|_{H^1(\Omega)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}$$

By definition of the correctors  $\chi_i$  we have

$$\int_{\Omega \times Y} A \left( \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \nabla_y \widehat{u}_\varepsilon = 0.$$

Besides, from the definitions of the function  $\phi_{g_\varepsilon}$  and the homogenized matrix  $\mathcal{A}$  we have

$$0 = \int_{\Omega} \mathcal{A} \nabla \phi_{g_\varepsilon} \nabla(\rho U) = \int_{\Omega \times Y} A \left( \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \nabla(\rho U).$$

The above inequality and equalities yield

$$\left| \int_{\Omega \times Y} A \left( \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \mathcal{T}_\varepsilon(\nabla(\rho U)) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \quad (6.17)$$

We have

$$\nabla(\rho U) = \rho \left( \nabla U + \nabla \rho \frac{U}{\rho} \right).$$

Then since  $U/\rho \in L^2(\Omega)$  and  $\|U/\rho\|_{L^2(\Omega)} \leq C\|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)}$  and due to (3.2) we get

$$\left\| \mathcal{T}_\varepsilon(\nabla(\rho U)) - \rho \mathcal{T}_\varepsilon \left( \nabla U + \nabla \rho \frac{U}{\rho} \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\varepsilon \left\| \nabla U + \nabla \rho \frac{U}{\rho} \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\varepsilon \|U\|_{H^1(\Omega)}.$$

From (6.17) and the above inequalities we deduce that

$$\left| \int_{\Omega \times Y} A \left( \rho \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \mathcal{T}_\varepsilon \left( \nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

We recall that  $\rho \nabla \phi_{g_\varepsilon} \in H_0^1(\Omega; \mathbb{R}^n)$ , hence from (2.14)<sub>2</sub>, (2.17)<sub>1</sub> and (6.15) we get

$$\begin{aligned} & \left| \int_{\Omega \times Y} A \left( \rho \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \nabla_y \chi_i \right) \mathcal{T}_\varepsilon \left( \nabla U + \nabla \rho \frac{U}{\rho} \right) \right. \\ & \left. - \int_{\Omega \times Y} A \left( \mathcal{T}_\varepsilon(\rho \nabla \phi_{g_\varepsilon}) + \sum_{i=1}^n \mathcal{M}_\varepsilon \left( \rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \right) \mathcal{T}_\varepsilon \left( \nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

Then transforming by inverse unfolding we obtain

$$\left| \int_{\widehat{\Omega}_\varepsilon} A_\varepsilon \left( \rho \nabla \phi_{g_\varepsilon} + \sum_{i=1}^n \mathcal{M}_\varepsilon \left( \rho \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \left( \nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C\varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

Now, thanks to (2.28) and (6.15) we get

$$\left| \int_{\Omega} A_{\varepsilon} \rho \left( \nabla \phi_{g_{\varepsilon}} + \sum_{i=1}^n \mathcal{M}_{\varepsilon} \left( \frac{\partial \phi_{g_{\varepsilon}}}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \left( \nabla U + \nabla \rho \frac{U}{\rho} \right) \right| \leq C \varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_{\varepsilon}\|_{H^{1/2}(\partial\Omega)}.$$

Then using (2.29)<sub>1</sub> it leads to

$$\left| \int_{\Omega} A_{\varepsilon} \left( \nabla \phi_{g_{\varepsilon}} + \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left( \frac{\partial \phi_{g_{\varepsilon}}}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \nabla(\rho U) \right| \leq C \varepsilon \|\nabla U\|_{L^2(\Omega; \mathbb{R}^n)} \|g_{\varepsilon}\|_{H^{1/2}(\partial\Omega)}.$$

We recall that  $\int_{\Omega} A_{\varepsilon} \nabla \phi^{\varepsilon} \nabla(\rho U) = 0$ . We choose  $U = \rho \left( \phi^{\varepsilon} - \phi_{g_{\varepsilon}} - \varepsilon \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left( \frac{\partial \phi_{g_{\varepsilon}}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right)$  which belongs to  $H_0^1(\Omega)$ . Due to the second estimate in (6.2), the third one in (6.2) follows immediately.

The estimate (6.3) is the consequence of (2.29)<sub>2</sub>, (6.2)<sub>2</sub>, (6.2)<sub>3</sub>, (6.4)<sub>2</sub> and (6.15).  $\square$

**Corollary 6.2.** *Let  $(\phi^{\varepsilon})_{\varepsilon > 0}$  be a sequence of functions belonging to  $H^1(\Omega)$  and satisfying (6.1). We set  $g_{\varepsilon} = \phi_{\partial\Omega}^{\varepsilon}$ , if we have*

$$g_{\varepsilon} \rightharpoonup g \quad \text{weakly in } H^{1/2}(\partial\Omega)$$

then we obtain

$$\begin{aligned} \phi^{\varepsilon} &\rightharpoonup \phi_g \quad \text{weakly in } H^1(\Omega), \\ \phi^{\varepsilon} - \phi_g - \varepsilon \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left( \frac{\partial \phi_g}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) &\longrightarrow 0 \quad \text{strongly in } H_{\rho}^1(\Omega). \end{aligned} \quad (6.18)$$

Moreover, if

$$g_{\varepsilon} \longrightarrow g \quad \text{strongly in } H^{1/2}(\partial\Omega) \quad (6.19)$$

then we have

$$\phi^{\varepsilon} - \phi_g - \varepsilon \sum_{i=1}^n \mathcal{Q}_{\varepsilon} \left( \frac{\partial \phi_g}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in } H^1(\Omega). \quad (6.20)$$

*Proof.* Thanks to (6.2)<sub>1</sub> the sequence  $(\phi^{\varepsilon})_{\varepsilon > 0}$  is uniformly bounded in  $H^1(\Omega)$ . Then due to Lemma 5.1 and Remark 5.8 we get

$$\|\phi_g - \phi_{g_{\varepsilon}}\|_{\rho} \leq C \|g - g_{\varepsilon}\|_{H^{-1/2}(\partial\Omega)}$$

which with (6.2)<sub>2</sub> (resp. (6.2)<sub>3</sub>) give the convergence (6.18)<sub>1</sub> (resp. (6.18)<sub>2</sub>).

Under the assumption (6.19), we use (5.2) and we proceed as in the proof of the Theorem 6.1 of [10] in order to obtain the strong convergence (6.20).  $\square$

**Theorem 6.3.** *Let  $\phi^\varepsilon$  be the solution of the following homogenization problem:*

$$-\operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) = f \quad \text{in } \Omega, \quad \phi^\varepsilon = g \quad \text{on } \partial\Omega$$

where  $f \in L^2(\Omega)$  and  $g \in H^{1/2}(\partial\Omega)$ . We have

$$\begin{aligned} \|\phi^\varepsilon - \Phi\|_{L^2(\Omega)} &\leq C\{\varepsilon\|f\|_{L^2(\Omega)} + \varepsilon^{1/2}\|g\|_{H^{1/2}(\partial\Omega)}\}, \\ \left\| \rho \left( \nabla \phi^\varepsilon - \nabla \Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \nabla_y \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} &\leq C\{\varepsilon\|f\|_{L^2(\Omega)} + \varepsilon^{1/2}\|g\|_{H^{1/2}(\partial\Omega)}\} \end{aligned}$$

where  $\Phi$  is the solution of the homogenized problem

$$-\operatorname{div}(\mathcal{A} \nabla \Phi) = f \quad \text{in } \Omega, \quad \Phi = g \quad \text{on } \partial\Omega.$$

Moreover we have

$$\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in } H^1(\Omega). \quad (6.21)$$

*Proof.* Let  $\tilde{\phi}^\varepsilon$  be the solution of the homogenization problem

$$\tilde{\phi}^\varepsilon \in H_0^1(\Omega), \quad -\operatorname{div}(A_\varepsilon \nabla \tilde{\phi}^\varepsilon) = f \quad \text{in } \Omega$$

and  $\tilde{\Phi}$  the solution of the homogenized problem

$$\tilde{\Phi} \in H_0^1(\Omega), \quad -\operatorname{div}(\mathcal{A} \nabla \tilde{\Phi}) = f \quad \text{in } \Omega.$$

The Theorem 3.2 in [16] gives the following estimate:

$$\|\tilde{\phi}^\varepsilon - \tilde{\Phi}\|_{L^2(\Omega)} + \left\| \rho \nabla \left( \tilde{\phi}^\varepsilon - \tilde{\Phi} - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \tilde{\Phi}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\varepsilon \|f\|_{L^2(\Omega)} \quad (6.22)$$

while the Theorem 4.1 in [15] gives

$$\left\| \tilde{\phi}^\varepsilon - \tilde{\Phi} - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \tilde{\Phi}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \|f\|_{L^2(\Omega)}. \quad (6.23)$$

The function  $\phi^\varepsilon - \tilde{\phi}^\varepsilon$  satisfies

$$\operatorname{div}(A_\varepsilon \nabla (\phi^\varepsilon - \tilde{\phi}^\varepsilon)) = 0 \quad \text{in } \Omega, \quad \phi^\varepsilon - \tilde{\phi}^\varepsilon = g \quad \text{on } \partial\Omega.$$

Thanks to the inequalities (6.2) and (6.22) we deduce the estimates of the theorem. The strong convergence (6.21) is a consequence of (6.23) and the strong convergence (6.20) after having observed that  $\Phi - \tilde{\Phi} = \phi_g$ .  $\square$

**Remark 6.4.** *In Theorem 6.3, if  $g \in H^{3/2}(\partial\Omega)$  then in the estimates therein, we can replace  $\varepsilon^{1/2}\|g\|_{H^{1/2}(\partial\Omega)}$  with  $\varepsilon\|g\|_{H^{3/2}(\partial\Omega)}$ . Moreover we have the following  $H^1$ -global error estimate:*

$$\left\| \phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \Phi}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \{ \|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)} \}.$$

## 7 A first result with strongly oscillating boundary data

In this section we consider the solution  $\phi^\varepsilon$  of the homogenization problem

$$\begin{aligned} \operatorname{div}(A_\varepsilon \nabla \phi^\varepsilon) &= 0 & \text{in } \Omega \\ \phi^\varepsilon &= g_\varepsilon & \text{on } \partial\Omega \end{aligned} \quad (7.1)$$

where  $g_\varepsilon \in H^{1/2}(\partial\Omega)$ . As a consequence of the Theorem 6.1 we obtain the following result:

**Theorem 7.1.** *Let  $\phi^\varepsilon$  be the solution of the problem (7.1). If we have*

$$g_\varepsilon \rightharpoonup g \quad \text{weakly in } H^{-1/2}(\partial\Omega)$$

and

$$\varepsilon^{1/2} g_\varepsilon \longrightarrow 0 \quad \text{strongly in } H^{1/2}(\partial\Omega) \quad (7.2)$$

then

$$\phi^\varepsilon \rightharpoonup \phi_g \quad \text{weakly in } H_\rho^1(\Omega). \quad (7.3)$$

Furthermore, if we have

$$g_\varepsilon \longrightarrow g \quad \text{strongly in } H^{-1/2}(\partial\Omega)$$

then

$$\phi^\varepsilon - \phi_g - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \longrightarrow 0 \quad \text{strongly in } H_\rho^1(\Omega). \quad (7.4)$$

*Proof.* Due to (6.3) the sequence  $(\phi^\varepsilon)_{\varepsilon>0}$  is uniformly bounded in  $H_\rho^1(\Omega)$ . From the estimates (6.2)<sub>3</sub> and (6.4)<sub>2</sub> we get

$$\left\| \phi^\varepsilon - \phi_{g_\varepsilon} - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \right\|_{H_\rho^1(\Omega)} \leq C \varepsilon^{1/2} \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)}.$$

Then using the variational problem (5.5) and estimate (6.4)<sub>2</sub> we obtain

$$\phi_{g_\varepsilon} \rightharpoonup \phi_g \quad \text{weakly in } H_\rho^1(\Omega).$$

Since the sequence  $\varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right)$  is uniformly bounded in  $H_\rho^1(\Omega)$  and strongly converges to 0 in  $L^2(\Omega)$ , we have  $\varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon \left( \frac{\partial \phi_{g_\varepsilon}}{\partial x_i} \right) \chi_i \left( \frac{\cdot}{\varepsilon} \right) \rightharpoonup 0$  weakly in  $H_\rho^1(\Omega)$ . Therefore the weak convergence (7.3) is proved.

In the case  $g_\varepsilon \longrightarrow g$  strongly in  $H^{-1/2}(\partial\Omega)$ , the estimates (5.4) and (5.8) lead to

$$\|\phi_{g_\varepsilon} - \phi_g\|_{H_\rho^1(\Omega)} \leq C \|g_\varepsilon - g\|_{H^{-1/2}(\partial\Omega)}.$$

Hence with (2.29)<sub>2</sub> they yield (7.4).  $\square$

In a forthcoming paper we will show that in both cases (weak or strong convergence of the sequence  $(g_\varepsilon)_{\varepsilon>0}$  towards  $g$  in  $H^{-1/2}(\partial\Omega)$ ) the assumption (7.2) is essential in order to obtain at least (7.3).

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