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# *Electro-magneto-optical Effects*

R. A. TOUPIN & R. S. RIVLIN

## 1. Introduction

Electro-magneto-optical effects in stationary materials are observed when the materials are placed in strong, static electric or magnetic fields and an electromagnetic wave (light) traverses the medium. The Faraday effect is an example of the class of phenomena we have in mind. Clearly, the existence of electro-magneto-optical effects in material media is direct evidence that the equations for the electromagnetic field in a material medium, unlike the equations for it in vacuum, are non-linear since the sum of two solutions generally fails to be a solution. An analysis of electro-magneto-optical effects, though based on non-linear equations for which all but the simplest exact solutions are difficult to exhibit, may be simplified and made tractable by assuming that the dynamical part of the solution (the light wave) has such weak intensity that the vectors which describe it may be treated as infinitesimals.

Accordingly, in this paper we assume that the electric displacement and magnetic intensity fields are functions of the static electric and magnetic induction fields and linear functionals of the small dynamical electric and magnetic induction fields. This starting point is derived, in the Appendix, from the assumption that for a more general class of fields, not necessarily representable as the superposition of small dynamic fields on large static fields, the electric displacement and magnetic intensity fields are non-linear functionals of the electric and magnetic induction fields.

From our initial assumption regarding the variables which enter into the constitutive equations, we derive the limitations which are imposed on them by the assumption that the material is isotropic. The isotropic constitutive equations are then applied to the study of the manner in which a plane electromagnetic wave is propagated in an isotropic material to which static electric and magnetic induction fields are simultaneously applied.

From these results we consider in greater detail the cases when the applied electric field is zero, leading to the magneto-optical effects, and that when the applied magnetic induction field is zero, leading to the electro-optical effects.

The case when both the static electric and magnetic induction fields are non-zero will be given more detailed consideration in a later paper.

## 2. The constitutive equations for small varying fields superposed on large constant fields

We assume that there exist in a body an electric field  $\mathbf{E}^*$  and a magnetic induction field  $\mathbf{B}^*$ . Each of these is assumed to be the resultant of a time-

independent field and a small field which may vary with time  $\tau$ . Thus,

$$\begin{aligned} \mathbf{E}^*(\tau) &= \mathcal{E} + \varepsilon \mathbf{E}(\tau) \\ \mathbf{B}^*(\tau) &= \mathcal{B} + \varepsilon \mathbf{B}(\tau), \end{aligned} \quad (2.1)$$

where  $\mathcal{E}$  and  $\mathcal{B}$  denote the time-independent fields and  $\varepsilon \mathbf{E}(\tau)$  and  $\varepsilon \mathbf{B}(\tau)$  the small additional time-dependent fields.  $\varepsilon$  is a small constant. The electric displacement field  $\mathbf{D}^*(\tau)$  and the magnetic intensity field  $\mathbf{H}^*(\tau)$  may also be expressed in the forms

$$\begin{aligned} \mathbf{D}^*(\tau) &= \mathcal{D} + \varepsilon \mathbf{D}(\tau) \\ \mathbf{H}^*(\tau) &= \mathcal{H} + \varepsilon \mathbf{H}(\tau), \end{aligned} \quad (2.2)$$

where  $\mathcal{D}$  and  $\mathcal{H}$  are the values of  $\mathbf{D}^*(\tau)$  and  $\mathbf{H}^*(\tau)$  corresponding to  $\varepsilon=0$ .  $\mathcal{D}$  and  $\mathcal{H}$  are, of course, independent of time. For brevity we shall employ the notation

$$\mathbf{D}(t) = \mathcal{D}, \quad \mathbf{H}(t) = \mathcal{H}, \quad \mathbf{E}(t) = \mathbf{E} \quad \text{and} \quad \mathbf{B}(t) = \mathbf{B}, \quad (2.3)$$

where  $t$  is a particular instant of time.

In general, we may expect that for a given material the values of  $\varepsilon \mathbf{D}$  and  $\varepsilon \mathbf{H}$  will depend on  $\mathcal{E}$  and  $\mathcal{B}$  and on the values taken by  $\varepsilon \mathbf{E}(\tau)$  and  $\varepsilon \mathbf{B}(\tau)$  throughout the time interval  $-\infty \leq \tau \leq t$ . We denote the components of  $\mathbf{D}$  and  $\mathbf{H}$  in a rectangular Cartesian coordinate system  $x$  by  $D_i$  and  $H_i$  respectively, those of  $\mathbf{E}(\tau)$  and  $\mathbf{B}(\tau)$  by  $E_i(\tau)$  and  $B_i(\tau)$  respectively and so on. Also we use the notation

$$E_i(t) = E_i \quad \text{and} \quad B_i(t) = B_i. \quad (2.4)$$

Then, each of the vector components  $\varepsilon D_i$  and  $\varepsilon H_i$  may be expressed as a functional of  $\varepsilon E_p(\tau)$  and  $\varepsilon B_p(\tau)$  over the range  $-\infty \leq \tau \leq t$  and an ordinary function of  $\mathcal{E}_p$  and  $\mathcal{B}_p$ , thus

$$\begin{aligned} \varepsilon D_i &= \Phi_i[\varepsilon E_p(\tau), \varepsilon B_p(\tau); \mathcal{E}_p, \mathcal{B}_p] \\ \varepsilon H_i &= \Psi_i[\varepsilon E_p(\tau), \varepsilon B_p(\tau); \mathcal{E}_p, \mathcal{B}_p]. \end{aligned} \quad (2.5)$$

and

### 3. Restrictions on the constitutive equations for hemihedral isotropic materials

If the material is hemihedral isotropic in the absence of any applied fields, then the constructive equations (2.5) must be form-invariant under the proper orthogonal group of transformations. Following a procedure similar to that adopted in previous papers [1, 2, 3], it can be shown that the restrictions imposed on  $\Phi_i$  and  $\Psi_i$  by this form-invariance may be expressed in the forms

$$\begin{aligned} \Phi_i[\varepsilon E'_p(\tau), \varepsilon B'_p(\tau); \mathcal{E}'_p, \mathcal{B}'_p] &= a_i, \quad \Phi_j[\varepsilon E_p(\tau), \varepsilon B_p(\tau); \mathcal{E}_p, \mathcal{B}_p] \\ \Psi_i[\varepsilon E'_p(\tau), \varepsilon B'_p(\tau); \mathcal{E}'_p, \mathcal{B}'_p] &= a_i, \quad \Psi_j[\varepsilon E_p(\tau), \varepsilon B_p(\tau); \mathcal{E}_p, \mathcal{B}_p], \end{aligned} \quad (3.4)$$

and

where

$$\begin{aligned} E'_p(\tau) &= a_{pq} E_q(\tau), & B'_p(\tau) &= a_{pq} B_q(\tau), \\ \mathcal{E}'_p &= a_{pq} \mathcal{E}_q & \text{and} & \quad \mathcal{B}'_p = a_{pq} \mathcal{B}_q, \end{aligned} \quad (3.2)$$

and  $a_{pq}$  are real constants satisfying the equations

$$a_{ps} a_{qs} = a_{sp} a_{sq} = \delta_{pq}, \quad |a_{pq}| = 1. \quad (3.3)$$

We introduce the notation

$$\begin{aligned} E_p(t) &= E_p = E_p^{(0)}, & B_p(t) &= B_p = B_p^{(0)}, \\ \frac{d^\alpha E_p(t)}{dt^\alpha} &= E_p^{(\alpha)}, & \frac{d^\alpha B_p(t)}{dt^\alpha} &= B_p^{(\alpha)} \quad (\alpha = 1, 2, \dots, \mu). \end{aligned} \quad (3.4)$$

In order to obtain the restrictions imposed on  $\Phi_i$  and  $\Psi_i$  by the relations (3.1) in more explicit form, we assume that, in (2.5),  $\Phi_i$  and  $\Psi_i$  are continuous hereditary functionals of  $\varepsilon E_p(\tau)$  and  $\varepsilon B_p(\tau)$  over the range  $-\infty \leq \tau \leq t$  and polynomials in  $\varepsilon E_p^{(\alpha)}$  and  $\varepsilon B_p^{(\alpha)}$  ( $\alpha=0, 1, 2, \dots, \mu$ ) and in  $\mathcal{E}_p$  and  $\mathcal{B}_p$ . We also assume that  $\varepsilon$  is small enough so that we may neglect terms in  $\Phi_i$  and  $\Psi_i$  which are non-linear in  $\varepsilon$ . We use the notation

$$\begin{aligned} \Phi &= (\Phi_i), & \Psi &= (\Psi_i), & E^{(\alpha)} &= (E_i^{(\alpha)}) \\ \text{and} & & \mathbf{B}^{(\alpha)} &= (B_j^{(\alpha)}) \quad (\alpha = 0, 1, 2, \dots, \mu). \end{aligned} \quad (3.5)$$

Then, following a procedure similar to that adopted in previous papers and bearing in mind that  $\Phi = \Psi = 0$  when  $\varepsilon = 0$ , it can be shown that the vectors  $\Phi$  and  $\Psi$  and hence the vectors  $\varepsilon \mathbf{D}$  and  $\varepsilon \mathbf{B}$  may be expressed as linear combinations of vectors of the forms

$$\begin{aligned} \varepsilon \mathbf{E}^{(\alpha)}, & \quad \varepsilon \int_{-\infty}^t \chi_1(t-\tau) \mathbf{E}(\tau) d\tau, \\ \varepsilon \mathcal{L} \times \mathbf{B}^{(\alpha)}, & \quad \varepsilon \mathcal{B} \times \mathbf{E}^{(\alpha)}, \\ \varepsilon \mathcal{L} \times \int_{-\infty}^t \chi_2(t-\tau) \mathbf{B}(\tau) d\tau, & \\ \varepsilon \mathcal{B} \times \int_{-\infty}^t \chi_3(t-\tau) \mathbf{E}(\tau) d\tau; & \end{aligned} \quad (3.6)$$

vectors of the forms

$$\begin{aligned} \varepsilon \mathbf{B}^{(\alpha)}, & \quad \varepsilon \int_{-\infty}^t \chi_4(t-\tau) \mathbf{B}(\tau) d\tau, \\ \varepsilon \mathcal{L} \times \mathbf{E}^{(\alpha)}, & \quad \varepsilon \mathcal{B} \times \mathbf{B}^{(\alpha)}, \\ \varepsilon \mathcal{L} \times \int_{-\infty}^t \chi_5(t-\tau) \mathbf{E}(\tau) d\tau, & \\ \varepsilon \mathcal{B} \times \int_{-\infty}^t \chi_6(t-\tau) \mathbf{B}(\tau) d\tau; & \end{aligned} \quad (3.7)$$

$$\mathcal{L}, \quad \mathcal{L} \times \mathcal{B} \quad (3.8)$$

and

$$\mathcal{B}, \quad (3.9)$$

where  $\alpha = 0, 1, 2, \dots, \mu$ .

The coefficients of the vectors (3.6) and (3.7) are polynomials in

$$\mathcal{L} \cdot \mathcal{L}, \quad \mathcal{B} \cdot \mathcal{B} \quad (3.10)$$

and

$$\mathcal{L} \cdot \mathcal{B}. \quad (3.11)$$

The coefficients of the vectors (3.8) and (3.9) are polynomials in (3.10) and (3.11) and are linear in the quantities

$$\varepsilon \mathcal{L} \cdot \mathbf{E}^{(\alpha)}, \quad \varepsilon \mathcal{B} \cdot \mathbf{B}^{(\alpha)}, \quad \varepsilon [\mathbf{E}^{(\alpha)}, \mathcal{L}, \mathcal{B}] \quad (3.12)$$

and

$$\varepsilon \mathcal{L} \cdot \mathbf{B}^{(\alpha)}, \quad \varepsilon \mathcal{B} \cdot \mathbf{E}^{(\alpha)}, \quad \varepsilon [\mathbf{B}^{(\alpha)}, \mathcal{L}, \mathcal{B}], \quad (3.13)$$

and in expressions obtained from (3.12) and (3.13) by substituting vectors of the forms  $\int_{-\infty}^t \chi(t-\tau) \mathbf{E}(\tau) d\tau$  and  $\int_{-\infty}^t \chi(t-\tau) \mathbf{B}(\tau)$  for the vectors  $\mathbf{E}^{(\alpha)}$  and  $\mathbf{B}^{(\alpha)}$  respectively.

#### 4. Restrictions on the constitutive equations for holohedral isotropic materials

We note that the vectors (3.6) and (3.8) are changed in sign by the central inversion transformation while the vectors (3.7) and (3.9) are unchanged by this transformation. Similarly, the scalars (3.11) and (3.13) are changed in sign by the central inversion transformation, while the scalars (3.10) and (3.12) are unchanged.

Since the constitutive equations for a holohedral isotropic material must be form-invariant under the central inversion transformation, in the constitutive equation for  $\varepsilon \mathbf{D}$ , each of the vectors (3.6) must have as its coefficient a polynomial in

$$\mathcal{L} \cdot \mathcal{L}, \quad \mathcal{B} \cdot \mathcal{B} \quad \text{and} \quad (\mathcal{L} \cdot \mathcal{B})^2; \quad (4.1)$$

each of the vectors (3.7) must have as its coefficient an expression of the form

$$\alpha \mathcal{L} \cdot \mathcal{B}, \quad (4.2)$$

where  $\alpha$  is a polynomial in the quantities (4.1); each of the vectors (3.8) must have as its coefficient an expression of the form

$$\begin{aligned} \varepsilon \sum_{\alpha} \{ & A_1^{(\alpha)} \mathcal{L} \cdot \mathbf{E}^{(\alpha)} + A_2^{(\alpha)} \mathcal{B} \cdot \mathbf{B}^{(\alpha)} + A_3^{(\alpha)} [\mathbf{E}^{(\alpha)}, \mathcal{L}, \mathcal{B}] \} + \\ & + \varepsilon \mathcal{L} \cdot \mathcal{B} \sum_{\alpha} \{ C_1^{(\alpha)} \mathcal{L} \cdot \mathbf{B}^{(\alpha)} + C_2^{(\alpha)} \mathcal{B} \cdot \mathbf{E}^{(\alpha)} + C_3^{(\alpha)} [\mathbf{B}^{(\alpha)}, \mathcal{L}, \mathcal{B}] \}, \end{aligned} \quad (4.3)$$

where the  $A$ 's and  $C$ 's are polynomials in  $\mathcal{L} \cdot \mathcal{L}$ ,  $\mathcal{B} \cdot \mathcal{B}$  and  $(\mathcal{L} \cdot \mathcal{B})^2$ , the vector (3.9) has as its coefficient an expression of the form

$$\begin{aligned} \varepsilon \mathcal{L} \cdot \mathcal{B} \sum_{\alpha} \{ & F_1^{(\alpha)} \mathcal{L} \cdot \mathbf{E}^{(\alpha)} + F_2^{(\alpha)} \mathcal{B} \cdot \mathbf{B}^{(\alpha)} + F_3^{(\alpha)} [\mathbf{E}^{(\alpha)}, \mathcal{L}, \mathcal{B}] \} + \\ & + \varepsilon \sum_{\alpha} \{ G_1^{(\alpha)} \mathcal{L} \cdot \mathbf{B}^{(\alpha)} + G_2^{(\alpha)} \mathcal{B} \cdot \mathbf{E}^{(\alpha)} + G_3^{(\alpha)} [\mathbf{B}^{(\alpha)}, \mathcal{L}, \mathcal{B}] \}, \end{aligned} \quad (4.4)$$

where the  $F$ 's and  $G$ 's are polynomials in  $\mathcal{L} \cdot \mathcal{L}$ ,  $\mathcal{B} \cdot \mathcal{B}$  and  $(\mathcal{L} \cdot \mathcal{B})^2$ .

Again, in the constitutive equation for  $\varepsilon \mathbf{H}$ , each of the vectors (3.6) must have as its coefficient an expression of the form (4.2); each of the vectors (3.7)

must have as its coefficient a polynomial in the quantities (4.1); each of the vectors (3.8) must have as its coefficient an expression of the form (4.4) and the vector (3.9) must have as its coefficient an expression of the form (4.3).

### 5. Constitutive equations for sinusoidal time-dependent fields

We now consider that the fields  $\varepsilon \mathbf{E}(\tau)$  and  $\varepsilon \mathbf{B}(\tau)$  vary sinusoidally with time with angular frequency  $\omega$ , so that we may write

$$\mathbf{E}(\tau) = \mathcal{R} \bar{\mathbf{E}}(\mathbf{x}) e^{-i\omega\tau} \quad \text{and} \quad \mathbf{B}(\tau) = \mathcal{R} \bar{\mathbf{B}}(\mathbf{x}) e^{-i\omega\tau}, \quad (5.1)$$

where  $\mathbf{x}$  is the position vector of a generic point of space. We shall, for the remainder of this paper, follow the common convention by omitting the symbol  $\mathcal{R}$ . Then, introducing (5.1) into (2.1), we obtain

$$\begin{aligned} \mathbf{E}^*(\tau) &= \mathcal{L} + \varepsilon \bar{\mathbf{E}}(\mathbf{x}) e^{-i\omega\tau} \\ \mathbf{B}^*(\tau) &= \mathcal{B} + \varepsilon \bar{\mathbf{B}}(\mathbf{x}) e^{-i\omega\tau}. \end{aligned} \quad (5.2)$$

Introducing (5.1) into the constitutive equations for a holohedral isotropic material discussed in §4, we obtain

$$\mathbf{D}(\tau) = \bar{\mathbf{D}}(\mathbf{x}) e^{-i\omega\tau} \quad \text{and} \quad \mathbf{H}(\tau) = \bar{\mathbf{H}}(\mathbf{x}) e^{-i\omega\tau}, \quad (5.3)$$

where  $\bar{\mathbf{D}}$  and  $\bar{\mathbf{H}}$  are given by expressions of the form

$$\begin{aligned} \bar{\mathbf{D}} &= \alpha_1 \bar{\mathbf{E}} + \alpha_2 \mathcal{B} \times \bar{\mathbf{E}} + \alpha_3 \mathcal{L} \times \bar{\mathbf{B}} + \\ &+ \mathcal{L} \cdot \mathcal{B} (\alpha_4 \bar{\mathbf{B}} + \alpha_5 \mathcal{L} \times \bar{\mathbf{E}} + \alpha_6 \mathcal{B} \times \bar{\mathbf{B}}) + \\ &+ \{\alpha_7 \mathcal{L} \cdot \bar{\mathbf{E}} + \alpha_8 \mathcal{B} \cdot \bar{\mathbf{B}} + \alpha_{19} [\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} + \\ &+ \{\alpha_9 \mathcal{L} \cdot \bar{\mathbf{E}} + \alpha_{10} \mathcal{B} \cdot \bar{\mathbf{B}} + \alpha_{20} [\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} \times \mathcal{B} + \\ &+ \mathcal{L} \cdot \mathcal{B} \{\alpha_{11} \mathcal{L} \cdot \bar{\mathbf{B}} + \alpha_{12} \mathcal{B} \cdot \bar{\mathbf{E}} + \alpha_{21} [\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} + \\ &+ \mathcal{L} \cdot \mathcal{B} \{\alpha_{13} \mathcal{L} \cdot \bar{\mathbf{B}} + \alpha_{14} \mathcal{B} \cdot \bar{\mathbf{E}} + \alpha_{22} [\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} \times \mathcal{B} + \\ &+ \{\alpha_{15} \mathcal{L} \cdot \bar{\mathbf{B}} + \alpha_{16} \mathcal{B} \cdot \bar{\mathbf{E}} + \alpha_{23} [\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}]\} \mathcal{B} + \\ &+ \mathcal{L} \cdot \mathcal{B} \{\alpha_{17} \mathcal{L} \cdot \bar{\mathbf{E}} + \alpha_{18} \mathcal{B} \cdot \bar{\mathbf{B}} + \alpha_{24} [\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}]\} \mathcal{B} \end{aligned} \quad (5.4)$$

and

$$\begin{aligned} \bar{\mathbf{H}} &= \beta_1 \bar{\mathbf{B}} + \beta_2 \mathcal{L} \times \bar{\mathbf{E}} + \beta_3 \mathcal{B} \times \bar{\mathbf{B}} + \\ &+ \mathcal{L} \cdot \mathcal{B} (\beta_4 \bar{\mathbf{E}} + \beta_5 \mathcal{B} \times \bar{\mathbf{E}} + \beta_6 \mathcal{L} \times \bar{\mathbf{B}}) + \\ &+ \{\beta_7 \mathcal{L} \cdot \bar{\mathbf{B}} + \beta_8 \mathcal{B} \cdot \bar{\mathbf{E}} + \beta_{19} [\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} + \\ &+ \{\beta_9 \mathcal{L} \cdot \bar{\mathbf{B}} + \beta_{10} \mathcal{B} \cdot \bar{\mathbf{E}} + \beta_{20} [\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} \times \mathcal{B} + \\ &+ \mathcal{L} \cdot \mathcal{B} \{\beta_{11} \mathcal{L} \cdot \bar{\mathbf{E}} + \beta_{12} \mathcal{B} \cdot \bar{\mathbf{B}} + \beta_{21} [\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} + \\ &+ \mathcal{L} \cdot \mathcal{B} \{\beta_{13} \mathcal{L} \cdot \bar{\mathbf{E}} + \beta_{14} \mathcal{B} \cdot \bar{\mathbf{B}} + \beta_{22} [\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}]\} \mathcal{L} \times \mathcal{B} + \\ &+ \{\beta_{15} \mathcal{L} \cdot \bar{\mathbf{E}} + \beta_{16} \mathcal{B} \cdot \bar{\mathbf{B}} + \beta_{23} [\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}]\} \mathcal{B} + \\ &+ \mathcal{L} \cdot \mathcal{B} \{\beta_{17} \mathcal{L} \cdot \bar{\mathbf{B}} + \beta_{18} \mathcal{B} \cdot \bar{\mathbf{E}} + \beta_{24} [\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}]\} \mathcal{B}, \end{aligned} \quad (5.5)$$

where the  $\alpha$ 's and  $\beta$ 's are functions of  $i\omega$  and polynomials in  $\mathcal{L} \cdot \mathcal{L}$ ,  $\mathcal{B} \cdot \mathcal{B}$  and  $(\mathcal{L} \cdot \mathcal{B})^2$ .

The following identities hold:

$$\begin{aligned}
[\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}] \mathcal{L} &\equiv -(\mathcal{L} \cdot \mathcal{B}) \mathcal{L} \times \bar{\mathbf{E}} + (\mathcal{L} \cdot \mathcal{L}) \mathcal{B} \times \bar{\mathbf{E}} + (\mathcal{L} \cdot \bar{\mathbf{E}}) \mathcal{L} \times \mathcal{B}, \\
[\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}] \mathcal{L} &\equiv -(\mathcal{L} \cdot \mathcal{B}) \mathcal{L} \times \bar{\mathbf{B}} - (\mathcal{L} \cdot \mathcal{L}) \mathcal{B} \times \bar{\mathbf{B}} + (\mathcal{L} \cdot \bar{\mathbf{B}}) \mathcal{L} \times \mathcal{B}, \\
[\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}] \mathcal{B} &\equiv (\mathcal{L} \cdot \mathcal{B}) \mathcal{B} \times \bar{\mathbf{E}} - (\mathcal{B} \cdot \mathcal{B}) \mathcal{L} \times \bar{\mathbf{E}} - (\mathcal{B} \cdot \bar{\mathbf{E}}) \mathcal{L} \times \mathcal{B}, \\
[\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}] \mathcal{B} &\equiv (\mathcal{L} \cdot \mathcal{B}) \mathcal{B} \times \bar{\mathbf{B}} + (\mathcal{B} \cdot \mathcal{B}) \mathcal{L} \times \bar{\mathbf{B}} - (\mathcal{B} \cdot \bar{\mathbf{B}}) \mathcal{L} \times \mathcal{B}.
\end{aligned} \tag{5.6}$$

From the third of these we obtain

$$\begin{aligned}
[\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}] \mathcal{L} \times \mathcal{B} &\equiv (\mathcal{L} \cdot \mathcal{B}) \mathcal{L} \times (\mathcal{B} \times \bar{\mathbf{E}}) - (\mathcal{B} \cdot \mathcal{B}) \mathcal{L} \times (\mathcal{L} \times \bar{\mathbf{E}}) - \\
&\quad - (\mathcal{B} \cdot \bar{\mathbf{E}}) \mathcal{L} \times (\mathcal{L} \times \mathcal{B}).
\end{aligned} \tag{5.7}$$

Using identities of the type

$$\mathcal{L} \times (\mathcal{B} \times \bar{\mathbf{E}}) \equiv (\mathcal{L} \cdot \bar{\mathbf{E}}) \mathcal{B} - (\mathcal{L} \cdot \mathcal{B}) \bar{\mathbf{E}}, \tag{5.8}$$

we obtain from (5.7),

$$\begin{aligned}
[\bar{\mathbf{E}}, \mathcal{L}, \mathcal{B}] \mathcal{L} \times \mathcal{B} &\equiv (\mathcal{L} \cdot \mathcal{B}) [(\mathcal{L} \cdot \bar{\mathbf{E}}) \mathcal{B} - (\mathcal{L} \cdot \mathcal{B}) \bar{\mathbf{E}}] - \\
&\quad - (\mathcal{B} \cdot \mathcal{B}) [(\mathcal{L} \cdot \bar{\mathbf{E}}) \mathcal{L} - (\mathcal{L} \cdot \mathcal{L}) \bar{\mathbf{E}}] - \\
&\quad - (\mathcal{B} \cdot \bar{\mathbf{E}}) [(\mathcal{L} \cdot \mathcal{B}) \mathcal{L} - (\mathcal{L} \cdot \mathcal{L}) \mathcal{B}].
\end{aligned} \tag{5.9}$$

Similarly, we have

$$\begin{aligned}
[\bar{\mathbf{B}}, \mathcal{L}, \mathcal{B}] \mathcal{L} \times \mathcal{B} &\equiv (\mathcal{L} \cdot \mathcal{B}) [(\mathcal{L} \cdot \bar{\mathbf{B}}) \mathcal{B} - (\mathcal{L} \cdot \mathcal{B}) \bar{\mathbf{B}}] - \\
&\quad - (\mathcal{B} \cdot \mathcal{B}) [(\mathcal{L} \cdot \bar{\mathbf{B}}) \mathcal{L} - (\mathcal{L} \cdot \mathcal{L}) \bar{\mathbf{B}}] - \\
&\quad - (\mathcal{B} \cdot \bar{\mathbf{B}}) [(\mathcal{L} \cdot \mathcal{B}) \mathcal{L} - (\mathcal{L} \cdot \mathcal{L}) \mathcal{B}].
\end{aligned} \tag{5.10}$$

Using the relations (5.6), (5.9) and (5.10), it is clear that we may, without loss of generality, take  $\alpha_{19} = \alpha_{20} = \dots = \alpha_{24} = 0$  and  $\beta_{19} = \beta_{20} = \dots = \beta_{24} = 0$  in equations (5.4) and (5.5).

## 6. Constitutive equations for a plane wave

We now consider that the time-dependent fields are those for an infinite plane wave, with angular frequency  $\omega$  and wave number  $k$ , in the direction of the unit vector  $\mathbf{n}$ , and are given by

$$\begin{aligned}
\mathbf{E} &= \mathbf{e} e^{i(k\mathbf{n} \cdot \mathbf{x} - \omega t)}, \\
\mathbf{B} &= \mathbf{b} e^{i(k\mathbf{n} \cdot \mathbf{x} - \omega t)}, \\
\mathbf{D} &= \mathbf{d} e^{i(k\mathbf{n} \cdot \mathbf{x} - \omega t)}, \\
\mathbf{H} &= \mathbf{h} e^{i(k\mathbf{n} \cdot \mathbf{x} - \omega t)},
\end{aligned} \tag{6.1}$$

where the complex vectors  $\mathbf{e}$ ,  $\mathbf{b}$ ,  $\mathbf{d}$  and  $\mathbf{h}$  are independent of position and time.  $\mathbf{x}$  denotes the position vector of a generic point of space.

In this case, we may replace  $\bar{\mathbf{E}}$ ,  $\bar{\mathbf{B}}$ ,  $\bar{\mathbf{D}}$  and  $\bar{\mathbf{H}}$  by  $\mathbf{e}$ ,  $\mathbf{b}$ ,  $\mathbf{d}$  and  $\mathbf{h}$  in the constitutive equations (6.1), in which we take  $\alpha_{19} = \alpha_{20} = \dots = \alpha_{24} = 0$  and  $\beta_{19} = \beta_{20} = \dots = \beta_{24} = 0$ . These equations may then be re-written as

$$\begin{aligned}
\mathbf{d} &= \Phi \cdot \mathbf{e} + \Psi \cdot \mathbf{b}, \\
\mathbf{h} &= \Omega \cdot \mathbf{e} + \Lambda \cdot \mathbf{b},
\end{aligned} \tag{6.2}$$

where  $\Phi$ ,  $\Psi$ ,  $\Omega$  and  $\Lambda$  are the  $3 \times 3$  matrices defined by  $\Phi = \|\Phi_{ij}\|$ ,  $\Psi = \|\Psi_{ij}\|$ , ... and

$$\begin{aligned}
\Phi_{ij} &= \alpha_1 \delta_{ij} - \alpha_2 \varepsilon_{ijk} \mathcal{B}_k - \alpha_5 \mathcal{E}_m \mathcal{B}_m \varepsilon_{ijk} \mathcal{E}_k + \alpha_7 \mathcal{E}_i \mathcal{E}_j + \\
&\quad + \alpha_9 \varepsilon_{imn} \mathcal{E}_m \mathcal{B}_n \mathcal{E}_j + \alpha_{12} \mathcal{E}_m \mathcal{B}_m \mathcal{E}_i \mathcal{B}_j + \\
&\quad + \alpha_{14} \mathcal{E}_m \mathcal{B}_m \varepsilon_{ipq} \mathcal{E}_p \mathcal{B}_q \mathcal{B}_j + \alpha_{16} \mathcal{B}_i \mathcal{B}_j + \\
&\quad + \alpha_{17} \mathcal{E}_m \mathcal{B}_m \mathcal{B}_i \mathcal{E}_j, \\
\Psi_{ij} &= -\alpha_3 \varepsilon_{ijk} \mathcal{E}_k + \alpha_4 \mathcal{E}_m \mathcal{B}_m \delta_{ij} - \alpha_8 \mathcal{E}_m \mathcal{B}_m \varepsilon_{ijk} \mathcal{B}_k + \\
&\quad + \alpha_8 \mathcal{E}_i \mathcal{B}_j + \alpha_{10} \varepsilon_{imn} \mathcal{E}_m \mathcal{B}_n \mathcal{B}_j + \\
&\quad + \alpha_{11} \mathcal{E}_m \mathcal{B}_m \mathcal{E}_i \mathcal{E}_j + \alpha_{13} \mathcal{E}_m \mathcal{B}_m \varepsilon_{ipq} \mathcal{E}_p \mathcal{B}_q \mathcal{E}_j + \\
&\quad + \alpha_{15} \mathcal{B}_i \mathcal{E}_j + \alpha_{18} \mathcal{E}_m \mathcal{B}_m \mathcal{B}_i \mathcal{B}_j,
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
\Omega_{ij} &= -\beta_2 \varepsilon_{ijk} \mathcal{E}_k + \beta_4 \mathcal{E}_m \mathcal{B}_m \delta_{ij} - \beta_5 \mathcal{E}_m \mathcal{B}_m \varepsilon_{ijk} \mathcal{B}_k + \\
&\quad + \beta_8 \mathcal{E}_i \mathcal{B}_j + \beta_{10} \varepsilon_{imn} \mathcal{E}_m \mathcal{B}_n \mathcal{B}_j + \beta_{11} \mathcal{E}_m \mathcal{B}_m \mathcal{E}_i \mathcal{E}_j + \\
&\quad + \beta_{13} \mathcal{E}_m \mathcal{B}_m \varepsilon_{ipq} \mathcal{E}_p \mathcal{B}_q \mathcal{E}_i + \beta_{15} \mathcal{B}_i \mathcal{E}_j + \\
&\quad + \beta_{18} \mathcal{E}_m \mathcal{B}_m \mathcal{B}_i \mathcal{B}_j,
\end{aligned}$$

$$\begin{aligned}
\Lambda_{ij} &= \beta_1 \delta_{ij} - \beta_3 \varepsilon_{ijk} \mathcal{B}_k - \beta_6 \mathcal{E}_m \mathcal{B}_m \varepsilon_{ijk} \mathcal{E}_k + \beta_7 \mathcal{E}_i \mathcal{E}_j + \\
&\quad + \beta_9 \varepsilon_{imn} \mathcal{E}_m \mathcal{B}_n \mathcal{E}_j + \beta_{12} \mathcal{E}_m \mathcal{B}_m \mathcal{E}_i \mathcal{B}_j + \\
&\quad + \beta_{14} \mathcal{E}_m \mathcal{B}_m \varepsilon_{ipq} \mathcal{E}_p \mathcal{B}_q \mathcal{B}_j + \beta_{16} \mathcal{B}_i \mathcal{B}_j + \beta_{17} \mathcal{E}_m \mathcal{B}_m \mathcal{B}_i \mathcal{E}_j.
\end{aligned}$$

$\delta_{ij}$  denotes the Kronecker delta and  $\varepsilon_{ijk}$  the alternating symbol, and the  $\alpha$ 's and  $\beta$ 's are functions of  $\iota\omega$  and polynomials in  $\mathcal{L} \cdot \mathcal{L}$ ,  $\mathcal{B} \cdot \mathcal{B}$  and  $(\mathcal{L} \cdot \mathcal{B})^2$ .

## 7. The propagation of a plane electromagnetic wave

The electromagnetic field equations in the absence of free currents and charges are

$$\begin{aligned}
\frac{\partial \mathbf{B}^*}{\partial t} + \text{curl } \mathbf{E}^* &= 0, & \text{div } \mathbf{B}^* &= 0, \\
\text{curl } \mathbf{H}^* - \frac{\partial \mathbf{D}^*}{\partial t} &= 0, & \text{div } \mathbf{D}^* &= 0.
\end{aligned} \tag{7.1}$$

We now assume that  $\mathbf{E}^*$  and  $\mathbf{B}^*$  are given by (2.1) and that the static portions of these fields,  $\mathcal{L}$  and  $\mathcal{B}$ , are independent of position. We then see that  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$  and  $\mathbf{H}$  must satisfy the equations

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} + \text{curl } \mathbf{E} &= 0, & \text{div } \mathbf{B} &= 0, \\
\text{curl } \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= 0, & \text{div } \mathbf{D} &= 0.
\end{aligned} \tag{7.2}$$

For the infinite plane wave (6.1), we obtain

$$\mathbf{k} \mathbf{n} \times \mathbf{e} - \omega \mathbf{b} = 0 \quad \text{and} \quad \mathbf{k} \mathbf{n} \times \mathbf{h} + \omega \mathbf{d} = 0. \tag{7.3}$$



Introducing (6.2) into the second of equations (7.3), we obtain, with the notation  $\star \eta = k/\omega$ ,

$$\eta \mathbf{n} \times (\boldsymbol{\Omega} \cdot \mathbf{e} + \boldsymbol{\Lambda} \cdot \mathbf{b}) + \boldsymbol{\Phi} \cdot \mathbf{e} + \boldsymbol{\Psi} \cdot \mathbf{b} = 0. \quad (7.4)$$

Using the first of equations (7.3) to substitute for  $\mathbf{b}$  in (7.4), we obtain

$$\eta \mathbf{n} \times [\boldsymbol{\Omega} \cdot \mathbf{e} + \eta \boldsymbol{\Lambda} \cdot (\mathbf{n} \times \mathbf{e})] + \eta \boldsymbol{\Psi} \cdot (\mathbf{n} \times \mathbf{e}) + \boldsymbol{\Phi} \cdot \mathbf{e} = 0. \quad (7.5)$$

(7.5) is a system of three homogeneous linear equations in the components of  $\mathbf{e}$ . We may therefore write it as

$$\chi_{ij} e_j = 0, \quad (7.6)$$

where

$$\chi_{ij} = \Phi_{ij} + \eta (\varepsilon_{jkl} \Psi_{ik} n_l - \varepsilon_{ikl} \Omega_{kj} n_l) + \eta^2 \varepsilon_{jkl} \varepsilon_{ipq} A_{qk} n_p n_l. \quad (7.7)$$

The secular equation for  $\eta$  is

$$|\chi_{ij}| = 0. \quad (7.8)$$

If the wave is propagated in the direction of the  $x_3$ -axis of a rectangular Cartesian coordinate system  $x$ , we may take  $n_i = \delta_{3i}$  in (7.7). We then obtain from (7.8),

$$\begin{vmatrix} \Phi_{11} + \eta(\Psi_{12} - \Omega_{21}) - \eta^2 A_{22}, & \Phi_{12} - \eta(\Psi_{11} + \Omega_{22}) + \eta^2 A_{21}, & \Phi_{13} - \eta^2 \Omega_{23} \\ \Phi_{21} + \eta(\Psi_{22} + \Omega_{11}) + \eta^2 A_{12}, & \Phi_{22} + \eta(\Psi_{21} + \Omega_{12}) - \eta^2 A_{11}, & \Phi_{23} + \eta \Omega_{13} \\ \Phi_{31} + \eta \Psi_{32}, & \Phi_{32} - \eta \Psi_{31}, & \Phi_{33} \end{vmatrix} = 0. \quad (7.9)$$

This is the secular equation for an infinite plane electromagnetic wave propagated in a holohedral isotropic material to which static electric and magnetic induction fields are applied. We note that it is an equation of fourth degree in  $\eta$ . In general the solutions of this equation are complex.

Denoting  $\eta$  by

$$\eta = \eta^+ + i\eta^-, \quad (7.10)$$

where  $\eta^+$  and  $\eta^-$  are real, we see that a solution with  $\eta^+ > 0$  corresponds to a wave travelling in the positive direction of the unit vector  $\mathbf{n}$ , while a solution with  $\eta^+ < 0$  corresponds to a wave travelling in the negative direction of  $\mathbf{n}$ . The amplitude of the wave will increase in the direction of propagation of the wave for the first of these waves if  $\eta^- < 0$  and for the second of the waves if  $\eta^- > 0$ . Consequently, if either of these situations arises for any value of  $\omega$ , the system will be unstable. We shall in our subsequent discussion assume that such situations do not arise.

Thus, the four solutions of (7.9) correspond, in general, to four elliptically-polarized, skew  $\mathbf{E}$ -waves travelling with different complex velocities. The manner in which the plane of polarization and ellipticity can be calculated is shown in the next section.

## 8. Analysis of a skew, elliptically-polarized plane wave

We consider a plane wave given by

$$\mathbf{E} = \mathbf{e} e^{i\varphi}, \quad (8.1)$$

where

$$\varphi = k\mathbf{n} \cdot \mathbf{x} - \omega t. \quad (8.2)$$

\*  $1/\eta$  is then the complex velocity.

We have\* 
$$\mathbf{E}^+ = \mathbf{e}^+ \cos \varphi - \mathbf{e}^- \sin \varphi. \quad (8.3)$$

The vector  $\mathbf{E}^+$  therefore executes an ellipse. Let  $\mathbf{l}$  be a unit vector which is at all times perpendicular to the plane of the ellipse.  $\mathbf{l}$  is taken in the same sense in relation to the plane of polarization as  $\mathbf{n}$ , *i.e.*  $\mathbf{l} \cdot \mathbf{n} \geq 0$ . Then,  $\mathbf{l} \cdot \mathbf{E}^+ = 0$ . From (8.3), we then have

$$\mathbf{l} \cdot \mathbf{e}^+ = \mathbf{l} \cdot \mathbf{e}^- = 0, \quad (8.4)$$

whence

$$\mathbf{l} = \pm \frac{\mathbf{e}^+ \times \mathbf{e}^-}{[(\mathbf{e}^+ \times \mathbf{e}^-) \cdot (\mathbf{e}^+ \times \mathbf{e}^-)]^{\frac{1}{2}}} = \pm \frac{\mathbf{e} \times \mathbf{e}^*}{[(\mathbf{e} \times \mathbf{e}^*) \cdot (\mathbf{e} \times \mathbf{e}^*)]^{\frac{1}{2}}}, \quad (8.5)$$

the positive or negative signs being chosen to make  $\mathbf{l} \cdot \mathbf{n} \geq 0$ . From (8.3), we have

$$\mathbf{E}^+ \cdot \mathbf{E}^+ = \mathbf{e}^+ \cdot \mathbf{e}^+ \cos^2 \varphi - 2\mathbf{e}^+ \cdot \mathbf{e}^- \cos \varphi \sin \varphi + \mathbf{e}^- \cdot \mathbf{e}^- \sin^2 \varphi. \quad (8.6)$$

We note that  $\mathbf{E}^+ \cdot \mathbf{E}^+$  is the square of the radius vector to a point on the ellipse. The vector  $\mathbf{E}^+$  is in the direction of an axis of the ellipse when  $d(\mathbf{E}^+ \cdot \mathbf{E}^+)/d\varphi = 0$ , *i.e.* when

$$\tan 2\varphi = -\frac{2\mathbf{e}^+ \cdot \mathbf{e}^-}{\mathbf{e}^+ \cdot \mathbf{e}^+ - \mathbf{e}^- \cdot \mathbf{e}^-} = \frac{\iota(\mathbf{e} \cdot \mathbf{e} - \mathbf{e}^* \cdot \mathbf{e}^*)}{\mathbf{e} \cdot \mathbf{e} + \mathbf{e}^* \cdot \mathbf{e}^*}. \quad (8.7)$$

From (8.6) and (8.7), we see that the semi-axes  $L_1$  and  $L_2$  of the ellipse are given by

$$\begin{aligned} L_1^2, L_2^2 &= \frac{1}{2}(\mathbf{e}^+ \cdot \mathbf{e}^+ + \mathbf{e}^- \cdot \mathbf{e}^-) \pm \frac{1}{2}\{(\mathbf{e}^+ \cdot \mathbf{e}^+ - \mathbf{e}^- \cdot \mathbf{e}^-)^2 + 4(\mathbf{e}^+ \cdot \mathbf{e}^-)^2\}^{\frac{1}{2}} \\ &= \frac{1}{2}\mathbf{e} \cdot \mathbf{e}^* \pm \frac{1}{2}\{(\mathbf{e} \cdot \mathbf{e})(\mathbf{e}^* \cdot \mathbf{e}^*)\}^{\frac{1}{2}}. \end{aligned} \quad (8.8)$$

$\mathbf{l}$  and  $\mathbf{e}$  may satisfy certain special relations which correspond to the  $\mathbf{E}^+$ -wave having a particular character. These relations together with the nature of the wave to which they correspond are listed below:

$$[\mathbf{e}^+, \mathbf{e}^-, \mathbf{l}] > 0 \quad \text{right-handed wave,} \quad (8.9)$$

$$[\mathbf{e}^+, \mathbf{e}^-, \mathbf{l}] < 0 \quad \text{left-handed wave,} \quad (8.10)$$

$$\mathbf{e} \cdot \mathbf{e} = 0 \quad \text{circularly-polarized wave,} \quad (8.11)$$

$$\mathbf{e}^* \times \mathbf{e} = 0 \quad \text{linearly-polarized wave,} \quad (8.12)$$

$$\mathbf{l} = \mathbf{n}, \quad \text{and hence } \mathbf{e} \cdot \mathbf{n} = 0, \quad \text{transverse wave,} \quad (8.13)$$

$$\mathbf{e} \times \mathbf{n} = 0 \quad \text{longitudinal wave.} \quad (8.14)$$

### 9. Magneto-optical effects in holohedral isotropic materials

In this section, we shall discuss the interaction between a plane electromagnetic wave in a holohedral isotropic material and a static magnetic field applied to the material. We accordingly take  $\mathcal{L} = 0$ . From equations (6.3), we then obtain

$$\begin{aligned} \Phi_{i,j} &= \alpha_1 \delta_{i,j} - \alpha_2 \varepsilon_{ijk} \mathcal{B}_k + \alpha_{18} \mathcal{B}_i \mathcal{B}_j, \\ A_{i,j} &= \beta_1 \delta_{i,j} - \beta_3 \varepsilon_{ijk} \mathcal{B}_k + \beta_{18} \mathcal{B}_i \mathcal{B}_j, \end{aligned} \quad (9.1)$$

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\* Here and subsequently, we shall denote the real and imaginary parts of a complex quantity by using the superscripts + and -.

and

$$\Psi_{ij} = \Omega_{ij} = 0,$$

the  $\alpha$ 's and  $\beta$ 's being functions of  $\omega$  and polynomials in  $\mathcal{B}$ ,  $\mathcal{B}_m$ .

From (7.7) and (9.1), we obtain

$$\chi_{ij} = \Phi_{ij} + \eta^2 \varepsilon_{jkl} \varepsilon_{ipq} A_{qk} n_p n_l. \quad (9.2)$$

a) *Longitudinal magnetic field*

If the magnetic induction field  $\mathcal{B}$  is parallel to the direction of propagation of the wave, then we may without loss of generality take both of these directions as the 3-direction of a rectangular Cartesian system  $x$ . We then have

$$n_i = \delta_{3i} \quad \text{and} \quad \mathcal{B}_i = \delta_{3i}. \quad (9.3)$$

With (9.3), the expressions (9.1) for  $\Phi_{ij}$  and  $A_{ij}$  become

$$\begin{aligned} \Phi_{ij} &= \alpha_1 \delta_{ij} - \alpha_2 \mathcal{B} \varepsilon_{ijs} + \alpha_{16} \mathcal{B}^2 \delta_{3i} \delta_{3j} \\ \text{and} \quad A_{ij} &= \beta_1 \delta_{ij} - \beta_3 \mathcal{B} \varepsilon_{ijs} + \beta_{16} \mathcal{B}^2 \delta_{3i} \delta_{3j}, \end{aligned} \quad (9.4)$$

where the  $\alpha$ 's and  $\beta$ 's are polynomials in  $\mathcal{B}^2$ . From (9.2) and (9.4), we obtain

$$\chi_{ij} = \alpha_1 \delta_{ij} - \alpha_2 \mathcal{B} \varepsilon_{ijs} + \alpha_{16} \mathcal{B}^2 \delta_{3i} \delta_{3j} + \eta^2 [-\beta_1 (\delta_{ij} - \delta_{3i} \delta_{3j}) + \beta_3 \mathcal{B} \varepsilon_{ijs}]. \quad (9.5)$$

Whence,

$$\|\chi_{ij}\| = \begin{vmatrix} \alpha_1 - \beta_1 \eta^2, & -(\alpha_2 - \beta_3 \eta^2) \mathcal{B}, & 0 \\ (\alpha_2 - \beta_3 \eta^2) \mathcal{B}, & \alpha_1 - \beta_1 \eta^2, & 0 \\ 0, & 0, & \alpha_1 + \alpha_{16} \mathcal{B}^2 \end{vmatrix}. \quad (9.6)$$

The secular equation (7.8) then yields

$$\begin{aligned} \alpha_1 + \alpha_{16} \mathcal{B}^2 &= 0 \\ \text{or} \quad (\alpha_1 - \beta_1 \eta^2)^2 + (\alpha_2 - \beta_3 \eta^2)^2 \mathcal{B}^2 &= 0. \end{aligned} \quad (9.7)$$

Discounting the first alternative, we obtain

$$\eta^2 = \frac{\alpha_1 + i \alpha_2 \mathcal{B}}{\beta_1 + i \beta_3 \mathcal{B}} \quad \text{or} \quad \frac{\alpha_1 - i \alpha_2 \mathcal{B}}{\beta_1 - i \beta_3 \mathcal{B}}. \quad (9.8)$$

From (9.6) and (7.6), it is easily seen that  $e_3 = 0$ . Consequently, since  $n_i = \delta_{3i}$ , the relation (8.13) is satisfied. Also, unless  $(\alpha_2 - \beta_3 \eta^2) \mathcal{B} = 0$  or  $\alpha_1 - \beta_1 \eta^2 = 0$ , it is seen that the relation (8.11) is satisfied. Thus each of the four solutions for  $\eta$ , given by (9.8), corresponds to a transverse, circularly-polarized plane wave. Two of these travel with speeds, which are in general different, in the positive direction of the  $x_3$ -axis and the remaining two travel with these speeds in the negative direction. If  $\mathcal{B} > 0$ , the wave in the positive direction of the  $x_3$ -axis given by the first of the solutions (9.8) is a left-handed wave and that given by the second of the solutions (9.8) is a right-handed wave; on the other hand the waves in the negative direction of the  $x_3$ -axis given by the first and second of the solutions (9.8) are right-handed and left-handed respectively.

It is thus seen that the right and left circularly polarized  $E^+$  waves, with a given direction of propagation parallel to the magnetic induction field  $\mathcal{B}$ , travel with different speeds. This is equivalent to showing that the plane of polarization

of a linearly polarized wave rotates at a rate proportional to the distance travelled. This is the well-known Faraday effect. In (9.8),  $\alpha_1$ ,  $\beta_1$ ,  $\alpha_2$  and  $\beta_3$  are, in general, polynomials in  $\mathcal{B}^2$ .

b) *Transverse magnetic field*

We now consider that the magnetic induction field  $\mathcal{B}$  is perpendicular to the direction of propagation  $\mathbf{n}$  of the plane electromagnetic wave. We choose a rectangular Cartesian reference system with its 1-axis parallel to  $\mathcal{B}$  and its 3-axis parallel to  $\mathbf{n}$ . Then, in (9.1) and (9.2), we take

$$\mathcal{B}_i = \mathcal{B} \delta_{1i} \quad \text{and} \quad n_i = \delta_{3i}. \quad (9.9)$$

We obtain

$$\Phi_{ij} = \alpha_1 \delta_{ij} - \alpha_2 \mathcal{B} \varepsilon_{ij1} + \alpha_{16} \mathcal{B}^2 \delta_{1i} \delta_{1j}, \quad (9.10)$$

$$\Lambda_{ij} = \beta_1 \delta_{ij} - \beta_3 \mathcal{B} \varepsilon_{ij1} + \beta_{16} \mathcal{B}^2 \delta_{1i} \delta_{1j},$$

and

$$\begin{aligned} \chi_{ij} = & \alpha_1 \delta_{ij} - \alpha_2 \mathcal{B} \varepsilon_{ij1} + \alpha_{16} \mathcal{B}^2 \delta_{1i} \delta_{1j} + \\ & + \eta^2 [-\beta_1 (\delta_{ij} - \delta_{3i} \delta_{3j}) - \beta_{16} \mathcal{B}^2 \varepsilon_{i13} \varepsilon_{j13}]. \end{aligned} \quad (9.11)$$

Whence,

$$\|\chi_{ij}\| = \begin{vmatrix} \alpha_1 + \alpha_{16} \mathcal{B}^2 - \beta_1 \eta^2, & 0, & 0 \\ 0, & \alpha_1 - \eta^2 (\beta_1 + \beta_{16} \mathcal{B}^2), & -\alpha_2 \mathcal{B} \\ 0, & \alpha_2 \mathcal{B}, & \alpha_1 \end{vmatrix}. \quad (9.12)$$

The secular equation (7.8) then yields

$$\eta^2 = \frac{\alpha_1 + \alpha_{16} \mathcal{B}^2}{\beta_1} \quad (9.13)$$

and

$$\eta^2 = \frac{\alpha_1^2 + \alpha_2^2 \mathcal{B}^2}{\alpha_1 (\beta_1 + \beta_{16} \mathcal{B}^2)}. \quad (9.14)$$

If we take the solution (9.13) for  $\eta^2$ , we see from (9.11) and (7.6) that, in general,  $e_2 = e_3 = 0$ . Thus the two waves corresponding to this solution satisfy the conditions (8.12) and (8.13) and are therefore transverse waves, linearly-polarized in the  $x_1$ -direction, travelling with equal speeds in the positive and negative directions of the  $x_3$ -axis.

The solution (9.14) for  $\eta^2$  yields, with (9.11) and (7.6),  $e_1 = 0$ . It therefore corresponds, in general, to two elliptically-polarized skew waves, the planes of polarization of which are normal to  $\mathcal{B}$ , travelling with equal and opposite velocities. For these waves,

$$e_3/e_2 = -\alpha_2 \mathcal{B}/\alpha_1, \quad (9.15)$$

so that  $\mathbf{e}$  has the form

$$\mathbf{e} = \xi (0, 1, \zeta), \quad (9.16)$$

where

$$\zeta = -\alpha_2 \mathcal{B}/\alpha_1 \quad (9.17)$$

and  $\xi$  is an arbitrary complex number. The  $E^+$ -waves whose velocities are determined by (9.14) are linearly-polarized if and only if  $\mathbf{e}^* \times \mathbf{e} = 0$  (cf. equation (8.12)). This condition is satisfied if and only if  $\zeta$  is real. The waves are transverse if and only if  $l_3 = 0$  (cf. equation (8.13)), i.e. if and only if  $\zeta = 0$ . They are circularly-polarized if and only if  $\mathbf{e} \cdot \mathbf{e} = 0$  (cf. equation (8.11)), i.e. if and only if  $\zeta = \pm i$ . For all other values of  $\zeta$  corresponding to the solution (9.14) for  $\eta^2$ ,

the  $\mathbf{E}^+$ -wave is an elliptically-polarized skew wave. The locus of  $\mathbf{E}^+$  at a fixed point in the material is an ellipse in the plane normal to  $\mathcal{B}$ . From (8.8) the semi-major and semi-minor axes  $L_1$  and  $L_2$  of this ellipse are given by

$$L_1^2, L_2^2 = \frac{1}{2} \xi \xi^* \{1 + \zeta \zeta^* \pm [(1 + \zeta^2)(1 + \zeta^{*2})]^{1/2}\}, \quad (9.18)$$

the positive sign applying to the semi-major axis  $L_1$  and the negative sign to the semi-minor axis  $L_2$ .

Introducing (9.16) into equation (8.7), we obtain

$$\tan 2\varphi = \frac{\xi[\xi^2(1 + \zeta^2) - \xi^{*2}(1 + \zeta^{*2})]}{\xi^2(1 + \zeta^2) + \xi^{*2}(1 + \zeta^{*2})}. \quad (9.19)$$

VOIGT [5] explained the double refraction of light propagating normal to a magnetic field by the difference between the speeds as determined by (9.13) and (9.14) and observed the effect in vapours. COTTON and MOUTON (see, for example [6]) observed a very strong effect of this nature in certain liquids. The skewness and ellipticity of the  $\mathbf{E}^+$ -wave propagated with speed given by (9.14) seems not to have been noticed.

### 10. Electro-optical effects in holohedral isotropic materials

We now discuss the interaction between a plane electro-magnetic wave and a static electric field in the material through which the wave is propagated. We accordingly take  $\mathcal{B} = 0$ . From (6.3), we then obtain

$$\begin{aligned} \Phi_{ij} &= \alpha_1 \delta_{ij} + \alpha_7 \mathcal{E}_i \mathcal{E}_j, \\ \Psi_{ij} &= -\alpha_3 \varepsilon_{ijk} \mathcal{E}_k, \\ \Omega_{ij} &= -\beta_2 \varepsilon_{ijk} \mathcal{E}_k, \\ A_{ij} &= \beta_1 \delta_{ij} + \beta_7 \mathcal{E}_i \mathcal{E}_j, \end{aligned} \quad (10.1)$$

the  $\alpha$ 's and  $\beta$ 's being functions of  $i\omega$  and polynomials in  $\mathcal{E}_m \mathcal{E}_m$ .

From (7.7) and (10.1), we obtain

$$\begin{aligned} \chi_{ij} &= (\alpha_1 \delta_{ij} + \alpha_7 \mathcal{E}_i \mathcal{E}_j) - \eta [(\beta_2 + \alpha_3) \mathcal{E}_m n_m \delta_{ij} - (\alpha_3 \mathcal{E}_j n_i + \beta_2 \mathcal{E}_i n_j)] + \\ &+ \eta^2 [\beta_1 (n_i n_j - \delta_{ij}) + \beta_7 \varepsilon_{jkl} \varepsilon_{ipq} \mathcal{E}_q \mathcal{E}_k n_p n_l]. \end{aligned} \quad (10.2)$$

#### a) Longitudinal electric field

If the electric field  $\mathcal{E}$  is parallel to the direction of propagation of the wave, then we take this common direction as the  $x_3$ -axis of the coordinate system  $x$ . Introducing

$$n_i = \delta_{3i} \quad \text{and} \quad \mathcal{E}_i = \mathcal{E} \delta_{3i} \quad (10.3)$$

into (10.2), we obtain

$$\chi_{ij} = (\alpha_1 \delta_{ij} + \alpha_7 \mathcal{E}^2 \delta_{3i} \delta_{3j}) - \eta \mathcal{E} (\beta_2 + \alpha_3) (\delta_{ij} - \delta_{3i} \delta_{3j}) - \eta^2 \beta_1 (\delta_{ij} - \delta_{3i} \delta_{3j}). \quad (10.4)$$

Whence,

$$\|\chi_{ij}\| = \left\| \begin{array}{ccc} \alpha_1 - (\beta_2 + \alpha_3) \mathcal{E} \eta - \beta_1 \eta^2, & 0, & 0 \\ 0, & \alpha_1 - (\beta_2 + \alpha_3) \mathcal{E} \eta - \beta_1 \eta^2, & 0 \\ 0, & 0, & \alpha_1 + \alpha_7 \mathcal{E}^2 \end{array} \right\|. \quad (10.5)$$

Discounting the possibility that  $\alpha_1 + \alpha_7 \mathcal{E}^2 = 0$ , the secular equation (7.8) then yields

$$\eta = \frac{-\mathcal{E}(\beta_2 + \alpha_3) \pm [(\beta_2 + \alpha_3)^2 \mathcal{E}^2 + 4\alpha_1 \beta_1]^{\frac{1}{2}}}{2\beta_1}. \quad (10.6)$$

Unless  $\beta_2 + \alpha_3 = 0$ , these two solutions for  $\eta$  correspond to waves travelling with different complex velocities parallel to the  $x_3$ -axis. We may expect that one of these waves will travel in the positive direction of the  $x_3$ -axis and the other in the negative direction.

Introducing (10.5) into (7.6), we obtain  $e_3 = 0$ . Consequently, for each of the waves the relation (8.13) is satisfied and the wave is accordingly transverse. Also, it is easily seen that the solutions (10.6) for  $\eta$  give the inverse complex velocities for waves linearly-polarized in an arbitrary direction normal to the direction of propagation.

We note, as was pointed out in §7, that if  $\eta^+ > 0$  and  $\eta^- < 0$ , or if  $\eta^+ < 0$  and  $\eta^- > 0$ , the system is unstable. Consequently,  $\alpha_1$ ,  $\beta_1$ ,  $\beta_2$  and  $\alpha_3$  must be such that this situation does not arise for either wave. A number of conclusions regarding the values of  $\alpha_1$ ,  $\beta_1$ ,  $\beta_2$  and  $\alpha_3$  might be drawn from this fact by purely algebraic reasoning. For example, the situation cannot exist in which  $[(\beta_2 + \alpha_3)^2 \mathcal{E}^2 + 4\alpha_1 \beta_1]^{\frac{1}{2}}/\beta_1$  is real and  $\mathcal{E}(\beta_2 + \alpha_3)/\beta_1$  is purely imaginary.

#### b) Transverse electric field

We may, without loss of generality, choose the reference system with the  $x_3$ -axis parallel to the direction of wave propagation and the  $x_2$ -axis parallel to  $\mathcal{L}$ . We then write

$$n_i = \delta_{3i} \quad \text{and} \quad \mathcal{E}_i = \mathcal{E} \delta_{2i}. \quad (10.7)$$

Introducing these relations into (10.2), we obtain

$$\chi_{ij} = (\alpha_1 \delta_{ij} + \alpha_7 \mathcal{E}^2 \delta_{2i} \delta_{2j}) + \eta \mathcal{E} (\alpha_2 \delta_{3i} \delta_{2j} + \beta_2 \delta_{2i} \delta_{3j}) + \eta^2 [\beta_1 (\delta_{3i} \delta_{3j} - \delta_{ij}) + \beta_2 \mathcal{E}^2 (\delta_{2i} \delta_{2j} + \delta_{3i} \delta_{3j} - \delta_{ij})]. \quad (10.8)$$

Thus,

$$\|\chi_{ij}\| = \begin{vmatrix} \alpha_1 - \eta^2(\beta_1 + \beta_7 \mathcal{E}^2), & 0, & 0 \\ 0, & \alpha_1 + \alpha_7 \mathcal{E}^2 - \beta_1 \eta^2, & \beta_2 \mathcal{E} \eta \\ 0, & \alpha_3 \mathcal{E} \eta, & \alpha_1 \end{vmatrix}. \quad (10.9)$$

The solutions for  $\eta^2$  of the secular equation (7.8) are therefore

$$\eta^2 = \frac{\alpha_1}{\beta_1 + \beta_7 \mathcal{E}^2}, \quad (10.10)$$

or

$$\eta^2 = \frac{\alpha_1(\alpha_1 + \alpha_7 \mathcal{E}^2)}{\alpha_1 \beta_1 + \alpha_3 \beta_2 \mathcal{E}^2}. \quad (10.11)$$

The solutions (10.10) for  $\eta$  correspond to waves travelling with equal speeds in the positive and negative directions of the  $x_3$ -axis. Unless

$$\alpha_7 \beta_7 \mathcal{E}^2 = \alpha_3 \beta_2 - \beta_1 \alpha_7 - \alpha_1 \beta_7 \quad (10.12)$$

$e_2 = e_3 = 0$  for each of these waves and consequently they are transverse waves, linearly polarized in the  $x_1$ -direction.

The solutions (10.11) for  $\eta$  also correspond to waves travelling with equal speeds in the positive and negative directions of the  $x_3$ -axis. For these waves, unless the relation (10.12) is satisfied, it is seen from (7.6) and (10.9) that

$$\begin{aligned} e_1 &= 0 \\ e_3/e_2 &= -\alpha_3 \mathcal{E} \eta / \alpha_1. \end{aligned} \quad (10.13)$$

The waves are consequently skew and elliptically polarized in the  $x_2 x_3$ -plane. From (10.13), we see that  $\mathbf{e}$  has the form

$$\mathbf{e} = \xi(0, 1, \zeta), \quad (10.14)$$

where

$$\zeta = -\alpha_3 \mathcal{E} \eta / \alpha_1 \quad (10.15)$$

and  $\xi$  is an arbitrary complex number. Following the discussion in §9, we again see that the semi-axes  $L_1$  and  $L_2$  of the ellipse executed by the vector  $\mathbf{E}^*$  are given by (9.18) and the values of the parameter  $\varphi$  determining their direction are given by (9.19), where  $\zeta$  is now given by (10.15).

## 11. Appendix

We suppose that the magnetic intensity and electric displacement fields,  $\mathbf{H}^*$  and  $\mathbf{D}^*$ , at time  $t$  are functionals of the electric and magnetic induction fields,  $\mathbf{E}^*(\tau)$  and  $\mathbf{B}^*(\tau)$ , over all times preceding the instant  $t$ . Thus,

$$\mathbf{D}^* = \Phi[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}]_{\tau=-\infty}$$

and

$$\mathbf{H}^* = \Psi[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}]_{\tau=-\infty}.$$

We assume that the functionals  $\Phi$  and  $\Psi$  satisfy the relations

$$\Phi[\overset{t}{\mathbf{E}^*(\tau-h)}, \overset{t}{\mathbf{B}^*(\tau-h)}] = \Phi[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}]_{\tau=-\infty}$$

and

$$\Psi[\overset{t}{\mathbf{E}^*(\tau-h)}, \overset{t}{\mathbf{B}^*(\tau-h)}] = \Psi[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}]_{\tau=-\infty}.$$

The material considered then possesses invariable heredity.

We also assume that each of the functionals  $\Phi$  and  $\Psi$  may be particularly dependent on the values of the arguments and of their time derivatives of various orders at time  $\tau=t$  and is continuous for  $-\infty < \tau < t$ . Using the notation

$$\begin{aligned} \mathbf{E}^*(t) &= \mathbf{E}^* = \mathbf{E}^{*(0)}, & \mathbf{E}^{*(\mu)} &= [d^\mu \mathbf{E}^*(\tau) / d\tau^\mu]_{\tau=t}, \\ \mathbf{B}^*(t) &= \mathbf{B}^* = \mathbf{B}^{*(0)}, & \mathbf{B}^{*(\mu)} &= [d^\mu \mathbf{B}^*(\tau) / d\tau^\mu]_{\tau=t}, \end{aligned} \quad (11.3)$$

( $\mu = 1, 2, \dots$ )

we suppose that the functionals  $\Phi$  and  $\Psi$  may be expressed in the forms

$$\begin{aligned} \Phi[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}] &= \sum f \cdot F[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}]_{\tau=-\infty}, \\ \Psi[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}] &= \sum g \cdot G[\overset{t}{\mathbf{E}^*(\tau)}, \overset{t}{\mathbf{B}^*(\tau)}]_{\tau=-\infty}, \end{aligned} \quad (11.4)$$

where  $f$  and  $g$  are differentiable functions of  $\mathbf{E}^{*(\mu)}$  and  $\mathbf{B}^{*(\mu)}$  ( $\mu = 0, 1, 2, \dots$ ) and  $F$  and  $G$  are continuous functionals of their arguments.

We now assume that the fields  $\mathbf{E}^*(\tau)$  and  $\mathbf{B}^*(\tau)$  are the resultants of static fields  $\mathcal{E}$  and  $\mathcal{B}$  and small varying fields  $\varepsilon \mathbf{E}(\tau)$  and  $\varepsilon \mathbf{B}(\tau)$ , where  $\varepsilon$  is a small constant, as given by equations (2.1). We have, neglecting terms of higher degree than the first in  $\varepsilon$ ,

$$f(\mathbf{E}^{*(\mu)}, \mathbf{B}^{*(\mu)}) = f(\mathbf{E}^{*(\mu)}, \mathbf{B}^{*(\mu)})|_{\varepsilon=0} + \varepsilon \sum_{\mu=0}^{\mu} \left\{ \frac{\partial f}{\partial \mathbf{E}^{*(\mu)}} \Big|_{\varepsilon=0} \mathbf{E}^{(\mu)} + \frac{\partial f}{\partial \mathbf{B}^{*(\mu)}} \Big|_{\varepsilon=0} \mathbf{B}^{(\mu)} \right\}$$

and (11.5)

$$\mathbf{F}[\mathbf{E}^*(\tau), \mathbf{B}^*(\tau)]_{\tau=-\infty}^t = \mathbf{F}(\mathcal{E}, \mathcal{B}) + \varepsilon \int_{-\infty}^t \boldsymbol{\varphi}(t-\tau) \mathbf{E}(\tau) d\tau + \varepsilon \int_{-\infty}^t \boldsymbol{\psi}(t-\tau) \mathbf{B}(\tau) d\tau,$$

where  $\boldsymbol{\varphi}$  and  $\boldsymbol{\psi}$  are functions of  $\mathcal{E}$  and  $\mathcal{B}$ . Here, we use the notation  $\mathbf{E}^{(0)} = \mathbf{E}$ ,  $\mathbf{B}^{(0)} = \mathbf{B}$ ,  $\mathbf{E}^{(\mu)} = [d^\mu \mathbf{E}(\tau)/d\tau^\mu]_{\tau=t}$ ,  $\mathbf{B}^{(\mu)} = [d^\mu \mathbf{B}(\tau)/d\tau^\mu]_{\tau=t}$ . We also use the notation

$$\mathbf{F}(\mathcal{E}, \mathcal{B}) = \mathbf{F}[\mathbf{E}^*(\tau), \mathbf{B}^*(\tau)]_{\tau=-\infty}^t, \quad (11.6)$$

observing that  $\mathcal{E} (= \mathbf{E}^*(\tau)|_{\varepsilon=0})$  and  $\mathcal{B} (= \mathbf{B}^*(\tau)|_{\varepsilon=0})$  are time-independent fields. The second of the results (11.5) follows directly from the integral representation of a continuous functional. We note that  $\partial f / \partial \mathbf{E}^{*(\mu)}|_{\varepsilon=0}$  and  $\partial f / \partial \mathbf{B}^{*(\mu)}|_{\varepsilon=0}$  are functions of  $\mathcal{E}$  and  $\mathcal{B}$ .

It follows from (11.4) and (11.5) that, neglecting terms of higher degree than the first in  $\varepsilon$ , the functional  $\Phi[\mathbf{E}^*(\tau), \mathbf{B}^*(\tau)]$  may be expressed in the form

$$\Phi[\mathbf{E}^*(\tau), \mathbf{B}^*(\tau)]_{\tau=-\infty}^t = \Phi(\mathcal{E}, \mathcal{B}) + \varepsilon \sum_{\mu=0}^{\mu} (\alpha_\mu \mathbf{E}^{(\mu)} + \beta_\mu \mathbf{B}^{(\mu)}) + \int_{-\infty}^t \boldsymbol{\varphi}(t-\tau) \mathbf{E}(\tau) d\tau + \int_{-\infty}^t \boldsymbol{\psi}(t-\tau) \mathbf{B}(\tau) d\tau, \quad (11.7)$$

where the  $\alpha$ 's,  $\beta$ 's,  $\boldsymbol{\varphi}$  and  $\boldsymbol{\psi}$  are functions of  $\mathcal{E}$  and  $\mathcal{B}$  which are, in general, different from those in (11.5).

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## References

- [1] PIPKIN, A. C., & R. S. RIVLIN: Arch. Rational Mech. Anal. **4**, 129 (1959).
- [2] RIVLIN, R. S.: Arch. Rational Mech. Anal. **4**, 262 (1960).
- [3] PIPKIN, A. C., & R. S. RIVLIN: J. Math. Phys. (in the press).
- [4] TOUPIN, R. A., & R. S. RIVLIN: Arch. Rational Mech. Anal. **6**, 188 (1960).
- [5] VOIGT, W.: Wied. Ann. d. Phys. **67**, 359 (1899).
- [6] JENKINS, F. A., & H. E. WHITE: Fundamentals of physical optics. New York McGraw-Hill 1937.

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