Linear functional electromagnetic constitutive relations and plane waves in a hemihedral isotropic material

R. A. Toupin, R. S. Rivlin

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1. Introduction

In any Lorentz frame of reference (x^i, t), where x^i $(i=1, 2, 3)$ are rectangular Cartesian spatial coordinates and t is the time, MAXWELL'S equations in a *stationary*, polarizable and magnetizable medium can be written in the form

$$\frac{\partial}{\partial t}\int_{\mathscr{A}} \boldsymbol{B}\cdot d\boldsymbol{a} + \oint_{\mathscr{L}} \boldsymbol{E}\cdot d\boldsymbol{l} = 0, \tag{1.1}$$

$$\oint_{\mathscr{A}} \boldsymbol{B}\cdot d\boldsymbol{a} = 0, \tag{1.2}$$

$$\oint_{\mathscr{L}} \boldsymbol{H}\cdot d\boldsymbol{l} - \frac{\partial}{\partial t}\int_{\mathscr{A}} \boldsymbol{D}\cdot d\boldsymbol{a} = \int_{\mathscr{A}} \boldsymbol{J}_{\mathrm{F}}\cdot d\boldsymbol{a}, \tag{1.3}$$

$$\oint \boldsymbol{D}\cdot d\boldsymbol{a} = \int Q_{\mathrm{F}}\, dv, \tag{1.4}$$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \qquad \boldsymbol{H} = \mu_0^{-1}\boldsymbol{B} - \boldsymbol{M}, \tag{1.5}$$

where $\boldsymbol{B}$ is the *magnetic flux density*, $\boldsymbol{E}$ the *electric field*, $\boldsymbol{H}$ the *magnetic intensity*, $\boldsymbol{D}$ the *electric displacement*, $\boldsymbol{J}_{\mathrm{F}}$ the *density of free current*, Q_{F} the *density of free charge*, $\boldsymbol{P}$ the *polarization*, $\boldsymbol{M}$ the *magnetization*, and ε_0 and μ_0 are fundamental constants whose product $\varepsilon_0\mu_0 = c^{-2}$ where c is the velocity of light in vacuum. $d\boldsymbol{l}$ denotes an element of an arbitrary closed contour $\mathscr{L}$ in space and $d\boldsymbol{a}$ an element of a surface $\mathscr{A}$ bounded by $\mathscr{L}$.

The system of equations (1.1) to (1.5) is underdetermined, and to obtain a determinate system of equations amongst the twenty-two unknown field components $(B_i, E_i, H_i, D_i, P_i, M_i, J_{\mathrm{F}}, Q_{\mathrm{F}})$ it is necessary to supplement the basic Maxwell equations with certain *constitutive relations*. The form of these constitutive relations depends on the nature of the *material medium* in which the electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ resides. The great mathematical variety of these electromagnetic constitutive relations makes possible a great variety of physical phenomena embraced by and consistent with MAXWELL'S framework of equations (1.1) to (1.5).

The simplest of all media, *vacuum*, is characterized by the constitutive relations

$$\boldsymbol{J}_{\mathrm{F}} = Q_{\mathrm{F}} = \boldsymbol{M} = \boldsymbol{P} = 0. \tag{1.6}$$

The next simplest medium is the rigid, linear, stationary, non-conducting dielectric, for which the constitutive relations

$$\begin{gathered} Q_F = \boldsymbol{J}_F = 0, \\ \boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E}, \quad \boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H}, \end{gathered} \tag{1.7}$$

were given by MAXWELL [1873, § 784]. Here $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are constant second order tensors, proportional to the unit tensor if the medium be isotropic. As is known, constitutive relations of this simple type do not account for the observed absorption and dispersion of electromagnetic waves in non-conductors, nor does MAXWELL's device of including a linear law of conduction, $\boldsymbol{J}_F = \boldsymbol{C} \cdot \boldsymbol{E}$, replacing $(1.7)_2$, suffice to account for the observed magnitude of the "dielectric losses" when a material is placed in a variable electric field (*Cf.* WHITEHEAD [1927, Lecture 1]).

HOPKINSON [1877], following a suggestion of MAXWELL, proposed a constitutive relation for the electric displacement in a dielectric having the form

$$\boldsymbol{D}(t) = \varepsilon \boldsymbol{E}(t) + \int_{-\infty}^{t} \varphi(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau, \tag{1.8}$$

where the function $\varphi(u)$, $u \geqq 0$, is a decreasing function of u. By suitable adjustment of the *memory function* φ, HOPKINSON was able to correlate his data on the residual charge of Leyden jars. In proposing (1.8), HOPKINSON was guided by the earlier mechanical constitutive relation of BOLTZMANN [1874] in which the torque and twist of a wire are related by a formula identical with (1.8).

MAXWELL (*cf.* RAYLEIGH [1899]; WHITTAKER [1951]) initiated also the method which enjoys current favor of deriving electromagnetic constitutive relations from a molecular model of a material medium and the laws of mechanics. Elaborate theories of electromagnetic absorption and dispersion of this general type were developed by DRUDE [1893, 1902], VOIGT [1899], and many others. A detailed and modern theory of electromagnetic constitutive relations as derived from the dynamics of an ionic crystal lattice, including an appropriate application of quantum mechanics, is given in the book by BORN & HUANG [1954].

VOLTERRA [1912] extended the HOPKINSON's relation (1.8) to the nonlinear, anisotropic, and magnetic case. VOLTERRA's most general expression of the idea took the form of the equations:

$$\begin{gathered} \boldsymbol{D}(\boldsymbol{x}, t) = \boldsymbol{\epsilon} \cdot \boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{F}\left[\overset{t}{\underset{-\infty}{\boldsymbol{E}}}(\boldsymbol{x}, \tau)\right], \\ \boldsymbol{\mathcal{B}}(\boldsymbol{x}, t) = \boldsymbol{\mu} \cdot \boldsymbol{H}(\boldsymbol{x}, t) + \boldsymbol{\Phi}\left[\overset{t}{\underset{-\infty}{\boldsymbol{H}}}(\boldsymbol{x}, \tau)\right], \end{gathered} \tag{1.9}$$

where $(1.9)_1$ reduces to (1.8) if the functional $\boldsymbol{F}$ is linear, isotropic, and satisfies certain other conditions of which we shall say more later. VOLTERRA's theory was developed further by GRAFFI [1927, 1928].

VOLTERRA's paper, though it treats the general anisotropic case and expresses a very general view, makes the traditional *a priori* separation of electric and magnetic effects characteristic of MAXWELL's simple relations (1.7). That this separation is inadequate to account for numerous known phenomena such as optical activity and the rotation of the plane of polarization of light by a strong magnetic field (Faraday effect) should have been apparent from the earlier

theory of VOIGT cited above. In this paper we wish to explore some of the more obvious consequences of electromagnetic constitutive relations having the general form

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{x}, t) &= \boldsymbol{\Phi}\overset{t}{\underset{-\infty}{[}}\boldsymbol{E}(\boldsymbol{x}, \tau), \boldsymbol{B}(\boldsymbol{x}, \tau)], \\ \boldsymbol{H}(\boldsymbol{x}, t) &= \boldsymbol{\Psi}\overset{t}{\underset{-\infty}{[}}\boldsymbol{E}(\boldsymbol{x}, \tau), \boldsymbol{B}(\boldsymbol{x}, \tau)]. \end{aligned} \tag{1.10}$$

We shall show that if VOLTERRA'S relations (1.9) and the relations (1.10) are both linearized, then, for *holohedral isotropic materials*, they coincide so that there is, for these materials, a separation of effects; however, we shall show that the phenomenon of optical activity, which does not occur in holohedral isotropic materials, depends in an essential way upon the appearance of $\boldsymbol{B}$ in $(1.10)_1$ or of $\boldsymbol{E}$ in $(1.10)_2$. We shall show that, so far as the dispersion and absorption of plane electromagnetic waves is concerned, *linear* functional constitutive relations of the form (1.10) which are consistent with VOLTERRA'S *principe du cycle fermé* (explained in § 2) lead to dispersion formulae similar to those obtained by BORN & HUANG from a molecular model. The simple and direct mathematical reasoning we apply to the relations (1.10) should have application in any theory of the electromagnetic field in stationary matter which leads ultimately to relations consistent with (1.10). The physical idea expressed by these relations is simply that, in a stationary medium, the values of the electric field $\boldsymbol{E}(\boldsymbol{x}, \tau)$ and the magnetic flux density $\boldsymbol{B}(\boldsymbol{x}, \tau)$ for all times $\tau \leqq t$ preceding the instant t uniquely determine the values of the electric displacement $\boldsymbol{D}(\boldsymbol{x}, t)$ and the magnetic intensity $\boldsymbol{H}(\boldsymbol{x}, t)$ at the time t.

2. Restrictions imposed on the constitutive relations by material symmetry

If the material considered has some symmetry, the constitutive relations (1.10) must be form invariant under the group of transformations $\{\boldsymbol{S}\}$ describing the symmetry. For solids, the group of *material symmetry transformations* $\{\boldsymbol{S}\}$ is either the full orthogonal group or a subgroup of it. Let $\boldsymbol{S} = \|s_{ij}\|$ be a generic transformation of this group. Then the functionals on the right-hand side of (1.10) must satisfy the equations

$$\begin{aligned} \Phi_i\overset{t}{\underset{-\infty}{[}}\bar{\boldsymbol{E}}(\tau), \bar{\boldsymbol{B}}(\tau)] &= s_{ij}\,\Phi_j\overset{t}{\underset{-\infty}{[}}\boldsymbol{E}(\tau), \boldsymbol{B}(\tau)], \\ \Psi_i\overset{t}{\underset{-\infty}{[}}\bar{\boldsymbol{E}}(\tau), \bar{\boldsymbol{B}}(\tau)] &= \det \boldsymbol{S}\, s_{ij}\,\Psi_j\overset{t}{\underset{-\infty}{[}}\boldsymbol{E}(\tau), \boldsymbol{B}(\tau)], \end{aligned} \tag{2.1}$$

where

$$\bar{E}_i = s_{ij} E_j, \qquad \bar{B}_i = \det \boldsymbol{S}\, s_{ij} B_j, \tag{2.2}$$

for every element $\boldsymbol{S}$ of $\{\boldsymbol{S}\}$. The factor $\det \boldsymbol{S} = \pm 1$ appears in $(2.1)_2$ and $(2.2)_2$ because, as one can recall from elementary electromagnetic theory, the invariance of MAXWELL'S equations requires that $\boldsymbol{D}$ and $\boldsymbol{E}$ transform as absolute vectors under time-independent orthogonal transformations if the current and charge are absolute quantities, while $\boldsymbol{B}$ and $\boldsymbol{H}$ must transform as axial vectors under this group.

In this paper, we shall treat only the case where Φ and Ψ are *linear* functionals. This will exclude the more interesting non-linear phenomena such as the Faraday effect, but we choose to dispense with the simpler linear theory first. In the literature on dielectrics (*cf.*, *e.g.*, WHITEHEAD [1927]), linearity of the functional Φ is justified by what is called the *principle of superposition*. Actually, we shall make the more restrictive assumption that Φ and Ψ are linear functionals having the form

$$\boldsymbol{D}(t) = \sum_{\nu=0}^{p} \boldsymbol{a}_\nu \cdot \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} \boldsymbol{c}_\nu \cdot \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} \boldsymbol{\varphi}_1(t,\tau) \cdot \boldsymbol{E}(\tau)\, d\tau + \int_{-\infty}^{t} \boldsymbol{\varphi}_2(t,\tau) \cdot \boldsymbol{B}(\tau)\, d\tau,$$

$$\boldsymbol{H}(t) = \sum_{\nu=0}^{p} \boldsymbol{d}_\nu \cdot \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} \boldsymbol{b} \cdot \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} \boldsymbol{\psi}_1(t,\tau) \cdot \boldsymbol{B}(\tau)\, d\tau + \int_{-\infty}^{t} \boldsymbol{\psi}_2(t,\tau) \cdot \boldsymbol{E}(\tau)\, d\tau, \quad (2.3)$$

$$A^{(\nu)} \equiv \frac{d^\nu A}{dt^\nu},$$

where $\boldsymbol{a}_\nu$, $\boldsymbol{b}_\nu$, $\boldsymbol{c}_\nu$, and $\boldsymbol{d}_\nu$ are constant tensors and the kernels $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\psi}_1$, and $\boldsymbol{\psi}_2$ are continuous tensor functions of t and τ such that each satisfies an order condition of the form

$$\boldsymbol{\varphi}_1(t,\tau) < \frac{\boldsymbol{C}}{(t-\tau)^{1+\varepsilon}}, \qquad \varepsilon > 0. \quad (2.4)$$

Thus, in VOLTERRA's terminology [1930, Chap. I], Φ and Ψ are linear functionals with order of continuity p having t as an exceptional point. Now VOLTERRA [1912] has proven that, with the order condition (2.4), $\boldsymbol{D}(t)$ and $\boldsymbol{H}(t)$ will be periodic functions of t whenever $\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ are periodic functions (principe du cycle fermé) *if and only if* the kernel functions $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\psi}_2$ depend on the two variables t and τ only through their difference $t-\tau$. Again, we shall delimit the class of materials under consideration by assuming (2.4) and that $\boldsymbol{D}$ and $\boldsymbol{H}$ are periodic when $\boldsymbol{E}$ and $\boldsymbol{B}$ are periodic. This latter assumption rules out something like an "ageing" or deterioration of the material. Heredity of the nature we assume here is also called *invariable heredity*.

For the linear functionals (2.3), the restrictions of symmetry (2.1) are satisfied if and only if the second-order tensors which completely determine the functional relations (2.3) satisfy the restrictions:

$$\begin{aligned} &\boldsymbol{S}\,\boldsymbol{a}_\nu\,\boldsymbol{S}^{-1} = \boldsymbol{a}_\nu, \quad \boldsymbol{S}\,\boldsymbol{b}_\nu\,\boldsymbol{S}^{-1} = \boldsymbol{b}_\nu, \quad \boldsymbol{S}\,\boldsymbol{\varphi}_1\,\boldsymbol{S}^{-1} = \boldsymbol{\varphi}_1, \quad \boldsymbol{S}\,\boldsymbol{\psi}_1\,\boldsymbol{S}^{-1} = \boldsymbol{\psi}_1 \\ &\qquad (\det \boldsymbol{S})\,\boldsymbol{S}\,\boldsymbol{c}_\nu\,\boldsymbol{S}^{-1} = \boldsymbol{c}_\nu, \qquad (\det \boldsymbol{S})\,\boldsymbol{S}\,\boldsymbol{d}_\nu\,\boldsymbol{S}^{-1} = \boldsymbol{d}_\nu, \\ &\qquad (\det \boldsymbol{S})\,\boldsymbol{S}\,\boldsymbol{\varphi}_2\,\boldsymbol{S}^{-1} = \boldsymbol{\varphi}_2, \qquad (\det \boldsymbol{S})\,\boldsymbol{S}\,\boldsymbol{\psi}_2\,\boldsymbol{S}^{-1} = \boldsymbol{\psi}_2, \end{aligned} \quad (2.5)$$

for each element $\boldsymbol{S}$ of the group of orthogonal transformations $\{\boldsymbol{S}\}$. In other words, $\boldsymbol{a}_\nu$, $\boldsymbol{b}_\nu$, $\boldsymbol{\varphi}_1$, and $\boldsymbol{\psi}_1$ must be invariant absolute second-order tensors under transformations of $\{\boldsymbol{S}\}$ and $\boldsymbol{c}_\nu$, $\boldsymbol{d}_\nu$, $\boldsymbol{\varphi}_2$, and $\boldsymbol{\psi}_2$ must be invariant *axial* second-order tensors under transformations of $\{\boldsymbol{S}\}$. One can, in principle, determine the most general solution of the invariant theoretic problem posed by (2.5) for any group of orthogonal transformations by methods which are known.

If $\{\boldsymbol{S}\}$ is the full orthogonal group, the material is called *holohedral isotropic*, there are no invariant axial second-order tensors, and the absolute invariant second-order tensors are all proportional to the unit matrix. Thus, for holohedral isotropic materials, the constitutive relations (2.3) must reduce to VOLTERRA'S

form

$$\boldsymbol{D}(t) = \sum_{\nu=0}^{p} a_\nu \boldsymbol{E}^{(\nu)}(t) + \int_{-\infty}^{t} \varphi_1(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau,$$
$$\boldsymbol{H}(t) = \sum_{\nu=0}^{p} b_\nu \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} \psi_1(t-\tau)\, \boldsymbol{B}(\tau)\, d\tau. \tag{2.6}$$

Thus we see that linearity *and* holohedral isotropy are sufficient conditions to eliminate $\boldsymbol{B}$ from $(1.10)_1$ and $\boldsymbol{E}$ from $(1.10)_2$, as had been *assumed* possible by VOLTERRA for a material of general symmetry.

If $\{\boldsymbol{S}\}$ is the *proper* orthogonal group, consisting of all orthogonal transformations with positive determinant, then all of the tensors in (2.5) may be arbitrary multiples of the unit matrix and the constitutive relations (2.3) reduce to the form

$$\boldsymbol{D}(t) = \sum_{\nu=0}^{p} a_\nu \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} c_\nu \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} [\varphi_1(t-\tau)\, \boldsymbol{E}(\tau) + \varphi_2(t-\tau)\, \boldsymbol{B}(\tau)]\, d\tau,$$
$$\boldsymbol{H}(t) = \sum_{\nu=0}^{p} d_\nu \boldsymbol{E}^{(\nu)}(t) + \sum_{\nu=0}^{p} b_\nu \boldsymbol{B}^{(\nu)}(t) + \int_{-\infty}^{t} [\psi_1(t-\tau)\, \boldsymbol{B}(\tau) + \psi_2(t-\tau)\, \boldsymbol{E}(\tau)]\, d\tau. \tag{2.7}$$

A material with this symmetry is called *hemihedral isotropic*. A sugar solution containing unequal amounts of right- and left-handed sugar molecules may serve as an example of a hemihedral isotropic material. The examples (2.6) and (2.7) suffice to indicate how one may proceed to obtain explicit representations for the linear functionals (2.3) for the case of an arbitrary symmetry group $\{\boldsymbol{S}\}$.

3. The propagation of an infinite plane electromagnetic wave in a hemihedral isotropic material

An infinite plane progressive vector wave $\boldsymbol{A}(\boldsymbol{x}, t)$ is a time-dependent vector field having the form

$$\boldsymbol{A}(\boldsymbol{x}, t) = \mathscr{R}\, \boldsymbol{a}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \tag{3.1}$$

where $\boldsymbol{a} = \boldsymbol{a}^+ + i\boldsymbol{a}^-$ ($i = \sqrt{-1}$, $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$ real vectors) is a complex vector called the *amplitude* of $\boldsymbol{A}$, $k = k^+ + ik^-$, $k^+ > 0$, is the *complex wave number*, $\boldsymbol{n}$ is a real unit vector called the *direction of propagation*, and $\omega > 0$ is the *angular frequency* measured in radians per unit time. The symbol $\mathscr{R}$ denotes the real part of the complex quantity placed after it.

For each fixed value of $\boldsymbol{x}$, $\boldsymbol{A}$ is a periodic vector function of time and the locus of $\boldsymbol{A}$ is an ellipse or one of its degenerate forms, a circle, or a straight line. The major axes of the ellipse, the radius of the circle, or the length and direction of the straight line are determined by the complex vector amplitude $\boldsymbol{a}$. If $\boldsymbol{a}\cdot\boldsymbol{a} = \boldsymbol{a}^+\cdot\boldsymbol{a}^+ - \boldsymbol{a}^-\cdot\boldsymbol{a}^- + 2i\boldsymbol{a}^+\cdot\boldsymbol{a}^- = 0$, the real and imaginary parts of $\boldsymbol{a}$ are perpendicular to each other and of equal length, and the locus of $\boldsymbol{A}$ is a circle. If $\boldsymbol{a}^* = \boldsymbol{a}^+ - i\boldsymbol{a}^-$ denotes the complex conjugate of $\boldsymbol{a}$, and $\boldsymbol{a}^*\times\boldsymbol{a} = 2i\,\boldsymbol{a}^+\times\boldsymbol{a}^- = 0$, the real and imaginary parts of $\boldsymbol{a}$ are parallel and the locus of $\boldsymbol{A}$ is a straight line. In all other cases, the locus of $\boldsymbol{A}$ is an ellipse.

In electromagnetic theory (*cf.* CONDON & ODISHAW 1958, Chap. VI), if the electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ is a plane wave, then its amplitude $\boldsymbol{e}$ determines its *polarization*. An electromagnetic plane wave is said to be *circularly*, *linearly*, or *elliptically polarized* according to whether the locus of $\boldsymbol{E}$ is a circle, a straight

line, or an ellipse*. Circularly and elliptically polarized waves are further classified as *left-handed* or *right-handed* according to the rule

$$\boldsymbol{a}^+ \cdot (\boldsymbol{a}^- \times \boldsymbol{n}) = [\boldsymbol{a}^+, \boldsymbol{a}^-, \boldsymbol{n}] \begin{matrix} > 0 \rightarrow \text{right-handed}, \\ < 0 \rightarrow \text{left-handed}. \end{matrix}$$

If $\boldsymbol{a} \cdot \boldsymbol{n} = 0$, the wave is called *transverse*, and if $\boldsymbol{a} \times \boldsymbol{n} = 0$, the wave is called *longitudinal*. All other waves will be called *skew*.

The field equations which follow from MAXWELL's equations (1.1) to (1.4) are

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{E} = 0, \qquad & \operatorname{div} \boldsymbol{B} = 0, \\ \operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}_{\mathrm{F}}, \qquad & \operatorname{div} \boldsymbol{D} = Q_{\mathrm{F}}. \end{aligned} \tag{3.2}$$

In this section, we shall consider plane electromagnetic waves in a non-conducting hemihedral isotropic medium for which $\boldsymbol{J}_{\mathrm{F}} = Q_{\mathrm{F}} = 0$, and for which the linear functional constitutive relations (2.7) are valid. Thus we seek plane wave solutions,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}, t) &= \mathscr{R}\, \boldsymbol{e}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{B}(\boldsymbol{x}, t) &= \mathscr{R}\, \boldsymbol{b}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{D}(\boldsymbol{x}, t) &= \mathscr{R}\, \boldsymbol{d}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{H}(\boldsymbol{x}, t) &= \mathscr{R}\, \boldsymbol{h}\, e^{i(k\boldsymbol{n}\cdot\boldsymbol{x} - \omega t)}, \end{aligned} \tag{3.3}$$

of the system of integral and differential equations (2.7) and (3.2).

Introducing vector fields of the form (3.3) into (2.7) and (3.2), we find that they will constitute a solution if and only if the amplitudes $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, and $\boldsymbol{h}$, the wave number k, and the frequency ω satisfy jointly the system of equations

$$\boldsymbol{n} \cdot \boldsymbol{b} = 0, \qquad \boldsymbol{n} \cdot \boldsymbol{d} = 0, \tag{3.4}$$

$$k\, \boldsymbol{n} \times \boldsymbol{e} - \omega\, \boldsymbol{b} = 0, \tag{3.5}$$

$$k\, \boldsymbol{n} \times \boldsymbol{h} + \omega\, \boldsymbol{d} = 0, \tag{3.6}$$

$$\boldsymbol{d} = \alpha(\omega)\, \boldsymbol{e} + \gamma(\omega)\, \boldsymbol{b}, \tag{3.7}$$

$$\boldsymbol{h} = \beta(\omega)\, \boldsymbol{b} + \delta(\omega)\, \boldsymbol{e}, \tag{3.8}$$

where

$$\begin{aligned} \alpha(\omega) &\equiv \sum_{\nu=0}^{p} a_\nu (-i\omega)^\nu + \int_0^\infty \varphi_1(u)\, e^{i\omega u}\, du, \\ \gamma(\omega) &\equiv \sum_{\nu=0}^{p} c_\nu (-i\omega)^\nu + \int_0^\infty \varphi_2(u)\, e^{i\omega u}\, du, \\ \beta(\omega) &\equiv \sum_{\nu=0}^{p} b_\nu (-i\omega)^\nu + \int_0^\infty \psi_1(u)\, e^{i\omega u}\, du, \\ \delta(\omega) &\equiv \sum_{\nu=0}^{p} d_\nu (-i\omega)^\nu + \int_0^\infty \psi_2(u)\, e^{i\omega u}\, du. \end{aligned} \tag{3.9}$$

* Linearly polarized electromagnetic waves are also called *plane* polarized waves, but this seems to overwork the word "plane".

From (3.4), (3.7), and (3.8) it follows that $\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{D}$, and $\boldsymbol{H}$ are transverse. Since we assume $k^+ > 0$, $\omega > 0$, so that the wave travels in the positive direction of $\boldsymbol{n}$, equations (3.4) will be satisfied as a consequence of (3.5) and (3.6). Thus it is sufficient that we consider the four homogeneous vector equations (3.5) to (3.8) in the four unknowns $\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{h}$, and $\boldsymbol{d}$. In order for there to exist non-trivial solutions, the determinant $\Delta(k, \omega)$ of the 12×12 matrix of coefficients must vanish. Thus the complex wave number k is some function of the angular frequency ω, and this function, whatever it may be, is called the *dispersion*

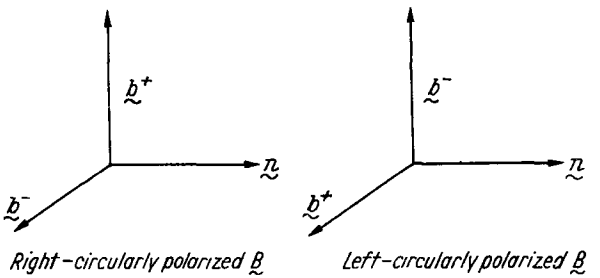

Fig. 1

relation. To determine this relation and other properties of the solution we proceed as follows. First eliminate $\boldsymbol{d}$ and $\boldsymbol{h}$ from (3.6) using (3.7) and (3.8) to get

$$k\,\boldsymbol{n} \times (\beta\,\boldsymbol{b} + \delta \boldsymbol{e}) + \omega(\alpha\,\boldsymbol{e} + \gamma\,\boldsymbol{b}) = 0. \tag{3.10}$$

Now eliminate $\boldsymbol{e}$ from (3.10) using (3.5) and $\boldsymbol{n} \cdot \boldsymbol{e} = 0$ to get

$$\left(\beta - \alpha \frac{\omega^2}{k^2}\right) \boldsymbol{n} \times \boldsymbol{b} + \left(\frac{\omega}{k}\right)(\delta + \gamma)\,\boldsymbol{b} = 0. \tag{3.11}$$

The scalar product of (3.11) with $\boldsymbol{b}$ then yields the condition

$$(\delta + \gamma)\,\boldsymbol{b} \cdot \boldsymbol{b} = 0. \tag{3.12}$$

Thus *either the coefficient* $\delta + \gamma = 0$ *or the magnetic flux* $\boldsymbol{B}$ must be *circularly polarized.* But if $\boldsymbol{B}$ is circularly polarized so also are $\boldsymbol{E}, \boldsymbol{D}$, and $\boldsymbol{H}$, as follows from (3.5) to (3.8).

Let us assume that $(\gamma + \delta) \neq 0$ so that $\boldsymbol{B}$ is circularly polarized. Since it is also transverse, $\boldsymbol{b}^+$, $\boldsymbol{b}^-$, and $\boldsymbol{n}$ are mutually orthogonal, so that one of the two geometrical relations in Fig. 1 holds.

From Figure 1 it is easy to see that

$$\boldsymbol{n} \times \boldsymbol{b} = \begin{cases} -i\,\boldsymbol{b} & \text{for right-circular polarization,} \\ +i\,\boldsymbol{b} & \text{for left-circular polarization.} \end{cases}$$

Using this result and (3.11) we then see that the dispersion relation is

$$\frac{k^2}{\omega^2} \pm \frac{i\,k}{\omega} \frac{\gamma + \delta}{\beta} - \frac{\alpha}{\beta} = 0, \tag{3.13}$$

where the upper sign in the linear term holds for right-circular polarization and the lower sign for left-circular polarization.

Thus far we have imposed no restrictions on the real constants a_ν, b_ν, c_ν, d_ν, or on the real memory functions $\varphi_1(u)$, $\varphi_2(u)$, $\psi_1(u)$, and $\psi_2(u)$, other than the order condition (2.4). We shall now assume that $k^+>0$ and $k^-\geqq 0$. It has already been pointed out that the first assumption implies that the wave travels in the positive direction of $\boldsymbol{n}$. Subject to $k^+>0$, the second assumption implies that a wave of frequency ω and small amplitude does not build up, resulting in an unstable situation. We may then express the root of (3.13) which satisfies $k^+>0$, $k^-\geqq 0$ in the form

$$\frac{k}{\omega}=\sqrt{\frac{\alpha}{\beta}-\left(\frac{\gamma+\delta}{2\beta}\right)^2}\mp i\,\frac{\gamma+\delta}{2\beta}, \tag{3.14}$$

where the upper sign in the bracket holds for right-circularly polarized $\boldsymbol{B}$ and the lower sign for left-circularly polarized $\boldsymbol{B}$, and the square root taken in each case denotes the determination for which $k^+>0$, $k^-\geqq 0$.

The *wave length* is then given by $\lambda=2\pi/k^+$, the *speed* of the wave by $v=\omega/k^+$, and the *absorption coefficient* by k^-. From (3.14) we see that, unless the quantity

$$\vartheta(\omega)\equiv\mathscr{I}\left(\frac{\gamma+\delta}{\beta}\right) \tag{3.15}$$

vanishes, the speeds of the right and left-circularly polarized waves will be different so that the medium is *optically active* (see DRUDE [1902]). Unless the quantity

$$\varkappa(\omega)\equiv\mathscr{R}\left(\frac{\gamma+\delta}{\beta}\right) \tag{3.16}$$

vanishes, the right and left-circularly polarized waves will have different absorption coefficients, as has been observed in some real materials.

4. Comparison with the Born-Huang theory

Beginning with a model of an ionic solid consisting of a set of charged mass points, the dynamics of which may be treated either classically or quantum mechanically, BORN & HUANG [1954, 1, Chap. VII] derive a relation between $\boldsymbol{d}$ and $\boldsymbol{e}$ having the form

$$\boldsymbol{d}=\boldsymbol{\epsilon}(\boldsymbol{k},\omega)\cdot\boldsymbol{e}\,,\qquad \boldsymbol{k}\equiv k\,\boldsymbol{n}\,, \tag{4.1}$$

where the *dielectric tensor* $\boldsymbol{\epsilon}$ is Hermitian.

$$\varepsilon_{ij}(\boldsymbol{k},\omega)=\varepsilon^*_{ji}(\boldsymbol{k},\omega)\,, \tag{4.2}$$

and satisfies also the relations

$$\varepsilon_{ij}(\boldsymbol{k},\omega)=\varepsilon^*_{ij}(-\boldsymbol{k},\omega)\,. \tag{4.3}$$

Thus if we put

$$\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^+ + i\,\boldsymbol{\epsilon}^-,$$

the conditions (4.2) and (4.3) are satisfied if and only if

$$\begin{aligned}
\varepsilon^+_{ij}(\boldsymbol{k},\omega)&=\varepsilon^+_{ji}(\boldsymbol{k},\omega)\,,\\
\varepsilon^-_{ij}(\boldsymbol{k},\omega)&=-\varepsilon^-_{ji}(\boldsymbol{k},\omega)\,,\\
\varepsilon^+_{ij}(\boldsymbol{k},\omega)&=\varepsilon^+_{ij}(-\boldsymbol{k},\omega)\,,\\
\varepsilon^-_{ij}(\boldsymbol{k},\omega)&=-\varepsilon^-_{ij}(-\boldsymbol{k},\omega)\,.
\end{aligned} \tag{4.4}$$

Now from our equations (3.5) and (3.7), we get a relation like (4.1) with

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \alpha\, \delta_{ij} + \frac{\gamma k}{\omega} e_{ikj} n_k, \tag{4.5}$$

where e_{ijk} is the completely antisymmetric axial tensor with $e_{123} = 1$. Thus we get

$$\begin{aligned} \varepsilon_{ij}^{+} &= \alpha^{+} \delta_{ij} + \frac{(\gamma k)^{+}}{\omega} e_{ikj} n_k, \\ \varepsilon_{ij}^{-} &= \alpha^{-} \delta_{ij} + \frac{(\gamma k)^{-}}{\omega} e_{ikj} n_k. \end{aligned} \tag{4.6}$$

The four conditions (4.4) are satisfied if and only if

$$(\gamma k)^{+} = 0, \qquad \alpha^{-} = 0. \tag{4.7}$$

It is obvious that these conditions are not met by an arbitrary assignment of the constants $a_\nu, b_\nu, c_\nu, d_\nu$, and the memory functions $\varphi_1, \varphi_2, \psi_1$, and ψ_2. In the special case of our relations for which $\alpha(\omega)$, $\beta(\omega)$ are real and positive and the functions $\gamma(\omega)$ and $\delta(\omega)$ are pure imaginary, k is real, there is no absorption, and the conditions (4.7) are both satisfied.

Further comparison of the two theories is not warranted since even the partial overlapping of results outlined above is hardly more than accidental. In the Born-Huang treatment, absorption is represented by a transfer of electromagnetic energy to mechanical energy of lattice vibrations, while in the simple phenomenological theory presented here, any motion or deformation of the medium has been ignored; moreover, we have assumed that the medium is homogeneous, while the treatment of BORN & HUANG is based on a highly inhomogeneous material medium, namely, a system of mass points.

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