Stress Tensors in Elastic Dielectrics

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1. Introduction

In a previous paper [*1*] we presented a theory of the finite deformation of an elastic, electrically polarizable material. As originally presented, the theory divides a symmetric total stress into two asymmetric components which we called the local stress and the Maxwell stress. Such a decomposition is not necessary but is made for the purpose of introducing a constitutive relation between the local stress and the deformation and polarization of the material. It is the purpose of this note to clarify the theory presented in [*1*] and to exhibit a number of equivalent formulations of its basic equations. In these equivalent formulations, a variety of "stress" tensors appear, and we shall attempt to make clear the relations between them and the stress tensors introduced in [*1*].

The problem of elastic dielectrics affords a simple model of a classical field theory based on a variational principle. Results on this model serve as a guide to the more complex and general dynamical theory of the electromagnetic field in a moving and deforming material which must contain the present statical theory as a special case. MINKOWSKI [*2*] and ABRAHAM [*3*] many years ago proposed different expressions for the electromagnetic stress, energy, momentum, and energy flux in material media. We do not cite the extensive literature which supports one or the other of these expressions. As emphasized by MØLLER [*5*] and GYORGYI [*4*], in a material medium, the electromagnetic field forms only part of the physical system. Any division of energy, momentum, stress, and energy flux into electromagnetic and mechanical components is bound to be somewhat arbitrary, and it is fruitless to attempt an independent theory of either component. We trust that the considerations to follow will illustrate these ideas concretely in a simple case where the physical principles are easy to grasp and intuition is trustworthy.

2. Some preliminary formalism

a) The divergence theorem. If $\mathfrak{f}^i$ is a contravariant vector density, *i.e.*, a relative tensor of weight 1, then its *natural divergence* is a scalar density*.

$$\operatorname{div}\mathfrak{f} \equiv \partial_i \mathfrak{f}^i, \qquad \partial_i \equiv \frac{\partial}{\partial x^i}. \tag{2.1}$$

* The natural divergence is defined independently of any definition of covariant differentiation. In metric or affinely connected spaces with symmetric affine connection, $V_i \mathfrak{f}^i = \partial_i \mathfrak{f}^i$ provided $\mathfrak{f}$ is a density. *Cf.* SCHOUTEN [*6*, Ch. II].

If $\mathfrak{s}$ is a surface enclosing a region $\mathfrak{v}$ where $\mathfrak{f}$ is continuously differentiable, then

$$\oint_{\mathfrak{s}} \mathfrak{f}^i \, d\mathfrak{s}_i = \int_{\mathfrak{v}} \operatorname{div} \mathfrak{f} \, d\mathfrak{v}, \tag{2.2}$$

which is the divergence theorem [6, p. 97].

Suppose $\mathfrak{v}$ is a region divided by a surface $\mathfrak{s}$ across which $\mathfrak{f}$ has a discontinuity $[\mathfrak{f}] = \mathfrak{f}^+ - \mathfrak{f}^-$ where $\mathfrak{f}^+$ and $\mathfrak{f}^-$ are the limiting values of $\mathfrak{f}$ on $\mathfrak{s}$. Then if $\mathfrak{f}$ satisfies the smoothness conditions of the divergence theorem in the two parts of $\mathfrak{v}$, and if the limits $\mathfrak{f}^+$ and $\mathfrak{f}^-$ exist, we have

$$\oint_{\mathfrak{s}} \mathfrak{f}^i \, d\mathfrak{s}_i = \int_{\mathfrak{v}} \operatorname{div} \mathfrak{f} \, d\mathfrak{v} + \int_{\mathfrak{s} \cap \mathfrak{v}} [\mathfrak{f}^i] \, d\mathfrak{s}_i, \tag{2.3}$$

where $d\mathfrak{s}_i$ points into $\mathfrak{v}^+$.

b) Motions. Let

$$\begin{gathered} x^i = x^i(X^A, \lambda), \qquad X^A = X^A(x^i, \lambda) \\ (\boldsymbol{x}/\boldsymbol{X}) \equiv \det \left\| \frac{\partial x^i}{\partial X^A} \right\| \neq 0 \end{gathered} \tag{2.4}$$

be a one parameter family of non-singular continuously differentiable mappings (point transformations) where the indices $i, j, \ldots A, B, \ldots$ have the same range $1, 2, \ldots, n$. In the applications, the variables X^A will be *material coordinates*, the x^i will be spatial coordinates, and n will equal three. We may think of λ as the *time* though this is not a necessary nor even desirable interpretation in purely statical considerations. It is preferable to regard λ simply as an index set which labels the configurations of a given set of material points $\{\boldsymbol{X}\}$. In these preliminaries, we need not assume that the dimension of the space is 3 nor that the space with coordinates x^i is metric in the sense of Riemannian geometry. This will allow application of the general formulae to be derived to problems other than the particular problem of elastic dielectrics considered in §4.

We set

$$\begin{aligned} v^i &\equiv \frac{\partial x^i}{\partial \lambda}, & \dot{X}^A &\equiv \frac{\partial X^A}{\partial \lambda}, \\ x^i_A &\equiv \frac{\partial x^i}{\partial X^A}, & X^A_i &\equiv \frac{\partial X^A}{\partial x^i} \end{aligned} \tag{2.5}$$

and record the identities*

$$\begin{aligned} v^i &= - x^i_A \, \dot{X}^A, & \dot{X}^A &= - X^A_i \, v^i, \\ x^i_A \, X^A_j &= \delta^i_j, & x^i_A \, X^B_i &= \delta^B_A, \\ \frac{\partial (\boldsymbol{x}/\boldsymbol{X})}{\partial x^i_A} &= (\boldsymbol{x}/\boldsymbol{X}) \, X^A_i, & \frac{\partial (\boldsymbol{x}/\boldsymbol{X})}{\partial X^A_i} &= - (\boldsymbol{x}/\boldsymbol{X}) \, x^i_A, \\ \frac{\partial X^A_i}{\partial x^j_B} &= - X^A_j \, X^B_i, & \frac{\partial x^i_A}{\partial X^B_j} &= - x^i_B \, x^j_A. \end{aligned} \tag{2.6}$$

c) Invariant derivatives. Let $f^{ij\cdots}_{k\cdots}(\boldsymbol{x}, \lambda)$ denote a tensor field of arbitrary rank and weight whose components may be functions of the parameter λ as

* *Cf.* TRUESDELL [7].

well as the coordinates x. Let

$$\dot{f}^{ij\cdots}_{k\cdots} \equiv \frac{\partial f^{ij\cdots}_{k\cdots}}{\partial \lambda}. \tag{2.7}$$

The *Lie derivative* of $\boldsymbol{f}$ with respect to v^i is a tensor field of the same type as $\boldsymbol{f}$ with components given by*

$$\underset{v}{\pounds} f^{ij\cdots}_{k\cdots} \equiv v^m \partial_m f^{ij\cdots}_{k\cdots} - \partial_m v^i f^{mj\cdots}_{k\cdots} - \partial_m v^j f^{im\cdots}_{k\cdots} - \cdots + \partial_k v^m f^{ij\cdots}_{m\cdots} + \cdots w\, \partial_m v^m f^{ij\cdots}_{k\cdots}, \tag{2.8}$$

where w is the weight of $\boldsymbol{f}$.

Any set of quantities f^Ω, $\Omega = 1, 2, \ldots, N$ which transform by a linear law with coefficients determined by the transformation and its derivatives is called a *geometric quantity* [6, Ch. II, §3]. The definition of the Lie derivative may be extended to arbitrary geometric quantities and has the general form**

$$\underset{v}{\pounds} f^\Omega = v^m \partial_m f^\Omega - F^{\Omega\, m}_{\Delta\, n} \partial_m v^n f^\Delta, \tag{2.9}$$

where the $F^{\Omega\, m}_{\Delta\, n}$ are constants determined by the law of transformation of the f^Ω. Formula (2.8) is a special case of (2.9). For example, when $\boldsymbol{f}$ is a contravariant vector density, $F^{i\, n}_{j\, m} = \delta^n_j \delta^i_m - \delta^i_j \delta^n_m$. The notation of (2.9) is that used by BERGMANN [8].

In a space with symmetric affine connection Γ^i_{jk} and when f^Ω is a set of *tensor* components, we have

$$V_i f^\Omega = \partial_i f^\Omega + F^{\Omega\, n}_{\Delta\, m} \Gamma^m_{in} f^\Delta, \tag{2.10}$$

where the $F^{\Omega\, m}_{\Delta\, n}$ will have the same values as in (2.9) for corresponding tensors or sets of tensor components f^Ω.

In affinely connected spaces we define the *material derivative* of $\boldsymbol{f}$ by

$$\dot{f}^\Omega \equiv \dot{f}^\Omega + v^m V_m f^\Omega. \tag{2.11}$$

The *convective* λ *derivative* of $\boldsymbol{f}$ is defined by

$$\overset{*}{f}{}^\Omega \equiv \dot{f}^\Omega + \underset{v}{\pounds} f^\Omega. \tag{2.12}$$

In a space with symmetric affine connection, the partial derivatives of v^i and f^Ω which occur in the definition (2.8) or (2.9) of the Lie derivative may be replaced by covariant derivatives [6, p. 152],

$$\underset{v}{\pounds} f^\Omega = v^m V_m f^\Omega - F^{\Omega\, n}_{\Delta\, m} V_n v^m f^\Delta, \tag{2.13}$$

provided, of course, that f^Ω is a tensor field or set of tensor fields for which the covariant derivative is defined. We then have the identity

$$\overset{*}{f}{}^\Omega \equiv \dot{f}^\Omega - F^{\Omega\, n}_{\Delta\, m} V_n v^m f^\Delta. \tag{2.14}$$

A geometric quantity f^Ω together with its first partial derivatives $\partial_i f^\Omega$ is a geometric quantity $(f^\Omega, \partial_i f^\Omega)$. In the sense explained by SCHOUTEN [6, p. 105],

* [6, Ch. II, § 10].
** *Cf.* YANO [10].

Lie differentiation and ordinary partial differentiation are commutative:

$$\underset{v}{\pounds}\,\partial_i f^\Omega = \partial_i \underset{v}{\pounds} f^\Omega. \tag{2.15}$$

Thus, so also do the operators ∂_i and (*) commute:

$$\overset{*}{\partial_i f^\Omega} = \partial_i \overset{*}{f}{}^\Omega. \tag{2.16}$$

However, the Lie derivative of an affine connection is given by

$$\underset{v}{\pounds}\,\Gamma^m_{in} = \nabla_i \nabla_n v^m + R_{ki\,\cdot n}^{\;\cdot\;\cdot\;m}\, v^k, \tag{2.17}$$

where $\boldsymbol{R}$ is the curvature tensor based on $\boldsymbol{\Gamma}$ and we assume $\Gamma^i_{[jk]}=0$. Taking the Lie derivative of equation (2.10) and using (2.15), we get

$$\underset{v}{\pounds}\,\nabla_i f^\Omega - \nabla_i \underset{v}{\pounds} f^\Omega = F^{\Omega\,n}_{\Delta\,m}\,\underset{v}{\pounds}\,\Gamma^m_{in}, \tag{2.18}$$

from which it is apparent that, even in flat spaces where $R_{ki\,\cdot n}^{\;\cdot\;\cdot\;m}=0$, Lie differentiation and covariant differentiation do not commute. It follows that covariant differentiation and convected λ differentiation do not commute:

$$\overset{*}{\nabla_k f^\Omega} - \nabla_k \overset{*}{f}{}^\Omega = F^{\Omega\,n}_{\Delta\,m}\,\overset{*}{\Gamma}{}^m_{in}\, f^\Delta. \tag{2.19}$$

As a consequence of the identities (2.6) and the definitions (2.9) and (2.12) we have the useful identities,

$$\dot{X}^K = \overset{*}{X}{}^K \equiv 0, \quad \overset{*}{X}{}^K_i \equiv 0, \quad \overset{*}{x}{}^i_A \equiv 0, \tag{2.20}$$

where the fields X^K are treated as absolute scalars.

It follows from (2.20) that if we define the scalars $F^{AB\cdots}_{C\cdots}$ by

$$F^{AB\cdots}_{C\cdots} \equiv |(\boldsymbol{x}/\boldsymbol{X})|^w\, X^A_i\, X^B_j \ldots x^k_C \ldots f^{ij\cdots}_{k\cdots}, \tag{2.21}$$

then

$$f^{ij\cdots}_{k\cdots} = |(\boldsymbol{x}/\boldsymbol{X})|^{-w}\, x^A_i\, x^B_j \ldots X^C_k \ldots \dot{F}^{AB\cdots}_{C\cdots} \tag{2.22}$$

where $\boldsymbol{f}$ is a tensor of weight w, and

$$\overset{*}{f}{}^{ij\cdots}_{k\cdots} = |(\boldsymbol{x}/\boldsymbol{X})|^{-w}\, x^i_A\, x^j_B \ldots X^C_k \ldots \dot{F}^{AB\cdots}_{C\cdots}, \tag{2.23}$$

since, for scalars, $\overset{*}{F}=\dot{F}$. If the $F^{AB\cdots}_{C\cdots}$ are expressed as functions of the material coordinates X^A and of λ, then $\dot{F}=\dfrac{\partial F}{\partial \lambda}$. It is easy to see that the formulas (2.21) to (2.23) can be written in the abbreviated form

$$\begin{aligned} f^\Omega &= A^\Omega{}_\Delta F^\Delta, \quad F^\Delta = \overset{-1}{A}{}^\Delta{}_\Omega f^\Omega, \\ \overset{*}{f}{}^\Omega &= A^\Omega{}_\Delta \dot{F}^\Delta, \end{aligned} \tag{2.24}$$

where the coefficients $A^\Omega{}_\Delta$ are certain rational functions of the $x^i{}_A$. In (2.24), the f^Ω may also stand for the ordered set of components of more than one tensor field.

d) Lagrange identities. The field equations which follow from a variational principle are determined by assigning a scalar density $\mathscr{L}$ as a function of the components $(f^\Omega, \partial_i f^\Omega, \partial_i \partial_j f^\Omega, \ldots)$ of a geometric quantity:

$$\mathscr{L} = \mathscr{L}(f^\Omega, \partial_i f^\Omega, \ldots). \tag{2.25}$$

The condition that $\mathscr{L}$ transform as a scalar density, *i.e.*,

$$\mathscr{L}' = |(\boldsymbol{x}/\boldsymbol{x})|^{-1} \mathscr{L} \tag{2.26}$$

under general transformations of the coordinates $\boldsymbol{x}$ leads to a set of algebraic and differential identities satisfied jointly by $\mathscr{L}$, $\partial\mathscr{L}/\partial f^\Omega$, $\partial\mathscr{L}/\partial(\partial_i f^\Omega), \ldots, f^\Omega$, $\partial_i f^\Omega, \ldots$ as shown by SCHOUTEN [*6*, Ch. II, §11]*. We restrict attention here to the case where $\mathscr{L}$ depends only on the f^Ω and their first partial derivatives $\partial_i f^\Omega$. This implies no loss in generality since the geometric quantity $(f^\Omega, \partial_i f^\Omega, \ldots, \partial_{i_1}\partial_{i_2}\ldots\partial_{i_n} f^\Omega)$ can always be regarded as the geometric quantity $(g^{\Omega'}, \partial_i g^{\Omega'})$ where $g^{\Omega'} = (f^\Omega, \partial_i f^\Omega, \ldots, \partial_{i_1}\partial_{i_2}\ldots\partial_{i_{n-1}} f^\Omega)$. However, in the applications, the geometric quantity f^Ω which appears in (2.27) below is assumed to have a linear homogeneous law of transformation corresponding to the special case where f^Ω stands for the ordered set of components of a set of *tensor* fields. Thus we assume that $\mathscr{L}$ has the form

$$\mathscr{L} = \mathscr{L}(f^\Omega, \partial_i f^\Omega). \tag{2.27}$$

We then have

$$\underset{v}{\mathfrak{L}}\, \mathscr{L} = \partial_k(v^k \mathscr{L}) = \frac{\partial \mathscr{L}}{\partial f^\Omega} \underset{v}{\mathfrak{L}}\, f^\Omega + \frac{\partial \mathscr{L}}{\partial f_i^\Omega} \underset{v}{\mathfrak{L}}\, f_i^\Omega \tag{2.28}$$

which must hold for arbitrary v^k. In (2.28) and henceforth we use the notation $f_i^\Omega = \partial_i f^\Omega$. By equating coefficients of v^k, $\partial_m v^k$, and $\partial_m \partial_n v^k$ on both sides of $(2.28)_2$ we get the identities**:

$$\partial_k \mathscr{L} = \frac{\partial \mathscr{L}}{\partial f^\Omega} + \frac{\partial \mathscr{L}}{\partial f_i^\Omega} \partial_k f_i^\Omega, \tag{2.29}$$

$$\delta_j^i \mathscr{L} = \frac{\partial \mathscr{L}}{\partial f^\Omega} f_j^\Omega - F_{\Delta\, j}^{\Omega\, i} \left(\frac{\partial \mathscr{L}}{\partial f^\Omega} f^\Delta + \frac{\partial \mathscr{L}}{\partial f_k^\Omega} f_k^\Delta \right), \tag{2.30}$$

$$0 = \frac{\partial \mathscr{L}}{\partial f_{(k}^\Omega} F_{\Delta\, j)}^{\Omega\, i} f^\Delta. \tag{2.31}$$

In field theories based on a variational principle with Lagrangean $\mathscr{L}$ the set of quantities,

$$\mathfrak{c}t_j^i \equiv \delta_j^i \mathscr{L} - \frac{\partial \mathscr{L}}{\partial f_i^\Omega} f_j^\Omega, \tag{2.32}$$

is called the *canonical stress matrix*. In dynamical theories where the underlying space of points $\boldsymbol{x}$ includes one dimension of time, $\mathfrak{c}t_j^i$ is called the canonical *stress-energy-momentum matrix*. Only in special cases will $\mathfrak{c}t_j^i$ transform as a mixed tensor density of weight 1 under general transformations of the coordinates x^i.

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* *Cf.* also THOMAS [*9*, Chs. VI, VII].

** The non-trivial identities (2.30) and (2.31) were obtained by EINSTEIN [*13*] for the case when f^Ω is a symmetric, non-singular, second order absolute tensor. The generalization of these identities to an arbitrary geometric quantity f^Ω upon which the remaining identities of this section rest was given by BERGMANN [*8*]. Various special cases of these identities are well known in differential geometry. See, *e.g.*, THOMAS [*9*] and SCHOUTEN [*6*].

On computing the natural divergence of $\mathfrak{c}\mathfrak{t}^i_j$ and using the identity (2.29), we find that

$$\partial_i(\mathfrak{c}\mathfrak{t}^i_j) \equiv L_\Omega f^\Omega_j, \tag{2.33}$$

where L_Ω is the *Lagrange derivative* of $\mathscr{L}$ with respect to f^Ω defined by

$$L_\Omega \equiv \frac{\partial \mathscr{L}}{\partial f^\Omega} - \partial_i\left(\frac{\partial \mathscr{L}}{\partial f^\Omega_i}\right). \tag{2.34}$$

The Lagrange derivative of $\mathscr{L}$ with respect to a tensor of weight w is a tensor of weight $1-w$. Integrating the last term in the identity (2.30) by parts, introducing the Lagrange derivative (2.34), and making use of the identity (2.31), we get the identity

$$\mathfrak{c}\mathfrak{t}^i_j \equiv -L_\Omega F^{\Omega\, i}_{\Delta\, j} f^\Delta - \partial_k\left(\frac{\partial \mathscr{L}}{\partial f^\Omega_{[k}} F^{\Omega\, i]}_{\Delta\, j} f^\Delta\right). \tag{2.35}$$

If one now computes the divergence of $\mathfrak{c}\mathfrak{t}^i_j$, the last term contributes nothing because of its antisymmetry in the indices i and k and one gets using (2.33)

$$L_\Omega f^\Omega_j + \partial_i(L_\Omega F^{\Omega\, i}_{\Delta\, j} f^\Delta) \equiv 0, \tag{2.36}$$

which is BERGMANN'S *identity* [8]. Equivalently, if we put

$$\mathfrak{T}^i_j \equiv \mathfrak{c}\mathfrak{t}^i_j + L_\Omega F^{\Omega\, i}_{\Delta\, j} f^\Delta = -\partial_k\left(\frac{\partial \mathscr{L}}{\partial f^\Omega_{[k}} F^{\Omega\, i]}_{\Delta\, j} f^\Omega\right), \tag{2.37}$$

we see that (2.36) is equivalent to the identity

$$\partial_i \mathfrak{T}^i_j \equiv 0. \tag{2.38}$$

Given any scalar density function of a geometric quantity f^Ω and its derivatives f^Ω_k, formula (2.37) constitutes a general rule for the construction of an array of n^2 quantities whose divergence vanishes *identically* as a consequence of the assumed law of transformation for $\mathscr{L}$. The f^Ω need not be a set of tensor fields but may also include more general geometric quantities such as an affine connection.

Special interest is attached to the case where one of the quantities included in the set f^Ω is a symmetric, non-singular tensor g_{mn} which we may think of as the metric tensor for the underlying space, though this is not a necessary interpretation for the present analysis. In this case, we indicate the functional dependence of $\mathscr{L}$ by writing

$$\mathscr{L} = \mathscr{L}(g_{mn}, \partial_k g_{mn}, \varphi^{\Omega'}, \varphi^{\Omega'}_k) \tag{2.39}$$

and separating the tensor g_{mn} explicitly from the total set f^Ω. If f^Ω is a geometric quantity, the remaining set of components $\varphi^{\Omega'}$ is also a geometric quantity whose Lie derivative we denote by

$$\underset{v}{\pounds}\, \varphi^{\Omega'} = v^k \partial_k \varphi^{\Omega'} - \Phi^{\Omega'\, i}_{\Delta'\, j} \partial_i v^j \varphi^{\Delta'}. \tag{2.40}$$

We introduce the special notation $\mathfrak{t}^{mn}$ for twice the Lagrange derivative of $\mathscr{L}$ with respect to g_{mn}:

$$\mathfrak{t}^{mn} \equiv 2\left[\frac{\partial \mathscr{L}}{\partial g_{mn}} - \partial_k\left(\frac{\partial \mathscr{L}}{\partial(\partial_k g_{mn})}\right)\right]. \tag{2.41}$$

BERGMANN's identity (2.36) applied to a Lagrange function having the form (2.39) yields TRAUTMAN's identity [11]:

$$\nabla_i \mathfrak{t}^i_j = L_{\Omega'} \varphi^{\Omega'}_j + \partial_i (L_{\Omega'} \Phi^{\Omega' i}_{\Delta' j} \varphi^{\Delta'}), \tag{2.42}$$

where $\mathfrak{t}^i_j = g_{jk} \mathfrak{t}^{ik}$, and ∇_i denotes covariant differentiation based on the Christoffel symbols of the tensor g_{mn}.

The Lagrange derivative of $\mathscr{L}$ with respect to an absolute tensor field is a tensor field of weight 1, *i.e.*, a tensor density. For reasons to appear later in the application of these general formulae, we shall call the symmetric tensor density $\mathfrak{t}^{ij}$ defined in (2.41) and the associated absolute tensor $t^{ij} = \mathfrak{t}^{ij}/\sqrt{|\det g_{mn}|}$, the *Cauchy stress*. From (2.42) we see that a sufficient condition for the covariant divergence of the Cauchy stress to vanish is that $L_{\Omega'} = 0$, *i.e.*, that the Lagrange derivative of $\mathscr{L}$ with respect to each of the remaining field components $\varphi^{\Omega'}$ vanish. From (2.33) it is apparent that $L_\Omega = 0$, which, in the case of a Lagrangean having the form (2.39), includes the condition $\mathfrak{t}^{mn} = 0$, is a sufficient condition that the ordinary divergence of the canonical stress matrix vanish. We have previously noted that the ordinary divergence of the tensor $\mathfrak{T}^i_j$ vanishes *identically*.

Henceforth we restrict attention to the case where $\mathscr{L}$ is independent of the partial derivatives of g_{mn}:

$$\mathscr{L} = \mathscr{L}(g_{mn}, \varphi^{\Omega'}, \varphi^{\Omega'}_m). \tag{2.43}$$

When $\mathscr{L}$ has this form we can solve the identity (2.30) for the Cauchy stress.

$$\mathfrak{t}^i_j = {}_{\mathrm{C}}\mathfrak{t}^i_j + \Phi^{\Omega' i}_{\Delta' j} \left(\frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}} \varphi^{\Delta'} + \frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}_k} \varphi^{\Delta}_k \right). \tag{2.44}$$

Now, by definition, the Cauchy stress is symmetric.

$$\mathfrak{t}^{[ij]} \equiv 0, \tag{2.45}$$

which, by (2.44) implies that

$${}_{\mathrm{C}}\mathfrak{t}^{[ij]} + \Phi^{\Omega' [i}_{\Delta' k} g^{j]k} \left(\frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}} \varphi^{\Delta'} + \frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}_m} \varphi^{\Delta'}_m \right) = 0. \tag{2.46}$$

Thus, in general, the canonical stress ${}_{\mathrm{C}}\mathfrak{t}^{ij} \equiv g^{ik} {}_{\mathrm{C}}\mathfrak{t}^j_k$ is not symmetric. Equation (2.46) is an explicit formula for its antisymmetric part when the Lagrangean has the special form (2.43).

3. Field equations and boundary condition of a variational principle

Let $\mathscr{L}$ have the special form (2.43) and consider the integral

$$I(\lambda) \equiv \int_{v(\lambda)} \mathscr{L} \, dv(\lambda) \tag{3.1}$$

where $\varphi^{\Omega'} = \varphi^{\Omega'}(x, \lambda)$ but the metric components $g_{mn}(x)$ are assumed to be independent of λ. The region of integration $v(\lambda)$ in (3.1) corresponds to a fixed

set of material points X^K and thus depends on the parameter λ. Let $v(\lambda)$ be divided by a surface of discontinuity $\mathscr{S}(\lambda)$ where we assume that $\mathscr{S}(\lambda)$ corresponds also to a fixed set of points X^K. The metric will be assumed continuously differentiable throughout $v(\lambda)$, but the fields φ^Ω will be assumed continuously differentiable only in each of the regions v^+ and v^- and may have ordinary discontinuities at $\mathscr{S}(\lambda)$. Under these conditions we have

$$\frac{dI}{d\lambda} = \int \overset{*}{\mathscr{L}}\, dv = \int \left(\frac{\partial \mathscr{L}}{\partial g_{mn}} \overset{*}{g}_{mn} + \frac{\partial \mathscr{L}}{\partial \varphi^\Omega} \overset{*}{\varphi}{}^\Omega + \frac{\partial \mathscr{L}}{\partial \varphi^\Omega_m} \overset{*}{\varphi}{}^\Omega_m \right) dv. \tag{3.2}$$

Since $\overset{*}{\varphi}{}^\Omega_m = \partial_m \overset{*}{\varphi}{}^\Omega$, and $\overset{*}{g}_{mn} = \partial_m v^k g_{kn} + \partial_n v^k g_{mk}$, we can integrate certain of the terms in (3.2) by parts and apply the divergence theorem (2.3) to get

$$\begin{aligned} \frac{dI}{d\lambda} = \int_v (-\nabla_m t^m{}_k v^k + L_{\Omega'} \overset{*}{\varphi}{}^{\Omega'})\, dv + \oint_s \left(t^m{}_k v^k + \frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}_m} \overset{*}{\varphi}{}^{\Omega'} \right) d s_m - \\ - \int_{\mathscr{S} \cap v} \left[t^m{}_k v^k + \frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}_m} \overset{*}{\varphi}{}^{\Omega'} \right] d \mathscr{S}_m. \end{aligned} \tag{3.3}$$

Let the derivatives (variations) v^k and $\overset{*}{\varphi}{}^\Omega$ be subject to a set of N *constraints*

$$0 = C^\omega = a^\omega{}_k v^k + b^\omega{}_{\Omega'} \overset{*}{\varphi}{}^{\Omega'}, \qquad \omega = 1, 2, \ldots, N. \tag{3.4}$$

By introducing the Lagrange multipliers η_ω in the usual way we obtain the result:

A sufficient condition that the integral $I(\lambda)$ have a stationary value for arbitrary variations $(v^k, \overset{*}{\varphi}{}^\Omega)$ consistent with the constraints $C^\omega = 0$, vanishing on the boundary $s(\lambda)$ of $v(\lambda)$, and continuous at $\mathscr{S}(\lambda) = 0$, i.e., $[v^k] = 0$, $[\overset{*}{\varphi}{}^\Omega] = 0$, is that the field equations

$$\begin{aligned} -\nabla_m t^m{}_k + \eta_\omega a^\omega{}_k = 0, \\ L_{\Omega'} + \eta_\omega b^\omega{}_{\Omega'} = 0, \end{aligned} \tag{3.5}$$

be satisfied at each interior point of the regions v^+ and v^- and that, at the surface of discontinuity $\mathscr{S}$, the boundary conditions

$$\begin{aligned} [t^m{}_k]\, n_m - \eta_\omega a^\omega{}_k = 0, \\ \left[\frac{\partial \mathscr{L}}{\partial \varphi^{\Omega'}_m} \right] n_m + \eta_\omega b^\omega{}_{\Omega'} = 0 \end{aligned} \tag{3.6}$$

be satisfied at each point of $\mathscr{S}(\lambda)$ interior to $v(\lambda)$.

Remarks. Suppose that the φ^Ω are the components of a set of tensor fields and introduce the scalars $\Phi^\Omega = \overset{-1}{A}{}^\Omega{}_{A'} \varphi^{A'}$ as in (2.24). Then one has $\overset{*}{\varphi}{}^\Omega = A^\Omega{}_{A'} \overset{*}{\Phi}{}^{A'}$ and from $(2.6)_1$, $v^k = -x^k_A X^A$. Thus the assumption that the variations $(v^k, \overset{*}{\varphi}{}^\Omega)$ are continuous at $\mathscr{S}(\lambda)$ is seen to be equivalent to the assumption that the variations $(\overset{*}{\Phi}{}^\Omega, X^A)$ are continuous at $\mathscr{S}(\lambda)$ provided one assumes that $[A^\Omega{}_{A'}] = 0$ which holds when the X^A_i are continuous. A different set of boundary conditions

arises if we write $dI/d\lambda$ in the form given by WEISS [12],

$$\begin{aligned}\frac{dI}{d\lambda} &= \int_v \left\{\frac{\partial \mathscr{L}}{\partial \lambda} + \partial_k(v^k \mathscr{L})\right\} dv \\ &= \int_v \left(\frac{\partial \mathscr{L}}{\partial \varphi^\Omega}\dot{\varphi}^\Omega + \frac{\partial \mathscr{L}}{\partial \varphi^\Omega_k}\dot{\varphi}^\Omega_k\right) dv + \oint_s v^k \mathscr{L}\, ds_k - \int_{\mathscr{S}\cap v} [v^k \mathscr{L}]\, d\mathscr{S}_k \\ &= \int_v L_\Omega \dot{\varphi}^\Omega\, dv + \oint_s \left\{\frac{\partial \mathscr{L}}{\partial \varphi^\Omega_k}\dot{\varphi}^\Omega + v^k \mathscr{L}\right\} ds_k - \int_{\mathscr{S}\cap v}\left[\frac{\partial \mathscr{L}}{\partial \varphi^\Omega_k}\dot{\varphi}^\Omega + v^k \mathscr{L}\right] d\mathscr{S}_k \\ &= \int_v L_\Omega \dot{\varphi}^\Omega\, dv + \oint_s \left\{\frac{\partial \mathscr{L}}{\partial \varphi^\Omega_k}\dot{\varphi}^\Omega + \mathsf{c}t^k{}_j v^j\right\} ds_k - \int_{\mathscr{S}\cap v}\left\{\left[\frac{\partial \mathscr{L}}{\partial \varphi^\Omega_k}\dot{\varphi}^\Omega\right] + [\mathsf{c}t^k{}_j v^j]\right\} d\mathscr{S}_k,\end{aligned} \tag{3.7}$$

where, in the last equation of (3.7), the variation $\dot{\varphi}^\Omega \equiv \dot{\varphi}^\Omega + v^k \partial_k \varphi^\Omega$ is not a tensor unless all the φ^Ω are absolute scalars and therefore, in general, has no particular geometric significance*. One then sees from (3.7) that a sufficient condition that $I(\lambda)$ have a stationary value for all variations $(\dot{\varphi}^\Omega, v^k)$ which vanish on s and are continuous in $v(\lambda)$ is that $L_\Omega = 0$ in v^+ and v^- and $[\mathsf{c}t^i{}_j]n_i = 0$, $\left[\frac{\partial \mathscr{L}}{\partial \varphi^\Omega_k}\right] n_k = 0$ on $\mathscr{S}(\lambda)$, provided there are no constraints on the variations $(\dot{\varphi}^\Omega, v^k)$.

4. Elastic dielectrics

The formalism of §2 and §3 will now be applied to a special case to reproduce the equations of the theory of large elastic deformations of a solid dielectric material considered previously in [1].

Let the space of points x be Euclidean three-dimensional space, and let the Lagrangean function have the form

$$\mathscr{L} = \mathfrak{W}(X^K, X^K_i, \mathfrak{P}^i, g_{ij}) + \mathfrak{P}^i \partial_i \varphi - (\varepsilon_0 \sqrt{\mathfrak{g}}/2)\, g^{ij} \partial_i \varphi\, \partial_j \varphi \tag{4.1}$$

where g_{ij} are the components of the Euclidean metric tensor, φ is an absolute scalar field called the *electric potential*, $\mathfrak{P}^i$ is a contravariant vector density with the transformation law,

$$\mathfrak{P}^{i'} = |(x'/x)|^{-1} \frac{\partial x^{i'}}{\partial x^i} \mathfrak{P}^i, \tag{4.2}$$

called the *polarization density*, and $\mathfrak{W}$ is a scalar density we call the *energy density of elastic deformation and polarization*. Note that $\mathfrak{W}$ is independent of φ and its derivatives, the central term in (4.1) is independent of the *deformation gradients* X^A_i, and the last group of terms in (4.1) depends only on g_{ij} and $\partial_i \varphi$. ε_0 is a constant, and, as usual, $\mathfrak{g}$ denotes the determinant of the metric. Since we assume that the metric is Euclidean, there exist coordinate systems, the rectangular Cartesian frames, for which $g_{ij} = \delta_{ij}$ everywhere. However, it is to be noted that $\overset{*}{\delta}_{ij} \neq 0$ except for rigid variations. In some works on electromagnetic theory it is customary to represent the polarization field by the absolute vector field $P^i = \mathfrak{P}^i/\sqrt{\mathfrak{g}}$. Actually, it is more convenient to work with the vector density $\mathfrak{P}^i$

* WEISS [12, p. 105] states that the $\dot{\varphi}^\Omega$ are "more fundamental" than the variations φ^Ω and gives a reason we do not understand. The variations $\overset{*}{\varphi}{}^\Omega$, on the other hand, have an invariant significance even in non-metric or non-affinely connected spaces.

which has the transformation law (4.2). If one restricts the coordinate transformations to the unimodular ones, no distinction need be maintained between tensor densities and absolute tensors. The scalar density $\mathfrak{W}$ is related to the absolute scalar function Σ introduced in [1] by

$$\mathfrak{W} = \varrho_0 (\boldsymbol{x}/\boldsymbol{X})^{-1} \Sigma \tag{4.3}$$

where ϱ_0 is the density of mass in the undeformed or natural state of the material.

A material characterized by the field equations and boundary conditions (3.5) and (3.6) with a Lagrange function having the form (4.1) will be called a *perfectly elastic dielectric* provided the constraints do not include the condition $\mathfrak{P}=0$.

The canonical stress in an elastic dielectric is given by

$$\begin{aligned} {}_{\mathrm{C}}t^i_j &= \delta^i_j \mathscr{L} - \frac{\partial \mathscr{L}}{\partial \varphi_i} \varphi_j - \frac{\partial \mathscr{L}}{\partial X^K_i} X^K_j \\ &= \left(\delta^i_j \mathfrak{W} - \frac{\partial \mathfrak{W}}{\partial X^K_i} X^K_j\right) + \mathfrak{P}^i E_j + \varepsilon_0 \sqrt{\mathfrak{g}}\, E^i E_j - \delta^i_j \left(\mathfrak{P} \cdot E + \frac{\varepsilon_0}{2} \sqrt{\mathfrak{g}}\, E^2\right), \end{aligned} \tag{4.3}$$

where we have introduced the *electric field*, $E_i \equiv -\partial_i \varphi$, $E^i \equiv g^{ij} E_j$, $\mathfrak{P} \cdot \boldsymbol{E} \equiv \mathfrak{P}^i E_i$, *etc.*

Now in [1] we introduced the absolute scalar field π^i defined by

$$\pi^i \equiv (\boldsymbol{x}/\boldsymbol{X})\, \mathfrak{P}^i/\varrho_0 \tag{4.4}$$

and wrote Σ in the form

$$\Sigma = \hat{\Sigma}(X^K, x^i_A, \pi^i, g_{ij}). \tag{4.5}$$

We then introduced a *local stress tensor* ${}_{\mathrm{L}}t^i_j$ and a *local field* ${}_{\mathrm{L}}E_i$ defined by

$${}_{\mathrm{L}}t^i_j \equiv \varrho \frac{\partial \hat{\Sigma}}{\partial x^j_A} x^i_A, \qquad {}_{\mathrm{L}}E_i = -\frac{\partial \hat{\Sigma}}{\partial \pi^i}, \tag{4.6}$$

where $\varrho = |(\boldsymbol{x}/\boldsymbol{X})|^{-1} \varrho_0$ is the density of mass in the present configuration. From (4.2), (4.4), (4.5), (4.6) and the identities given in § 2 we derive the identities:

$$\begin{aligned} \delta^i_j \mathfrak{W} - \frac{\partial \mathfrak{W}}{\partial X^K_i} X^K_j &\equiv {}_{\mathrm{L}}t^i_j - \delta^i_j\, {}_{\mathrm{L}}\boldsymbol{E}\ \mathfrak{P}, \\ \frac{\partial \mathfrak{W}}{\partial \mathfrak{P}^i} &\equiv -\,{}_{\mathrm{L}}E_i. \end{aligned} \tag{4.7}$$

Substituting these relations into (4.3), we get

$${}_{\mathrm{C}}t^i_j = {}_{\mathrm{L}}t^i_j + \mathfrak{P}^i E_j + \varepsilon_0 \sqrt{\mathfrak{g}}\, E^i E_j - \delta^i_j\, (\boldsymbol{E} + {}_{\mathrm{L}}\boldsymbol{E}) \cdot \mathfrak{P} - (\varepsilon_0 \sqrt{\mathfrak{g}}/2)\, \delta^i_j E^2, \tag{4.8}$$

which exhibits an explicit relation between the canonical stress and the local stress.

For elastic dielectrics, the symmetric Cauchy stress as given by (2.44) turns out to have the form

$$\begin{aligned} t^i_j &= {}_{\mathrm{C}}t^i_j + (\delta^i_m \delta^n_j - \delta^i_j \delta^n_m) \frac{\partial \mathscr{L}}{\partial \mathfrak{P}^n_{,}} \mathfrak{P}^m, \\ &= {}_{\mathrm{C}}t^i_j - (E_j + {}_{\mathrm{L}}E_j)\, \mathfrak{P}^i + \delta^i_j (\boldsymbol{E} + {}_{\mathrm{L}}\boldsymbol{E}) \cdot \mathfrak{P}. \end{aligned} \tag{4.9}$$

Substituting from (4.8) for the canonical stress in (4.9), we get

$$t^i_j = {}_{\mathrm{L}}t^i_j - {}_{\mathrm{L}}E_j\,\mathfrak{P}^i + \varepsilon_0\sqrt{g}\,E^i E_j - (\varepsilon_0\sqrt{g}/2)\,\delta^i_j E^2 = 2\frac{\partial \mathscr{L}}{\partial g_{im}}g_{jm}. \tag{4.10}$$

The field equations and boundary conditions are obtained from the general formulae of §3 as follows. First note that $\overset{*}{X}{}^K = 0$, $\overset{*}{X}{}^A_i = 0$, so that the constaints (3.4) corresponding to these conditions have a simple form. We then consider a region $\mathscr{v}(\lambda)$ filled with elastic dielectric which might consist of two disimilar materials with the common boundary $\mathscr{S}(\lambda)$. In $\mathscr{v}^+$ and $\mathscr{v}^-$ we assume the variations $\overset{*}{\mathfrak{P}}{}^i$ and $\overset{*}{\varphi}$ are arbitrary. We then get the field equations and boundary conditions:

$$\begin{aligned} \nabla_j t^j_k &= 0, \\ {}_{\mathrm{L}}E_i + E_i &= 0, \\ \varepsilon_0\sqrt{g}\,\nabla^2\varphi - \operatorname{div}\mathfrak{P} &= 0, \end{aligned} \tag{4.11}$$

which must hold at each point of $\mathscr{v}^+$ and $\mathscr{v}^-$, and

$$[t^j_k]\,n_j = 0, \qquad [\varepsilon_0\sqrt{g}\,E^i + \mathfrak{P}^i]\,n_i = 0 \tag{4.12}$$

which must hold at each point of $\mathscr{S}$ contained in $\mathscr{v}$.

When the field equation $(4.11)_2$ is satisfied, the Cauchy stress tensor (4.10) reduces to the form

$$t^i_j = {}_{\mathrm{L}}t^i_j + E_j\,\mathfrak{P}^i + (\varepsilon_0\sqrt{g}/2)\,\delta^i_j E^2 \tag{4.13}$$

given in [*1*]. Thus the field equations and boundary conditions (4.11) and (4.12) are equivalent to those of [*1*]. When the field equation $(4.11)_2$ is satisfied, it follows from $(4.9)_2$ that the Cauchy stress and the canonical stress are equal in elastic dielectrics.

From the symmetry of the Cauchy stress and (4.10) we conclude that

$${}_{\mathrm{L}}t^{[ij]} - {}_{\mathrm{L}}E^{[i}\,\mathfrak{P}^{j]} = 0, \tag{4.14}$$

which is equation (10.36) of [*1*]. The derivation of (4.14) given in [*1*] shows how this identity follows from the invariance of Σ under rigid motions.

It is natural to introduce still another stress tensor in elastic dielectrics which we define by

$${}_{\mathrm{M}}t^{ij} \equiv {}_{\mathrm{L}}t^{ij} - {}_{\mathrm{L}}E^i\,\mathfrak{P}^j. \tag{4.15}$$

Let us call ${}_{\mathrm{M}}t^{ij}$, the *material stress*. By (4.14) it is always symmetric. Thus the Cauchy stress can be given the alternative decomposition

$$t^{ij} = {}_{\mathrm{M}}t^{ij} + \mathfrak{m}^{ij}, \tag{4.16}$$

where

$$\mathfrak{m}^{ij} = \varepsilon_0\sqrt{g}\,(E^i E^j - \tfrac{1}{2}g^{ij}E^2) \tag{4.17}$$

is a type of Maxwell stress tensor, or electromagnetic stress tensor, which is symmetric and has the same form in all materials.

As shown in [1], a stored energy function Σ which is invariant under rigid motions must reduce to a function of the quantities

$$\begin{aligned} \Sigma &= \bar{\Sigma}(C_{AB}, \Pi^A, X^A), \\ C_{AB} &\equiv g_{ij}\, x^i_A\, x^j_B, \\ \Pi^A &\equiv |(\boldsymbol{x}/\boldsymbol{X})|\, X^A_i\, \mathfrak{P}^i/\varrho_0, \end{aligned} \tag{4.18}$$

each of which is an absolute scalar under general transformations of the spatial coordinates x^i. One can then show that the material stress tensor ${}_{\mathrm{M}}\mathfrak{t}^{ij}$ is given by

$${}_{\mathrm{M}}\mathfrak{t}^{ij} = 2\varrho \frac{\partial \bar{\Sigma}}{\partial C_{AB}} x^i_A\, x^j_B, \tag{4.19}$$

analogous to the formula of ordinary finite elasticity theory (*cf.* TRUESDELL [7, Eq. (39.2)]).

Following this deluge of formalism and definitions of stress tensors in elastic dielectrics, perhaps it is best that we restate our position. The fundamental entity from the point of view of mechanics is the Cauchy stress t^{ij} which is a certain function of the fields X^K, X^K_i, φ_i, $\mathfrak{P}^i$, and g_{ij} fixed by the functional form of $\mathfrak{W}(X^K, X^K_i, \mathfrak{P}^i, g_{ij})$. The field equations and boundary conditions are (4.11) and (4.12). These equations are obviously independent of any decomposition of the Cauchy stress into a mechanical and an electromagnetic component. Decompositions of the type (4.13) and (4.16) may be used as intuitive guides in the construction of admissible constitutive equations for $\mathfrak{W}$ but are otherwise irrelevant. A distinguishing feature of the decomposition (4.16) is that $\mathfrak{m}^{ij}$ is independent of $\mathfrak{P}$ and of the deformation and has the same form in every material. It need not vanish in a vacuum where $\mathfrak{W} = \mathfrak{P} = 0$. On the other hand, the material stress ${}_{\mathrm{M}}\mathfrak{t}^{ij}$ is a function only of the polarization $\mathfrak{P}$ and the deformation and depends on these variables in a way which is characteristic of the material.

Consider next the Lagrange derivatives of $\mathscr{L}$ with respect to the field variables X^K, $\mathfrak{P}^i$, and φ. Let us denote them by

$$\begin{aligned} \mathfrak{L}_K &\equiv \frac{\partial \mathfrak{W}}{\partial X^K} - \partial_i\left(\frac{\partial \mathfrak{W}}{\partial X^K_i}\right), \qquad -\mathfrak{L}_i \equiv {}_{\mathrm{L}}E_i + E_i, \\ \mathfrak{L} &\equiv \varepsilon_0 \sqrt{g}\, \nabla^2 \varphi - \operatorname{div} \mathfrak{P}. \end{aligned} \tag{4.20}$$

For elastic dielectrics, TRAUTMAN'S identity (2.42) takes the form

$$\nabla_i \mathfrak{t}^i_j = \mathfrak{L}_K X^K_j - \mathfrak{L}_i\, \partial_j \mathfrak{P}^i - \mathfrak{L}\, E_j + (\delta^m_j\, \delta^i_k - \delta^i_j\, \delta^m_k)\, \partial_i(\mathfrak{L}_m\, \mathfrak{P}^k). \tag{4.21}$$

Since $\det X^K_j \neq 0$, it follows from (4.21) that the field equations (4.11) are equivalent to the set of Lagrange equations $\mathfrak{L}_K = 0$, $\mathfrak{L}_i = 0$, $\mathfrak{L} = 0$. That is, the field equation $\nabla_i \mathfrak{t}^i_j = 0$ may be replaced by the equation $\mathfrak{L}_K = 0$ in the set (4.11).

Finally, we consider briefly the problem treated in [1] of an elastic dielectric placed in a vacuum. The variational principle of §3 still applies provided we take into account the constraint $\mathfrak{W} = 0$, $\mathfrak{P} = 0$ outside the dielectric. Here we let the discontinuity surface $\mathscr{S}(\lambda)$ correspond to the boundary between the dielectric and vacuum. The field equations interior to the dielectric retain the form (4.10), but owing to the constraints $\mathfrak{P} = \mathfrak{W} = 0$, exterior to the dielectric,

we get the field equations

$$\nabla_j \mathfrak{t}^j_k = 0, \qquad \varepsilon_0 \sqrt{g}\, \nabla^2 \varphi = 0. \tag{4.22}$$

But when $\mathfrak{M} = \mathfrak{P} = 0$, $\mathfrak{t}^i_j$ has the form

$$\mathfrak{t}^i_j = \varepsilon_0 \sqrt{g}\,(E^i E_j - \tfrac{1}{2}\,\delta^i_j E^2) \tag{4.23}$$

and TRAUTMAN's identity reduces to the trivial one

$$\nabla_i \mathfrak{t}^i_j \equiv \varepsilon_0 \sqrt{g}\, \nabla^2 \varphi\, E_j. \tag{4.24}$$

Thus the field equations (4.22) are not independent, and we may take the simple scalar equation $(4.22)_2$ as the single vacuum field equation.

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U. S. Naval Research Laboratory
Washington D. C.