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Korn Inequalities for the Sphere and Circle

B. BERNSTEIN & R. A. TOUPIN

1. Introduction*

Let $\mathfrak{G}$ be a given region in either two or three dimensions. Let $\mathscr{C}^n$ denote the class of all vector fields $\boldsymbol{u}(\boldsymbol{x})$, continuous in the closure $\overline{\mathfrak{G}}$ of $\mathfrak{G}$ and n times continuously differentiable in $\mathfrak{G}$ such that the Dirichlet integral

$$D(\boldsymbol{u}) = \int_{\mathfrak{G}} u_{i,j}\, u_{i,j}\, dv \tag{1.1}$$

is finite. Let d_{ij} and ω_{ij} denote the symmetric and antisymmetric parts of the gradient of $\boldsymbol{u}$:

$$\begin{aligned} d_{ij} &= u_{(i,j)} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \\ \omega_{ij} &= u_{[i,j]} = \tfrac{1}{2}(u_{i,j} - u_{j,i}). \end{aligned} \tag{1.2}$$

We shall also use the abbreviated notation

$$d^2 = d_{ij}\, d_{ij}, \qquad \omega^2 = \omega_{ij}\, \omega_{ij}, \qquad \operatorname{tr} d = d_{kk}. \tag{1.3}$$

The summation convention, Cartesian coordinates, and Cartesian tensor notation will be used throughout.

A *Korn inequality* states the existence of a number K which depends only on the shape of the region $\mathfrak{G}$ such that

$$\int_{\mathfrak{G}} \omega^2\, dv \leqq K \int_{\mathfrak{G}} d^2\, dv \tag{1.4}$$

for all vector fields which satisfy one or more side conditions. That an inequality of the form (1.4) cannot hold for arbitrary vector fields in $\mathscr{C}^\infty$ follows from the existence of pure rotations, $u_i = e_{ijk}\, x_j\, h_k$, $\boldsymbol{h}$ = constant vector, $e_{123} = 1$, $e_{ijk} = e_{[ijk]}$.

* While this paper was being written, the authors were in touch with L. E. PAYNE & H. F. WEINBERGER, who, by using a variational approach, were able to obtain extrema for the expression (1.6) in the second case for the sphere and the circle. But they were not at that time able to prove that their largest extremum was actually the supremum of (1.6) and hence KORN'S constant in the second cese. However, after this paper had gone to press, the authors were delighted to learn that PAYNE & WEINBERGER had succeeded in formulating a proof of this, and thus KORN'S constant in the second case is now known to be 43/13 for the sphere and 3 for the circle. PAYNE & WEINBERGER will publish their work in a subsequent issue of this Archive.

The authors were dissuaded from withdrawing this paper in the light of the results of PAYNE & WEINBERGER because, although the numerical values of upper bounds for KORN'S constant contained herein are no longer of value, the remainder of the work remains a contribution. Also the adaptation of the method of FRIEDRICHS contained herein is applicable to more general regions, whereas PAYNE & WEINBERGER presently feel that their method is not.

We define, with FRIEDRICHS [1], $\mathscr{D}^{\bullet}$ to be that subset of $\mathscr{C}^1$ consisting of those fields ${}^{\bullet}\boldsymbol{u}$ which vanish outside some compact subset of $\mathfrak{G}$, and $\mathscr{D}^{\circ}$ to be the set of fields $\boldsymbol{u}$ in $\mathscr{C}^1$ for which fields ${}^{\bullet}\boldsymbol{u}$ in $\mathscr{D}^{\bullet}$ may be found to make $D(\boldsymbol{u}-{}^{\bullet}\boldsymbol{u})$ arbitrarily small.

Korn inequalities for a class of regions, called *Ω-domains* by FRIEDRICHS [1], are known to exist. For a definition of the class of Ω-domains the reader is referred to [1]. It suffices here merely to note that this class of regions includes the sphere and the circle. Moreover, for an Ω-domain, Korn inequalities exist under three types of side conditions. Following KORN and FRIEDRICHS, we consider separately the three sets of side conditions which, using the terminology of FRIEDRICHS, we call the *three cases:*

$$\begin{aligned} &\textit{First case:} && \boldsymbol{u}\in\mathscr{D}^{\circ}, \\ &\textit{Second case:} && \int_{\mathfrak{G}}\omega_{ij}\,dv=0, \quad \boldsymbol{u}\in\mathscr{C}^1, \\ &\textit{Main Case:} && \int\omega_{ij}\,dv=0, \quad \boldsymbol{u}\in\mathscr{C}^3, \quad \textit{and} \quad d_{ij,j}=0. \end{aligned} \tag{1.5}$$

If a Korn inequality exists for a given region $\mathfrak{G}$, subject to a given one of the side conditions (1.5), then the set of real numbers which are the values of

$$\frac{\int_{\mathfrak{G}}\omega^2\,dv}{\int_{\mathfrak{G}}d^2\,dv} \tag{1.6}$$

subject to the given side conditions, has an upper bound, and hence a least upper bound. We call this least upper bound *Korn's constant for the region* $\mathfrak{G}$ *in the first case, the second case, and the main case* respectively. We denote these dimensionless positive numbers by $K^1_{\mathfrak{G}}$, $K^2_{\mathfrak{G}}$, and $K^M_{\mathfrak{G}}$. Any number greater than or equal to $K^{\Omega}_{\mathfrak{G}}$, $\Omega=1, 2, M$, will be called an *upper bound for* $K^{\Omega}_{\mathfrak{G}}$, and any number smaller than or equal to $K^{\Omega}_{\mathfrak{G}}$ will be called a *lower bound* for $K^{\Omega}_{\mathfrak{G}}$. Substitution of any upper bound for a Korn constant in (1.4) yields an inequality valid for all vector fields satisfying the appropriate set of side conditions (1.5).

The proof of the existence of a Korn inequality in the first case for an Ω-domain follows immediately from the identity

$$\int_{\mathfrak{G}}\omega^2\,dv=\int_{\mathfrak{G}}d^2\,dv-\int_{\mathfrak{G}}(\operatorname{tr}d)^2\,dv, \tag{1.7}$$

which is shown by FRIEDRICHS [1] to hold for all $\boldsymbol{u}$ in $\mathscr{D}^{\circ}$ when $\mathfrak{G}$ is an Ω-domain. Clearly, then, for the first case

$$\int_{\mathfrak{G}}\omega^2\,dv\leqq\int_{\mathfrak{G}}d^2\,dv, \tag{1.8}$$

and $K^1_{\mathfrak{G}}\leqq 1$ for all $\mathfrak{G}$. Moreover, since it is easy to construct vector fields $\boldsymbol{u}\in\mathscr{D}^{\bullet}$ for which $\operatorname{tr}d=0$, we see from (1.7) that $K^1_{\mathfrak{G}}=1$ for any Ω-domain.

The useful role of the identity (1.7) and of KORN's inequality in the first case which follows from it in studies of the displacement boundary value problem of linear elasticity theory is made apparent in the papers of KORN [2, 3] and FRIEDRICHS [1].

To prove the existence of a Korn inequality in the second and main cases is not so easy. As shown by FRIEDRICHS and in Section 4, the second case follows easily from the other two, whence the name, the "main case". We base our understanding of the main case on the paper by FRIEDRICHS. With him we confess inability to follow KORN's original treatment.

The constructive proof in [*1*] of the existence of a Korn inequality in the second and main cases provides a technique for computing, at least in principle, upper bounds for the Korn constants $K_{\mathfrak{G}}^{M}$ and $K_{\mathfrak{G}}^{2}$ for any Ω-domain. But to our knowledge, no one has found a value for the Korn constants $K_{\mathfrak{G}}^{2}$ or $K_{\mathfrak{G}}^{M}$ for any region. Nor have we seen in the literature any upper bound for these numbers for any region whatsoever. Our curiosity was aroused as to just how large or how small the Korn constants might be for even a region so simple as a sphere or a circle. For these regions it turns out that FRIEDRICHS's construction for the second and main cases can be simplified somewhat, though we retain its essential ingredients. To the simplified proof for the sphere and circle we add a determination of the following upper and lower bounds:

$$\begin{aligned} &\text{For the sphere:} \quad 3 \leqq K_{\mathfrak{G}}^{2} < 20, \quad 3 \leqq K_{\mathfrak{G}}^{M} < 17 \\ &\text{For the circle:} \quad \tfrac{4}{3} \leqq K_{\mathfrak{G}}^{2} \leqq 6, \quad K_{\mathfrak{G}}^{M} \leqq 4. \end{aligned} \tag{1.9}$$

It is our belief that the method developed below for the sphere and the circle can be extended to more general regions.

2. Lower bounds for Korn's constants

a) The case of three dimensions. For the purposes of this section let $\mathfrak{G}$ be any region for which the centroid and moment of inertia tensor about the centroid (EULER's tensor) have meaning. Let the origin of the Cartesian coordinate system x_i coincide with the centroid of $\mathfrak{G}$, so that the moment of inertia tensor about the centroid is given by

$$I_{ij} = \int_{\mathfrak{G}} x_i x_j \, dv. \tag{2.1}$$

Let $\boldsymbol{h}$ be a constant unit vector field, $h_i h_i = 1$, and consider the vector field

$$u_i = (h_k x_k) \, e_{ipq} h_p x_q. \tag{2.2}$$

Viewed as a displacement field of an elastic body say, the field $\boldsymbol{u}$ represents a uniform twisting in which each particle in a plane normal to $\boldsymbol{h}$ remains in that plane but is displaced in a direction normal to $\boldsymbol{h}$ and its position vector $\boldsymbol{x}$.

We obtain for the vector field (2.2)

$$\begin{aligned} \int_{\mathfrak{G}} d^2 \, dv &= \tfrac{1}{2} (\delta_{ij} - h_i h_j) \, I_{ij}, \\ \int_{\mathfrak{G}} \omega^2 \, dv &= \int_{\mathfrak{G}} d^2 \, dv + 2 h_i h_j I_{ij}, \\ d_{ij,j} &= 0, \end{aligned}$$

from which it follows that

$$\frac{\int_{\mathfrak{G}} \omega^2 \, dv}{\int_{\mathfrak{G}} d^2 \, dv} = 1 + \frac{4 I_{ij} h_i h_j}{(\delta_{ij} - h_i h_j) \, I_{ij}}. \tag{2.3}$$

Also, since the derivatives of u_i are linear and homogeneous in x_k, and since the origin is at the centroid of $\mathfrak{G}$, $(1.5)_2$ is satisfied. Furthermore, since $d_{i,j,j}=0$, our example belongs to the main case.

In general, the right-hand side of (2.3) will vary with the choice of the direction $\boldsymbol{h}$. As can be shown by the method of Lagrange multipliers, it will have extrema when $\boldsymbol{h}$ is in the direction of the principal axes of the moment of inertia tensor for the region. The largest we can make the right-hand side by varying the unit vector $\boldsymbol{h}$ is

$$1+\frac{4I_{\max}}{I_{\text{med}}+I_{\min}}, \tag{2.4}$$

where $I_{\max}\geqq I_{\text{med}}\geqq I_{\min}$ are the eigenvalues of the moment of inertia tensor.

Considering different regions now, (2.4) will have the smallest value for a region for which the three eigenvalues are all equal, in which case its value will be 3. Thus for any region in three dimensions $K^M_{\mathfrak{G}}\geqq 3$, $K^2_{\mathfrak{G}}\geqq 3$. Since (2.4) yields a lower bound for $K^M_{\mathfrak{G}}$ (and $K^2_{\mathfrak{G}}$) for any given region we may make the following observations:

1. There is no single number K which will yield a Korn inequality in the second or main cases for all three-dimensional regions—*i.e.*, the set of Korn constants in the second and main cases for three-dimensional regions has no upper bound.

2. Given any region $\mathfrak{G}$ in three dimensions, one can find a sufficiently long and thin right circular cylinder $\mathfrak{G}'$ whose Korn constant $K^M_{\mathfrak{G}'}$ exceeds $K^M_{\mathfrak{G}}$.

b) The case of two dimensions. Next we establish a lower bound for KORN's constant $K^2_{\mathfrak{G}}$ for regions in the plane. For this purpose we compute the ratio (1.7) for the 2-dimensional vector field $\boldsymbol{u}$ having the form

$$u_i=(h_k\,x_k)\,e_{ij}\,x_j. \tag{2.5}$$

All indices now take the values 1 and 2, and $e_{12}=1$, $e_{[ij]}=e_{ij}$. Again we assume the origin of the coordinate system coincides with the centroid of the planar region and let

$$J_{ij}=\int_{\mathfrak{G}} x_i\,x_j\,dv \tag{2.6}$$

denote the moment of inertia tensor of the planar region $\mathfrak{G}$ about its centroid. For a vector field having the form (2.5) we get

$$2\int_{\mathfrak{G}} d^2\,dv=2\operatorname{tr}J-J_{ij}\,h_i\,h_j,$$
$$2\int_{\mathfrak{G}}\omega^2\,dv=4J_{ij}\,h_i\,h_j,$$

from which it follows that

$$\frac{\int_{\mathfrak{G}}\omega^2\,dv}{\int_{\mathfrak{G}} d^2\,dv}=\frac{4J_{ij}\,h_i\,h_j}{2\operatorname{tr}J-J_{ij}\,h_i\,h_j}. \tag{2.7}$$

The maximum value of the right-hand side of (2.7) is attained when $\boldsymbol{h}$ is an eigenvector of J_{ij} corresponding to its maximum eigenvalue. Thus the maximum value for the ratio in (2.7) is

$$\frac{4J_{\max}}{J_{\max}+2J_{\min}}, \tag{2.8}$$

which lies between $\frac{4}{3}$ and 4. It follows that for any region in the plane, $K^2_{\mathfrak{G}}\geqq\frac{4}{3}$. The lower bounds announced in Section 1 are established.

3. Some differential identities and algebraic inequalities

In what follows we shall make repeated use of the identities

$$\omega_{jk,l} = d_{lj,k} - d_{kl,j}, \tag{3.1}$$

$$d_{ij,kl} + d_{kl,ij} - d_{ik,jl} - d_{jl,ik} = 0, \tag{3.2}$$

and of the inequality,

$$(\operatorname{tr} d)^2 \leqq n\, d^2, \tag{3.3}$$

where n is the dimension of the space. The differential identities (3.1) and (3.2) are well known in elasticity theory. The latter are the familiar *compatibility conditions* that d_{ij} be expressible in the form $(1.2)_1$. Inequality (3.3) follows easily from CAUCHY'S inequality, $2ab \leqq a^2 + b^2$, in a frame of reference where d_{ij} is diagonal. Since both members of the inequality are invariant under orthogonal transformations, (3.3) is valid always.

From (3.1) we get

$$\omega_{ij,k}\,\omega_{ij,k} = 2\,(d_{ij,k}\,d_{ij,k} - d_{ij,k}\,d_{ki,j}). \tag{3.4}$$

We shall be concerned in particular with vector fields which satisfy the differential equations $d_{ij,j} = 0$ of the main case. For this class of vector fields we get from (3.1) and (3.2):

$$\begin{aligned} \nabla^2 \omega_{ij} &= \omega_{ij,kk} = 0, \\ \omega_{ij,j} &= -\,d_{jj,i}, \\ d_{ij,kk} &= -\,d_{kk,ij}. \end{aligned} \tag{3.5}$$

4. Upper bounds for $K^2_{\mathfrak{G}}$ in terms of an upper bound for $K^M_{\mathfrak{G}}$

That every upper bound for KORN'S constant in the main case determines a corresponding upper bound for KORN'S constant in the second case was shown for an Ω-domain by FRIEDRICHS [*1*]. We wish to present here a slight variation or refinement of his result which is necessary for us to establish upper bounds for $K^2_{\mathfrak{G}}$ as small as those announced in Section 1.

In this section we take $\mathfrak{G}$ to be an Ω-domain to which the divergence theorem applies. As shown by FRIEDRICHS, any vector field in $\mathscr{C}^1$, satisfying the side conditions $(1.5)_2$ of the second case, may be uniquely decomposed into a sum

$$\boldsymbol{u} = {}'\boldsymbol{u} + {}''\boldsymbol{u}, \tag{4.1}$$

where ${}'\boldsymbol{u} \in \mathscr{D}^\circ$, and ${}''\boldsymbol{u}$ satisfies the side conditions of the main case, namely,

$${}''\boldsymbol{u} \in \mathscr{C}^3, \qquad {}''d_{ij,j} = 0, \qquad \int_{\mathfrak{G}} {}''\omega_{ij}\, dv = 0, \tag{4.2}$$

where ${}''d_{ij}$ and ${}''\omega_{ij}$ are the deformation and rotation measures of ${}''\boldsymbol{u}$. This follows from the known uniqueness, existence, and differentiability properties of solutions of (4.2) with prescribed data for ${}''\boldsymbol{u}$ on the boundary, and the fact that $\int_{\mathfrak{G}} {}'\omega_{ij}\, dv = 0$ for all ${}'\boldsymbol{u}$.

Let K'' be an upper bound for $K^M_{\mathfrak{G}}$, so that we have

$$\int_{\mathfrak{G}} {}''\omega^2\, dv \leqq K'' \int_{\mathfrak{G}} {}''d^2\, dv. \tag{4.3}$$

Now given a field $\boldsymbol{u}\in\mathscr{C}^1$, decomposed as in (4.1), let ${}^{\bullet}\boldsymbol{u}$ be any field in $\mathscr{C}^2\cap\mathscr{D}^{\bullet}$ and define

$$^{*}\boldsymbol{u} = {}^{\bullet}\boldsymbol{u} + {}^{''}\boldsymbol{u}. \tag{4.4}$$

Then

$$\begin{aligned}\int_{\mathfrak{G}} {}^{*}\omega^2\, dv &= \int_{\mathfrak{G}} [{}^{\bullet}\omega^2 + {}^{''}\omega^2 + 2\,{}^{\bullet}\omega_{ij}\,{}^{''}\omega_{ij}]\, dv \\ &= \int_{\mathfrak{G}} [{}^{\bullet}d^2 - (\operatorname{tr}{}^{\bullet}d)^2 + K''\,{}^{''}d^2 + 2\,{}^{\bullet}u_{i,j}\,{}^{''}\omega_{ij}]\, dv.\end{aligned} \tag{4.5}$$

In (4.5) we have used (1.7) to eliminate ${}'\omega^2$ and (4.3) to eliminate ${}''\omega^2$. Using the identity $(3.5)_2$ and the fact that ${}^{\bullet}\boldsymbol{u}$ vanishes outside some compact subset of $\mathfrak{G}$, we can transform the last term in (4.5), with the aid of the divergence theorem, as follows:

$$\begin{aligned}2\int_{\mathfrak{G}} {}^{\bullet}u_{i,j}\,{}^{''}\omega_{ij}\, dv &= 2\int_{\mathfrak{G}} [({}^{\bullet}u_i\,{}^{''}\omega_{ij})_{,j} - {}^{\bullet}u_i\,{}^{''}\omega_{ij,j}]\, dv \\ &= 2\int_{\mathfrak{G}} {}^{\bullet}u_i\, d_{jj,i}\, dv = -2\int_{\mathfrak{G}} (\operatorname{tr}{}^{\bullet}d)\,(\operatorname{tr}{}^{''}d)\, dv \\ &\leqq \int_{\mathfrak{G}} [(\operatorname{tr}{}^{\bullet}d)^2 + (\operatorname{tr}{}^{''}d)^2]\, dv,\end{aligned} \tag{4.6}$$

where the last line follows from CAUCHY's inequality. Substituting (4.6) into (4.5) and applying (3.3) yields

$$\int_{\mathfrak{G}} {}^{*}\omega^2\, dv \leqq \int_{\mathfrak{G}} {}^{\bullet}d^2\, dv + (K'' + n)\int_{\mathfrak{G}} {}^{''}d^2\, dv. \tag{4.7}$$

FRIEDRICHS [*1*] has shown that

$$\int_{\mathfrak{G}} {}^{*}d^2\, dv = \int_{\mathfrak{G}} [{}^{\bullet}d^2 + {}^{''}d^2]\, dv, \tag{4.8}$$

and thus (4.7) and (4.8) yield

$$\int_{\mathfrak{G}} {}^{*}\omega^2\, dv \leqq (K'' + n)\int_{\mathfrak{G}} {}^{*}d^2\, dv. \tag{4.9}$$

Now, as remarked by FRIEDRICHS, any ${}^{\bullet}\boldsymbol{u}\in\mathscr{D}^{\bullet}$ as well as its first derivatives may be approximated uniformly by a ${}^{\bullet}\boldsymbol{u}\in\mathscr{D}^{\bullet}\cap\mathscr{C}^2$. Thus (4.9) holds for all fields (4.4), where ${}^{\bullet}\boldsymbol{u}\in\mathscr{D}^{\bullet}$. Moreover, since ${}'\boldsymbol{u}\in\mathscr{D}^{\circ}$, we may find a ${}^{\bullet}\boldsymbol{u}\in\mathscr{D}^{\bullet}$ to make $D({}^{*}\boldsymbol{u}-\boldsymbol{u}) = D({}^{\bullet}\boldsymbol{u} - {}'\boldsymbol{u})$ as small as desired, and since

$$D(\boldsymbol{u}) = \int_{\mathfrak{G}} d^2\, dv + \int_{\mathfrak{G}} \omega^2\, dv$$

for any $\boldsymbol{u}$, it follows from (4.9) that

$$\int_{\mathfrak{G}} \omega^2\, dv \leqq (K'' + n)\int_{\mathfrak{G}} d^2\, dv \tag{4.10}$$

for any $\boldsymbol{u}$ satisfying $(1.5)_2$.

Therefore if K'' is an upper bound for KORN's constant in the main case, then K given by

$$K = K'' + n \tag{4.11}$$

is an upper bound for KORN's constant in the second case.

FRIEDRICHS showed that if K'' is an upper bound in the main case, then K given by

$$K = 2\max(1, K'') \tag{4.12}$$

is an upper bound in the second case.

Since we have established that, in three dimensions, $K'' \geqq 3$, the upper bound for the second case as given by (4.11) is never worse than the upper bound (4.12) for any allowed value of K'' if the dimension of the space be three. In two dimensions, since we do not know if $K'' \geqq 2$, we are unable to make a similar comparison.

5. The main case for the sphere and the circle

In this section we consider only vector fields $\boldsymbol{u}$ satisfying the differential equation $d_{ij,j}=0$ of the main case.

Our proof of a Korn inequality in the main case and our calculation of upper bounds for the Korn constants $K_{\mathfrak{S}}^{M}$ for the sphere and the circle rest ultimately on the construction of a function ψ defined in the sphere $\mathfrak{S}$ (or circle $\mathfrak{C}$) of radius R with the following properties:

1° $\psi > 0$ in $\mathfrak{S}$ (in $\mathfrak{C}$).

2° ψ and its first derivatives are continuous in $\overline{\mathfrak{S}}$ (in $\overline{\mathfrak{C}}$).

3° There exists a sphere $\mathfrak{S}_\delta$ (a circle $\mathfrak{C}_\delta$) concentric with $\mathfrak{S}$ (with $\mathfrak{C}$) of radius $R_\delta < R$ such that

a) In $\mathfrak{S}_\delta$ (in $\mathfrak{C}_\delta$) ψ is equal to a function which is twice continuously differentiable in $\overline{\mathfrak{S}}_\delta$ (in $\overline{\mathfrak{C}}_\delta$).

b) In $\mathfrak{S} - \overline{\mathfrak{S}}_\delta$ (in $\mathfrak{C} - \overline{\mathfrak{C}}_\delta$) ψ is equal to a function which is twice continuously differentiable in the closure of $\mathfrak{S} - \overline{\mathfrak{S}}_\delta$ (of $\mathfrak{C} - \overline{\mathfrak{C}}_\delta$).

4° $\nabla^2 \psi = h(x)$ in $\mathfrak{S}$ (in $\mathfrak{C}$) where $h=1$ in $\mathfrak{S} - \mathfrak{S}_\delta$ (in $\mathfrak{C} - \mathfrak{C}_\delta$) and $h = -c_1$ in $\mathfrak{S}_\delta$ (in $\mathfrak{C}_\delta$) where c_1 is a positive constant.

5° ψ and its first derivatives vanish on $\dot{\mathfrak{S}}$, the boundary of $\mathfrak{S}$ (on $\dot{\mathfrak{C}}$, the boundary of $\mathfrak{C}$).

By GREEN's theorem and properties 4° and 5°, it will follow that the mean value of $h(\boldsymbol{x})$ over $\mathfrak{S}$ (over $\mathfrak{C}$) must be zero; hence, the constant c_1 is not at our disposal but is fixed by the radius R_δ.

For the sphere ψ is given by

$$\psi = \tfrac{1}{6}[r^2 + 2R^3/r - 3R^2] \quad \text{in } \mathfrak{S} - \mathfrak{S}_\delta, \tag{5.1}$$

$$\psi = \tfrac{1}{6}[(R_\delta^3 - R^3)\, r^2/R_\delta^3 + 3R^2(R - R_\delta)/R_\delta] \quad \text{in } \mathfrak{S}_\delta. \tag{5.2}$$

For the circle ψ is given by

$$\psi = \tfrac{1}{4}[r^2 - R^2 + R^2 \ln(R^2/r^2)] \quad \text{in } \mathfrak{C} - \mathfrak{C}_\delta, \tag{5.3}$$

$$\psi = \tfrac{1}{4}[(R_\delta^2 - R^2)\, r^2/R_\delta^2 + R^2 \ln(R^2/R_\delta^2)] \quad \text{in } \mathfrak{C}_\delta. \tag{5.4}$$

For the remainder of this section, and in Section 6 until (6.18) as well as in Section 7 until (7.9), we add to the requirements $(1.5)_3$ the restriction that the first and second derivatives of u_i be continuous in $\overline{\mathfrak{S}}$ (in $\overline{\mathfrak{C}}$). We remove this restriction after (6.18) and after (7.9).

The objective of the remaining calculations of this section is to obtain inequality (5.19), which we shall use in the following two sections to obtain Korn inequalities for the sphere and circle. We shall understand below that, although we ostensibly are treating the sphere, we may, in the rest of the calculations of this section, replace $\mathfrak{S}$, $\mathfrak{S}_\delta$, *etc.*, by $\mathfrak{C}$, $\mathfrak{C}_\delta$, *etc.*

Although any choice of R_δ between zero and R will yield functions ψ for (5.1) and (5.2) or for (5.3) and (5.4), we shall leave the choice of R_δ arbitrary in this section, for not until Sections 6 and 7 will it be apparent how R_δ may be most advantageously chosen.

Although we are dealing with the main case, not until after equation (5.15) shall we assume that the mean value of the vorticity (ω_{ij}) vanishes, and until then our relations will be independent of this condition, though they will depend on the condition $d_{11,1}=0$.

From 4° we have

$$\int_{\mathfrak{S}-\mathfrak{S}_\delta} \omega^2\, dv = c_1 \int_{\mathfrak{S}_\delta} \omega^2\, dv + \int_{\mathfrak{S}} h\,\omega^2\, dv. \tag{5.5}$$

By repeated use of the divergence theorem, dropping boundary terms because of 5°, and by using $(3.5)_1$, we get

$$\int_{\mathfrak{S}} h\,\omega^2\, dv = \int_{\mathfrak{S}} \psi_{,kk}\,\omega_{ij}\,\omega_{ij}\, dv = 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv. \tag{5.6}$$

Then adding $\int_{\mathfrak{S}_\delta} \omega^2\, dv$ to both sides of (5.5) and using (5.6), we obtain

$$\int_{\mathfrak{S}} \omega^2\, dv = (1+c_1)\int_{\mathfrak{S}_\delta} \omega^2\, dv + 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv. \tag{5.7}$$

Using POINCARÉ's inequality [4] in the form

$$\int_{\mathfrak{S}_\delta} \omega^2\, dv \leqq P_\delta \int_{\mathfrak{S}_\delta} \omega_{ij,k}\,\omega_{ij,k}\, dv + \overline{\omega}_\delta^2\, V_\delta, \tag{5.8}$$

where $P_\delta>0$ depends only on the region $\mathfrak{S}_\delta$, V_δ is the volume of $\mathfrak{S}_\delta$, and

$$\overline{\omega}_\delta^2 = \overline{\omega}_{ij}\,\overline{\omega}_{ij}, \qquad \overline{\omega}_{ij} = \frac{1}{V_\delta}\int_{\mathfrak{S}_\delta} \omega_{ij}\, dv, \tag{5.9}$$

we get from (5.7)

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq (1+c_1)\, P_\delta \int_{\mathfrak{S}_\delta} \omega_{ij,k}\,\omega_{ij,k}\, dv + 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + (1+c_1)\, V_\delta\, \overline{\omega}_\delta^2. \tag{5.10}$$

Now since $\psi>0$ in $\mathfrak{S}$, ψ has a positive minimum in $\overline{\mathfrak{S}}_\delta$, say $c_2 R_\delta^2$. Then

$$1 < \frac{\psi}{c_2 R_\delta^2} \text{ in } \mathfrak{S}_\delta,$$

and hence

$$\begin{aligned}\int_{\mathfrak{S}} \omega^2\, dv &\leqq \left[(1+c_1)\, P_\delta/c_2 R_\delta^2\right] \int_{\mathfrak{S}_\delta} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + 2\int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + \\ &+ (1+c_1)\, V_\delta\, \overline{\omega}_\delta^2 \leqq c_3 \int_{\mathfrak{S}} \psi\,\omega_{ij,k}\,\omega_{ij,k}\, dv + (1+c_1)\, V_\delta\, \overline{\omega}_\delta^2,\end{aligned} \tag{5.11}$$

where

$$c_3 = (1+c_1)\, P_\delta/c_2 R_\delta^2 + 2. \tag{5.12}$$

From (5.11) and (3.4) follows

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq 2c_3 \int_{\mathfrak{S}} \psi\,(d_{ij,k}\,d_{ij,k} - d_{ij,k}\,d_{ki,j})\, dv + \overline{\omega}_\delta^2\, V_\delta (1+c_1). \tag{5.13}$$

Using the divergence theorem, remembering that $\psi=0$ on $\dot{\mathfrak{S}}$, and $d_{ij,j}=0$, we get from (5.13)

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq 2c_3 \int_{\mathfrak{S}} \left[-\tfrac{1}{2}\psi_{,k}(d^2)_{,k} - \psi d_{ij,kk} d_{i1} + \psi_{,k} d_{ij} d_{ki,j}\right] dv + \overline{\omega}_\delta^2 V_\delta (1+c_1). \quad (5.14)$$

Next we use the identity $(3.5)_3$ and the fact that both ψ and $\psi_{,k}$ vanish on $\dot{\mathfrak{S}}$ to obtain from (5.14)

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq c_3 \int_{\mathfrak{S}} \left[\psi_{,kk} d^2 + 2\psi_{,ij}(d_{kk} d_{ij} - d_{ik} d_{kj})\right] dv + \overline{\omega}_\delta^2 V_\sigma (1+c_1). \quad (5.15)$$

Now suppose that $\boldsymbol{u}$ also satisfies the second side condition $(1.5)_3$ of the main case. If we define

$$^*u_i = u_i - x_j \overline{\omega}_{ij},$$

we shall have

$$\begin{gathered} {}^*d_{ij} = d_{ij}, \qquad {}^*\omega_{ij} = \omega_{ij} - \overline{\omega}_{ij}, \\ \int_{\mathfrak{S}_\delta} {}^*\omega_{ij}\, dv = 0. \end{gathered} \quad (5.16)$$

Thus ${}^*d_{ij,j}=0$, and we may substitute (5.16) into (5.15) to get

$$\int_{\mathfrak{S}} {}^*\omega^2\, dv \leqq c_3 \int_{\mathfrak{S}} \left[\psi_{,kk} d^2 + 2\psi_{,ij}(d_{kk} d_{i1} - d_{ik} d_{kj})\right] dv. \quad (5.17)$$

But since $(1.5)_3$ holds for ω_{ij},

$$\int_{\mathfrak{S}} \omega^2\, dv = \int_{\mathfrak{S}} {}^*\omega^2\, dv - V\overline{\omega}_\delta^2 \leqq \int_{\mathfrak{S}} {}^*\omega^2\, dv, \quad (5.18)$$

where V is the volume of $\mathfrak{S}$.

Thus from (5.17) and (5.18) we conclude that

$$\int_{\mathfrak{S}} \omega^2\, dv \leqq c_3 \int_{\mathfrak{S}} \left[\psi_{,kk} d^2 + 2\psi_{,ij}(d_{kk} d_{ij} - d_{ik} d_{kj})\right] dv, \quad (5.19)$$

which is the relation we sought in this section. An inequality of the form (5.19) holds also for the circle.

6. An upper bound for Korn's constant for the sphere in the main case

We proceed now to use the results of the previous sections to obtain an upper bound for KORN's constant for the sphere in the main case.

To simplify the notation we set

$$\lambda = R/R_\delta, \qquad \lambda > 1. \quad (6.1)$$

For the function ψ defined in (5.1) and (5.2) we have

$$\psi_{,ij} = \tfrac{1}{3}\left[\delta_{ij}(1 - R^3/r^3) + 3(R^3/r^3) n_i n_j\right] \quad \text{in } \mathfrak{S} - \overline{\mathfrak{S}}_\delta \quad (6.2)$$

where

$$n_i = x_i/r, \qquad n_i n_i = 1, \quad (6.3)$$

and

$$\psi_{,ij} = -\tfrac{1}{3}(\lambda^3 - 1)\,\delta_{ij} \quad \text{in } \mathfrak{S}_\delta. \quad (6.4)$$

From (6.4) and 4° we have

$$c_1 = -\psi_{,kk} = (\lambda^3 - 1) \quad \text{in } \mathfrak{S}_\delta. \quad (6.5)$$

Now the minimum value of ψ in $\overline{\mathfrak{S}}_\delta$, which is, by definition, $c_2 R_\delta^2$, is assumed on $\dot{\mathfrak{S}}_\delta$ and is given by

$$c_2 R_\delta^2 = \frac{2\lambda^3 - 3\lambda^2 + 1}{6} R_\delta^2. \tag{6.6}$$

Thus c_3 as given in (5.12) has the value

$$c_3 = \frac{6\lambda^3}{2\lambda^3 - 3\lambda^2 + 1} \frac{P_\delta}{R_\delta^2} + 2. \tag{6.7}$$

Insertion of (6.3), (6.4), and (6.5) into (5.19) yields

$$\int_{\mathfrak{S}} \omega^2 \, dv \leqq \frac{c_3}{3} \int_{\mathfrak{S}_\delta} [-c_1 d^2 - 2(\operatorname{tr} d)^2] \, dv + \frac{c_3}{3} \int_{\mathfrak{S}-\mathfrak{S}_\delta} \left[d^2 + 2(\operatorname{tr} d)^2 + 2\frac{R^3}{r^3} A_{ij} n_i n_j \right] dv, \tag{6.8}$$

where

$$A_{ij} = -(\operatorname{tr} d)^2 \delta_{ij} + d^2 \delta_{ij} + 3 d_{kk} d_{ij} - 3 d_{kj} d_{ki}. \tag{6.9}$$

Since the first integral on the right-hand side of (6.8) is negative, we may drop it to obtain, using inequality (3.3) in the second integrand,

$$\int_{\mathfrak{S}} \omega^2 \, dv \leqq \frac{c_3}{3} \int_{\mathfrak{S}-\mathfrak{S}_\delta} \left[7 d^2 + \frac{2R^3}{r^3} A_{ij} n_i n_j \right] dv. \tag{6.10}$$

Consider now the form $A_{ij} n_i n_j$ at one point in $\mathfrak{S} - \mathfrak{S}_\delta$. If there we diagonalize d_{ij}, getting diagonal components d_1, d_2, d_3, we have, remembering that $n_i n_i = 1$,

$$A_{ij} n_i n_j = d_1 d_2 (1 - 3 n_3^2) + d_2 d_3 (1 - 3 n_1^2) + d_3 d_1 (1 - 3 n_2^2). \tag{6.11}$$

By the method of Lagrange multipliers, we see that the extrema of the form (6.11) are attained when n_i has the form (1, 0, 0), (0, 1, 0), or (0, 0, 1). In the first case, using CAUCHY's inequality, we get

$$\begin{aligned} A_{ij} n_r n_j &= -2 d_1 d_2 + d_2 d_3 + d_3 d_1 \leqq d_1^2 + d_2^2 + \\ &+ \frac{d_2^2 + d_3^2}{2} + \frac{d_3^2 + d_1^2}{2} = \frac{3 d_1^2 + 3 d_2^2 + 2 d_3^2}{2} \leqq \frac{3}{2} d^2. \end{aligned} \tag{6.12}$$

Inequalities similar to (6.12) may be obtained for the other extrema of (6.11), so that for all n_i we have

$$A_{ij} n_i n_j \leqq \tfrac{3}{2} d^2. \tag{6.13}$$

But (6.13) is invariant to orthogonal transformations of the coordinates, and thus it holds for all Cartesian systems.

Putting (6.13) into (6.10) yields

$$\int_{\mathfrak{S}} \omega^2 \, dv \leqq \frac{c_3}{3} \int_{\mathfrak{S}-\mathfrak{S}_\delta} \left(7 + 3 \frac{R^3}{r^3} \right) d^2 \, dv. \tag{6.14}$$

Now in $\mathfrak{S} - \mathfrak{S}_\delta$ we have $R^3/r^3 \leqq \lambda^3$, so that from (6.14) we may conclude that

$$\int_{\mathfrak{S}} \omega^2 \, dv \leqq \frac{c_3}{3} (7 + 3\lambda^3) \int_{\mathfrak{S}-\mathfrak{S}_\delta} d^2 \, dv \leqq \frac{c_3}{3} (7 + 3\lambda^3) \int_{\mathfrak{S}} d^2 \, dv. \tag{6.15}$$

A Korn inequality for the sphere in the main case is proven under the restriction that the first and second derivatives of u_i are continuous in $\overline{\mathfrak{S}}$. There remains the problem of determining the best, *i.e.*, the smallest value of the coefficient in (6.15).

The coefficient c_3 in (6.15) is an increasing function of P_δ/R_δ^2 where P_δ is the number introduced into the calculation by our use of POINCARÉ's inequality (5.11). Now it can be shown that, for a sphere,

$$\frac{P_\delta}{R_\delta^2} < \frac{1}{20.19}. \tag{6.16}$$

We say a word about how this number is obtained. It can be shown [4] (Chap. VII, § 7) that P_δ^{-1}, which is the infimum of

$$\frac{\int\limits_{\mathfrak{S}_\delta} f_{,i} f_{,i}\, dv}{\int\limits_{\mathfrak{S}_\delta} f^2\, dv}$$

for all square-integrable functions f with average value zero and with piecewise continuous first derivatives such that $f_{,i} f_{,i}$ is integrable, is the smallest positive number μ^2 for which there exists a square-integrable function g, with piecewise first and second derivatives, such that $g_{,i} g_{,i}$ is integrable, satisfying

$$\nabla^2 g + \mu^2 g = 0$$

in $\mathfrak{S}_\delta$, with zero normal derivative on the boundary. By the method of eigenfunction expansions, it can be shown [5] that g is a product of three functions, one each of the three spherical coordinates, and in particular, the radial function, say $\varrho(r)$, satisfies the spherical Bessel equation

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\varrho}{dr} + \left[\mu^2 - \frac{n(n+1)}{r^2}\right]\varrho = 0$$

with n an integer, $d\varrho/dr = 0$ on the boundary and

$$\int\limits_0^{R_\delta} \varrho\, r^2\, dr < \infty, \qquad \int\limits_0^{R_\delta} r^2 (d\varrho/dr)^2\, dr < \infty.$$

From this it follows that μ is the smallest positive number satisfying $\tan(\mu R_\delta) = (\mu R_\delta)$, from which $\mu^2 R_\delta^2 < 20.19$, and hence (6.16).

Using (6.7) and (6.16), we get from (6.15),

$$\begin{gathered}\int\limits_{\mathfrak{S}} \omega^2\, dv \leqq c_4(\lambda) \int\limits_{\mathfrak{S}} d^2\, dv, \\ c_4(\lambda) = \frac{2}{3}\left[\frac{3\lambda^2}{(20.19)(2\lambda^3 - 3\lambda^2 + 1)} + 2\right](7 + 3\lambda^2).\end{gathered} \tag{6.17}$$

Inequality (6.17) holds for all $\lambda > 1$, since we have left R_δ arbitrary, but smaller than R. Thus, in particular, it will hold for the smallest value of $c_4(\lambda)$, $\lambda > 1$. The minimum of $c_4(\lambda)$, $\lambda > 1$, occurs at about $\lambda = 1.4$ and gives an inequality for the sphere of the form (4.3) for all $''u$ satisfying (4.2) whose second derivatives are continuous in $\overline{\mathfrak{S}}$, with

$$K'' = 17 \tag{6.18}$$

to the best integer.

Now we remove the restriction that the first and second derivatives of u_i be continuous in $\overline{\mathfrak{S}}$, maintaining the side conditions of the main case, $(1.5)_3$.

Let $\mathfrak{S}_\sigma$ be a sphere concentric with $\mathfrak{S}$ and of radius $R_\sigma < R$. Then the first and second derivatives of u_i are continuous in $\overline{\mathfrak{S}}_\sigma$. Thus defining

$$\overline{\omega}_\sigma^2 = \overline{\omega}_{\sigma i j}\overline{\omega}_{\sigma i j}, \qquad \overline{\omega}_{\sigma i j} = \frac{1}{V_\sigma}\int_{\mathfrak{S}_\sigma} \omega_{i j}\, dv, \qquad V_\sigma = \int_{\mathfrak{S}_\sigma} dv, \tag{6.19}$$

and defining

$$^{\#}u_i = u_i - \overline{\omega}_{\sigma i j}\, x_j, \tag{6.20}$$

we see that the corresponding strain and rotation measures are given by

$$^{\#}d_{i j} = d_{i j}, \qquad ^{\#}\omega_{i j} = \omega_{i j} - \overline{\omega}_{\sigma i j}, \tag{6.21}$$

and hence, using (6.19) and (6.21),

$$\int_{\mathfrak{S}_\sigma} {}^{\#}\omega_{i j}\, dv = 0. \tag{6.22}$$

Thus from our foregoing analysis we get

$$\int_{\mathfrak{S}_\sigma} {}^{\#}\omega^2\, dv \leqq K'' \int_{\mathfrak{S}_\sigma} d^2\, dv \leqq K'' \int_{\mathfrak{S}} d^2\, dv, \tag{6.23}$$

and hence, using (6.21), (6.22) and (6.23), we obtain

$$\begin{aligned} \int_{\mathfrak{S}} \omega^2\, dv = \int_{\mathfrak{S}_\sigma} \omega^2\, dv + \int_{\mathfrak{S}-\mathfrak{S}_\sigma} \omega^2\, dv \leqq \int_{\mathfrak{S}_\sigma} {}^{\#}\omega^2\, dv + V_\sigma\, \overline{\omega}_\sigma^2 + \int_{\mathfrak{S}-\mathfrak{S}_\sigma} \omega^2\, dv \\ \leqq K'' \int_{\mathfrak{S}} d^2\, dv + V\, \overline{\omega}_\sigma^2 + \int_{\mathfrak{S}-\mathfrak{S}_\sigma} \omega^2\, dv. \end{aligned} \tag{6.24}$$

But since $\int_{\mathfrak{S}} \omega_{i j}\, dv = 0$, $\overline{\omega}_\sigma^2$ may be made as small as desired by choosing R_σ close enough to R. Hence (6.23) implies an inequality of the form (4.3) for all $''\boldsymbol{u}_i$ satisfying (4.2) with $K'' = 17$. In other words, 17 is an upper bound for KORN's constant in the main case and, from (4.11), 20 is an upper bound for KORN's constant in the second case. The upper bounds announced in $(1.9)_1$ are established.

7. An upper bound for Korn's constant for the circle in the main case

For the circle, using (5.3) and (5.4), we obtain

$$\psi_{,ij} = \tfrac{1}{2}\left[(2R^2/r^2)\, n_i n_j + (1 - R^2)\, \delta_{ij}\right], \quad \text{in } \mathfrak{C} - \overline{\mathfrak{C}}_\delta \tag{7.1}$$

and

$$\psi_{,ij} = -\tfrac{1}{2}(\lambda^2 - 1)\, \delta_{ij}, \quad \text{in } \mathfrak{C}_\delta. \tag{7.2}$$

From (7.2) and 4° follows

$$c_1 = -\psi_{,kk} = \lambda^2 - 1, \quad \text{in } \mathfrak{C}_\delta. \tag{7.3}$$

As in the case of the sphere, the minimum of ψ in $\overline{\mathfrak{C}}_\delta$ is assumed on $\dot{\mathfrak{C}}_\delta$ and is given by

$$c_2 R_\delta^2 = (\tfrac{1}{4})\,(1 - \lambda^2 + \lambda^2 \ln \lambda^2)\, R_\delta^2. \tag{7.4}$$

Thus, using (5.12), c_3 becomes

$$c_3 = \frac{4\lambda^2}{1 - \lambda^2 + \lambda^2 \ln \lambda^2} P + 2, \tag{7.5}$$

where $P=P_\delta/R_\delta^2$. It may be shown that P is independent of R_δ, either by simple dimensional considerations or by evaluation of P_δ in a manner similar to that used for the sphere.

Substitution of (7.1), (7.2), and (7.3) into (5.19) yields

$$\begin{aligned}\int_{\mathfrak{C}} \omega^2\, dv \leqq c_3 \int_{\mathfrak{C}_\delta} \left[-(c_1/2)\, d^2 - (c_1/2)\,(\operatorname{tr} d)^2\right] dv + \\ + c_3 \int_{\mathfrak{C}-\mathfrak{C}_\delta} \left[(\operatorname{tr} d)^2 + (R^2/r^2)\, A_{ij}\, n_i\, n_j\right] dv,\end{aligned} \tag{7.6}$$

where

$$A_{ij} = -(\operatorname{tr} d)^2\, \delta_{ij} + d^2\, \delta_{ij} + 2 d_{ij}\, d_{kk} - d_{ik}\, d_{jk}. \tag{7.7}$$

Now to our surprise and delight, $A_{ij}=0$, which can be shown most easily by referring all quantities to a frame of reference in which d_{ij} is diagonal. Thus, observing that the first integrand on the right-hand side of (7.6) is non-positive and using (3.3), we get

$$\int_{\mathfrak{C}} \omega^2\, dv \leqq \left[\frac{8\lambda^2 P}{1-\lambda^2+\lambda^2 \ln \lambda^2} + 4\right] \int_{\mathfrak{C}} d^2\, dv. \tag{7.8}$$

Since (7.8) holds for all $\lambda>1$, it holds as $\lambda\to\infty$, in which case we get

$$\int_{\mathfrak{C}} \omega^2\, dv \leqq 4 \int_{\mathfrak{C}} d^2\, dv. \tag{7.9}$$

The restriction that the first and second derivatives of u_i be continuous in $\overline{\mathfrak{C}}$ may be removed in the same manner as for the sphere.

Thus we have proven that 4 is an upper bound for KORN's constant for the circle in the main case and, by (4.11), that 6 is an upper bound in the second case. All of the bounds announced in (1.9) are now proven.

We close with some remarks on the extension of the above analysis for the circle to more general plane regions, namely, those onto which $\mathfrak{C}$ is mapped conformally by a function $\zeta(z)=\xi+i\eta$, $z=x+iy$, which is regular in $\overline{\mathfrak{C}}$. If we denote such a region by $\zeta(\mathfrak{C})$ and express ψ, defined in the x, y plane by (5.3) and (5.4), as a function of ξ and η, we may note the following:

a) The boundary of $\zeta(\mathfrak{C})$ is $\zeta(\dot{\mathfrak{C}})$.

b) Since $\psi_{,\xi}$ and $\psi_{,\eta}$ are linear combinations of $\psi_{,x}$ and $\psi_{,y}$, it follows that ψ and its derivatives are both zero on the boundary of $\zeta(\mathfrak{C})$.

c) The Laplacian of ψ transforms according to

$$\underset{z}{\nabla^2}\psi = |\zeta'(z)|^2 \underset{\xi}{\nabla^2}\psi.$$

d) Because of the regularity of $\zeta(z)$ in $\overline{\mathfrak{C}}$, $|\zeta'(z)|^2$ has a positive maximum and minimum in $\overline{\mathfrak{C}}$. Thus there are positive numbers, say b and B, b_δ and B_δ such that

$$\begin{aligned} b < \nabla^2\psi < B \quad &\text{in} \quad \zeta(\mathfrak{C}-\mathfrak{C}_\delta) \\ -B_\delta < \nabla^2\psi < -b_\delta \quad &\text{in} \quad \zeta(\mathfrak{C}_\delta). \end{aligned}$$

Thus the function ψ defined in the ξ, η plane satisfies conditions 1°, 2°, 3°, and 5°, but 4° must be replaced by d°. Using $\psi(\xi,\eta)$, then, we may carry through an analysis for $\zeta(\mathfrak{C})$ similar to that in Section 5, except that equations and inequalities which follow from 4° must be replaced by equations and inequalities implied by d°.

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