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Variance estimation and goodness-of-fit test in a high-dimensional strict factor model

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Abstract

In the classic setting where the dimension p is small compared to the sample size n , an asymptotic likelihood estimation theory is well-known for the factor model by letting n tending to infinity while keeping p fixed. This theory is however no more valid for high-dimensional data where typically the dimension p is large compared to the sample size. In this paper, we develop new asymptotic results under the high-dimensional setting in a strict factor model with homoscedastic noise variance. For the maximum likelihood estimator of the noise variance, first we identify the reasons of a widely observed downward bias of the estimator. Second, a bias-corrected estimator is proposed using this knowledge. Third, we establish an asymptotically normal distribution for this corrected estimator under the high-dimensional setting. The second contribution of the paper concerns the correction of the likelihood-ratio statistic of the goodness-of-fit test to make it adapted to high-dimensional observations. The corrected statistic is proved asymptotically normal. Throughout the paper, Monte-Carlo experiments are conducted to assess the finite-sample behaviour of the methods. An application to returns of S&P 500 stock prices is also proposed.

Keywords. High-dimensional Factor model, noise variance estimator, goodness-of-fit test, likelihood ratio, random matrix theory.

1 Introduction

Recently, factor models have become a favoured tool for the analysis of high-dimensional data since these models are efficient for reduction of data dimension.

Important references on the topic include the early works of Forni et al. (2000), Bai (2003) and Bai and Ng (2002) where some main statistical questions on the models have been addressed. As for more recent developments, factor models are used in Fan et al. (2008) to approximate a general high-dimensional covariance matrix, and in Lam et al. (2011) and Lam and Yao (2012) to approximate dynamics of high-dimensional time series; finally Bai and Li (2012) and Bai and Liao (2012) have addressed the general problem of inference for large-dimensional factor models using the likelihood function and/or the method of large covariance matrix thresholding. Moreover, in the signal processing community, factor models are referred as “signal plus noise” model and recent statistical developments are reported in Kritchman and Nadler (2008), Bianchi et al. (2011) and Hachem et al. (2012).

More precisely, we consider the following factor model following the presentation of (Anderson, 2003, Chapter 14): the observation vectors $\{\mathbf{x}_i\}_{1 \leq i \leq n}$ are p -dimensional and satisfy the equation

$$\mathbf{x}_i = \mathbf{\Lambda} \mathbf{f}_i + \mathbf{e}_i + \boldsymbol{\mu}, \quad i = 1, \dots, n. \quad (1)$$

Here, \mathbf{f}_i is a m -dimensional *common factors* with $m \ll p$, $\mathbf{\Lambda}$ a $p \times m$ matrix of *factor loadings*, and $\boldsymbol{\mu}$ represents the general mean and (\mathbf{e}_i) is a sequence of independent idiosyncratic errors. The random factors \mathbf{f}_i and the noise \mathbf{e}_i have a Gaussian distribution and they are both unobserved.

To ensure the identification of the model constraints have to be introduced on the parameters. There are several possibilities for the choice of such constraints, see e.g. Table 1 in Bai and Li (2012). A traditional choice is the following:

- $\mathbb{E} \mathbf{f}_i = \mathbf{0}$ and $\mathbb{E} \mathbf{f}_i \mathbf{f}_i' = \mathbf{I}$; $\boldsymbol{\Psi} = \text{cov}(\mathbf{e}_i)$ is diagonal;
- The matrix $\boldsymbol{\Gamma} := \mathbf{\Lambda}' \boldsymbol{\Psi}^{-1} \mathbf{\Lambda}$ is diagonal with distinct diagonal elements.

Therefore, the population covariance matrix (PCM) of $\{\mathbf{x}_i\}$ is

$$\boldsymbol{\Sigma} = \mathbf{\Lambda} \mathbf{\Lambda}' + \boldsymbol{\Psi}. \quad (2)$$

Since the number of factor m is much smaller than the dimension p , $\boldsymbol{\Sigma}$ can be seen as a finite-rank perturbation of the error covariance matrix $\boldsymbol{\Psi}$. Finally, the

special case where $\Psi = \sigma^2 \mathbf{I}$ ($\sigma^2 > 0$) is called a *strict factor model*, i.e. the error is cross-sectionally homogeneous and uncorrelated.

The likelihood-based estimation theory in a factor model has been known since Lawley (1940) as follows. Let $\bar{\mathbf{x}}$ be the sample mean and define the sample covariance matrix

$$\mathbf{S}_n = \frac{1}{n-1} \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'$$

Let $\lambda_{n,1} \geq \lambda_{n,2} \geq \dots \geq \lambda_{n,p}$ be the eigenvalues of \mathbf{S}_n . The maximum likelihood estimator (m.l.e.) of $\boldsymbol{\mu}$ is $\bar{\mathbf{x}}$ and those of Λ and Ψ are obtained by solving the following implicit equations

$$\Lambda(\Gamma + \mathbf{I}_m) = \mathbf{S}_n \Psi^{-1} \Lambda, \quad (3)$$

$$\text{diag}(\Lambda \Lambda' + \Psi) = \text{diag}(\mathbf{S}_n) \quad , \quad \text{with } \Gamma = \Lambda' \Psi^{-1} \Lambda \text{ diagonal.} \quad (4)$$

These equations can be solved using EM-type algorithms, see Zhao et al. (2008) for a review. The asymptotic normality of the maximum likelihood estimators $\hat{\Lambda}$ (resp. $\hat{\Psi}$) of Λ (resp. Ψ) is established in Anderson and Amemiya (1988) (actually under a more general setting than assuming normal distributions, see Proposition 2).

For the strict factor model case, the estimation of $\Psi = \sigma^2 \mathbf{I}_p$ is simplified to that of σ^2 . The equations (3) and (4) become

$$\Lambda(\Gamma + \mathbf{I}_m) = \frac{1}{\sigma^2} \mathbf{S}_n \Lambda, \quad (5)$$

$$p\sigma^2 = \text{tr}(\mathbf{S}_n - \Lambda \Lambda'), \quad \text{with } \Gamma = \frac{1}{\sigma^2} \Lambda' \Lambda \text{ diagonal.} \quad (6)$$

In contrast to the general case, these equations can be solved explicitly (Anderson and Rubin, 1956) to obtain the m.l.e.:

$$\hat{\sigma}^2 = \frac{1}{p-m} \sum_{i=m+1}^p \lambda_{n,i}, \quad (7)$$

$$\hat{\Lambda}_k = (\lambda_{n,k} - \hat{\sigma}^2)^{\frac{1}{2}} v_{n,k}, \quad 1 \leq k \leq m, \quad (8)$$

where for $v_{n,k}$ is the normalized eigenvector of \mathbf{S}_n corresponding to $\lambda_{n,k}$ ($1 \leq k \leq p$).

In the classical setting, hereafter referred as the *low-dimensional setting*, the asymptotic likelihood theory is developed by fixing the dimension p while the

sample size $n \rightarrow \infty$. This theory assesses that the m.l.e.'s are asymptotically normal with the standard \sqrt{n} -convergence rate (Anderson and Amemiya, 1988). In particular, as $n \rightarrow \infty$,

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, s^2), \quad s^2 = \frac{2\sigma^4}{p - m}. \quad (9)$$

The situation is radically different when p is large compared to the sample size n . Recent advances in high-dimensional statistics indicate that the above asymptotic results are no more valid with high-dimensional data. First of all, it has been widely observed in the literature that $\hat{\sigma}^2$ seriously underestimates the true noise variance σ^2 in such situation. As all meaningful inference procedures in the model will unavoidably use this variance estimate, such a severe bias is more than disappointing and need to be corrected. Two corrected estimator have been reported in the literature: Kritchman and Nadler (2008) proposes an estimator by solving a system of implicit equations; and Ulfarsson and Solo (2008) introduces an estimator using the median of the sample eigenvalues $\{\lambda_{n,k}\}$. However, these estimators are assessed by Monte-Carlo experiments and their theoretic properties (bias, consistency or asymptotic normality) are unknown. As a first contribution of the paper, we are able to identify completely the aforementioned negative bias of the variance estimator σ^2 in the high-dimensional context. Using this identification, we further propose a bias-corrected estimator for the noise variance and prove its asymptotic normality (with explicit asymptotic mean and variance). An interesting feature that this new asymptotic limit coincide with the classical low-dimensional limit (9) when the dimension to sample size ratio p/n shrinks to zero. Therefore, the new asymptotic limit is a natural extension of the classical result to the high-dimensional context. This is done in Section 3.

Next, in Section 4, we consider the likelihood ratio test for the goodness-of-fit to a strict factor model. Under the low-dimensional scheme for small dimension p , the classical theory assesses a limiting chi-squared distribution for the test statistic (Amemiya and Anderson, 1990). Under the large dimensional scheme, this limit is no more valid and the realised size of the test becomes much larger than the nominal significance level. Following an approach devised in Bai et al. (2009), we propose a correction to this goodness-of-fit test statistic to cope with the high-dimensional effects and establishes an asymptotic normal limit. As mentioned earlier, this results uses the corrected estimator of the variance discussed above. As an application, we discuss the problem of determining the number of factors

for the Standard & Poor 500 stock returns.

The remaining sections are organized as follows. In Section 2, we recall some results from random matrix theory which will be used in establishing our main results. Technical details of secondary importance are presented in the Appendix. Throughout the paper, Monte-Carlo experiments are conducted to assess the quality of the proposed methods.

2 Useful results from random matrix theory

Random matrix theory has become a powerful tool to address new inference problems raised by high-dimensional data. For general background and references, we refer to review papers Johnstone (2007) and Johnstone and Titterton (2009). Here we recall some important results that will be used afterwards.

2.1 Results about spiked population model

The strict factor model is a *spiked population model* (Johnstone, 2001) since the eigenvalues of Σ are

$$\begin{aligned} \text{spec}(\Sigma) &= \underbrace{(\alpha_1, \dots, \alpha_1)}_{n_1}, \dots, \underbrace{(\alpha_K, \dots, \alpha_K)}_{n_K}, \underbrace{(0, \dots, 0)}_{p-m} + \sigma^2 \underbrace{(1, \dots, 1)}_p \\ &= \sigma^2 \underbrace{(\alpha_1^*, \dots, \alpha_1^*)}_{n_1}, \dots, \underbrace{(\alpha_K^*, \dots, \alpha_K^*)}_{n_K}, \underbrace{(1, \dots, 1)}_{p-m}, \end{aligned} \quad (10)$$

where (α_i) are non-null eigenvalues of $\Lambda\Lambda'$ with multiplicity numbers (n_i) satisfying $n_1 + \dots + n_K = m$ and we have used the notation $\alpha_i^* = \alpha_i/\sigma^2 + 1$.

In the low-dimensional setting, as $n \rightarrow \infty$, $\mathbf{S}_n \rightarrow \Sigma$ so that by continuity, for $1 \leq k \leq m$ $\lambda_{n,k} \rightarrow \alpha_i + \sigma^2$ for some i and for $m < k \leq p$, $\lambda_{n,k} \rightarrow \sigma^2$. Again the situation is different in the high-dimensional context. More precisely, assume that p and n are related so that when $n \rightarrow \infty$, $c_n = p/(n-1) \rightarrow c > 0$. Therefore, p can be large compared to the sample size n and for the asymptotic theory, p and

n tend to infinity proportionally. Define the function

$$\phi(\alpha) = \alpha + \frac{c\alpha}{\alpha - 1}, \quad \alpha \neq 1,$$

and set $s_0 = 0$ and $s_i = n_1 + \dots + n_i$ for $1 \leq i \leq K$. The set $J_i = \{s_{i-1} + 1, \dots, s_i\}$ is then the indexes among $\{1, \dots, p\}$ associated to α_i counting the multiplicities. Following Baik and Silverstein (2006), assumed that $\alpha_1^* \geq \dots \geq \alpha_m^* > 1 + \sqrt{c}$, i.e all the eigenvalues α_i are greater than $\sigma^2\sqrt{c}$. It is then known that, for the spiked eigenvalues $\lambda_{n,k}$, $1 \leq k \leq m$, almost surely if $k \in J_i$,

$$\lambda_{n,k} \longrightarrow \sigma^2\phi(\alpha_i^*) = \alpha_i + \sigma^2 + \sigma^2c \left(1 + \frac{\sigma^2}{\alpha_i}\right). \quad (11)$$

Moreover, the remaining sample eigenvalues $\{\lambda_{n,k}\}_{m < k \leq p}$, called noise eigenvalues, will converge to a continuous distribution with support interval $[a(c), b(c)]$ where $a(c) = \sigma^2(1 - \sqrt{c})^2$ and $b(c) = \sigma^2(1 + \sqrt{c})^2$. In particular, for all $1 \leq i \leq L$ with a prefixed range L and almost surely, $\lambda_{n,m+i} \rightarrow b(c)$. It is worth noticing that in (11), if we let $c \rightarrow 0$, we recover the low-dimensional limits $\lambda_{n,k} \rightarrow \alpha_i + \sigma^2$ (spike eigenvalues) and $\lambda_{n,k} \rightarrow \sigma^2$ (noise eigenvalues) discussed earlier.

In a further step, CLT for the spiked eigenvalues is established in Bai and Yao (2008): the n_i -dimensional vector

$$\{\sqrt{n}(\lambda_{n,k} - \sigma^2\phi(\alpha_k^*)), k \in J_i\} \quad (12)$$

converges to a well-determined n_i -dimensional limiting distribution. Moreover, this limiting distribution is Gaussian if and only if the corresponding population spike eigenvalue α_i is simple, i.e. $n_i = 1$.

2.2 Empirical spectral distribution and Marčenko-Pastur distributions

Let H be a probability measure on \mathbb{R}^+ and $c > 0$ a constant. We define the map

$$g(s) = g_{c,H}(s) = \frac{1}{s} + c \int \frac{t}{1 + ts} dH(t) \quad (13)$$

in the set $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$. The map g is a one-to-one map from \mathbb{C}^+ onto itself (see Bai and Silverstein (2010), chapter 6), and the inverse map $m = g^{-1}$

satisfies all the requirements of the Stieltjes transform of a probability measure on $[0, \infty)$. We call this measure $\mathbb{F}_{c,H}$. Next, a companion measure $F_{c,H}$ is introduced by the equation $cF_{c,H} = (c-1)\delta_0 + \mathbb{F}_{c,H}$ (note that in this equation, measures can be signed). The measure $F_{c,H}$ is referred as the generalized Marčenko-Pastur distribution with index (c, H) .

Let $F_n = \frac{1}{p} \sum_{i=1}^p \delta_{\lambda_{n,i}}$ be the empirical spectral distribution (ESD) of \mathbf{S}_n . Then, F_n converges to the Marčenko-Pastur distribution of index (c, δ_{σ^2}) , simply denoted as F_{c,σ^2} , with the following density function

$$p_{c,\sigma^2}(x) = \begin{cases} \frac{1}{2\pi x c \sigma^2} \sqrt{\{b(c) - x\}\{x - a(c)\}}, & a(c) \leq x \leq b(c), \\ 0, & \text{otherwise.} \end{cases}$$

The distribution has an additional mass $(1 - 1/c)$ at the origin if $c > 1$.

Let $H_n = F^\Sigma$ be the ESD of Σ . We have

$$H_n = \frac{p-m}{p} \delta_{\sigma^2} + \frac{1}{p} \sum_{i=1}^m \delta_{\alpha_i + \sigma^2}$$

and $H_n \rightarrow \delta_{\sigma^2}$.

2.3 CLT for linear spectral statistic of a high-dimensional covariance matrix

Define the normalized empirical process

$$G_n(f) = p \int_{\mathbb{R}} f(x) [F_n - F_{c_n, H_n}](dx), \quad f \in \mathcal{A},$$

where \mathcal{A} is the set of analytic functions $f : \mathcal{U} \rightarrow \mathbb{C}$, with \mathcal{U} an open set of \mathbb{C} such that $[1_{(0,1)}(c)a(c), b(c)] \subset \mathcal{U}$. We will need the following CLT which is a combination of Theorem 1.1 of Bai and Silverstein (2004) and a recent addition proposed in Zheng and Bai (2013).

Proposition 1. *Consider the strict factor model (1) with population covariance matrix $\Sigma = \Lambda\Lambda' + \sigma^2\mathbf{I}_p$ where both the random factors and the noise are Gaussian.*

Assume that $p \rightarrow \infty$, $n \rightarrow \infty$ and $c_n = p/(n-1) \rightarrow c > 0$. Then, for any functions $f_1, \dots, f_k \in \mathcal{A}$, the random vector $(G_n(f_1), \dots, G_n(f_k))$ converges to a k -dimensional Gaussian vector with mean vector

$$m(f_j) = \frac{f_j(a(c)) + f_j(b(c))}{4} - \frac{1}{2\pi} \int_{a(c)}^{b(c)} \frac{f_j(x)}{\sqrt{4c\sigma^4 - (x - \sigma^2 - c\sigma^2)^2}} dx, \quad j = 1, \dots, k,$$

and covariance function

$$v(f_j, f_l) = -\frac{1}{2\pi^2} \oint \oint \frac{f_j(z_1) f_l(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} d\underline{m}(z_1) d\underline{m}(z_2), \quad j, l = 1, \dots, k, \quad (14)$$

where $\underline{m}(z)$ is the Stieltjes transform of $F_{c,\sigma^2} = (1-c)\delta_0 + cF_{c,\sigma^2}$. The contours are non overlapping and both contain the support of F_{c,σ^2} .

An important and subtle point here is that the centering term in $G_n(f)$ in the above CLT is defined with respect to the Marcčenko-Pastur distribution F_{c_n, H_n} with ‘‘current’’ index (c_n, H_n) instead of the limiting distribution F_{c,σ^2} with index (c, σ^2) . In contrast, the limiting mean function $m(f_j)$ and covariance function $v(f_j, f_l)$ depend on the limiting distribution F_{c,σ^2} only.

3 Estimation of the homoscedastic noise variance

As explained in Introduction, when the dimension p is large compared to the sample size n , the m.l.e. $\hat{\sigma}^2$ in (7) has a negative bias. In this section, we identify this bias and establishes its asymptotic normality under the high-dimensional scheme.

Theorem 1. *We assume the same conditions as in Proposition 1. Then, we have*

$$\frac{(p-m)}{\sigma^2 \sqrt{2c}} (\hat{\sigma}^2 - \sigma^2) + b(\sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1),$$

where $b(\sigma^2) = \sqrt{\frac{c}{2}} \left(m + \sigma^2 \sum_{i=1}^m \frac{1}{\alpha_i} \right)$.

Therefore for high-dimensional data, the m.l.e. $\hat{\sigma}^2$ has an asymptotic bias $-b(\sigma^2)$ (after normalization). This bias is a complex function of the noise variance

and the m non-null eigenvalues of the loading matrix $\mathbf{\Lambda}\mathbf{\Lambda}'$. It is worth noticing that the above CLT is still valid if $\tilde{c}_n = (p - m)/n$ is substituted for c . Now if we let $p \ll n$ so tha $\tilde{c}_n \simeq 0$ and $b(\sigma^2) \simeq 0$, and hence

$$\frac{(p - m)}{\sigma^2 \sqrt{2c}} (\hat{\sigma}^2 - \sigma^2) + b(\sigma^2) \simeq \frac{\sqrt{p - m}}{\sigma^2 \sqrt{2}} (\hat{\sigma}^2 - \sigma^2) .$$

This is nothing but the CLT (9) for $\hat{\sigma}^2$ known under the classical low-dimensional scheme. From this point of view, Theorem 1 consitutes a natural extension of the classical CLT to the high-dimensional context.

Proof of Theorem 1. We have

$$(p - m)\hat{\sigma}^2 = \sum_{i=1}^p \lambda_{n,i} - \sum_{i=1}^m \lambda_{n,i}.$$

By (11),

$$\sum_{i=1}^m \lambda_{n,i} \longrightarrow \sum_{i=1}^m \left(\alpha_i + \frac{c\sigma^4}{\alpha_i} \right) + \sigma^2 m(1 + c) \text{ a.s.} \quad (15)$$

For the first term, we have

$$\begin{aligned} \sum_{i=1}^p \lambda_i &= p \int x dF_n(x) \\ &= p \int x d(F_n - F_{c_n, H_n})(x) + p \int x dF_{c_n, H_n}(x) \\ &= G_n(x) + p \int x dF_{c_n, H_n}(x). \end{aligned}$$

By Proposition 1, the first term is asymptotically normal

$$G_n(x) = \sum_{i=1}^p \lambda_{n,i} - p \int x dF_{c_n, H_n}(x) \xrightarrow{\mathcal{L}} \mathcal{N}(m(x), v(x)),$$

with asymptotic mean

$$m(x) = 0 , \quad (16)$$

and asymptotic variance

$$v(x) = 2c\sigma^4. \quad (17)$$

The derivation of these two formula are given in the appendix. Furthermore, by Lemma 1 of Bai et al. (2010),

$$\int x dF_{c_n, H_n}(x) = \int t dH_n(t) = \sigma^2 + \frac{1}{p} \sum_{i=1}^m \alpha_i.$$

So we have

$$\sum_{i=1}^p \lambda_{n,i} - p\sigma^2 - \sum_{i=1}^m \alpha_i \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2c\sigma^4). \quad (18)$$

By (15) and (18) and using Slutsky's lemma, we obtain

$$(p - m)(\hat{\sigma}^2 - \sigma^2) + c\sigma^2 \left(m + \sigma^2 \sum_{i=1}^m \frac{1}{\alpha_i} \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2c\sigma^4).$$

□

3.1 Monte-carlo experiments

We consider an i.i.d. Gaussian sample of size n in three different settings:

- Model 1: $\text{spec}(\Sigma) = (25, 16, 9, 0, \dots, 0) + \sigma^2(1, \dots, 1)$, $\sigma^2 = 4$, $c = 1$;
- Model 2: $\text{spec}(\Sigma) = (4, 3, 0, \dots, 0) + \sigma^2(1, \dots, 1)$, $\sigma^2 = 2$, $c = 0.2$;
- Model 3: $\text{spec}(\Sigma) = (12, 10, 8, 8, 0, \dots, 0) + \sigma^2(1, \dots, 1)$, $\sigma^2 = 3$, $c = 1.5$.

Figure 1 presents the histograms from 1000 replications of

$$\frac{(p - m)}{\sigma^2 \sqrt{2c}} (\hat{\sigma}^2 - \sigma^2) + b(\sigma^2)$$

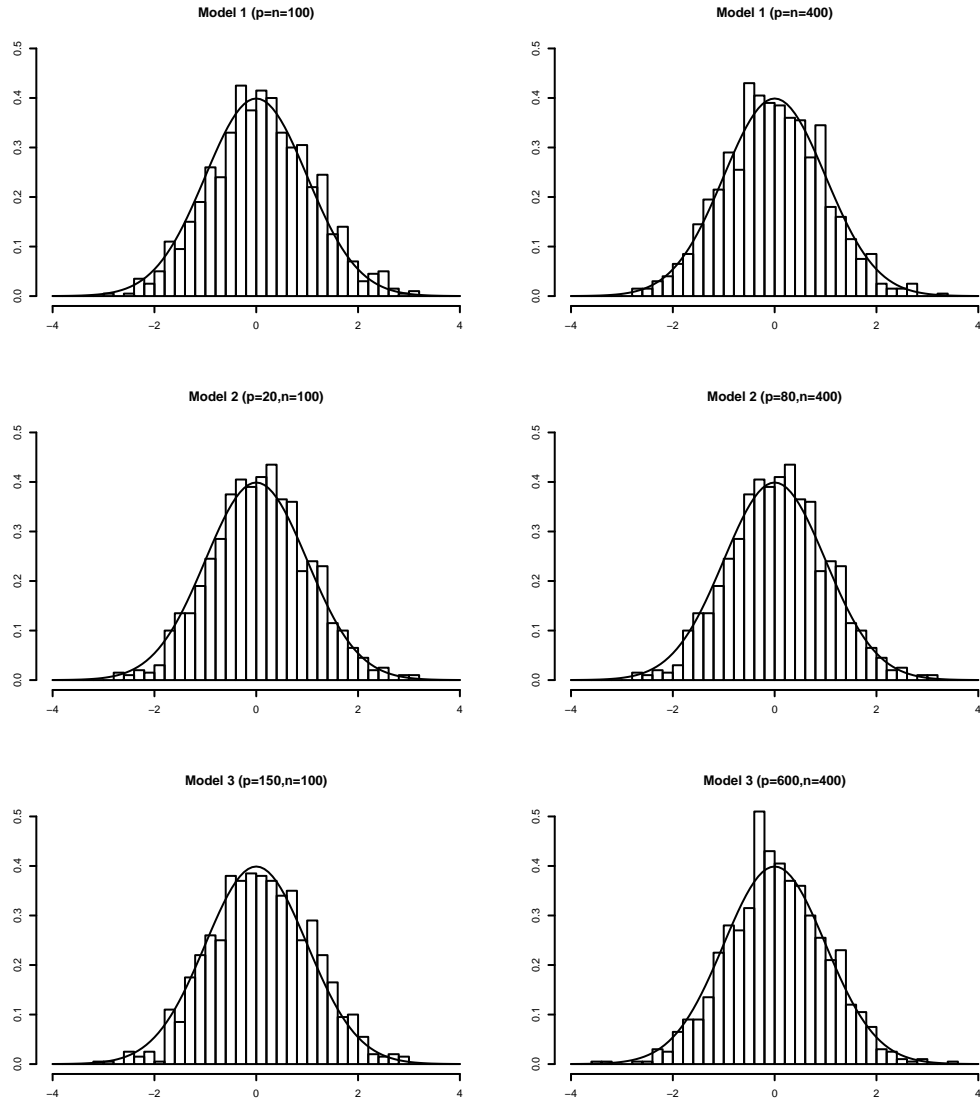


Figure 1: Histogram of $\frac{(p-m)}{\sigma^2\sqrt{2c}}(\hat{\sigma}^2 - \sigma^2) + b(\sigma^2)$ compared with the density of a standard Gaussian law.

for the three models above, with different sample size n and $p = c \times n$, compared to the density of the standard normal probability law. Even for a moderate sample size like $n = 100$, the distribution is almost normal.

In Table 1, we compare the empirical bias of $\widehat{\sigma}^2$ (i.e. the empirical mean of $\sigma^2 - \widehat{\sigma}^2 = \sigma^2 - \frac{1}{p-m} \sum_{i=m+1}^p \lambda_{n,i}$) over 1000 replications with the theoretical one $-\sigma^2 \sqrt{2cb(\sigma^2)}/(p-m)$ in different settings. In all the three models, the empirical and theoretical bias are close each other. As expected, their difference vanishes when p and n increase.

Table 1: Comparison between the empirical and the theoretical bias in various settings.

	Settings		Empirical bias	Theoretical bias	Difference
Model 1	$p = 100$	$n = 100$	-0.1556	-0.1589	0.0023
	$p = 400$	$n = 400$	-0.0379	-0.0388	0.0009
	$p = 800$	$n = 800$	-0.0189	-0.0193	0.0004
Model 2	$p = 20$	$n = 100$	-0.0654	-0.0704	0.0050
	$p = 80$	$n = 400$	-0.0150	-0.0162	0.0012
	$p = 200$	$n = 1000$	-0.0064	-0.0063	0.0001
Model 3	$p = 150$	$n = 100$	-0.0801	-0.0795	0.0006
	$p = 600$	$n = 400$	-0.0400	-0.0397	0.0003
	$p = 1500$	$n = 1000$	-0.0157	-0.0159	0.0002

3.2 A bias-corrected estimator

The previous theory recommends to correct the negative bias of $\widehat{\sigma}^2$. However, the bias $b(\sigma^2)$ depends on the number m and the values α_i of the spikes. These parameters could not be known in real-data applications and they need to be first estimated. In the literature, consistent estimators of m have been proposed, e.g. in Kritchman and Nadler (2008), Onatski (2009) and Passemier and Yao (2012).

For the values of the spikes α_i , it is easy to see that it can be done by inverting the function ϕ in (11) at the corresponding eigenvalues λ_j . Moreover, by applying the delta-method to (12), we can obtain the asymptotic distribution of this estimator, see Bai and Ding (2012).

As the bias depends on σ^2 which we want to estimate, a natural correction is

to use the plug-in estimator

$$\hat{\sigma}_*^2 = \hat{\sigma}^2 + \frac{b(\hat{\sigma}^2)}{p-m} \hat{\sigma}^2 \sqrt{2c}.$$

Notice that in this formula, the number of factors m can be replaced by any consistent estimate as discussed above without affecting its limiting distribution. Using Theorem 1 and the delta-method, we obtain the following CLT

Theorem 2. *We assume the same conditions as in Proposition 1. Then, we have*

$$\tilde{v}(c)^{-\frac{1}{2}} \left\{ \hat{\sigma}_*^2 - \sigma^2 + \tilde{b}(\sigma^2) \right\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1),$$

where

$$\tilde{b}(\sigma^2) = \frac{c\sqrt{2c}\sigma^2}{(p-m)^2} \left(m b(\sigma^2) + 2\sigma^2 b(\sigma^2) \sum_{i=1}^m \alpha_i^{-1} \right) - \frac{2c^2\sigma^4 b(\sigma^2)^2 \sum_{i=1}^m \alpha_i^{-1}}{(p-m)^3} = O\left(\frac{1}{p^2}\right),$$

and

$$\tilde{v}(c) = \frac{2c\sigma^4}{(p-m)^2} \left(1 + \frac{cm}{p-m} + \frac{4c^2\sigma^4}{(p-m)^3} \sum_{i=1}^m \alpha_i^{-1} \right)^2 = \frac{2c\sigma^4}{(p-m)^2} \left(1 + O\left(\frac{1}{p}\right) \right).$$

Basically, this theorem states that

$$\frac{p-m}{\sigma^2\sqrt{2c}} (\hat{\sigma}_*^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Compared to the m.l.e. $\hat{\sigma}^2$ in Theorem 1, the new estimator has no more a bias after normalization by $\frac{p-m}{\sigma^2\sqrt{2c}}$. The terms $\tilde{b}(\sigma^2)$ and $\tilde{v}(c)$ in the theorem give more details for the centering parameter and the normalization rate.

To assess the quality of this bias-corrected estimator $\hat{\sigma}_*^2$, we conduct some simulation experiments using the previous settings: Tables 2 and 3 give the empirical mean of $\hat{\sigma}_*^2$ over 1000 replications compared with the empirical mean of $\hat{\sigma}^2$, as well as the mean squared errors and mean absolute deviations. For comparison, the same statistics are also given for the estimator $\hat{\sigma}_{\text{KN}}^2$ of Kritchman and Nadler (2008) and the estimator $\hat{\sigma}_{\text{US}}^2$ of Ulfarsson and Solo (2008). These two estimators are defined as follow:

- $\hat{\sigma}_{\text{KN}}^2$ is the solution of the following non-linear system of $m + 1$ equations involving the $m + 1$ unknowns $\hat{\rho}_1, \dots, \hat{\rho}_m$ and $\hat{\sigma}_{\text{KN}}^2$

$$\hat{\sigma}_{\text{KN}}^2 - \frac{1}{p-m} \left[\sum_{j=m+1}^p \lambda_{n,j} + \sum_{j=1}^m (\lambda_{n,j} - \hat{\rho}_j) \right] = 0,$$

$$\hat{\rho}_j - \hat{\rho}_j \left(\lambda_{n,j} + \hat{\sigma}_{\text{KN}}^2 - \hat{\sigma}_{\text{KN}}^2 \frac{p-m}{n} \right) + \lambda_{n,j} \hat{\sigma}_{\text{KN}}^2 = 0.$$

We used the computing code available on the author's web-page to carry out the simulations.

- $\hat{\sigma}_{\text{US}}^2$ is defined as

$$\hat{\sigma}_{\text{US}}^2 = \frac{\text{median}(\lambda_{n,m+1}, \dots, \lambda_{n,p})}{p_{c,1}^{-1}(0.5)},$$

where $p_{c,1}^{-1}$ is quantile function of the Marčenko-Pastur distribution $F_{c,1}$.

Table 2: Empirical mean, MSE (between brackets) and mean absolute deviation of $\hat{\sigma}^2$ and $\hat{\sigma}_*^2$ in various settings.

Mod.	Settings			$\hat{\sigma}^2$	$ \sigma^2 - \hat{\sigma}^2 $	$\hat{\sigma}_*^2$	$ \sigma^2 - \hat{\sigma}_*^2 $
	p	n	σ^2				
1	100	100		3.8464 (0.0032)	0.1536	3.9979 (0.0035)	0.0021
	400	400	4	3.9616 (0.0002)	0.0384	4.0000 (0.0002)	$< 10^{-5}$
	800	800		3.9809 (0.0001)	0.0191	4.0002 (0.0001)	0.0002
2	20	100		1.9340 (0.0043)	0.0660	2.0012 (0.0047)	0.0012
	80	400	2	1.9841 (0.0003)	0.0159	2.0001 (0.0003)	0.0001
	200	1000		1.9939 ($< 10^{-5}$)	0.0061	2.0002 ($< 10^{-5}$)	0.0002
3	150	100		2.8400 (0.0011)	0.1600	2.9926 (0.0013)	0.0074
	600	400	3	2.9605 (0.0001)	0.0395	2.9999 (0.0001)	0.0001
	1500	1000		2.9839 ($< 10^{-5}$)	0.0161	2.9998 ($< 10^{-5}$)	0.0002

In all three models considered, the bias-corrected estimator $\hat{\sigma}_*^2$ is far much better than the original m.l.e. $\hat{\sigma}^2$: here mean absolute deviations are reduced by 95% at least. The performances of $\hat{\sigma}_*^2$ and $\hat{\sigma}_{\text{KN}}^2$ are similar. The estimator $\hat{\sigma}_{\text{US}}^2$ shows slightly better performance than the m.l.e. $\hat{\sigma}^2$, but performs poorly compared to $\hat{\sigma}_*^2$ and $\hat{\sigma}_{\text{KN}}^2$. Notice however the theoretic properties of $\hat{\sigma}_{\text{KN}}^2$ and $\hat{\sigma}_{\text{US}}^2$ are unknown and so far there have been checked via simulations only.

Table 3: Empirical mean, MSE (between brackets) and mean absolute deviation of $\hat{\sigma}_{\text{KN}}^2$ and $\hat{\sigma}_{\text{US}}^2$ in various settings.

Mod.	Settings			$\hat{\sigma}_{\text{KN}}^2$	$ \sigma^2 - \hat{\sigma}_{\text{KN}}^2 $	$\hat{\sigma}_{\text{US}}^2$	$ \sigma^2 - \hat{\sigma}_{\text{US}}^2 $
	p	n	σ^2				
1	100	100	4	4.0030 (0.0036)	0.0030	3.8384 (0.0154)	0.1616
	400	400		4.0003 (0.0002)	0.0003	3.9585 (0.0013)	0.0415
	800	800		4.0002 (0.0001)	0.0002	3.9794 (0.0004)	0.0206
2	20	100	2	1.9997 (0.0048)	0.0003	1.9400 (0.0087)	0.0600
	80	400		2.0001 (0.0003)	0.0001	1.9851 (0.0008)	0.0149
	200	1000		2.0002 ($< 10^{-5}$)	0.0002	1.9942 (0.0001)	0.0058
3	150	100	3	2.9935 (0.0016)	0.0065	2.7750 (0.0092)	0.2250
	600	400		3.0006 (0.0001)	0.0006	2.9450 (0.0007)	0.0550
	1500	1000		2.9999 ($< 10^{-5}$)	0.0001	2.9773 (0.0001)	0.0227

4 Corrected likelihood ratio test of the hypothesis that the factor model fits

As a second statistical problem in the high-dimensional factor model, we consider the following goodness-of-fit test. The null hypothesis is then

$$\mathcal{H}_0 : \Sigma = \Lambda \Lambda' + \sigma^2 \mathbf{I}_p,$$

where the number of factors m is specified. Following Anderson and Rubin (1956), the likelihood ratio test (LRT) statistic is

$$T_n = -nL^*,$$

where

$$L^* = \sum_{j=m+1}^p \log \frac{\lambda_{n,j}}{\hat{\sigma}^2},$$

and $\hat{\sigma}^2$ is the m.l.e. (7) of the variance.. Keeping p fixed while letting $n \rightarrow \infty$, the low-dimensional theory states that T_n converges to χ_q^2 , where $q = p(p+1)/2 + m(m-1)/2 - pm - 1$, see Anderson and Rubin (1956). However, this classical approximation is again useless in the large-dimensional setting. Indeed, it will be shown below that this criterion leads to a high false-positive rate. In particular, the test becomes biased since the size will be much higher than the nominal level (see Table 4).

In a way similar to Section 3, we now construct a corrected version of T_n using Proposition 1 and calculus done in Bai et al. (2009) and Zheng (2012). As we consider the logarithm of the eigenvalues of the sample covariance matrix, we will assume in the sequel that $c < 1$ to avoid null eigenvalues.

Theorem 3. *Assume the same conditions as in Proposition 1 and in addition $c < 1$. Then, we have*

$$v(c)^{-\frac{1}{2}} \{L^* - m(c) - ph(c_n) - \eta + (p - m) \log(\beta)\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1),$$

where

$$\begin{aligned} m(c) &= \frac{\log(1 - c)}{2}, & h(c_n) &= \frac{c_n - 1}{c_n} \log(1 - c_n) - 1, \\ \eta &= \sum_{i=1}^m \log(1 + c\sigma^2\alpha_i^{-1}), & \beta &= 1 - \frac{c}{p - m} (m + \sigma^2 \sum_{i=1}^m \alpha_i^{-1}), \\ v(c) &= -2 \log(1 - c) + \frac{2c}{\beta} \left(\frac{1}{\beta} - 2 \right). \end{aligned}$$

Note that the above statistic depends on the unknown variance σ^2 and the spike eigenvalues (α_i). First of all, as explained in Section 3.2, consistent estimates of (α_i) are available. By using these estimates and substituting bias-corrected estimate $\hat{\sigma}_*^2$ for σ^2 , we obtain consistent estimates $\hat{v}(c_n)$, $\hat{\eta}$ and $\hat{\beta}$ of $v(c)$, η and β , respectively. Therefore, to test \mathcal{H}_0 , it is natural to use the statistic

$$\Delta_n := \hat{v}(c_n)^{-\frac{1}{2}} (L^* - m(c_n) - ph(c_n) - \hat{\eta} + (p - m) \log(\hat{\beta})).$$

Since Δ_n is asymptotically standard normal, the critical region $\{\Delta_n > q_\alpha\}$ where q_α is the α th upper quantile of standard normal, will have an asymptotic size α . This test will be hereafter referred as the corrected likelihood ratio test (CLRT in short).

Proof of Theorem 3. We have

$$\begin{aligned}
L^* &= \sum_{i=m+1}^p \log \frac{\lambda_{n,i}}{\widehat{\sigma}^2} \\
&= \sum_{i=m+1}^p \log \frac{\lambda_{n,i}}{\sigma^2} - \sum_{i=m+1}^p \log \frac{\widehat{\sigma}^2}{\sigma^2} \\
&= \sum_{i=m+1}^p \log \frac{\lambda_{n,i}}{\sigma^2} - (p-m) \log \left(\frac{1}{p-m} \sum_{i=m+1}^p \frac{\lambda_{n,i}}{\sigma^2} \right) \\
&= L_1 - (p-m) \log \left(\frac{L_2}{p-m} \right),
\end{aligned}$$

where we have defined a two-dimensional vector $(L_1, L_2) = (\sum_{i=m+1}^p \log \frac{\lambda_{n,i}}{\sigma^2}, \sum_{i=m+1}^p \frac{\lambda_{n,i}}{\sigma^2})$.

CLT when $\sigma^2 = 1$. To start with, we consider the case $\sigma^2 = 1$. We have

$$\begin{aligned}
L_1 &= p \int \log(x) dF_n(x) - \sum_{i=1}^m \log \lambda_{n,i} \\
&= p \int \log(x) d(F_n - F_{c_n, H_n})(x) + p \int \log(x) dF_{c_n, H_n}(x) - \sum_{i=1}^m \log \lambda_{n,i}.
\end{aligned}$$

Similarly, we have

$$L_2 = p \int x d(F_n - F_{c_n, H_n})(x) + p \int x dF_{c_n, H_n}(x) - \sum_{i=1}^m \lambda_{n,i}.$$

By Proposition 1, we find that

$$p \begin{pmatrix} \int \log(x) d(F_n - F_{c_n, H_n})(x) \\ \int x d(F_n - F_{c_n, H_n})(x) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N} \left(\begin{pmatrix} m_1(c) \\ m_2(c) \end{pmatrix}, \begin{pmatrix} v_1(c) & v_{1,2}(c) \\ v_{1,2}(c) & v_2(c) \end{pmatrix} \right) \quad (19)$$

with $m_2(c) = 0$ and $v_2(c) = 2c$ and

$$m_1(c) = \frac{\log(1-c)}{2}, \quad (20)$$

$$v_1(c) = -2 \log(1-c), \quad (21)$$

$$v_{1,2}(c) = 2c. \quad (22)$$

Formulae of m_2 and v_2 have been established in the proof of Theorem 1 and the others are derived in Appendix A.

In Theorem 1, with $\sigma^2 = 1$, we found that

$$\int x dF_{c_n, H_n}(x) = 1 + \frac{1}{p} \sum_{i=1}^m \alpha_i,$$

and

$$\sum_{i=1}^m \lambda_{n,i} \xrightarrow{\text{a.s.}} \sum_{i=1}^m \left(\alpha_i + \frac{c}{\alpha_i} \right) + m(1+c).$$

For the last term of L_1 , by (11), we have

$$\log \lambda_{n,i} \longrightarrow \log(\phi(\alpha_i + 1)) = \log((\alpha_i + 1)(1 + c\alpha_i^{-1})) \text{ a.s.}$$

Furthermore, by Wang et al. (2013), we have

$$\int \log(x) dF_{c_n, H_n}(x) = \frac{1}{p} \sum_{i=1}^m \log(\alpha_i + 1) + h(c_n) + o\left(\frac{1}{p}\right),$$

where

$$h(c_n) = \int \log(x) dF_{c_n, \delta_1}(x) = \frac{c_n - 1}{c_n} \log(1 - c_n) - 1. \quad (23)$$

can be calculated using the density of the Marčenko-Pastur law (see appendix). Summarizing, we have obtained that

$$L_1 - m_1(c) - ph(c_n) - \eta(c, \alpha) \xrightarrow{\mathcal{L}} \mathcal{N}(0, v_1(c)),$$

where $h(c_n) = \frac{c_n - 1}{c_n} \log(1 - c_n) - 1$ and $\eta(c, \alpha) = \sum_{i=1}^m \log(1 + c\sigma^2 \alpha_i^{-1})$. Similarly, we have

$$L_2 - (p - m) + \rho(c, \alpha) \xrightarrow{\mathcal{L}} \mathcal{N}(0, v_2(c)),$$

where $\rho(c, \alpha) = c(m + \sum_{i=1}^m \alpha_i^{-1})$.

Using (19) and Slutsky's lemma,

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N} \left(\begin{pmatrix} m_1(c) + ph(c_n) + \eta(c, \alpha) \\ p - m - \rho(c, \alpha) \end{pmatrix}, \begin{pmatrix} v_1(c) & v_{1,2}(c) \\ v_{1,2}(c_n) & v_2(c_n) \end{pmatrix} \right),$$

with $h(c_n) = \frac{c_n-1}{c_n} \log(1 - c_n) - 1$, $\eta(c, \alpha) = \sum_{i=1}^m \log(1 + c\sigma^2\alpha_i^{-1})$ and $\rho(c, \alpha) = c(m + \sum_{i=1}^m \alpha_i^{-1})$.

CLT with general σ^2 . When $\sigma^2 = 1$,

$$\text{spec}(\Sigma) = (\alpha_1 + 1, \dots, \alpha_m + 1, 1, \dots, 1),$$

whereas in the general case

$$\begin{aligned} \text{spec}(\Sigma) &= (\alpha_1 + \sigma^2, \dots, \alpha_m + \sigma^2, \sigma^2, \dots, \sigma^2) \\ &= \sigma^2 \left(\frac{\alpha_1}{\sigma^2} + 1, \dots, \frac{\alpha_m}{\sigma^2} + 1, \dots, 1 \right). \end{aligned}$$

Thus, if we consider λ_i/σ^2 , we will find the same CLT by replacing the $(\alpha_i)_{1 \leq i \leq m}$ by α_i/σ^2 . Furthermore, we divide L_2 by $p - m$ to find

$$\left(\begin{array}{c} L_1 \\ \frac{L_2}{p-m} \end{array} \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left(\left(\begin{array}{c} m_1(c) + ph(c_n) + \eta(c, \alpha/\sigma^2) \\ 1 - \frac{\rho(c, \alpha/\sigma^2)}{p-m} \end{array} \right), \left(\begin{array}{cc} \frac{2c}{(p-m)^2} & \frac{2c}{p-m} \\ \frac{2c}{p-m} & -2 \log(1 - c) \end{array} \right) \right),$$

with $\eta(c, \alpha/\sigma^2) = \sum_{i=1}^m \log(1 + c\sigma^2\alpha_i^{-1})$, $\rho(c, \alpha/\sigma^2) = c(m + \sigma^2 \sum_{i=1}^m \alpha_i^{-1})$ and $h(c_n) = \frac{c_n-1}{c_n} \log(1 - c_n) - 1$.

Asymptotic distribution of L^* . We have $L^* = g(L_1, L_2/(p-m))$, with $g(x, y) = x - (p - m) \log(y)$. We will apply the multivariate delta-method on (24) with the function g . We have $\nabla g(x, y) = \left(1, -\frac{p-m}{y} \right)$ and

$$L^* \xrightarrow{\mathcal{L}} \mathcal{N}(\beta_1 - (p - m) \log(\beta_2), \nabla g(\beta_1, \beta_2) \text{cov}(L_1, L_2/(p - m)) \nabla g(\beta_1, \beta_2)'),$$

with $\beta_1 = m_1(c) + ph(c_n) + \eta(c, \alpha/\sigma^2)$ and $\beta_2 = 1 - \frac{\rho(c, \alpha/\sigma^2)}{p-m}$. After some standard calculation, we finally find

$$L^* \xrightarrow{\mathcal{L}} \mathcal{N} \left(m_1(c) + ph(c_n) + \eta \left(c, \frac{\alpha}{\sigma^2} \right) - (p - m) \log(\beta_2), -2 \log(1 - c) + \frac{2c}{\beta_2} \left(\frac{1}{\beta_2} - 2 \right) \right).$$

□

4.1 Monte-Carlo experiments

We consider again the models 1 and 2 described in Section 3.1, and a new one (model 4):

- Model 1: $\text{spec}(\Sigma) = (25, 16, 9, 0, \dots, 0) + \sigma^2(1, \dots, 1)$, $\sigma^2 = 4$, $c = 0.9$;
- Model 2: $\text{spec}(\Sigma) = (4, 3, 0, \dots, 0) + \sigma^2(1, \dots, 1)$, $\sigma^2 = 2$, $c = 0.2$;
- Model 4: $\text{spec}(\Sigma) = (8, 7, 0, \dots, 0) + \sigma^2(1, \dots, 1)$, $\sigma^2 = 1$, varying c .

Table 4 gives the realized sizes (i.e. the empirical probability of rejecting the null hypothesis) of the classical likelihood ratio test (LRT) and the new corrected likelihood ratio test (CLRT) above. For the LRT, we use the correction proposed by Bartlett (1950), that is replacing $T_n = -nL^*$ by $\tilde{T}_n = -(n - (2p+11)/6 - 2m/3)L^*$. The computations are done under 10000 independent replications and the nominal test level is 0.05.

Table 4: Comparison of the realized size of the classical likelihood ratio test (LRT) and the corrected likelihood ratio test (CLRT) in various settings.

Settings		Realized size of CLRT	Realized size of LRT
Model 1	$p = 90$ $n = 100$	0.0497	0.9995
	$p = 180$ $n = 200$	0.0491	1
	$p = 720$ $n = 800$	0.0496	1
Model 2	$p = 20$ $n = 100$	0.0324	0.0294
	$p = 80$ $n = 400$	0.0507	0.0390
	$p = 200$ $n = 1000$	0.0541	0.0552
Model 4	$p = 5$ $n = 500$	0.0108	0.0483
	$p = 10$ $n = 500$	0.0190	0.0465
	$p = 50$ $n = 500$	0.0424	0.0445
	$p = 100$ $n = 500$	0.0459	0.0461
	$p = 200$ $n = 500$	0.0491	0.2212
	$p = 250$ $n = 500$	0.0492	0.7395
	$p = 300$ $n = 500$	0.0509	0.9994

The sizes of the new CLRT are very close to the theoretical one, except when the ratio $c = p/n$ is very small (less than 0.1). On the contrary, the sizes produced

by the classical LRT are much higher than the nominal level especially when c is not too small, and the test will always reject the null hypothesis when p becomes large. In particular when $p/n \geq \frac{1}{2}$, the LRT test tends to reject automatically the null.

4.2 Application to S&P 500 stocks data

In this section, we will present a financial application of our goodness-of-fit test to a factor model with m factors. The considered data is a set of 488 U.S. stocks included in the S&P 500 index from September 2007 to September 2011, i.e. a period of 1001 trading days (12 stocks have been removed because of missing values). The final sample size is $n = 1000$ and the dimension of the observations is $p = 488$.

We will use our goodness-of-fit test to give an estimator \hat{m} of the number of factors, by calculating the p-value of the test for different values of the parameter m : higher the p-value of the goodness-of-fit test, more likely the proposed number m of factors. For m ranging from 0 to 100, the p-values are all very close to zero, except for $m = 71$ and $m = 72$, for which the p-values are 0.0514 and 0.1041 respectively. This method thus lead to the estimates $\hat{m} = 72$ or $\hat{m} = 73$. We have also run the estimators proposed by Nadler (2010) and Passemier and Yao (2012) on this data set: the estimates are respectively $\hat{m}_{KN} = 95$ and $\hat{m}_{PY} = 20$.

Inference on the number of factors for daily returns has been frequently reported in the literature. For example, Harding (2009) has analyzed a set of NYSE daily returns ($p = 300$ returns) over a period of 10 years, and has proposed an estimator of the number of factors. His estimates using a moving time frame from 1 year to 10 years range from 5 to 40. Assuming that the dynamics among NYSE returns and S&P returns are approximately the same, this method would give, for our data frame of 4 years and $p = 488$ stocks, an equivalent number of factors of about $\hat{m}_H = 45$. In summary, the four methods lead to quite different estimates, $(\hat{m}_{PY}, \hat{m}_H, \hat{m}, \hat{m}_{KN}) = (20, 45, 72, 95)$. These findings confirm a widely believed fact that if the dynamic of stock returns is indeed captured by a factor model, the number of these factors cannot be very small.

Meanwhile, it is reported in Bouchaud and Potters (2011); Li et al. (2013)

that, for return time series, the noise variance σ^2 cannot be constant over all stock returns even after normalization. We suspect this fact as one of the main reasons for the weakness of the results reported here. Further investigation on how to generalize the above discussed estimator and the estimation theory of this paper in a more complex noise structure would be highly valuable.

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A Technical proofs

Proof of (9)

The general theory of the m.l.e. for the factor model (1) in the classical setting has been developed in Anderson and Amemiya (1988) with in particular the following result.

Proposition 2. *Let $\Theta = (\theta_{ij})_{1 \leq i, j \leq p} = \Psi - \Lambda(\Lambda' \Psi^{-1} \Lambda)^{-1} \Lambda'$. If $(\theta_{ij}^2)_{1 \leq i, j \leq p}$ is nonsingular, if Λ and Ψ are identified by the condition that $\Lambda' \Psi \Lambda$ is diagonal and the diagonal elements are different and ordered, if $\mathbf{S}_n \rightarrow \Lambda \Lambda' + \Psi$ in probability and if $\sqrt{n}(\mathbf{S}_n - \Sigma)$ has a limiting distribution, then $\sqrt{n}(\widehat{\Lambda} - \Lambda)$ and $\sqrt{n}(\widehat{\Psi} - \Psi)$ have a limiting distribution. The covariance of $\sqrt{n}(\widehat{\Psi}_{ii} - \Psi_{ii})$ and $\sqrt{n}(\widehat{\Psi}_{jj} - \Psi_{jj})$ in the limiting distribution is $2\Psi_{ii}^2 \Psi_{jj}^2 \xi^{ij}$ ($1 \leq i, j \leq p$), where $(\xi^{ij}) = (\theta_{ij}^2)^{-1}$.*

To prove the CLT (9), by Proposition 2, we know that the inverse of the Fisher information matrix is $\mathcal{I}^{-1}(\psi_{11}, \dots, \psi_{pp}) = (2\psi_{ii}^2 \psi_{jj}^2 \xi^{ij})_{ij}$. We have to change the parametrization: in our case, we have $\psi_{11} = \dots = \psi_{pp}$. Let $g : \mathbb{R} \rightarrow \mathbb{R}^p$, $a \mapsto (a, \dots, a)$. The information matrix in this new parametrization becomes

$$\mathcal{I}(\sigma^2) = J' \mathcal{I}(g(\sigma^2)) J,$$

where J is the Jacobian matrix of g . As

$$\mathcal{I}(g(\sigma^2)) = \frac{1}{2\sigma^8} (\theta_{ij}^2)_{ij},$$

we have

$$\mathcal{I}(\sigma^2) = \frac{1}{2\sigma^8} \sum_{i,j=1}^p \theta_{ij}^2,$$

and

$$\begin{aligned} \Theta = (\theta_{ij})_{ij} &= \Psi - \Lambda(\Lambda' \Psi^{-1} \Lambda)^{-1} \Lambda' \\ &= \sigma^2 (\mathbf{I}_p - \Lambda(\Lambda' \Lambda)^{-1} \Lambda'). \end{aligned}$$

By hypothesis, we have $\Lambda' \Lambda = \text{diag}(d_1^2, \dots, d_m^2)$. Consider the Singular Value Decomposition of Λ , $\Lambda = \mathbf{U} \mathbf{D} \mathbf{V}$, where \mathbf{U} is a $p \times p$ matrix such that $\mathbf{U} \mathbf{U}' = \mathbf{I}_p$, \mathbf{V} is a $m \times m$ matrix such that $\mathbf{V}' \mathbf{V} = \mathbf{I}_m$, and \mathbf{D} is a $p \times m$ diagonal matrix with d_1, \dots, d_m as diagonal elements. As $\Lambda' \Lambda$ is diagonal, $\mathbf{V} = \mathbf{I}_m$, so $\Lambda = \mathbf{U} \mathbf{D}$. By elementary calculus, one can find that

$$\Lambda(\Lambda' \Lambda)^{-1} \Lambda' = \text{diag}(\underbrace{1, \dots, 1}_m, \underbrace{0, \dots, 0}_{p-m}),$$

so

$$\Theta = \sigma^2 \text{diag}(\underbrace{0, \dots, 0}_m, \underbrace{1, \dots, 1}_{p-m}).$$

Finally,

$$\mathcal{I}(\sigma^2) = \frac{1}{2\sigma^8} (p-m) \sigma^4 = \frac{p-m}{2\sigma^4},$$

and the asymptotic variance of $\hat{\sigma}^2$ is

$$s^2 = \mathcal{I}^{-1}(\sigma^2) = \frac{2\sigma^4}{p-m}.$$

Proof of (16)

By Proposition 1, for $g(x) = x$, by using the variable change $x = \sigma^2(1 + c - 2\sqrt{c} \cos \theta)$, $0 \leq \theta \leq \pi$, we have

$$\begin{aligned} m(g) &= \frac{g(a(c)) + g(b(c))}{4} - \frac{1}{2\pi} \int_{a(c)}^{b(c)} \frac{x}{\sqrt{4c\sigma^4 - (x - \sigma^2 - c\sigma^2)^2}} dx, \quad j = 1, \dots, k \\ &= \frac{\sigma^2(1 + c)}{2} - \frac{\sigma^2}{2\pi} \int_0^\pi (1 + c - 2\sqrt{c} \cos \theta) d\theta \\ &= 0. \end{aligned}$$

Proof of (17)

Let $\underline{s}(z)$ be the Stieltjes transform of $(1 - c)\mathbf{1}_{[0, \infty)} + cF_{c, \delta_1}$. One can show that

$$\underline{m}(z) = \frac{1}{\sigma^2} \underline{s} \left(\frac{z}{\sigma^2} \right).$$

Then, in Proposition 1, we have

$$v(f_j, f_l) = -\frac{1}{2\pi^2} \oint \oint \frac{f_j(\sigma^2 z_1) f_l(\sigma^2 z_2)}{(\underline{s}(z_1) - \underline{s}(z_2))^2} d\underline{s}(z_1) d\underline{s}(z_2), \quad j, l = 1, \dots, k. \quad (24)$$

For $g(x) = x$, we have

$$\begin{aligned} v(g) &= -\frac{1}{2\pi^2} \oint \oint \frac{g(\sigma^2 z_1) g(\sigma^2 z_2)}{(\underline{s}(z_1) - \underline{s}(z_2))^2} d\underline{s}(z_1) d\underline{s}(z_2) \\ &= -\frac{\sigma^4}{2\pi^2} \oint \oint \frac{z_1 z_2}{(\underline{s}(z_1) - \underline{s}(z_2))^2} d\underline{s}(z_1) d\underline{s}(z_2) \\ &= 2c\sigma^4, \end{aligned}$$

where $-\frac{1}{2\pi^2} \oint \oint \frac{z_1 z_2}{(\underline{s}(z_1) - \underline{s}(z_2))^2} d\underline{s}(z_1) d\underline{s}(z_2) = 2c$ is calculated in Bai et al. (2009) (it corresponds to $v(z_1, z_2)$, Section 5, proof of (3.4)).

Proof of (20)

By Proposition 1, for $\sigma^2 = 1$ and $g(x) = \log(x)$, by using the variable change $x = 1 + c - 2\sqrt{c} \cos \theta$, $0 \leq \theta \leq \pi$, we have

$$\begin{aligned}
 m(g) &= \frac{g(a(c)) + g(b(c))}{4} - \frac{1}{2\pi} \int_{a(c)}^{b(c)} \frac{x}{\sqrt{4c - (x - 1 - c)^2}} dx, \quad j = 1, \dots, k \\
 &= \frac{\log(1 - c)}{2} - \frac{1}{2\pi} \int_0^\pi \log(1 + c - 2\sqrt{c} \cos \theta) d\theta \\
 &= \frac{\log(1 - c)}{2} - \frac{1}{4\pi} \int_0^{2\pi} \log |1 - \sqrt{c}e^{i\theta}|^2 d\theta \\
 &= \frac{\log(1 - c)}{2},
 \end{aligned}$$

where $\int_0^{2\pi} \log |1 - \sqrt{c}e^{i\theta}|^2 d\theta = 0$ is calculated in Bai and Silverstein (2010).

Proof of (21)

By Proposition 1 and (24), for $\sigma^2 = 1$ and $g(x) = x$, we have

$$\begin{aligned}
 v(g) &= -\frac{1}{2\pi^2} \oint \oint \frac{g(z_1)g(z_2)}{(\underline{s}(z_1) - \underline{s}(z_2))^2} d\underline{s}(z_1) d\underline{s}(z_2) \\
 &= -\frac{1}{2\pi^2} \oint \oint \frac{\log(z_1) \log(z_2)}{(\underline{s}(z_1) - \underline{s}(z_2))^2} d\underline{s}(z_1) d\underline{s}(z_2) \\
 &= -2 \log(1 - c_n),
 \end{aligned}$$

where the last integral is calculated in Bai and Silverstein (2010).

Proof of (23)

F_{c_n, δ_1} is the Marčenko-Pastur distribution of index c_n . By using the variable change $x = 1 + c_n - 2\sqrt{c_n} \cos \theta$, $0 \leq \theta \leq \pi$, we have

$$\begin{aligned}
\int \log(x) dF_{c_n, \delta_1}(x) &= \int_{a(c_n)}^{b(c_n)} \frac{\log x}{2\pi x c_n} \sqrt{(b(c_n) - x)(x - a(c_n))} dx \\
&= \frac{1}{2\pi c_n} \int_0^\pi \frac{\log(1 + c_n - 2\sqrt{c_n} \cos \theta)}{1 + c_n - 2\sqrt{c_n} \cos \theta} 4c_n \sin^2 \theta d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{2 \sin^2 \theta}{1 + c_n - 2\sqrt{c_n} \cos \theta} \log |1 - \sqrt{c_n} e^{i\theta}|^2 d\theta \\
&= \frac{c_n - 1}{c_n} \log(1 - c_n) - 1,
\end{aligned}$$

where the last integral is calculated in Bai and Silverstein (2010).

Proof of (22)

In the normal case with $\sigma^2 = 1$, Zheng (2012) gives the following equivalent expression of (14):

$$v(f_j, f_l) = - \lim_{r \rightarrow 1^+} \frac{\kappa}{4\pi^2} \oint \oint_{|\xi_1|=|\xi_2|=1} f_j(|1 + h\xi_1|^2) f_l(|1 + h\xi_2|^2) \frac{1}{(\xi_1 - r\xi_2)^2} d\xi_1 d\xi_2,$$

where $\kappa = 2$ in the real case and $h = \sqrt{c}$ in our case. We take $f_j(x) = \log(x)$ and $f_l(x) = x$, so we need to calculate

$$v(\log(x), x) = - \lim_{r \rightarrow 1^+} \frac{1}{2\pi^2} \oint \oint_{|\xi_1|=|\xi_2|=1} |1 + \sqrt{c}\xi_2|^2 \frac{\log(|1 + \sqrt{c}\xi_1|^2)}{(\xi_1 - r\xi_2)^2} d\xi_1 d\xi_2.$$

We follow the calculations done in Zheng (2012): when $|\xi| = 1$, $|1 + \sqrt{c}\xi|^2 = (1 + \sqrt{c}\xi)(1 + \sqrt{c}\xi^{-1})$, so $\log(|1 + \sqrt{c}\xi|^2) = \frac{1}{2} (\log(1 + \sqrt{c}\xi)^2 + \log(1 + \sqrt{c}\xi^{-1})^2)$.

Consequently,

$$\begin{aligned}
\oint_{|\xi_1|=1} \frac{\log(|1 + \sqrt{c}\xi_1|^2)}{(\xi_1 - r\xi_2)^2} d\xi_1 &= \frac{1}{2} \oint_{|\xi_1|=1} \frac{\log(1 + \sqrt{c}\xi_1)^2}{(\xi_1 - r\xi_2)^2} d\xi_1 + \frac{1}{2} \oint_{|\xi_1|=1} \frac{\log(1 + \sqrt{c}\xi_1^{-1})^2}{(\xi_1 - r\xi_2)^2} d\xi_1 \\
&= \frac{1}{2} \oint_{|\xi_1|=1} \log(1 + \sqrt{c}\xi_1)^2 \left(\frac{1}{(\xi_1 - r\xi_2)^2} + \frac{1}{(1 - r\xi_1\xi_2)^2} \right) d\xi_1 \\
&= 0 + i\pi \left(\frac{1}{(r\xi_2)^2} \frac{2\sqrt{c}}{1 + \frac{\sqrt{c}}{r\xi_2}} \right) \\
&= 2i\pi \frac{\sqrt{c}}{r\xi_2(r\xi_2 + \sqrt{c})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
v(\log(x), x) &= \frac{1}{i\pi} \oint_{|\xi_2|=1} |1 + \sqrt{c}\xi_2|^2 \frac{\sqrt{c}}{\xi_2(\xi_2 + \sqrt{c})} d\xi_2 \\
&= \frac{1}{i\pi} \oint_{|\xi|=1} (1 + c + c(\xi + \xi^{-1})) \frac{\sqrt{c}}{\xi(\xi + \sqrt{c})} d\xi \\
&= \frac{1}{i\pi} \oint_{|\xi|=1} \left(\frac{\sqrt{c}(1+c)}{\xi(\xi + \sqrt{c})} + \frac{c}{\xi + \sqrt{c}} + \frac{c}{\xi^2(\xi + \sqrt{c})} \right) d\xi \\
&= 2(1 + c - (1 + c) + c + 1 - 1) \\
&= 2c.
\end{aligned}$$