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# EXPONENTIAL CONVERGENCE TO EQUILIBRIUM FOR THE HOMOGENEOUS LANDAU EQUATION WITH HARD POTENTIALS

KLEBER CARRAPATOSO

ABSTRACT. This paper deals with the long time behaviour of solutions to the spatially homogeneous Landau equation with hard potentials. We prove an exponential in time convergence towards the equilibrium with the optimal rate given by the spectral gap of the associated linearized operator. This result improves the polynomial in time convergence obtained by Desvillettes and Villani [5]. Our approach is based on new decay estimates for the semigroup generated by the linearized Landau operator in weighted (polynomial or stretched exponential)  $L^p$ -spaces, using a method developed by Gualdani, Mischler and Mouhot [7].

## 1. INTRODUCTION AND MAIN RESULTS

This work deals with the asymptotic behaviour of solutions to the spatially homogeneous Landau equation for hard potentials. It is well known that these solutions converge towards the Maxwellian equilibrium when time goes to infinity and we are interested in quantitative rates of convergence.

On the one hand, in the case of Maxwellian molecules, Villani [15] and Desvillettes-Villani [5] prove a linear functional inequality between the entropy and entropy dissipation by constructive methods, from which one deduces an exponential convergence (with quantitative rate) of the solution to the Landau equation towards the Maxwellian equilibrium in relative entropy, which in turn implies an exponential convergence in  $L^1$ -distance (thanks to the Csiszár-Kullback-Pinsker inequality). This kind of linear functional inequality entropy-entropy dissipation is known as Cercignani's Conjecture in Boltzmann and Landau theory, for more details and a review of results we refer to [3].

On the other hand, in the case of hard potentials, Desvillettes-Villani [5] proves a functional inequality for entropy-entropy dissipation that is not linear, from which one obtains a polynomial convergence of solutions towards the equilibrium, again in relative entropy, which implies the same type of convergence in  $L^1$ -distance.

Before going further on details of existing results and on the contributions of the present work, we shall introduce in a precise manner the problem addressed here. In kinetic theory, the Landau equation is a model in plasma physics that describes the evolution of the density in the phase space of all positions and velocities of particles. Assuming that the density function does not depend on the position, we obtain the *spatially homogeneous Landau equation* in the form

$$(1.1) \quad \begin{cases} \partial_t f &= Q(f, f) \\ f|_{t=0} &= f_0 \end{cases}$$

where  $f = f(t, v) \geq 0$  is the density of particles with velocity  $v$  at time  $t$ ,  $v \in \mathbb{R}^3$  and  $t \in \mathbb{R}^+$ . The Landau operator  $Q$  is a bilinear operator given by

$$(1.2) \quad Q(g, f) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) [g_* \partial_j f - f \partial_j g_*] dv_*,$$

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where here and below we shall use the convention of implicit summation over repeated indices and we use the shorthand  $g_* = g(v_*)$ ,  $\partial_j g_* = \partial_{v_{*j}} g(v_*)$ ,  $f = f(v)$  and  $\partial_j f = \partial_{v_j} f(v)$ .

The matrix  $a$  is nonnegative, symmetric and depends on the interaction between particles. If two particles interact with a potential proportional to  $1/r^s$ , where  $r$  denotes their distance,  $a$  is given by (see for instance [16])

$$(1.3) \quad a_{ij}(v) = |v|^{\gamma+2} \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right),$$

with  $\gamma = (s-4)/s$ . We usually call hard potentials if  $\gamma \in (0, 1]$ , Maxwellian molecules if  $\gamma = 0$ , soft potentials if  $\gamma \in (-3, 0)$  and Coulombian potential if  $\gamma = -3$ . Through this paper we shall only consider the case of hard potentials  $\gamma \in (0, 1]$ .

The Landau equation conserves mass, momentum and energy, indeed, at least formally, for any test function  $\varphi$  we have (see e.g. [14])

$$\int_{\mathbb{R}^3} Q(f, f) \varphi(v) dv = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v - v_*) f f_* \left( \frac{\partial_i f}{f} - \frac{\partial_i f_*}{f_*} \right) (\partial_j \varphi - \partial_j \varphi_*) dv dv_*$$

from which we deduce

$$(1.4) \quad \int Q(f, f) \varphi(v) = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2.$$

Moreover, the entropy  $H(f) = \int f \log f$  is nonincreasing, indeed, at least formally, since  $a_{ij}$  is nonnegative we have the following inequality for  $D(F)$  the entropy dissipation,

$$(1.5) \quad \begin{aligned} D(f) &:= -\frac{d}{dt} H(f) \\ &= \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f f_* a_{ij}(v - v_*) \left( \frac{\partial_i f}{f} - \frac{\partial_i f_*}{f_*} \right) \left( \frac{\partial_j f}{f} - \frac{\partial_j f_*}{f_*} \right) dv dv_* \geq 0. \end{aligned}$$

It follows that any equilibrium is a Maxwellian distribution

$$\mu_{\rho, u, T}(v) := \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}},$$

for some  $\rho > 0$ ,  $u \in \mathbb{R}^3$  and  $T > 0$ . This is the Landau version of the famous Boltzmann's  $H$ -theorem (for more details we refer to [5, 15] again), from which the solution  $f(t, \cdot)$  of the Landau equation is expected to converge towards the Maxwellian  $\mu_{\rho_f, u_f, T_f}$  when  $t \rightarrow +\infty$ , where  $\rho_f$  is the density of the gas,  $u_f$  the mean velocity and  $T_f$  the temperature, defined by

$$\rho_f = \int f(v), \quad u_f = \frac{1}{\rho} \int v f(v), \quad T_f = \frac{1}{3\rho} \int |v-u|^2 f(v),$$

and these quantities are defined by the initial datum  $f_0$  thanks to the conservation properties of the Landau operator (1.4).

We may only consider the case of initial datum  $f_0$  satisfying

$$(1.6) \quad \int_{\mathbb{R}^3} f_0(v) dv = 1, \quad \int_{\mathbb{R}^3} v f_0(v) dv = 0, \quad \int_{\mathbb{R}^3} |v|^2 f_0(v) dv = 3,$$

the general case being reduced to (1.6) by a simple change of coordinates (see [5]). Then, we shall denote  $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$  the standard Gaussian distribution in  $\mathbb{R}^3$ , which corresponds to the Maxwellian with  $\rho = 1$ ,  $u = 0$  and  $T = 1$ , i.e. the Maxwellian with same mass, momentum and energy of  $F_0$  (1.6).

We linearize the Landau equation around  $\mu$ , with the perturbation

$$f = \mu + h,$$

hence the equation satisfied by  $h = h(t, v)$  takes the form

$$(1.7) \quad \partial_t h = \mathcal{L}h + Q(h, h),$$

with initial datum  $h_0$  defined by  $h_0 = f_0 - \mu$ , and where the linearized Landau operator  $\mathcal{L}$  is given by

$$(1.8) \quad \mathcal{L}h = Q(\mu, h) + Q(h, \mu).$$

Furthermore, from the conservations properties (1.4), we observe that the null space of  $\mathcal{L}$  has dimension 5 and is given by (see e.g. [2, 8, 1, 11, 13])

$$(1.9) \quad \mathcal{N}(\mathcal{L}) = \text{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}.$$

**1.1. Known results.** We present here existing results concerning spectral gap estimates for the linearized operator and convergence to equilibrium for the nonlinear equation.

For any weight function  $m = m(v)$  ( $m : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ ) we define the weighted Lebesgue space  $L^p(m)$ , for  $p \in [1, +\infty]$ , associated to the norm

$$\|f\|_{L^p(m)} := \|mf\|_{L^p},$$

and the weighted Sobolev spaces  $W^{s,p}(m)$  for  $s \in \mathbb{N}$ , associated to the norm

$$\|f\|_{W^{s,p}(m)} := \left( \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^p(m)}^p \right)^{1/p}, \quad \text{if } p \in [1, +\infty),$$

$$\|f\|_{W^{s,\infty}(m)} := \sup_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^\infty(m)}.$$

We denote by  $\mathcal{D}$  the Dirichlet form associated to  $-\mathcal{L}$  on  $L^2_\gamma \mu^{-1/2}$ ,

$$\mathcal{D}(h) := \langle -\mathcal{L}h, h \rangle_{L^2(\mu^{-1/2})} := \int (-\mathcal{L}h)h\mu^{-1},$$

and we say that  $h \in \mathcal{N}(\mathcal{L})^\perp$ , where  $\mathcal{N}(\mathcal{L})$  denotes the nullspace of  $\mathcal{L}$ , if  $f$  is of the form  $h = h - \Pi h$ , where  $\Pi$  denotes the projection onto the null space.

We can now state the existing results on the spectral gap of  $\mathcal{L}$ . The spectral gap inequality for the linearized Landau operator for hard potentials  $\gamma \in (0, 1]$ ,

$$(1.10) \quad \mathcal{D}(h) \geq \lambda_0 \|h\|_{L^2(\mu^{-1/2})}^2, \quad \forall h \in \mathcal{N}(\mathcal{L})^\perp,$$

was proven by Baranger-Mouhot [1], for some constructive constant  $\lambda_0 > 0$ .

In the case of hard and soft potentials  $\gamma \in (-3, 1]$ , Mouhot [11] proved the following result

$$(1.11) \quad \mathcal{D}(h) \geq \lambda_0 \left\{ \|h\|_{H^1(\langle v \rangle^{\gamma/2} \mu^{-1/2})}^2 + \|h\|_{L^2(\langle v \rangle^{(\gamma+2)/2} \mu^{-1/2})}^2 \right\}, \quad \forall h \in \mathcal{N}(\mathcal{L})^\perp.$$

Furthermore, Guo [8], by nonconstructive arguments, and later Mouhot-Strain [13], by constructive arguments, proved a spectral gap inequality for an anisotropic norm for the linearized Landau operator (in all cases: hard, soft and Coulombien potentials)  $\gamma \in [-3, 1]$ ,

$$(1.12) \quad \mathcal{D}(h) \geq \lambda_0 \|h\|_*^2, \quad \forall h \in \mathcal{N}(\mathcal{L})^\perp,$$

with the anisotropic norm  $\|\cdot\|_*$  defined by

$$\|h\|_*^2 := \|\langle v \rangle^{\gamma/2} P_v \nabla h\|_{L^2(\mu^{-1/2})}^2 + \|\langle v \rangle^{(\gamma+2)/2} (I - P_v) \nabla h\|_{L^2(\mu^{-1/2})}^2 + \|\langle v \rangle^{(\gamma+2)/2} h\|_{L^2(\mu^{-1/2})}^2$$

where  $P_v$  denotes the projection onto the  $v$ -direction, more precisely  $P_v g = \left( \frac{v}{|v|} \cdot g \right) \frac{v}{|v|}$ . We also have from [8], the reverse inequality

$$(1.13) \quad \mathcal{D}(h) \leq C_2 \|h\|_*^2, \quad \forall h \in \mathcal{N}(\mathcal{L})^\perp,$$

which, together with (1.12), imply a spectral gap for  $\mathcal{L}$  in  $L^2(\mu^{-1/2})$  if and only if  $\gamma + 2 \geq 0$ .

Summarizing the results (1.10), (1.11) and (1.12), for  $\gamma \in [0, 1]$  there is a constructive constant  $\lambda_0 > 0$  (spectral gap) such that

$$(1.14) \quad \forall t \geq 0, \forall h \in L^2(\mu^{-1/2}), \quad \|e^{t\mathcal{L}}h - \Pi h\|_{L^2(\mu^{-1/2})} \leq e^{-\lambda_0 t} \|h - \Pi h\|_{L^2(\mu^{-1/2})},$$

where  $\Pi$  denotes the projection onto  $\mathcal{N}(\mathcal{L})$ , the null space of  $\mathcal{L}$  given by (1.9). Hence we have an exponential decay to equilibrium for the linearized equation  $\partial_t h = \mathcal{L}h$  on the space  $L^2(\mu^{-1/2})$ .

Another approach is to study directly the nonlinear equation, establishing functional inequalities between the entropy and entropy dissipation functionals. The following entropy dissipation inequality for the (nonlinear) Landau operator for Maxwellian molecules  $\gamma = 0$

$$(1.15) \quad D(f) \geq \delta_0 H(f|\mu), \quad \forall f \in L^1_{1,0,1}(\mathbb{R}^3) := \{f \in L^1(\mathbb{R}^3); \rho_f = 1, u_f = 0, T_f = 1\};$$

for some explicit constant  $\delta_0$ , was proven by Desvillettes-Villani [5] and Villani [15]. Here  $H(f|\mu) := \int f \log(f/\mu)$  denotes the relative entropy of  $f$  with respect to  $\mu$ , and this inequality implies an exponential decay to the equilibrium  $\mu$ . Taking  $f = \mu + \varepsilon h$ , they also deduce a degenerated spectral gap inequality for the linearized Landau operator for  $\gamma = 0$ ,

$$(1.16) \quad \mathcal{D}(h) \geq \bar{\delta}_0 \|\nabla h\|_{L^2(\mu^{-1/2})}^2 \quad \forall h \in \mathcal{N}(\mathcal{L})^\perp.$$

In the case of hard potentials  $\gamma \in (0, 1]$ , Desvillettes-Villani [5] proved the following entropy-entropy dissipation inequality, for some explicit  $\delta_1, \delta_2 > 0$ ,

$$(1.17) \quad D(f) \geq \min \left\{ \delta_1 H(f|\mu), \delta_2 H(f|\mu)^{1+\gamma/2} \right\} \quad \forall f \in L^1_{1,0,1}(\mathbb{R}^3),$$

which implies a polynomial decay to equilibrium in relative entropy (see theorem 3.2 for more details).

As we can see above, the result (1.17) tell us that the solution to the Landau equation converges to the equilibrium in polynomial time. Furthermore, from the exponential decay for the linearized equation (1.14), we might expect that the solution to the nonlinear equation also decays exponentially in time if it lies in some neighborhood of the equilibrium in which the linear part is dominant. One could then expect to prove an exponential convergence to equilibrium combining these two results: for small times one uses the polynomial decay, then for large times, when the solution enters in the appropriated neighborhood of the equilibrium (in  $L^2(\mu^{-1/2})$ -norm), one uses the exponential decay. However these two theories, linear and nonlinear, are not compatible in the sense that the spectral gap for the linearized operator holds in  $L^2(\mu^{-1/2})$  and the Cauchy theory [4] for the nonlinear Landau equation is constructed in  $L^1$ -spaces with polynomial weight, which means that in order to apply the strategy above, starting from some initial datum in weighted  $L^1$ -space, one would need the appearance of the  $L^2(\mu^{-1/2})$ -norm of the solution in positive time to be able to use (1.14), and this is not known to be true (one does not know even if the  $L^2(\mu^{-1/2})$ -norm is propagated). Hence, in order to be able to "connect" the linearized theory with the nonlinear one, we need to enlarge the functional space of semigroup decay estimates generated by the linearized operator  $\mathcal{L}$ .

Our goal in this paper is to prove an *(optimal) exponential in time convergence* of solutions to the Landau equation towards the equilibrium and our strategy is based on:

- (1) New decay estimates for the semigroup generated by the linearized Landau operator  $\mathcal{L}$  in various  $L^p$ -spaces with polynomial and stretched exponential weight, using a method developed in [7].
- (2) The well-known Cauchy theory for the nonlinear equation developed in [4, 5]: the appearance and uniform propagation of  $L^1$ -moments, smoothing effect and the polynomial in time convergence to equilibrium.

- (3) The strategy of connecting the linearized theory with the nonlinear one, roughly presented in the above paragraph.

**1.2. Statement of the main result.** Let us state our main result, which proves a sharp exponential decay to equilibrium for the spatially homogeneous Landau equation with hard potentials.

First of all we define the notion of weak solutions that we shall use.

**Definition 1.1** (Weak solutions [4]). Let  $\gamma \in (0, 1]$  and consider a nonnegative initial data  $f_0 \in L^1_2$ . We say that  $f$  is a weak solution of the Cauchy problem (1.1) if the following conditions are fulfilled:

- (i)  $f \geq 0$ ,  $f \in C([0, \infty); \mathcal{D}') \cap L^\infty([0, \infty); L^1_2) \cap L^1_{loc}([0, \infty); L^1_{2+\gamma})$ ;
- (ii) for any  $t \geq 0$

$$\int f(t)|v|^2 \leq \int f_0|v|^2$$

- (iii)  $f$  verifies (1.1) in the distributional sense: for any  $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^3_v)$ , for any  $t \geq 0$ ,

$$\int f(t)\varphi(t) - \int f_0\varphi(0) - \int_0^t \int f(\tau)\partial_t\varphi(\tau) = \int_0^t \int Q(f(\tau), f(\tau))\varphi(\tau),$$

where the last integral in the right-hand side is defined by

$$\int Q(f, f)\varphi = \frac{1}{2} \iint a_{ij}(v - v_*)(\partial_{ij}\varphi + \partial_{ij}\varphi_*) f_* f + \iint b_i(v - v_*)(\partial_i\varphi - \partial_i\varphi_*) f_* f$$

It is proven in [4] that if  $f_0 \in L^1_{2+\delta}$  for some  $\delta > 0$ , then there exists a global weak solution.

Our main theorem reads:

**Theorem 1.2** (Exponential decay to equilibrium). *Let  $\gamma \in (0, 1]$  and a nonnegative  $f_0 \in L^1(\langle v \rangle^{2+\delta})$  for some  $\delta > 0$ , satisfying (1.6). Then, for any weak solution  $(f_t)_{t \geq 0}$  to the spatially homogeneous Landau equation (1.1) with initial datum  $f_0$ , there exists a constant  $C > 0$  such that*

$$\forall t \geq 0, \quad \|f_t - \mu\|_{L^1} \leq Ce^{-\lambda_0 t},$$

where  $\lambda_0 > 0$  is the spectral gap (1.14) of the linearized operator  $\mathcal{L}$  on  $L^2(\mu^{-1/2})$ .

As mentioned above, in the case of hard potentials  $\gamma \in (0, 1]$ , a polynomial decay to equilibrium was proven by Desvillettes and Villani [5] and in the case of maxwellian molecules  $\gamma = 0$  an exponential decay to equilibrium was proven by Villani [15] and also by Desvillettes and Villani [5]. The proof of theorem 1.2 relies on coupling the polynomial decay from [5] for small times and the exponential decay for the linearized operator in weighted  $L^p$ -spaces from theorem 2.2 for large times, when the linearized dynamics is dominant. This method was first used by Mouhot [12] where is proved the exponential decay to equilibrium for the spatially homogeneous Boltzmann equation for hard potentials with cut-off. Later, the same approach was used by Gualdani, Mischler and Mouhot [7] to prove the exponential decay to the equilibrium for the inhomogeneous Boltzmann equation for hard spheres on the torus, and also by Mischler and Mouhot [9] for Fokker-Planck equations.

**1.3. Organization of the paper.** We start Section 2 presenting some properties of the linearized equation and then we state and prove the "spectral gap/semigroup decay" extension theorem (Theorem 2.13), which is a key ingredient of the proof of the main theorem. Finally, in Section 3, we prove estimates for the (nonlinear) Landau operator and then prove theorem 1.2.

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## 2. THE LINEARIZED EQUATION

We define (see e.g. [4, 14, 15]) in 3-dimension the following quantities

$$(2.1) \quad b_i(z) = \partial_j a_{ij}(z) = -2|z|^\gamma z_i, \quad c(z) = \partial_{ij} a_{ij}(z) = -2(\gamma + 3)|z|^\gamma.$$

Hence, we can rewrite the Landau operator (1.2) in the following way

$$(2.2) \quad Q(g, f) = (a_{ij} * g) \partial_{ij} f - (c * g) f = \partial_i [(a_{ij} * g) \partial_j f - (b_i * g) f].$$

We also denote

$$(2.3) \quad \bar{a}_{ij}(v) = a_{ij} * \mu, \quad \bar{b}_i(v) = b_i * \mu, \quad \bar{c}(v) = c * \mu.$$

Furthermore we have the following results concerning  $\bar{a}_{ij}(v)$ .

**Lemma 2.1.** *The following properties hold:*

- (a) *The matrix  $\bar{a}(v)$  has a simple eigenvalue  $\ell_1(v) > 0$  associated with the eigenvector  $v$  and a double eigenvalue  $\ell_2(v) > 0$  associated with the eigenspace  $v^\perp$ . Moreover,*

$$\begin{aligned} \ell_1(v) &= \int_{\mathbb{R}^3} \left( 1 - \left( \frac{v}{|v|} \cdot \frac{w}{|w|} \right)^2 \right) |w|^{\gamma+2} \mu(v-w) dw \\ \ell_2(v) &= \int_{\mathbb{R}^3} \left( 1 - \frac{1}{2} \left| \frac{v}{|v|} \times \frac{w}{|w|} \right|^2 \right) |w|^{\gamma+2} \mu(v-w) dw. \end{aligned}$$

When  $|v| \rightarrow +\infty$  we have

$$\begin{aligned} \ell_1(v) &\sim 2|v|^\gamma \\ \ell_2(v) &\sim |v|^{\gamma+2}. \end{aligned}$$

If  $\gamma \in (0, 1]$  there exists  $\ell_0 > 0$  such that, for all  $v \in \mathbb{R}^3$ ,  $\min\{\ell_1(v), \ell_2(v)\} \geq \ell_0$ .

- (b) *The function  $\bar{a}_{ij}$  is smooth, for any multi-index  $\beta \in \mathbb{N}^3$*

$$|\partial^\beta \bar{a}_{ij}(v)| \leq C_\beta \langle v \rangle^{\gamma+2-|\beta|}$$

and

$$\begin{aligned} \bar{a}_{ij}(v) \xi_i \xi_j &= \ell_1(v) |P_v \xi|^2 + \ell_2(v) |(I - P_v) \xi|^2, \\ \bar{a}_{ij}(v) v_i v_j &= \ell_1(v) |v|^2, \end{aligned}$$

where  $P_v$  is the projection on  $v$ , i.e.

$$P_v \xi = \left( \xi \cdot \frac{v}{|v|} \right) \frac{v}{|v|}.$$

- (c) *We have*

$$\bar{a}_{ii}(v) = 2 \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} \mu(v_*) dv_* \quad \text{and} \quad \bar{b}_i(v) = -\ell_1(v) v_i.$$

*Proof.* We just give the proof of item (c) since (a) comes from [2, Propositions 2.3 and 2.4, Corollary 2.5] and (b) is [8, Lemma 3].

Hence, for item (c) we write

$$\bar{a}_{ii}(v) = \sum_{i=1}^3 \int_{\mathbb{R}^3} a_{ii}(v - v_*) \mu(v_*) dv_*.$$

Using (1.3) we obtain that

$$a_{ii}(z) = \sum_{i=1}^3 |z|^{\gamma+2} \left( 1 - \frac{z_i^2}{|z|^2} \right) = 2|z|^{\gamma+2}$$

and then

$$\bar{a}_{ii}(v) = 2 \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} \mu(v_*) dv_*.$$

Moreover, we compute

$$\bar{b}_i(v) = (\partial_j a_{ij} * \mu)(v) = (a_{ij} * \partial_j \mu)(v) = - \int_{\mathbb{R}^3} a_{ij}(v - v_*) v_{*j} \mu(v_*) dv_*,$$

and using that  $a_{ij}(z)z_j = 0$  we obtain

$$\begin{aligned} \bar{b}_i(v) &= - \int_{\mathbb{R}^3} a_{ij}(v - v_*) v_{*j} \mu(v_*) dv_* \\ &= - \int_{\mathbb{R}^3} a_{ij}(v_*) (v_j - v_{*j}) \mu(v - v_*) dv_* \\ &= - \left( \int_{\mathbb{R}^3} a_{ij}(v_*) \mu(v - v_*) dv_* \right) v_j = -\bar{a}_{ij}(v) v_j = -\ell_1(v) v_i. \end{aligned}$$

□

Using the form (2.2) of the operator  $Q$ , we decompose the linearized Landau operator  $\mathcal{L}$  defined in (1.8) as  $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$ , where we define

$$(2.4) \quad \begin{aligned} \mathcal{A}_0 f &:= Q(f, \mu) = (a_{ij} * f) \partial_{ij} \mu - (c * f) \mu, \\ \mathcal{B}_0 f &:= Q(\mu, f) = (a_{ij} * \mu) \partial_{ij} f - (c * \mu) f. \end{aligned}$$

Consider a smooth nonnegative function  $\chi \in C_c^\infty(\mathbb{R}^3)$  such that  $0 \leq \chi(v) \leq 1$ ,  $\chi(v) \equiv 1$  for  $|v| \leq 1$  and  $\chi(v) \equiv 0$  for  $|v| > 2$ . For any  $R \geq 1$  we define  $\chi_R(v) := \chi(R^{-1}v)$  and in the sequel we shall consider the function  $M\chi_R$ , for some constant  $M > 0$ . Then, we make the final decomposition of the operator  $\mathcal{L}$  as  $\mathcal{L} = \mathcal{A} + \mathcal{B}$  with

$$(2.5) \quad \mathcal{A} := \mathcal{A}_0 + M\chi_R, \quad \mathcal{B} := \mathcal{B}_0 - M\chi_R,$$

where  $M$  and  $R$  will be chosen later (see lemma 2.6).

Let us now make our assumptions on the weight functions  $m = m(v)$ . We define the polynomial weight, for all  $p \in [1, +\infty)$ ,

$$(2.6) \quad m = \langle v \rangle^k, \quad \text{with } k > \max\{\gamma(1 - 1/p), \gamma + 2\} + 3(1 - 1/p)$$

and the abscissa

$$(2.7) \quad \begin{aligned} a_{m,p} &:= 2[(\gamma + 3)(1 - 1/p) - k], & \text{if } \gamma = 0, \\ a_{m,p} &:= -\infty, & \text{if } \gamma \in (0, 1]. \end{aligned}$$

Moreover, we define the exponential weight, for  $p \in [1, +\infty)$ ,

$$(2.8) \quad m = \exp(r\langle v \rangle^s), \quad \text{with } \begin{cases} r > 0, & \text{if } s \in (0, 2), \\ 0 < r < \frac{1}{2p}, & \text{if } s = 2, \end{cases}$$

and we define the abscissa, for all cases,

$$(2.9) \quad a_{m,p} := -\infty.$$

We are able now to state the following result that extends to various weighted  $L^p$ -spaces the exponential decay for the semigroup associated to  $\mathcal{L}$  known to hold on  $L^2(\mu^{-1/2})$  as presented in (1.14).



**Theorem 2.2.** *Let  $\gamma \in [0, 1]$ ,  $p \in [1, 2]$ , a weight function  $m = m(v)$  satisfying (2.6) or (2.8) and their respective abscissa  $a_{m,p}$  given by (2.7) or (2.9). Consider the linearized Landau operator (1.8), then for any positive  $\lambda \leq \min\{\lambda_0, |a_{m,p}| - 0\}$  there exists  $C_\lambda > 0$  such that*

$$(2.10) \quad \forall t \geq 0, \forall h \in L^p(m), \quad \|e^{t\mathcal{L}}h - \Pi h\|_{L^p(m)} \leq C_\lambda e^{-\lambda t} \|h - \Pi h\|_{L^p(m)},$$

where  $\Pi$  is the projection onto the null space of  $\mathcal{L}$ , and  $\lambda_0 > 0$  is the spectral gap of  $\mathcal{L}$  in  $L^2(\mu^{-1/2})$  given by (1.14).

*Remark 2.3.* As we can see in the definition of  $a_{m,p}$  in (2.7) and (2.9), we conclude that:

- (1) If  $\gamma \in [0, 1]$  and  $m$  is the stretched exponential weight (2.8) or if  $\gamma \in (0, 1]$  and  $m$  is the polynomial weight (2.6), then  $\lambda = \lambda_0$  since  $a_{m,p} := -\infty$ .
- (2) If  $\gamma = 0$  and  $m = \langle v \rangle^k$  is the polynomial weight (2.6), then  $\lambda = \lambda_0$  if  $k$  is big enough such that  $a_{m,p} := 2[(\gamma+3)(1-1/p) - k] < -\lambda_0$ , otherwise  $\lambda = 2[k - (\gamma+3)(1-1/p)] - 0$ .

This theorem extends the exponential semigroup decay to weighted  $L^p$  spaces using a method developed by Gualdani, Mischler and Mouhot [7] (see theorem 2.13 below) for Boltzmann and Fokker-Planck equations (see also Mischler and Mouhot [9] for other results on Fokker-Planck equations).

**2.1. Hypodissipativity properties.** In this subsection we shall investigate the hypodissipativity in  $L^p(m)$  spaces of the operator  $\mathcal{B}$  defined in (2.5) in order to prove assumption (i) of theorem 2.13. Before proving the desired result in lemma 2.6, we give the following lemma that will be useful in the sequel.

**Lemma 2.4.** *Let  $J_\alpha(v) := \int_{\mathbb{R}^3} |v - w|^\alpha \mu(w) dw$ , for  $0 \leq \alpha \leq 3$ , and denote  $M_\alpha(\mu) := \int |v|^\alpha \mu$ . Then it holds:*

- (a)  $J_0(v) = 1$ .
- (b)  $J_\alpha(v) \leq |v|^\alpha + M_\alpha(\mu)$ , for  $0 < \alpha \leq 1$ .
- (c)  $J_\alpha(v) \leq |v|^\alpha + M_2(\mu)^{\alpha/2}$ , for  $1 < \alpha < 2$ .
- (d)  $J_2(v) = |v|^2 + M_2(\mu)$ .
- (e)  $J_\alpha(v) \leq |v|^\alpha + 10^{\alpha/4}|v|^{\alpha/2} + M_4(\mu)^{\alpha/4}$ , for  $2 < \alpha \leq 3$ .

*Remark 2.5.* As we will see in the proof of lemma 2.6, the important point here is that, for all  $0 \leq \alpha \leq 3$ , the dominant part of the upper bound of  $J_\alpha$  has coefficient 1.

*Proof.* Items (a) and (d) are evident. For (b) we see that  $|v - w|^\alpha \leq |v|^\alpha + |w|^\alpha$  and it implies  $J_\alpha(v) \leq |v|^\alpha + M_\alpha(\mu)$ . To prove item (c) we use  $\alpha/2 < 1$  and Jensen's inequality to write

$$J_\alpha(v) \leq \left( \int_{\mathbb{R}^3} |v - w|^2 \mu(dw) \right)^{\alpha/2} = (|v|^2 + M_2(\mu))^{\alpha/2} \leq |v|^\alpha + M_2(\mu)^{\alpha/2}.$$

Finally, item (e) can be proven in the same way as (d). Firstly, for  $\alpha = 4$  explicit computation gives  $J_4(v) = |v|^4 + 10|v|^2 + M_4(\mu)$ . Then, from  $\alpha/4 < 1$  and Jensen's inequality we obtain

$$\begin{aligned} J_\alpha(v) &\leq \left( \int_{\mathbb{R}^3} |v - w|^4 \mu(dw) \right)^{\alpha/4} = (|v|^4 + 10|v|^2 + M_4(\mu))^{\alpha/4} \\ &\leq |v|^\alpha + 10^{\alpha/4}|v|^{\alpha/2} + M_4(\mu)^{\alpha/4}. \end{aligned}$$

□

With the help of the result above, we are able to state the hypodissipativity result for  $\mathcal{B}$ .

**Lemma 2.6.** *Let  $\gamma \in [0, 1]$ ,  $p \in [1, +\infty)$  and consider a weight function  $m = m(v)$  satisfying (2.6) or (2.8) with the corresponding definitions of the abscissa (2.7) or (2.9), respectively. Then, for any  $a > a_{m,p}$  we can choose  $M$  and  $R$  large enough such that the operator  $\mathcal{B} - a$  is dissipative in  $L^p(m)$ .*

*Proof.* We split the proof into four steps.

*Step 1.* Let us denote  $\Phi'(z) = |z|^{p-1} \text{sign}(z)$  and consider the equation

$$\partial_t f = \mathcal{B}f = \mathcal{B}_0 f - M\chi_R f.$$

For all  $1 \leq p < +\infty$ , we have

$$(2.11) \quad \begin{aligned} \frac{d}{dt} \|f\|_{L^p(m)} &= \|f\|_{L^p(m)}^{1-p} \left\{ \int (\mathcal{B}f) \Phi'(f) m^p \right\} \\ &= \|f\|_{L^p(m)}^{1-p} \left\{ \int (\mathcal{B}_0 f) \Phi'(f) m^p - \int (M\chi_R f) \Phi'(f) m^p \right\} \end{aligned}$$

with, from (2.4) and (2.2),

$$\int (\mathcal{B}_0 f) \Phi'(f) m^p = \int \bar{a}_{ij} \partial_{ij} f \Phi'(f) m^p - \int \bar{c} m^p |f|^p$$

Let us denote  $h = m^\theta f$ , for some  $\theta$  to be chosen later. For the first term, using  $\Phi'(f) = \Phi'(h) m^{-\theta(p-1)}$ , we have

$$\begin{aligned} T_1 &= \int \bar{a}_{ij} \partial_{ij} (hm^{-\theta}) \Phi'(h) m^{p+\theta(1-p)} \\ &= - \int \partial_j (hm^{-\theta}) \partial_i (\bar{a}_{ij} \Phi'(h) m^{p+\theta(1-p)}) \\ &= - \int \partial_j (hm^{-\theta}) \bar{a}_{ij} \partial_i (\Phi'(h) m^{p+\theta(1-p)}) - \int \partial_j (hm^{-\theta}) \bar{b}_j \Phi'(h) m^{p+\theta(1-p)} \\ &=: T_{11} + T_{12}. \end{aligned}$$

We also have

$$\begin{aligned} &\partial_j (hm^{-\theta}) \partial_i (\Phi'(h) m^{p+\theta(1-p)}) \\ &= (p-1) \partial_i h \partial_j h m^{p(1-\theta)} |h|^{p-2} + \frac{[p+\theta(1-p)]}{p} \partial_i m \partial_j (|h|^p) m^{p(1-\theta)-1} \\ &\quad - \frac{\theta(p-1)}{p} \partial_i (|h|^p) \partial_j m m^{p(1-\theta)-1} - \theta[p-\theta(p-1)] \partial_i m \partial_j m m^{p(1-\theta)-2} |h|^p, \end{aligned}$$

then, since  $\bar{a}_{ij}$  is symmetric, it follows

$$\begin{aligned} T_{11} &= -(p-1) \int \bar{a}_{ij} \partial_i h \partial_j h m^{p(1-\theta)} |h|^{p-2} \\ &\quad + \left[ 2\theta \frac{(p-1)}{p} - 1 \right] \int \bar{a}_{ij} \partial_i m \partial_j (|h|^p) m^{p(1-\theta)-1} \\ &\quad + \theta[p-\theta(p-1)] \int \bar{a}_{ij} \partial_i m \partial_j m m^{p(1-\theta)-2} |h|^p. \end{aligned}$$

Performing an integration by parts, we obtain

$$(2.12) \quad \begin{aligned} T_{11} &= -(p-1) \int \bar{a}_{ij} \partial_i h \partial_j h m^{p(1-\theta)} |h|^{p-2} \\ &\quad + \delta_1(p, \theta) \int \bar{b}_i \partial_i m m^{p(1-\theta)-1} |h|^p \\ &\quad + \delta_1(p, \theta) \int \bar{a}_{ij} \partial_{ij} m m^{p(1-\theta)-1} |h|^p \\ &\quad + \delta_2(p, \theta) \int \bar{a}_{ij} \partial_i m \partial_j m m^{p(1-\theta)-2} |h|^p \end{aligned}$$

where

$$(2.13) \quad \delta_1(p, \theta) := 1 - 2\theta(1 - 1/p), \quad \delta_2(p, \theta) := \delta_1(p, \theta)[p(1 - \theta) - 1] + \theta[p - \theta(p - 1)].$$

For th term  $T_{12}$  we have

$$(2.14) \quad \begin{aligned} T_{12} &= - \int \partial_j(hm^{-\theta})\bar{b}_j \Phi'(h)m^{p+\theta(1-p)} \\ &= - \int \partial_j h \Phi'(h)\bar{b}_j m^{p(1-\theta)} + \theta \int h \Phi'(h)\bar{b}_j \partial_j m m^{p(1-\theta)-1} \\ &= -\frac{1}{p} \int \partial_j(|h|^p)\bar{b}_j m^{p(1-\theta)} + \theta \int \bar{b}_j \partial_j m m^{p(1-\theta)-1} |h|^p \\ &= \frac{1}{p} \int \bar{c} m^{p(1-\theta)} |h|^p + \int \bar{b}_j \partial_j m m^{p(1-\theta)-1} |h|^p. \end{aligned}$$

Gathering (2.12) and (2.14) one obtains

$$(2.15) \quad \int (\mathcal{B}_0 f) \Phi'(g) m^p = -(p-1) \int \bar{a}_{ij} \partial_i(m^\theta f) \partial_j(m^\theta f) m^{p-2\theta} |f|^{p-2} + \int \varphi_{m,p,\theta}(v) m^p |f|^p,$$

with

$$(2.16) \quad \begin{aligned} \varphi_{m,p,\theta} &:= \delta_1(p, \theta) \left( \bar{a}_{ij} \frac{\partial_{ij} m}{m} \right) + \delta_2(p, \theta) \left( \bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} \right) \\ &\quad + (1 + \delta_1(p, \theta)) \left( \bar{b}_i \frac{\partial_i m}{m} \right) + \left( \frac{1}{p} - 1 \right) \bar{c}, \end{aligned}$$

where  $\delta_1$  and  $\delta_2$  are defined by (2.13).

Let us now split the proof into two steps, corresponding to the polynomial weight (2.6) and to the stretched exponential weight (2.8).

*Step 2. Polynomial weight.* Consider  $m = \langle v \rangle^k$  defined in (2.6). On the one hand, we have

$$\begin{aligned} \frac{\partial_i m}{m} &= k v_i \langle v \rangle^{-2}, & \frac{\partial_i m}{m} \frac{\partial_j m}{m} &= k^2 v_i v_j \langle v \rangle^{-4}, \\ \frac{\partial_{ij} m}{m} &= \delta_{ij} k \langle v \rangle^{-2} + k(k-2) v_i v_j \langle v \rangle^{-4}. \end{aligned}$$

Hence, from the definitions (2.1)-(2.3) and lemma 2.1 we obtain

$$(2.17) \quad \begin{aligned} \bar{a}_{ij} \frac{\partial_{ij} m}{m} &= (\delta_{ij} \bar{a}_{ij}) k \langle v \rangle^{-2} + (\bar{a}_{ij} v_i v_j) k(k-2) \langle v \rangle^{-4} \\ &= \bar{a}_{ii} k \langle v \rangle^{-2} + \ell_1(v) k(k-2) |v|^2 \langle v \rangle^{-4}, \end{aligned}$$

where we recall that the eigenvalue  $\ell_1(v) > 0$  is defined in lemma 2.1. Moreover, arguing exactly as above we obtain

$$(2.18) \quad \bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} = (\bar{a}_{ij} v_i v_j) k^2 \langle v \rangle^{-4} = \ell_1(v) k^2 |v|^2 \langle v \rangle^{-4}$$

and also, using the fact that  $\bar{b}_i(v) = -\ell_1(v) v_i$  from lemma 2.1,

$$(2.19) \quad \bar{b}_i \frac{\partial_i m}{m} = -\ell_1(v) v_i k v_i \langle v \rangle^{-2} = -\ell_1(v) k |v|^2 \langle v \rangle^{-2}.$$

On the other hand, from item (c) of lemma 2.1 and definitions (2.1)-(2.3) we obtain that

$$(2.20) \quad \bar{a}_{ii} = 2J_{\gamma+2}(v) \quad \text{and} \quad \bar{c} = -2(\gamma+3)J_\gamma(v),$$

where  $J_\alpha$  is defined in lemma 2.4. It follows from (2.16)–(2.20) that

$$(2.21) \quad \begin{aligned} \varphi_{m,p,\theta}(v) &= \delta_1(p,\theta) 2k J_{\gamma+2}(v) \langle v \rangle^{-2} + \delta_1(p,\theta) k(k-2) \ell_1(v) |v|^2 \langle v \rangle^{-4} \\ &\quad + \delta_2(p,\theta) k^2 \ell_1(v) |v|^2 \langle v \rangle^{-4} - [1 + \delta_1(p,\theta)] k \ell_1(v) |v|^2 \langle v \rangle^{-2} \\ &\quad + 2(\gamma+3) \left(1 - \frac{1}{p}\right) J_\gamma(v). \end{aligned}$$

Since  $\ell_1(v) \sim 2\langle v \rangle^\gamma$  and  $J_\alpha(v) \sim \langle v \rangle^\alpha$  when  $|v| \rightarrow +\infty$  by lemmas 2.1 and 2.4, the dominant terms in (2.21) are the first, fourth and fifth one, all of order  $\langle v \rangle^\gamma$ .

For  $p \in (1, +\infty)$  we choose  $\theta = p/[2(p-1)]$ , then  $\delta_1(p,\theta) = 0$ ,  $\delta_2(p,\theta) = p^2/[4(p-1)]$  and

$$\varphi_{m,p,\theta}(v) = \frac{p^2}{4(p-1)} k^2 \ell_1(v) |v|^2 \langle v \rangle^{-4} - k \ell_1(v) |v|^2 \langle v \rangle^{-2} + 2(\gamma+3) \left(1 - \frac{1}{p}\right) J_\gamma(v).$$

Using lemma 2.4 to bound  $J_\gamma$ , we obtain that  $\varphi_{m,p,\theta}(v) \leq \bar{\varphi}_{m,p,\theta}(v)$  where

$$\bar{\varphi}_{m,p,\theta}(v) \underset{|v| \rightarrow +\infty}{\sim} -2k \langle v \rangle^\gamma + 2(\gamma+3) \left(1 - \frac{1}{p}\right) \langle v \rangle^\gamma.$$

Hence it yields, since  $k > (\gamma+3)(1-1/p)$  from (2.6),

$$(2.22) \quad \begin{cases} \bar{\varphi}_{m,p,\theta}(v) \underset{|v| \rightarrow +\infty}{\sim} -2[k - (\gamma+3)(1-1/p)] & \xrightarrow{|v| \rightarrow +\infty} k_c < 0, & \text{if } \gamma = 0, \\ \bar{\varphi}_{m,p,\theta}(v) \underset{|v| \rightarrow +\infty}{\sim} -2[k - (\gamma+3)(1-1/p)] \langle v \rangle^\gamma & \xrightarrow{|v| \rightarrow +\infty} -\infty, & \text{if } \gamma \in (0, 1]. \end{cases}$$

If  $p = 1$ , for all  $\theta$ , we have  $\delta_1(1,\theta) = 1$  and  $\delta_2(1,\theta) = 0$  which gives

$$\varphi_{m,1,\theta}(v) = 2k J_{\gamma+2}(v) \langle v \rangle^{-2} + k(k-2) \lambda(v) |v|^2 \langle v \rangle^{-4} - 2k \ell_1(v) |v|^2 \langle v \rangle^{-2},$$

and the dominant terms are the first and last one, both of order  $\langle v \rangle^\gamma$ . Using lemma 2.4 to bound  $J_{\gamma+2}$ , we have  $\varphi_{m,1,\theta}(v) \leq \bar{\varphi}_{m,1,\theta}(v)$  where

$$\bar{\varphi}_{m,1,\theta}(v) \underset{|v| \rightarrow +\infty}{\sim} 2k \langle v \rangle^\gamma - 4k \langle v \rangle^\gamma = -2k \langle v \rangle^\gamma.$$

Then we obtain, since  $k > 0$  from (2.6),

$$(2.23) \quad \begin{cases} \bar{\varphi}_{m,1,\theta}(v) \underset{|v| \rightarrow +\infty}{\sim} -2k & \xrightarrow{|v| \rightarrow +\infty} -2k, & \text{if } \gamma = 0, \\ \bar{\varphi}_{m,1,\theta}(v) \underset{|v| \rightarrow +\infty}{\sim} -2k \langle v \rangle^\gamma & \xrightarrow{|v| \rightarrow +\infty} -\infty, & \text{if } \gamma \in (0, 1]. \end{cases}$$

*Step 3. Exponential weight.* We consider now  $m = \exp(r\langle v \rangle^s)$  given by (2.8). In this case we have

$$\begin{aligned} \frac{\partial_i m}{m} &= r s v_i \langle v \rangle^{s-2}, & \frac{\partial_i m}{m} \frac{\partial_j m}{m} &= r^2 s^2 v_i v_j \langle v \rangle^{2s-4}, \\ \frac{\partial_{ij} m}{m} &= r s \langle v \rangle^{s-2} \delta_{ij} + r s (s-2) v_i v_j \langle v \rangle^{s-4} + r^2 s^2 v_i v_j \langle v \rangle^{2s-4}. \end{aligned}$$

It follows from last equation that

$$(2.24) \quad \begin{aligned} \bar{a}_{ij} \frac{\partial_{ij} m}{m} &= (\delta_{ij} \bar{a}_{ij}) r s \langle v \rangle^{s-2} + (\bar{a}_{ij} v_i v_j) r s (s-2) \langle v \rangle^{s-4} + (\bar{a}_{ij} v_i v_j) r^2 s^2 \langle v \rangle^{2s-4} \\ &= \bar{a}_{ii} r s \langle v \rangle^{s-2} + \ell_1(v) r s (s-2) |v|^2 \langle v \rangle^{s-4} + \ell_1(v) r^2 s^2 |v|^2 \langle v \rangle^{2s-4}, \end{aligned}$$

where we used lemma 2.1,

$$(2.25) \quad \bar{a}_{ij} \frac{\partial_i m}{m} \frac{\partial_j m}{m} = (\bar{a}_{ij} v_i v_j) r^2 s^2 \langle v \rangle^{2s-4} = \ell_1(v) r^2 s^2 |v|^2 \langle v \rangle^{2s-4}$$

and also, using the fact that  $\bar{b}_i(v) = -\ell_1(v) v_i$ ,

$$(2.26) \quad \bar{b}_i \frac{\partial_i m}{m} = -\ell_1(v) v_i r s v_i \langle v \rangle^{s-2} = -\ell_1(v) r s |v|^2 \langle v \rangle^{s-2}.$$

Gathering together (2.16), (2.24), (2.25) and (2.26), and thanks to lemma 2.1, it yields

$$(2.27) \quad \begin{aligned} \varphi_{m,p,\theta}(v) &= \delta_1(p, \theta) 2rs J_{\gamma+2}(v) \langle v \rangle^{s-2} + \delta_1(p, \theta) rs(s-2) \ell_1(v) |v|^2 \langle v \rangle^{s-4} \\ &\quad + \delta_1(p, \theta) r^2 s^2 \ell_1(v) |v|^2 \langle v \rangle^{2s-4} + \delta_2(p, \theta) r^2 s^2 \ell_1(v) |v|^2 \langle v \rangle^{2s-4} \\ &\quad - [1 + \delta_1(p, \theta)] rs \ell_1(v) |v|^2 \langle v \rangle^{s-2} + 2(\gamma+3) \left(1 - \frac{1}{p}\right) J_\gamma(v) \end{aligned}$$

where we recall that  $J_\alpha$  is given in lemma 2.4.

Let us choose  $\theta = 0$  for all cases  $p \in [1, +\infty)$ . Then  $\delta_1(p, 0) = 1$ ,  $\delta_2(p, 0) = p - 1$  and

$$(2.28) \quad \begin{aligned} \varphi_{m,p,0}(v) &= 2rs J_{\gamma+2}(v) \langle v \rangle^{s-2} + rs(s-2) \ell_1(v) |v|^2 \langle v \rangle^{s-4} + pr^2 s^2 \ell_1(v) |v|^2 \langle v \rangle^{2s-4} \\ &\quad - 2rs \ell_1(v) |v|^2 \langle v \rangle^{s-2} + 2(\gamma+3) \left(1 - \frac{1}{p}\right) J_\gamma(v), \end{aligned}$$

and we recall that  $\ell_1(v) \sim 2\langle v \rangle^\gamma$  and  $J_\alpha(v) \sim \langle v \rangle^\alpha$  when  $|v| \rightarrow +\infty$  by lemmas 2.1 and 2.4.

If  $0 < s < 2$ , the dominant terms in (2.28) is the fourth one, of order  $\langle v \rangle^{\gamma+s}$ . Then we obtain the asymptotic behaviour

$$(2.29) \quad \varphi_{m,p,0}(v) \underset{|v| \rightarrow +\infty}{\approx} -4rs \langle v \rangle^{s+\gamma} \xrightarrow{|v| \rightarrow +\infty} -\infty$$

since  $s + \gamma > 0$ . If  $s = 2$ , the dominant terms in (2.28) are the first, third and fourth one, all of order  $\langle v \rangle^{\gamma+2}$ . Hence, using lemma 2.4 to bound  $J_{\gamma+2}$  and lemma 2.1, we have  $\varphi_{m,p,0}(v) \leq \bar{\varphi}_{m,p,0}(v)$  where the asymptotic behaviour of  $\bar{\varphi}$  is given by

$$(2.30) \quad \bar{\varphi}_{m,p,0}(v) \underset{|v| \rightarrow +\infty}{\approx} 4r(2pr-1) \langle v \rangle^{\gamma+2} \xrightarrow{|v| \rightarrow +\infty} -\infty$$

since  $r < 1/(2p)$  from (2.8).

*Remark 2.7.* We could also, for  $p \in (1, +\infty)$ , chose  $\theta = p/[2(p-1)]$  as we did for the polynomial weight. This would not change anything for  $0 < s < 2$ , however for the case  $s = 2$  we would obtain

$$\varphi_{m,p,\theta}(v) \underset{|v| \rightarrow +\infty}{\approx} \left( \frac{2p^2 r^2}{p-1} - 4r \right) \langle v \rangle^{\gamma+2}$$

which goes to  $-\infty$  when  $|v| \rightarrow +\infty$  if  $r < 2(p-1)/p^2$ , modifying then the conditions on  $r$  defined in (2.8). Using these two computations, a more general condition on  $r$  defined in (2.8) in the case  $s = 2$  would be  $r < \max\left\{\frac{1}{2p}, \frac{2(p-1)}{p^2}\right\}$ .

*Step 4.* Finally, gathering steps 1, 2 and 3, for any  $p \in [1, +\infty)$ , for any  $a > a_{m,p}$ , thanks to the asymptotic behaviour (2.22)-(2.23)-(2.29)-(2.30), we can choose  $M$  and  $R$  large enough such that  $\varphi_{m,p,\theta}(v) - M\chi_R(v) \leq a$  for all  $v \in \mathbb{R}^3$ . It follows that the operator  $\mathcal{B} - a = \mathcal{B}_0 - M\chi_R - a$  is dissipative in  $L^p(m)$  for all  $a > a_{m,p}$ , and we have, for all  $f \in L^p(m)$ ,

$$(2.31) \quad \|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^p(m)} \leq e^{at} \|f\|_{L^p(m)}.$$

Indeed, from (2.11) and (2.15) we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|f\|_{L^p(m)}^p &\leq -(p-1) \int \bar{a}_{ij} \partial_i(m^\theta f) \partial_j(m^\theta f) m^{p-2\theta} |f|^{p-2} + \int (\varphi_{m,p,\theta} - M\chi_R) m^p |f|^p \\ &\leq \int (\varphi_{m,p,\theta} - M\chi_R) m^p |f|^p \\ &\leq a \int m^p |f|^p \end{aligned}$$

which yields (2.31). □

**2.2. Regularization properties.** We are now interested in regularization properties of the operator  $\mathcal{A}$  and the iterated convolutions of  $\mathcal{AS}_\mathcal{B}$  to prove assumptions (ii) and (iii) of theorem 2.13. Let us recall the operator  $\mathcal{A}$  defined in (2.5),

$$\mathcal{A}g = \mathcal{A}_0g + M\chi_Rg = (a_{ij} * g)\partial_{ij}\mu - (c * g)\mu + M\chi_Rg,$$

for  $M$  and  $R$  large enough chosen before.

Thanks to the function  $\chi_R$ , for any  $q \in [1, +\infty)$ ,  $p \geq q$  and any weight function  $m_0$ , we have

$$(2.32) \quad \|M\chi_Rg\|_{L^q(m_0)} \leq C\|\chi_Rm_0m^{-1}\|_{L^{pq/(p-q)}}\|g\|_{L^p(m)} \leq C\|g\|_{L^p(m)},$$

from which we deduce that  $M\chi_R \in \mathcal{B}(L^p(m), L^q(m_0))$ .

Let us now focus on regularization estimates for the operator  $\mathcal{A}_0$ . First of all we give the following result, which will be useful in the sequel.

**Lemma 2.8.** *Let  $\gamma \in [0, 1]$  and  $\beta \in \mathbb{N}^3$  be a multindex such that  $|\beta| \leq 2$ . Then*

$$|\partial_\beta(a_{ij} * g)(v)| \lesssim \langle v \rangle^{\gamma+2} \|\partial_\beta g\|_{L^1(\langle v \rangle^{\gamma+2})} \quad \text{and} \quad |\partial_\beta(a_{ij} * g)(v)| \lesssim \langle v \rangle^{\gamma+2-|\beta|} \|g\|_{L^1(\langle v \rangle^{\gamma+2-|\beta|})}$$

*Proof.* First of all, we write  $\partial_\beta(a_{ij} * g) = a_{ij} * \partial_\beta g$  and then

$$|(a_{ij} * \partial_\beta g)(v)| \leq \int |a_{ij}(v - v_*)| |\partial_\beta g_*| dv_*.$$

For  $\gamma \in [0, 1]$  we have  $|a_{ij}(v - v_*)| \leq |v - v_*|^{\gamma+2} \leq C\langle v \rangle^{\gamma+2} \langle v_* \rangle^{\gamma+2}$ , which yields

$$|(a_{ij} * \partial_\beta g)(v)| \lesssim \langle v \rangle^{\gamma+2} \|\partial_\beta g\|_{L^1(\langle v \rangle^{\gamma+2})}.$$

Finally, writing  $\partial_\beta(a_{ij} * g) = \partial_\beta a_{ij} * g$  and using that

$$|\partial_\beta a_{ij}(v - v_*)| \lesssim |v - v_*|^{\gamma+2-|\beta|} \lesssim \langle v \rangle^{\gamma+2-|\beta|} \langle v_* \rangle^{\gamma+2-|\beta|}$$

from lemma 2.1 and because  $\gamma + 2 - |\beta| \geq 0$ , it follows

$$|(\partial_\beta a_{ij} * g)(v)| \lesssim \int \langle v \rangle^{\gamma+2-|\beta|} \langle v_* \rangle^{\gamma+2-|\beta|} |g_*| dv_* \lesssim \langle v \rangle^{\gamma+2-|\beta|} \|g\|_{L^1(\langle v \rangle^{\gamma+2-|\beta|})},$$

which finishes the proof.  $\square$

**Lemma 2.9.** *Let  $\gamma \in [0, 1]$  and  $p \in [1, +\infty]$ . Then we have*

$$(2.33) \quad \|\mathcal{A}_0g\|_{L^p(m)} \leq C_\mu (\|g\|_{L^1(\langle v \rangle^{\gamma+2})} + \|g\|_{L^1(\langle v \rangle^\gamma)}).$$

*As a consequence,  $\mathcal{A}_0 \in \mathcal{B}(L^p(m), L^1(\langle v \rangle^{\gamma+2}))$  and also  $\mathcal{A}_0 \in \mathcal{B}(L^p(m))$ .*

*Proof.* For the first inequality, we write

$$\|\mathcal{A}_0g\|_{L^p(m)} \leq \|(a_{ij} * g)\partial_{ij}\mu\|_{L^p(m)} + \|(c * g)\mu\|_{L^p(m)}.$$

For the first term, using lemma 2.8, we compute

$$\begin{aligned} \|(a_{ij} * g)\partial_{ij}\mu\|_{L^p(m)}^p &\leq C \|g\|_{L^1(\langle v \rangle^{\gamma+2})}^p \int \langle v \rangle^{(\gamma+2)p} |\partial_{ij}\mu(v)|^p m^p(v) dv \\ &\leq C_\mu \|g\|_{L^1(\langle v \rangle^{\gamma+2})}^p. \end{aligned}$$

Arguing in the same way, we also obtain

$$\begin{aligned} \|(c * g)\mu\|_{L^p(m)}^p &\leq C \|g\|_{L^1(\langle v \rangle^\gamma)}^p \int \langle v \rangle^{\gamma p} |\mu(v)|^p m^p(v) dv \\ &\leq C_\mu \|g\|_{L^1(\langle v \rangle^\gamma)}^p, \end{aligned}$$

which completes the proof of the first inequality of the lemma.

Then we compute, for some  $\sigma > 0$  and using Hölder's inequality,

$$\begin{aligned} \|g\|_{L^1(\langle v \rangle^{\gamma+2})} &\leq \left( \int \langle v \rangle^{-\sigma p/(p-1)} \right)^{(p-1)/p} \|g\|_{L^p(\langle v \rangle^{\gamma+2+\sigma})} \\ &\leq C \|g\|_{L^p(\langle v \rangle^{\gamma+2+\sigma})}, \end{aligned}$$

if  $\sigma > 3(1 - 1/p)$ . This implies that  $\|\mathcal{A}_0 g\|_{L^p(m)} \leq C_\mu \|g\|_{L^p(m)}$  since  $k > \gamma + 2 + 3(1 - 1/p)$  when  $m = \langle v \rangle^k$  satisfies (2.6) or  $m = e^{r\langle v \rangle^\sigma}$  satisfies (2.8).  $\square$

**Corollary 2.10.** *Let  $p \in [2, +\infty]$ . Then  $\mathcal{A} \in \mathcal{B}(L^p(m), L^2(\mu^{-1/2}))$  and for any  $a > a_{m,p}$  we have*

$$\|\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(L^p(m), L^2(\mu^{-1/2}))} \leq C_a e^{at}.$$

*Proof.* From lemma 2.9 and equation (2.32) it follows that  $\mathcal{A} \in \mathcal{B}(L^p(m), L^2(\mu^{-1/2}))$  for all  $p \in [2, +\infty]$ . Then we compute using lemma 2.6,

$$\|\mathcal{A}\mathcal{S}_\mathcal{B}(t)f\|_{L^2(\mu^{-1/2})} \leq \|\mathcal{A}\|_{\mathcal{B}(L^p(m), L^2(\mu^{-1/2}))} \|\mathcal{S}_\mathcal{B}(t)f\|_{L^p(m)} \leq C e^{at} \|f\|_{L^p(m)},$$

which concludes the proof.  $\square$

Let us denote  $m_0 = e^{r\langle v \rangle^2}$  with  $r \in (0, 1/4)$ , then  $L^2(\mu^{-1/2}) \subset L^q(m_0)$  for any  $1 \leq q \leq 2$ .

**Lemma 2.11.** *There exists  $C > 0$  such that for all  $1 \leq p < 2$ ,*

$$(2.34) \quad \|\mathcal{S}_\mathcal{B}(t)f\|_{L^2(m_0)} \leq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{at} \|f\|_{L^p(m_0)}, \quad \forall t \geq 0.$$

*As a consequence, for all  $1 \leq p < 2$  and  $m$  satisfying (2.6) or (2.8), for any  $a' > a$  we have*

$$(2.35) \quad \|(\mathcal{A}\mathcal{S}_\mathcal{B})^{*2}(t)f\|_{L^2(\mu^{-1/2})} \leq C e^{a't} \|f\|_{L^p(m)}, \quad \forall t \geq 0.$$

*Proof of Lemma 2.11.* Consider the equation  $\partial_t f = \mathcal{B}f$ . Then from (2.11) and (2.15) we have

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(m_0)}^2 = - \int \bar{a}_{ij} \partial_i(m_0 f) \partial_j(m_0 f) + \int (\varphi_{m_0, 2, 1} - M\chi_R) m_0^2 f^2$$

From lemma 2.1 there exists  $\ell_0 > 0$  such that  $\bar{a}_{ij} \xi_i \xi_j \geq \ell_0 |\xi|^2$ . We obtain

$$(2.36) \quad \frac{1}{2} \frac{d}{dt} \|f\|_{L^2(m_0)}^2 \leq -\ell_0 \int |\nabla(m_0 f)|^2 + \int (\varphi_{m_0, 2, 1} - M\chi_R) m_0^2 f^2$$

The weight function  $m_0$  satisfies (2.8), then Lemma 2.6 holds, more precisely

$$(2.37) \quad \|\mathcal{S}_\mathcal{B}(t)f\|_{L^p(m_0)} \leq e^{at} \|f\|_{L^p(m_0)}, \quad \forall t \geq 0.$$

Applying Nash's inequality in 3-dimension:  $\|g\|_{L^2}^2 \leq c_1 \|\nabla g\|_{L^2}^{6/5} \|g\|_{L^1}^{4/5}$  with  $g = m_0 f$  we obtain

$$c_1^{-1} \|m_0 f\|_{L^2}^{10/3} \|m_0 f\|_{L^1}^{-4/3} \leq \int |\nabla(m_0 f)|^2.$$

Putting together last inequality with (2.38), it follows

$$(2.38) \quad \frac{1}{2} \frac{d}{dt} \|f\|_{L^2(m_0)}^2 \leq -C \|f\|_{L^2(m_0)}^{10/3} \|f\|_{L^1(m_0)}^{-4/3} + a \|f\|_{L^2(m_0)}^2.$$

Let us denote  $x(t) := \|f(t)\|_{L^2(m_0)}^2$  and  $y(t) := \|f(t)\|_{L^1(m_0)}$ , then we have the following inequality  $\dot{x}(t) \leq -C_1 x(t)^{5/3} y(t)^{-4/3} + 2ax(t)$ . From (2.37) we have  $y(t) \leq y_0$  and then

$$\dot{x}(t) \leq -C_1 x(t)^{5/3} y_0^{-4/3} + 2ax(t).$$

If  $x_0 \leq C y_0$  by (2.37) we have  $x(t) \leq C e^{at} y_0$ . If  $x_0$  is such that  $x_0 > [C_1/4a] y_0$  then  $x(t) \leq C(y_0^{-4/3} t)^{-3/2}$ , and we obtain

$$\|\mathcal{S}_\mathcal{B}(t)f\|_{L^2(m_0)} \leq C t^{-\frac{3}{4}} e^{at} \|f\|_{L^1(m_0)}.$$

Using Riesz-Thorin interpolation theorem to  $\mathcal{S}_{\mathcal{B}}(t)$  which acts from  $L^2 \rightarrow L^2$  with estimate (2.37) and from  $L^1 \rightarrow L^2$  with the estimate above, we obtain (2.34).

Let us prove now (2.35). From lemma 2.9 and equation (2.32) we have the following estimates, for any  $p \in [1, +\infty]$ ,

$$(2.39) \quad \|\mathcal{A}g\|_{L^2(\mu^{-1/2})} \lesssim \|g\|_{L^2(m_0)}, \quad \|\mathcal{A}g\|_{L^p(m_0)} \lesssim \|g\|_{L^p(m)}.$$

Hence, by (2.39) and (2.34), for  $1 \leq p \leq 2$ , it follows

$$(2.40) \quad \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(\mu^{-1/2})} \lesssim \|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(m_0)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{at} \|f\|_{L^p(m_0)}.$$

Computing the convolution of  $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)$  we have

$$\begin{aligned} \|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)f\|_{L^2(\mu^{-1/2})} &\lesssim \int_0^t \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{S}_{\mathcal{B}}(s)f\|_{L^2(\mu^{-1/2})} ds \\ &\lesssim \int_0^t \|\mathcal{S}_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{S}_{\mathcal{B}}(s)f\|_{L^2(m_0)} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{a(t-s)} \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(s)f\|_{L^p(m_0)} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{a(t-s)} \|\mathcal{S}_{\mathcal{B}}(s)f\|_{L^p(m)} ds \\ &\lesssim \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})} e^{as} \|f\|_{L^p(m)} ds \\ &\lesssim t^{\frac{1}{2}(\frac{7}{2}-\frac{3}{p})} e^{at} \|f\|_{L^p(m)} \\ &\lesssim e^{a't} \|f\|_{L^p(m)}, \end{aligned}$$

where we have used successively (2.39), (2.34), (2.39), lemma 2.6 and the fact that  $(\frac{7}{2} - \frac{3}{p}) > 0$  for  $1 \leq p < 2$ . Hence for all  $t \geq 0$  we have  $\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)\|_{\mathcal{B}(L^p(m), L^2(\mu^{-1/2}))} \lesssim e^{a't}$ , for any  $a' > a$  (where  $a > a_{m,p}$  is fixed in lemma 2.6).  $\square$

**2.3. Abstract theorem.** We shall present in this subsection an abstract theorem from [7, 9], which will be used to prove theorem 2.2.

Let us introduce some notation before state the theorem. Consider two Banach spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$ . We denote by  $\mathcal{B}(X, Y)$  the space of bounded linear operators from  $X$  to  $Y$  and by  $\|\cdot\|_{\mathcal{B}(X, Y)}$  its operator norm. Moreover we write  $\mathcal{C}(X, Y)$  the space of closed unbounded linear operators from  $X$  to  $Y$  with dense domain. When  $X = Y$  we simply denote  $\mathcal{B}(X) = \mathcal{B}(X, X)$  and  $\mathcal{C}(X) = \mathcal{C}(X, X)$ .

Given a Banach space  $X$  and a operator  $\Lambda : X \rightarrow X$ , we denote  $\mathcal{S}_{\Lambda}(t)$  or  $e^{t\Lambda}$  the semigroup generated by  $\Lambda$ . We also denote  $\mathcal{N}(\Lambda)$  its null space,  $\text{dom}(\Lambda)$  its domain,  $\Sigma(\Lambda)$  its spectrum and  $\text{R}(\Lambda)$  its range. Recall that for any  $z$  in the resolvent set  $\rho(\Lambda) := \mathbb{C} \setminus \Sigma(\Lambda)$ , the operator  $\Lambda - z$  is invertible, moreover the resolvent operator  $(\Lambda - z)^{-1} \in \mathcal{B}(X)$  and its range equals  $\text{dom}(\Lambda)$ . An eigenvalue  $\xi \in \Sigma(\Lambda)$  is isolated if

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C}; |z - \xi| \leq r\} = \{\xi\} \quad \text{for some } r > 0.$$

Then for an isolated eigenvalue  $\xi$  we define the associated spectral projector  $\Pi_{\Lambda, \xi} \in \mathcal{B}(X)$  by

$$(2.41) \quad \Pi_{\Lambda, \xi} := -\frac{1}{2i\pi} \int_{|z-\xi|=r'} (\Lambda - z)^{-1} dz \quad \text{with } 0 < r' < r.$$

If moreover the algebraic eigenspace  $\text{R}(\Pi_{\mathcal{L}, \xi})$  is finite dimensional, we say that  $\xi$  is a discrete eigenvalue and write  $\xi \in \Sigma_d(\Lambda)$ .



**Definition 2.12.** Let  $X_1, X_2$  and  $X_3$  be Banach spaces and  $\mathcal{S}_1 \in L^1(\mathbb{R}_+, \mathcal{B}(X_1, X_2))$ ,  $\mathcal{S}_2 \in L^1(\mathbb{R}_+, \mathcal{B}(X_2, X_3))$ . We define the convolution  $\mathcal{S}_2 * \mathcal{S}_1 \in L^1(\mathbb{R}_+, \mathcal{B}(X_1, X_3))$  by

$$\forall t \geq 0, \quad \mathcal{S}_2 * \mathcal{S}_1(t) := \int_0^t \mathcal{S}_2(s) \mathcal{S}_1(t-s) ds.$$

If  $X_1 = X_2 = X_3$  and  $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$ , we define  $\mathcal{S}^1 = \mathcal{S}$  and  $\mathcal{S}^{*n} = \mathcal{S} * \mathcal{S}^{*(n-1)}$  for all  $n \geq 2$ .

We can now state the following theorem from [7, Theorem 2.13].

**Theorem 2.13.** Let  $E$  and  $\mathcal{E}$  be Banach spaces such that  $E \subset \mathcal{E}$  is dense with continuous embedding. Consider the operators  $L \in \mathcal{C}(E)$ ,  $\mathcal{L} \in \mathcal{C}(\mathcal{E})$  with  $L = \mathcal{L}|_E$  and  $a \in \mathbb{R}$ . Assume that :

(1)  $L$  generates a semigroup  $e^{tL}$  on  $E$ ,  $L - a$  is hypodissipative on  $R(\text{Id} - \Pi_{L,a})$  and

$$\Sigma(L) \cap \Delta_a := \{\xi_1, \dots, \xi_k\} \subset \Sigma_d(L) \quad (\text{dinstinct discrete eigenvalues})$$

(2) There are  $\mathcal{A}, \mathcal{B} \in \mathcal{C}(\mathcal{E})$  such that  $\mathcal{L} = \mathcal{A} + \mathcal{B}$ , with the corresponding restrictions  $A$  and  $B$  on  $E$ , some  $n \in \mathbb{N}^*$  and some constant  $C_a > 0$  such that

- (i)  $\mathcal{B} - a$  is hypodissipative on  $\mathcal{E}$ ;
- (ii)  $A \in \mathcal{B}(E)$  and  $\mathcal{A} \in \mathcal{B}(\mathcal{E})$ ;
- (iii) we have

$$\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*n}(t)\|_{\mathcal{B}(\mathcal{E}, E)} \leq C_a e^{at}.$$

Then  $\mathcal{L}$  is hypodissipative on  $\mathcal{E}$  and we have

$$(2.42) \quad \forall t \geq 0, \quad \left\| \mathcal{S}_{\mathcal{L}}(t) - \sum_{j=1}^k \mathcal{S}_{\mathcal{L}}(t) \Pi_{\mathcal{L}, \xi_j} \right\|_{\mathcal{B}(\mathcal{E})} \leq C'_a t^n e^{at},$$

where  $C'_a > 0$  is an explicit constant depending on the constants from the assumptions.

This theorem permits us to enlarge the space of the spectral estimates of a given operator. More precisely, the knowledge of the spectral information in the "small space" (1) permit us to extend this information to a "big space" (2.42). Hence, the strategy to prove theorem 2.2 is to consider  $E = L^2(\mu^{-1/2})$ , in which space the spectral gap is known to hold (assumption (1)), and prove assumptions (2i), (2ii) and (2iii) with  $\mathcal{E} = L^p(m)$  to obtain spectral gap estimates in  $\mathcal{E}$  applying theorem 2.13.

**2.4. Proof of theorem 2.2.** With the results of subsections 2.1 and 2.2 and theorem 2.13 we are able to prove the spectral gap for the linearized Landau operator.

Let  $E = L^2(\mu^{-1/2})$ , in which space we already know the spectral gap (1.14), and  $\mathcal{E} = L^p(m)$ , for any  $p \in [1, 2]$  and  $m$  satisfying (2.6) or (2.8). We consider the decomposition  $\mathcal{L} = \mathcal{A} + \mathcal{B}$  as in (2.5). For any  $a > a_{m,p}$ , the operator  $\mathcal{B} - a$  is hypodissipative in  $\mathcal{E}$  from lemma 2.6 (assumption (i) of theorem 2.13);  $\mathcal{A} \in \mathcal{B}(\mathcal{E})$  and  $A \in \mathcal{B}(E)$  from lemma 2.9 and equation (2.32) (assumption (ii) of theorem 2.13). Hence we only need to prove assumption (iii) to conclude applying theorem 2.13.

We split the proof in different cases.

*Case  $p = 2$ .* First of all, in this case we have  $E \subset \mathcal{E}$ . Moreover,  $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t) \in \mathcal{B}(\mathcal{E}, E)$  with exponential decay rate from corollary 2.10, which proves assumption (iii) with  $n = 1$ .

*Case  $p \in [1, 2)$ .* Here  $E \subset \mathcal{E}$  and from lemma 2.11 we have  $(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t) \in \mathcal{B}(\mathcal{E}, E)$  with exponential decay rate, which gives assumption (iii) with  $n = 2$ .

## 3. PROOF OF THE MAIN RESULT

Recall the Landau operator (2.2)

$$Q(g, h) = (a_{ij} * g) \partial_{ij} h - (c * g) h.$$

We shall prove some estimates for the nonlinear operator  $Q$  before proving the theorem 1.2.

**Proposition 3.1.** *Let  $\gamma \in [0, 1]$  and  $p \in [1, +\infty]$ . Then*

$$\|Q(g, h)\|_{L^p(m)} \lesssim \|g\|_{L^1(\langle v \rangle^{\gamma+2})} \|\partial_{ij} h\|_{L^p(m \langle v \rangle^{\gamma+2})} + \|g\|_{L^1(\langle v \rangle^\gamma)} \|h\|_{L^p(m \langle v \rangle^\gamma)}$$

*Proof.* We write

$$\|Q(g, h)\|_{L^p(m)} \leq \|(a_{ij} * g) \partial_{ij} h\|_{L^p(m)} + \|(c * g) h\|_{L^p(m)}.$$

Thanks to lemma 2.8

$$\|(a_{ij} * g) \partial_{ij} h\|_{L^p(m)} \lesssim \|g\|_{L^1(\langle v \rangle^{\gamma+2})} \|\partial_{ij} h\|_{L^p(m \langle v \rangle^{\gamma+2})}$$

Moreover, by lemma 2.8 one obtains, since  $c = \partial_{ij} a_{ij}$  and  $|(c * g)(v)| \leq C \langle v \rangle^\gamma \|g\|_{L^1(\langle v \rangle^\gamma)}$ ,

$$\|(c * g) h\|_{L^p(m)} \lesssim \|g\|_{L^1(\langle v \rangle^\gamma)} \|h\|_{L^p(m \langle v \rangle^\gamma)},$$

and the proof is complete.  $\square$

The proof of theorem 1.2 relies on known results by Desvillettes and Villani [4, 5] concerning the polynomial decay rate to equilibrium, together with the spectral estimates from theorem 2.2 and some estimates on the nonlinear operator from 3.1. We follow the strategy of [12].

Let us first summarize the results on the Cauchy theory for the Landau equation with hard potentials from [4, Theorems 3, 6 and 7] and [5, Theorem 8], with a improvement of [6] concerning the smoothness effect.

**Theorem 3.2.** *Consider  $\gamma \in (0, 1]$ .*

(1) *Let  $f_0 \in L^1(\langle v \rangle^{2+\delta})$  for some  $\delta > 0$  and consider a weak solution  $f$  to (1.1), then:*

(a) *for all  $t_0 > 0$ , all integer  $k > 0$  and all  $\theta > 0$ , there exists  $C_{t_0} > 0$  such that*

$$\sup_{t \geq t_0} \|f(t, \cdot)\|_{H^k(\langle v \rangle^\theta)} \leq C_{t_0}.$$

(b) *for all  $t_0 > 0$ ,  $f \in C^\infty([t_0, +\infty); \mathcal{S}(\mathbb{R}_v^3))$ .*

(2) *Let  $f$  be any weak solution of (1.1) with initial datum  $f_0 \in L^1(\langle v \rangle^2)$  satisfying the decay of energy, then for all  $t_0 > 0$  and all  $\theta > 0$ , there is a constant  $C_{t_0} > 0$  such that*

$$\sup_{t \geq t_0} \|f(t, \cdot)\|_{L^1(\langle v \rangle^\theta)} \leq C_{t_0}.$$

(3) *If  $f$  is a smooth solution of (1.1) (in the sense of (1) above), then for all  $t \geq 0$  there is  $C > 0$  such that*

$$H(f_t | \mu) := \int_{\mathbb{R}^3} f_t \log \frac{f_t}{\mu} dv \leq C(1+t)^{-2/\gamma}$$

**Corollary 3.3.** *For all  $t_0 > 0$  and all  $\ell > 0$ , there exists  $C_{t_0} > 0$  such that*

$$\forall t \geq t_0, \quad \|f_t - \mu\|_{L^1(\langle v \rangle^\ell)} \leq C_{t_0} (1+t)^{-\frac{1}{2\gamma}}.$$

*Proof.* Let us fixe some  $t_0 > 0$ . First of all, from theorem 3.2 and the Csiszár-Kullback-Pinsker inequality (see e.g. [17, Remark 22.12])

$$\|f - \mu\|_{L^1} \leq C \sqrt{H(F|\mu)},$$

we obtain

$$\forall t \geq 0, \quad \|f_t - \mu\|_{L^1} \leq C(1+t)^{-1/\gamma}.$$

Then, using the bounds of theorem 3.2 and Hölder's inequality we obtain

$$\forall t \geq t_0, \quad \|f_t - \mu\|_{L^1(\langle v \rangle^\ell)} \leq \|f_t - \mu\|_{L^1(\langle v \rangle^{2\ell})}^{1/2} \|f_t - \mu\|_{L^1}^{1/2} \leq C_{t_0} (1+t)^{-\frac{1}{2\gamma}}.$$

□

*Proof of Theorem 1.2.* Let  $f = \mu + h$  and consider the equation

$$(3.1) \quad \begin{cases} \partial_t h &= \mathcal{L}h + Q(h, h) \\ h|_{t=0} &= h_0. \end{cases}$$

Since  $f_0$  has same mass, momentum and energy than  $\mu$ , we have  $\Pi h_0 = 0$  and for all  $t \geq 0$ , thanks to the conservation of these quantities, we also have  $\Pi h_t = \Pi Q(h_t, h_t) = 0$ .

**Lemma 3.4.** *Consider  $m = \langle v \rangle^k$  satisfying (2.6). There exists  $\epsilon > 0$  such that, if the solution  $h$  of (3.1) satisfies*

$$\|h_0\|_{L^1(\langle v \rangle^k)} \leq \epsilon \quad \text{and} \quad \|h_t\|_{L^1(\langle v \rangle^\ell)} \leq \epsilon, \quad \forall t \geq 0,$$

with  $\ell := 2\gamma + 8 + k$ , and if

$$\forall t \geq 0, \quad \|h_t\|_{H^4(\langle v \rangle^\ell)} \leq C,$$

then there is  $C' > 0$  such that

$$\forall t \geq 0, \quad \|h_t\|_{L^1(\langle v \rangle^k)} \leq C' e^{-\lambda_0 t} \|h_0\|_{L^1(\langle v \rangle^k)},$$

where  $\lambda_0 > 0$  is the spectral gap in (1.14).

*Proof of lemma 3.4.* By Duhamel's formula for the solution of (3.1), we write,

$$h_t = e^{t\mathcal{L}} h_0 + \int_{t_0}^t e^{(t-s)\mathcal{L}} Q(h_s, h_s) ds.$$

Using theorem 2.2 (remark that we can take  $\lambda = \lambda_0$  in that theorem) and proposition 3.1, one deduces

$$\begin{aligned} \|h_t\|_{L^p(m)} &\leq \|e^{t\mathcal{L}} h_0\|_{L^p(m)} + \int_0^t \|e^{(t-s)\mathcal{L}} Q(h_s, h_s)\|_{L^p(m)} ds \\ &\leq C e^{-\lambda_0 t} \|h_0\|_{L^p(m)} + C \int_0^t e^{-\lambda_0(t-s)} \|Q(h_s, h_s)\|_{L^p(m)} ds \\ &\leq C e^{-\lambda_0 t} \|h_0\|_{L^p(m)} + C \int_0^t e^{-\lambda_0(t-s)} \left( \|h_s\|_{L^1(\langle v \rangle^\gamma)} \|h_s\|_{L^p(m\langle v \rangle^\gamma)} \right. \\ &\quad \left. + \|h_s\|_{L^1(\langle v \rangle^{\gamma+2})} \|\nabla^2 h_s\|_{L^p(m\langle v \rangle^{\gamma+2})} \right) ds. \end{aligned}$$

For  $L^p(m) = L^1(\langle v \rangle^k)$  we obtain

$$(3.2) \quad \|h_t\|_{L^1(\langle v \rangle^k)} \leq C e^{-\lambda_0 t} \|h_0\|_{L^1(\langle v \rangle^k)} + C \int_0^t e^{-\lambda_0(t-s)} \left( \|h_s\|_{L^1(\langle v \rangle^\gamma)} \|h_s\|_{L^1(\langle v \rangle^{\gamma+k})} \right. \\ \left. + \|h_s\|_{L^1(\langle v \rangle^{\gamma+2})} \|\nabla^2 h_s\|_{L^1(\langle v \rangle^{\gamma+2+k})} \right) ds.$$

We recall the following interpolation inequality from [10, Lemma B.1]

$$\|u\|_{W^{q,1}(\langle v \rangle^\alpha)} \leq C \|u\|_{W^{q_1,1}(\langle v \rangle^{\alpha_1})}^{1-\theta} \|u\|_{W^{q_2,1}(\langle v \rangle^{\alpha_2})}^\theta$$

with  $\theta \in (0, 1)$ ,  $\alpha \geq \alpha_1$  and  $q \geq q_1$ ,  $q = (1 - \theta)q_1 + \theta q_2$  and  $\alpha = (1 - \theta)\alpha_1 + \theta\alpha_2$  with  $q, q_1, q_2, \alpha, \alpha_1, \alpha_2 \in \mathbb{Z}$ . From this we get

$$\|\nabla^2 h\|_{L^1(\langle v \rangle^{\gamma+2+k})} \lesssim \|h\|_{L^1(\langle v \rangle^k)}^{1/2} \|h\|_{W^{4,1}(\langle v \rangle^{2\gamma+4+k})}^{1/2} \lesssim \|h\|_{L^1(\langle v \rangle^k)}^{1/2} \|h\|_{H^4(\langle v \rangle^{2\gamma+6+k})}^{1/2},$$

where we used Hölder's inequality in last step. Gathering last inequality with (3.2) and using Hölder's inequality again to write

$$\|h\|_{L^1(\langle v \rangle^\gamma)} \|h\|_{L^1(\langle v \rangle^{\gamma+k})} \leq \|h\|_{L^1(\langle v \rangle^{2\gamma+k})}^{1/2} \|h\|_{L^1(\langle v \rangle^k)}^{3/2},$$

it follows that

$$\begin{aligned} \|h_t\|_{L^1(\langle v \rangle^k)} &\leq C e^{-\lambda_0 t} \|h_0\|_{L^1(\langle v \rangle^k)} + C \int_0^t e^{-\lambda_0(t-s)} \|h_s\|_{L^1(\langle v \rangle^{2\gamma+k})}^{1/2} \|h_s\|_{L^1(\langle v \rangle^k)}^{3/2} ds \\ &\quad + C \int_0^t e^{-\lambda_0(t-s)} \|h_s\|_{H^4(\langle v \rangle^{2\gamma+6+k})}^{1/2} \|h_s\|_{L^1(\langle v \rangle^k)}^{3/2} ds. \end{aligned}$$

Denoting  $x(t) := \|h_t\|_{L^1(\langle v \rangle^k)}$  and using the assumptions of the lemma we hence obtain the following inequality

$$x(t) \leq C e^{-\lambda_0 t} x(0) + C \epsilon^{1/4} \int_0^t e^{-\lambda_0(t-s)} x(s)^{1+1/4} ds,$$

and arguing as in [12, Lemma 4.5], if  $x(0)$  and  $\epsilon$  are small enough we obtain, for all  $t \geq 0$ ,  $x(t) \leq C' e^{-\lambda_0 t} x(0)$ , i.e.

$$\|h_t\|_{L^1(\langle v \rangle^k)} \leq C' e^{-\lambda_0 t} \|h_0\|_{L^1(\langle v \rangle^k)}.$$

□

We can now complete the proof of theorem 1.2. From corollary 3.3 we pick  $t_0 > 0$  such that

$$\forall t \geq t_0, \quad \|f_t - \mu\|_{L^1(\langle v \rangle^\epsilon)} = \|h_t\|_{L^1(\langle v \rangle^\epsilon)} \leq \epsilon,$$

where  $\epsilon$  is chosen in lemma 3.4. From theorem 3.2 we have that, for all  $t \geq t_0$ ,  $\|h_t\|_{H^4(\langle v \rangle^\epsilon)} \leq \|f_t\|_{H^4(\langle v \rangle^\epsilon)} + \|\mu\|_{H^4(\langle v \rangle^\epsilon)} \leq C$ . We can then apply lemma 3.4 to  $h$  starting from  $t_0$ , then

$$\forall t \geq t_0, \quad \|f_t - \mu\|_{L^1(\langle v \rangle^k)} = \|h_t\|_{L^1(\langle v \rangle^k)} \leq C' e^{-\lambda_0 t} \|h_{t_0}\|_{L^1(\langle v \rangle^k)} \leq C'' e^{-\lambda_0 t},$$

which completes the proof.

□

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