Geodesic Completeness for Sobolev H^s-metrics on the Diffeomorphisms Group of the Circle

Joachim Escher, Boris Kolev

GEODESIC COMPLETENESS FOR SOBOLEV H^s-METRICS ON THE DIFFEOMORPHISM GROUP OF THE CIRCLE

JOACHIM ESCHER AND BORIS KOLEV

Abstract. We prove that the weak Riemannian metric induced by the fractional Sobolev norm H^s on the diffeomorphism group of the circle is geodesically complete, provided that $s > 3/2$.

1. Introduction

The interest in right-invariant metrics on the diffeomorphism group of the circle started when it was discovered by Kouranbaeva [17] that the Camassa–Holm equation [3] can be recast as the Euler equation of the right-invariant metric on $\mathrm{Diff}^{\infty}(\mathbb{S}^1)$ induced by the H^1 Sobolev inner product on the corresponding Lie algebra $\mathrm{C}^{\infty}(\mathbb{S}^1)$. The well-posedness of the geodesics flow for the right-invariant metric induced by the H^k inner product was obtained by Constantin and Kolev [5], for $k \in \mathbb{N}$, $k \geq 1$, following the pioneering work of Ebin and Marsden [8]. These investigations have been extended to the case of fractional order Sobolev spaces H^s with $s \in \mathbb{R}_+$, $s \geq 1/2$ by Escher and Kolev [10]. The method used to establish local existence of geodesics is to extend the metric and its spray to the Hilbert approximation $\mathcal{D}^q(\mathbb{S}^1)$ (the Hilbert manifold of diffeomorphisms of class H^q) and then to show that the (extended) spray is smooth. This was proved to work in [10] for $s \geq 1/2$ provided we choose $q > 3/2$ and $q \geq 2s$. The well-posedness on $\mathrm{Diff}^{\infty}(\mathbb{S}^1)$ follows as $q \to \infty$ from a regularity preserving result of the geodesic flow.

A Riemannian metric is *strong* if at each point it induces a topological isomorphism between the tangent space and the cotangent space. It is *weak* if it defines merely an injective linear mapping between the tangent space and the cotangent space. Note that on $\mathrm{Diff}^{\infty}(\mathbb{S}^1)$ only weak metrics exist. Furthermore we also mention that the extended metric on $\mathcal{D}^q(\mathbb{S}^1)$ is not strong but only weak as soon as $q > 2s$.

On a Banach manifold equipped with a strong metric, the geodesic semi-distance induced by the metric is in fact a distance [19]. This is no longer true for weak metrics. It was shown by Bauer, Bruveris, Harms, and Michor [2] that this semi-distance identically vanishes for the H^s metric if $0 \leq s \leq 1/2$, whereas it is a distance for $s > 1/2$. This distance is nevertheless probably not complete on $T\mathcal{D}^q(\mathbb{S}^1)$. Indeed, although for a strong metric topological completeness implies geodesic completeness, this is generally not true for a weak metric. Finally, we recall that the metric induced by the H^1-norm (or equivalently by $A := I - D^2$) is not geodesically complete, c.f. [4]. The main result of this paper is the following.

Date: May 24, 2014.

2010 *Mathematics Subject Classification.* 58D05, 35Q53.

Theorem 1.1. *Let $s > 3/2$ be given. Then the geodesic flow on $\mathcal{D}^q(\mathbb{S}^1)$ for $q \geq 2s+1$ and on $\mathrm{Diff}^\infty(\mathbb{S}^1)$, respectively, is complete for the weak Riemannian metric induced by the $H^s(\mathbb{S}^1)$-inner product.*

Completeness results for groups of diffeomorphisms on $\mathbb{R}^n$ have been studied in [26] and in [21]. In both papers, stronger conditions on s had been presupposed: Compared to our setting s has to be larger than $7/2$ in [26] and an *integer* larger than 2 in [21], respectively. Additionally, in the work of [26, 21], the phenomenon that the diffeomorphisms of an orbit with finite extinction time may degenerate in the sense of the remarks following Corollary 4.3 is not reported on.

Let us briefly give an outline of the paper. In Section 2, we introduce basic facts on right-invariant metrics on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ and we recall a well-posedness result for related geodesic flows. In Section 3, we introduce a complete metric structure on suitable Banach approximations of $\mathrm{Diff}^\infty(\mathbb{S}^1)$, which allows us to describe the precise blow-up mechanism of finite time geodesics. This is the subject matter of Section 4. In Section 5, we prove our main result, Theorem 1.1. In Appendix A, we recall the material on Friedrichs mollifier that have been used throughout the paper.

2. Right-invariant metrics on $\mathrm{Diff}^\infty(\mathbb{S}^1)$

Let $\mathrm{Diff}^\infty(\mathbb{S}^1)$ be the group of all smooth and orientation preserving diffeomorphism on the circle. This group is naturally equipped with a *Fréchet manifold* structure; it can be covered by charts taking values in the *Fréchet vector space* $\mathrm{C}^\infty(\mathbb{S}^1)$ and in such a way that the change of charts are smooth mappings (a smooth atlas with only two charts may be constructed, see for instance [14]).

Since both the composition and the inversion are smooth for this structure we say that $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a *Fréchet-Lie group*, c.f. [15]. Its Lie algebra, $\mathrm{Vect}(\mathbb{S}^1)$, is the space of smooth vector fields on the circle. It is isomorphic to $\mathrm{C}^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[u, v] = u_x v - u v_x.$$

From an analytic point of view, the Fréchet Lie group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ may be viewed as an inverse limit of *Hilbert manifolds*. More precisely, recall that the Sobolev space $H^q(\mathbb{S}^1)$ is defined as the completion of $\mathrm{C}^\infty(\mathbb{S}^1)$ for the norm

$$\|u\|_{H^q(\mathbb{S}^1)} := \left(\sum_{n\in\mathbb{Z}} (1+n^2)^q \, |\hat{u}_n|^2 \right)^{1/2},$$

where $q \in \mathbb{R}^+$ and where $\hat{u}_n$ stands for the $n-$th Fourier coefficient of $u \in L^2(\mathbb{S}^1)$. Let $\mathcal{D}^q(\mathbb{S}^1)$ denote the set of all orientation preserving homeomorphisms φ of the circle $\mathbb{S}^1$, such that both φ and φ^{-1} belong to the fractional Sobolev space $H^q(\mathbb{S}^1)$. For $q > 3/2$, $\mathcal{D}^q(\mathbb{S}^1)$ is a *Hilbert manifold* and a *topological group* [8]. It is however not a *Lie group* because neither composition, nor inversion in $\mathcal{D}^q(\mathbb{S}^1)$ are *smooth*, see again [8]. We have

$$\mathrm{Diff}^\infty(\mathbb{S}^1) = \bigcap_{q>\frac{3}{2}} \mathcal{D}^q(\mathbb{S}^1).$$

Remark 1. Like any Lie group, $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a *parallelizable* manifold:

$$T\mathrm{Diff}^\infty(\mathbb{S}^1) \sim \mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1).$$

What is less obvious, however, is that $T\mathcal{D}^q(\mathbb{S}^1)$ is also a *trivial bundle*. Indeed, let

$$\mathfrak{t} : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$$

be a *smooth trivialisation* of the tangent bundle of $\mathbb{S}^1$. Then

$$T\mathcal{D}^q(\mathbb{S}^1) \to \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), \qquad \xi \mapsto \mathfrak{t} \circ \xi$$

is a *smooth vector bundle isomorphism* (see [8, p. 107]).

A *right-invariant* metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is defined by an inner product on the Lie algebra $\mathrm{Vect}(\mathbb{S}^1) = \mathrm{C}^\infty(\mathbb{S}^1)$. In the following we assume that this inner product is given by

$$\langle u, v \rangle = \int_{\mathbb{S}^1} (Au) v \, dx,$$

where $A : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$ is a L^2-symmetric, positive definite, invertible *Fourier multiplier* (i.e. a continuous linear operator on $\mathrm{C}^\infty(\mathbb{S}^1)$ which commutes with $D := d/dx$). For historical reasons going back to Euler's work [12], A is called the *inertia operator*.

By translating the above inner product, we obtain an inner product on each tangent space $T_\varphi \mathrm{Diff}^\infty(\mathbb{S}^1)$

$$\langle \eta, \xi \rangle_\varphi = \langle \eta \circ \varphi^{-1}, \xi \circ \varphi^{-1} \rangle_{id} = \int_{\mathbb{S}^1} \eta (A_\varphi \xi) \varphi_x \, dx, \tag{2.1}$$

where $\eta, \xi \in T_\varphi \mathrm{Diff}^\infty(\mathbb{S}^1)$ and $A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}$, and $R_\varphi(v) := v \circ \varphi$. This defines a smooth weak Riemannian metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$.

This weak Riemannian metric admits the following *geodesic spray*[1]

$$F : (\varphi, v) \mapsto (\varphi, v, v, S_\varphi(v)) \tag{2.2}$$

where

$$S_\varphi(v) := \left(R_\varphi \circ S \circ R_{\varphi^{-1}} \right)(v),$$

and S is a quadratic operator on the Lie algebra given by:

$$S(u) := A^{-1} \left\{ [A, u] u_x - 2(Au) u_x \right\}.$$

A *geodesic* is an integral curve of this second order vector field, that is a solution (φ, v) of

$$\begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases} \tag{2.3}$$

Given a geodesic (φ, v), we define the *Eulerian velocity* as

$$u := v \circ \varphi^{-1}.$$

Then u solves

$$u_t = -A^{-1} \left[u (Au)_x + 2(Au) u_x \right], \tag{2.4}$$

[1]A Riemannian metric on a manifold M defines a smooth function on TM, given by half the square norm of a tangent vector. The corresponding Hamiltonian vector field on TM, relatively to the pullback of the canonical symplectic structure on T^*M is called the *geodesic spray*.

called the *Euler equation* defined by the inertia operator A.

Remark 2. When A is a *differential operator* of order $r \geq 1$ then the quadratic operator

$$S(u) = A^{-1}\left\{[A,u]u_x - 2(Au)u_x\right\}$$

is of order 0 because the commutator $[A,u]$ is of order not higher than $r-1$. One might expect, that for a larger class of operators A, the quadratic operator S to be of order 0 and consequently the second order system (2.3) can be viewed as an ODE on $T\mathcal{D}^q(\mathbb{S}^1)$.

Definition 2.1. A Fourier multiplier $A = \mathbf{op}\,(a(k))$ with symbol a is of *order* $r \in \mathbb{R}$ if there exists a constant $C > 0$ such that

$$|a(k)| \leq C\left(1+k^2\right)^{r/2},$$

for every $k \in \mathbb{Z}$. In that case, for each $q \geq r$, the operator A extends to a bounded linear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$. In this paper we only consider symmetric operators, i.e. $a(k) \in \mathbb{R}$ for all $k \in \mathbb{Z}$.

When A is a differential operator of order $r \geq 1$, the map

$$\varphi \mapsto A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)) \tag{2.5}$$

is smooth (it is in fact *real analytic*) for $q > 3/2$ and $q \geq r$. Indeed, in this case A_φ is a linear differential operator with coefficients consisting of polynomial expressions of $1/\varphi_x$ and of the derivatives of φ up to order r. Unfortunately, this argument *does not apply* to a general *Fourier multiplier* $A = \mathbf{op}\,(p(k))$. In that case, even if A extends to a bounded linear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$, one cannot conclude directly that the mapping $\varphi \mapsto A_\varphi$ is smooth, because the mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^\sigma(\mathbb{S}^1), H^\sigma(\mathbb{S}^1))$$

is *not even continuous*[2], for any choice of $\sigma \in [0,q]$.

Let us now precisely formulate the conditions that will be required on the inertia operator subsequently.

Presupposition 2.2. The following conditions will be assumed on the inertia operator A:

(a) $A = \mathbf{op}\,(a(k))$ is a Fourier multiplier of order $r \geq 1$, or equivalently, $a(k) = \mathcal{O}(|k|^r)$;

(b) For all $q \geq r$, $A : H^q(\mathbb{S}^1) \to H^{q-r}(\mathbb{S}^1)$ is a bounded isomorphism, or equivalently, for all $k \in \mathbb{Z}$, $a(k) \neq 0$ and $1/a(k) = \mathcal{O}(|k|^{-r})$;

(c) For each $q > 3/2$ with $q \geq r$, the mapping

$$\varphi \mapsto A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth.

[2]The map $(\varphi, u) \mapsto u \circ \varphi$ is however continuous but not differentiable.

In [10] we have specified conditions on the symbol of A which guarantee that A satisfies presupposition 2.2. Particularly, inertia operators of the form of Bessel potentials, i.e.

$$\Lambda^{2s} := \mathbf{op}\left(\left(1+k^2\right)^s\right),$$

which generate the inner product of the fractional order Sobolev space $H^s(\mathbb{S}^1)$

$$(u,v) \mapsto \langle \Lambda^s u | \Lambda^s v\rangle_{L^2}, \quad u,\, v \in H^s(\mathbb{S}^1),$$

meet these conditions, provided that $s \geq 1/2$.

If the conditions 2.2 are satisfied, then expression (2.1) defines a smooth, weak Riemannian metric on $\mathcal{D}^q(\mathbb{S}^1)$, provided that $q > 3/2$ and $q \geq r$. Moreover, it can be shown that the spray F defined by equation (2.2) extends to a smooth vector field F_q on $T\mathcal{D}^q(\mathbb{S}^1)$, which is the geodesic spray of the metric, c.f. [10, Theorem 3.10]. In that case, the *Picard-Lindelöf Theorem* on the Banach manifold $T\mathcal{D}^q(\mathbb{S}^1)$ ensures that, given any initial data $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$, there is a unique *non-extendable solution* (φ, v) of (2.3), defined on a maximal interval $I_q(\varphi_0, v_0)$, satisfying the initial condition

$$(\varphi(0), v(0)) = (\varphi_0, v_0).$$

A remarkable observation due to Ebin and Marsden (see [8, Theorem 12.1]) states that, if the initial data (φ_0, v_0) is smooth, then the maximal time interval of existence $I_q(\varphi_0, v_0)$ is independent of the parameter q. This is an essential ingredient in the proof of the local existence theorem for geodesics on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ (see [10]).

Theorem 2.3. *Suppose that presupposotion 2.2 hold true. Then, given any $(\varphi_0, v_0) \in T\mathrm{Diff}^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution*

$$(\varphi, v) \in C^\infty(J, T\mathrm{Diff}^\infty(\mathbb{S}^1))$$

of (2.3), with initial data (φ_0, v_0), defined on the maximal interval of existence $J_{max} = (t^-, t^+)$. Moreover, the solution depends smoothly on the initial data.

As a corollary, we get well-posedness for the corresponding Euler equation (2.4).

Theorem 2.4. *Assume that the operator A satisfies presupposition 2.2. Let $v_0 \in \mathrm{Diff}^\infty(\mathbb{S}^1)$ be given and denote by J_{max} the maximal interval of existence for (2.3) with the initial datum $(id_{\mathbb{S}^1}, v_0)$. Set $u := v \circ \varphi^{-1}$. Then $u \in C^\infty(J_{max}, \mathrm{C}^\infty(\mathbb{S}^1))$ is the unique non-extendable solution of the Euler equation*

$$\begin{cases} u_t = -A^{-1}\left[u(Au)_x + 2(Au)u_x\right], \\ u(0) = v_0. \end{cases} \tag{2.6}$$

It is also worth to recall that the metric norm along the flow is conserved.

Lemma 2.5. *Let u be a solution to (2.4) on the time interval J, then*

$$\|u(t)\|_A = \left(\int_{\mathbb{S}^1} (Au)u\,dx\right)^{1/2} \tag{2.7}$$

is constant on J.

3. A complete metric structure on $\mathcal{D}^q(\mathbb{S}^1)$

We recall that in what follows, $\mathbb{S}^1$ is the unit circle of the complex plane and that $\mathrm{Diff}^\infty(\mathbb{S}^1)$ and $\mathcal{D}^q(\mathbb{S}^1)$ may be considered as subset of the set $C^0(\mathbb{S}^1,\mathbb{S}^1)$ of all continuous maps of the circle. Besides the Banach manifold $\mathcal{D}^q(\mathbb{S}^1)$ may be covered by two charts (see [9] for instance). We let

$$d_0(\varphi_1,\varphi_2) := \max_{x\in\mathbb{S}^1} |\varphi_2(x)-\varphi_1(x)|$$

be the C^0-distance between continuous maps of the circle. Endowed with this distance $C^0(\mathbb{S}^1,\mathbb{S}^1)$ is a complete metric space. Let $\mathrm{Homeo}^+(\mathbb{S}^1)$ be the group of orientation preserving homeomorphisms of the circle. Equipped with the induced topology, $\mathrm{Homeo}^+(\mathbb{S}^1)$ is a topological group, and each right translation R_φ is an isometry for the distance d_0.

Definition 3.1. Given $q > 3/2$, we introduce the following distance on $\mathcal{D}^q(\mathbb{S}^1)$

$$d_q(\varphi_1,\varphi_2) := d_0(\varphi_1,\varphi_2) + \|\varphi_{1x}-\varphi_{2x}\|_{H^{q-1}} + \|1/\varphi_{1x}-1/\varphi_{2x}\|_\infty .$$

Lemma 3.2. *Let $q > 3/2$ be given and assume that B is a bounded subset of $(\mathcal{D}^q(\mathbb{S}^1), d_q)$. Then*

$$\inf_{\varphi\in B}\left(\min_{y\in\mathbb{S}^1}\varphi_x(y)\right) > 0.$$

Proof. Let $M := \operatorname{diam} B$, fix $\varphi_0 \in B$ and put $\varepsilon := 1/(M + \|1/\varphi_{0x}\|_\infty)$. By hypothesis $M < \infty$, thus $\varepsilon > 0$. Assume now by contradiction that

$$\inf_{\varphi\in B}\left(\min_{y\in\mathbb{S}^1}\varphi_x(y)\right) = 0.$$

Then there is a $\varphi_1 \in B$ such that $\min_{y\in\mathbb{S}^1}\varphi_{1x}(y) < \varepsilon$. Using

$$\|1/\varphi_{1x}\|_\infty = \max_{y\in\mathbb{S}^1}\left(\frac{1}{\varphi_{1x}(y)}\right) = \left(\min_{y\in\mathbb{S}^1}\varphi_{1x}(y)\right)^{-1} > \frac{1}{\varepsilon},$$

we find by the definition of ε the contradiction:

$$M \geq d(\varphi_1,\varphi_0) \geq \|1/\varphi_{1x} - 1/\varphi_{0x}\|_\infty \geq \|1/\varphi_{1x}\|_\infty - \|1/\varphi_{0x}\|_\infty > M,$$

which completes the proof. □

Proposition 3.3. *Let $q > 3/2$. Then $(\mathcal{D}^q(\mathbb{S}^1), d_q)$ is a complete metric space and its topology is equivalent to the Banach manifold topology on $\mathcal{D}^q(\mathbb{S}^1)$.*

Proof. Let τ be the Banach manifold topology on $\mathcal{D}^q(\mathbb{S}^1)$ and τ_d be the metric topology. Then

$$id : (\mathcal{D}^q(\mathbb{S}^1),\tau) \to (\mathcal{D}^q(\mathbb{S}^1),\tau_d)$$

is continuous because $\varphi \mapsto \varphi^{-1}$ is a homeomorphism of $\mathcal{D}^q(\mathbb{S}^1)$ (equipped with the manifold topology) and the fact that

$$d_0(\varphi_1,\varphi_2) \lesssim \|\tilde\varphi_1 - \tilde\varphi_2\|_{H^q}$$

if $\tilde\varphi_1$ and $\tilde\varphi_2$ are lifts of φ_1 and φ_2 respectively. Conversely

$$id : (\mathcal{D}^q(\mathbb{S}^1),\tau_d) \to (\mathcal{D}^q(\mathbb{S}^1),\tau)$$

is continuous because given φ_0, there exists $\delta > 0$ such that if $d_0(\varphi_0, \varphi) < \delta$, then φ belongs to the same chart as φ_0 and in a local chart we have

$$\|\tilde{\varphi} - \tilde{\varphi_0}\|_{H^q} \lesssim d_q(\varphi, \varphi_0).$$

This shows the equivalence of the two topologies.

Let now (φ_n) be a Cauchy sequence for the distance d_q. We observe first that (φ_n) converges in $C^0(\mathbb{S}^1, \mathbb{S}^1)$ to a map φ, that this map is C^1 and that $\varphi_{nx} \to \varphi_x$ in $H^{q-1}(\mathbb{S}^1)$, because for n large enough, all φ_n belong to a same chart. Invoking Lemma 3.2, we know that

$$\inf_{n\in\mathbb{N}} \left(\min_{y\in\mathbb{S}^1} \varphi_{nx}(y) \right) > 0.$$

This implies that $\varphi_x > 0$ and hence that φ is a C^1-diffeomorphism of class H^q, and finally that $d_q(\varphi_n, \varphi) \to 0$. □

Lemma 3.4. *Let $\varphi \in C^1(I, \mathcal{D}^q(\mathbb{S}^1))$ be a path in $\mathcal{D}^q(\mathbb{S}^1)$ and let $v := \varphi_t$ be its velocity. Then*

$$d_q(\varphi(t), \varphi(s)) \lesssim |t-s| \max_{[s,t]} \|v\|_{H^q} \left(1 + \max_{[s,t]} \|1/\varphi_x\|_\infty^2 \right)$$

for all $t, s \in I$.

Proof. Let $\widetilde{\varphi} \in C^1(I, H^q(\mathbb{R}))$ be a lift of the path φ. Given $s, t \in I$ with $s < t$, we have first

$$\begin{aligned} d_0(\varphi(t), \varphi(s)) &\lesssim \|\widetilde{\varphi}(t) - \widetilde{\varphi}(s)\|_\infty \\ &\leq \int_s^t \|\varphi_t(\tau)\|_\infty \, d\tau \lesssim |t-s| \max_{[s,t]} \|v\|_{H^q} . \end{aligned} \tag{3.1}$$

Next, we have

$$\varphi_x(t) - \varphi_x(s) = \int_s^t \varphi_{tx}(\tau) d\tau,$$

in $H^{q-1}(\mathbb{S}^1)$ and hence

$$\|\varphi_x(t) - \varphi_x(s)\|_{H^{q-1}} \leq \int_s^t \|\varphi_{tx}(\tau)\|_{H^{q-1}} \, d\tau \leq |t-s| \max_{[s,t]} \|v\|_{H^q} . \tag{3.2}$$

Finally we have

$$\begin{aligned} \|1/\varphi_x(t) - 1/\varphi_x(s)\|_\infty &\leq \left(\max_{[s,t]} \|1/\varphi_x\|_\infty \right)^2 \int_s^t \|\varphi_{tx}(\tau)\|_\infty \\ &\lesssim |t-s| \max_{[s,t]} \|v\|_{H^q} \left(\max_{[s,t]} \|1/\varphi_x\|_\infty \right)^2 . \end{aligned} \tag{3.3}$$

Fusing (3.1), (3.2), and (3.3) completes the proof. □

4. The blow-up scenario for geodesics

In the sequel a bounded set in $\mathcal{D}^q(\mathbb{S}^1)$ will always mean *bounded relative to the distance d_q* and a bounded set in $T\mathcal{D}^q(\mathbb{S}^1) = \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ will mean *bounded relative to the product distance*

$$d_q(\varphi_1, \varphi_2) + \|v_1 - v_2\|_{H^q} .$$

The main result of this section is the following.

Theorem 4.1. *Let $q > 3/2$ be given with $q \geq r$. Then the geodesic spray*

$$F_q : (\varphi, v) \mapsto (v, S_\varphi(v))$$

is bounded on bounded sets of $\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$.

The proof of this theorem is based on Lemma 4.2, which is itself a corollary of the following estimates obtained in [10, Appendix B].

$$\|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1),H^\rho(\mathbb{S}^1))} \leq C_\rho^1 \left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}\right), \tag{4.1}$$

for $0 \leq \rho \leq 1$,

$$\|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1),H^\rho(\mathbb{S}^1))} \leq C_\rho^2 \left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}}\right), \tag{4.2}$$

for $0 \leq \rho \leq 2$,

$$\|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1),H^\rho(\mathbb{S}^1))} \leq C_\rho^3 \left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}\right) \|\varphi_x\|_{H^{\rho-1}}, \tag{4.3}$$

for $3/2 < \rho \leq 3$,

$$\|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1),H^\rho(\mathbb{S}^1))} \leq C_\rho^4 \left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{\rho-2}}\right) \|\varphi_x\|_{H^{\rho-1}}, \tag{4.4}$$

for $\rho > 5/2$, and

$$\left\|(\varphi^{-1})_x\right\|_{H^{\rho-1}} \lesssim C_\rho^5(\|1/\varphi_x\|_\infty, \|\varphi_x\|_{H^{\rho-1}}), \tag{4.5}$$

for $\rho > 3/2$, where C_ρ^k is a positive, continuous function on $(\mathbb{R}^+)^2$, for $k = 1, \dots, 5$.

Lemma 4.2. *Let $q > 3/2$ and $0 \leq \rho \leq q$ be given. Then the mappings*

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))$$

and

$$\varphi \mapsto R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))$$

are bounded on bounded subsets of $\mathcal{D}^q(\mathbb{S}^1)$.

Proof of Theorem 4.1. Recall that $S_\varphi(v) = R_\varphi \circ S \circ R_{\varphi^{-1}}$ where

$$S(u) := A^{-1}\left\{[A, u]u_x - 2(Au)u_x\right\}.$$

In particular, $S_\varphi(v)$ is quadratic in v and

$$\|S_\varphi(v)\|_{H^q} \leq \|R_\varphi\|_{\mathcal{L}(H^q,H^q)} \|S\|_{\mathcal{L}(H^q \times H^q, H^q)} \left\|R_{\varphi^{-1}}\right\|^2_{\mathcal{L}(H^q,H^q)} \|v\|^2_{H^q}.$$

Now, S is a bounded bilinear operator and R_φ and $R_{\varphi^{-1}}$ are bounded on bounded subsets of $\mathcal{D}^q(\mathbb{S}^1)$ by Lemma 4.2. This completes the proof. □

Our next goal is to study the behaviour of geodesics which do not exists globally, i.e. $t^+ < \infty$ or $t^- > -\infty$. We have the following result, which is a consequence of Theorem 4.1.

Corollary 4.3. *Assume that presupposition 2.2 are satisfied and let*

$$(\varphi, v) \in C^\infty((t^-, t^+), T\mathcal{D}^q(\mathbb{S}^1))$$

denote the non-extendable solution of the geodesic flow (2.3), emanating from

$$(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1).$$

If $t^+ < \infty$, then

$$\lim_{t \uparrow t^+} \left[d_q(\varphi_0, \varphi(t)) + \|v(t)\|_{H^q}\right] = +\infty.$$

A similar statement holds true if $t^- > -\infty$.

Proof. Suppose that $t^+ < \infty$ and set

$$f(t) := d_q(\varphi_0, \varphi(t)) + \|v(t)\|_{H^q}$$

where $(\varphi(t), v(t)) \in T\mathcal{D}^q(\mathbb{S}^1)$ is the solution of (2.3) at time $t \in (t^-, t^+)$, emanating from (φ_0, v_0).

(i) Note first that f cannot be bounded on $[0, t^+)$. Otherwise, the spray $F_q(\varphi(t), v(t))$ would be bounded on $[0, t^+)$ by Theorem 4.1. In that case, given any sequence (t_k) in $[0, t^+)$ converging to t^+, we would conclude, invoking Lemma 3.4, that $(\varphi(t_k))$ is a Cauchy sequence in the complete metric space $(\mathcal{D}^q(\mathbb{S}^1), d_q)$. Similarly, we would conclude that the sequence $(v(t_k))$ is a Cauchy sequence in the Hilbert space $H^q(\mathbb{S}^1)$. Then, by the Picard-Lindelöf theorem, we would deduce that the solution could be extended beyond t^+, which would contradict the maximality of t^+.

(ii) We are going to show now that

$$\lim_{t \nearrow t^+} f(t) = +\infty.$$

If this was wrong, then we would have

$$\liminf_{t \nearrow t^+} f < +\infty \quad \text{and} \quad \limsup_{t \nearrow t^+} f = +\infty.$$

But then, using the continuity of f, we could find $r > 0$ and two sequences (s_k) and (t_k) in $[0, t^+)$, each converging to t^+, with

$$s_k < t_k, \quad f(s_k) = r, \quad f(t_k) = 2r$$

and such that

$$f(t) \le 2r, \quad \forall t \in \bigcup_k [s_k, t_k].$$

However, by Theorem 4.1, we can find a positive constant M such that

$$\|S_\varphi(v)\|_{H^q} \le M,$$

for all $(\varphi, v) \in T\mathcal{D}^q(\mathbb{S}^1)$ satisfying

$$d_q(\varphi_0, \varphi) + \|v\|_{H^q} \le 2r.$$

We would get therefore, using again Lemma 3.4, that

$$r = f(t_k) - f(s_k) \le C\,|t_k - s_k|\,, \quad \forall k \in \mathbb{N},$$

for some positive constant C, which would lead to a contradiction and completes the proof. □

Assume that $t^+ < \infty$. Then Corollary 4.3 makes it clear that there are only two possible blow-up scenarios: either the solution $(\varphi(t), v(t))$ becomes large in the sense that

$$\lim_{t \to t^+} \left(\|\varphi_x(t)\|_{H^{q-1}} + \|v(t)\|_{H^q}\right) = \infty,$$

or the family of diffeomorphisms $\{\varphi(t)\,;\, t \in (t^-, t^+)\}$ becomes singular in the sense that

$$\lim_{t \to t^+} \left(\min_{x \in \mathbb{S}^1} \{\varphi_x(t, x)\} \right) = 0.$$

It is however worth emphasizing that the blow-up result in Corollary 4.3 only represents a necessary condition. Indeed, for $A = I - D^2$, i.e. for the Camassa–Holm equation the precise blow-up mechanism is known (see [4]): a classical solution u blows up in finite time if and only if

$$\lim_{t \to t^+} \left(\min_{x \in \mathbb{S}^1} \{u_x(t,x)\} \right) = -\infty, \tag{4.6}$$

which is somewhat weaker than blow up in $H^2(\mathbb{S}^1)$. Since it is known that any (classical) solution to the Camassa–Holm equation preserves the H^1 norm and thus stays bounded, one says that the blow up occurs as a *wave breaking*. Note also that

$$u_x(t,x) = v_x \circ \varphi(t,x) \cdot \frac{1}{\varphi_x(t,x)} \quad \text{for} \quad (t,x) \in (t^-, t^+) \times \mathbb{S}^1.$$

Hence in the case of a wave breaking, either $|v_x|$ becomes unbounded or v_x becomes negative and φ_x tends to 0 as $t \uparrow t^+$.

On the other hand there are several evolution equations, different from the Camassa–Holm equation, e.g. the *Constantin-Lax-Majda equation* [6, 27], which corresponds to the case $A = \mathcal{H}D$, where $\mathcal{H}$ denotes the Hilbert transform, cf. [11] for which the blow up mechanism is much less understood and so far no sharper results than blow up in $H^{1+\sigma}(\mathbb{S}^1)$ for any $\sigma > 1/2$ or pointwise vanishing of φ_x seem to be known.

5. Global solutions

Throughout this section, we suppose that the inertia operator A satisfies conditions 2.2. We fix some $q \geq r + 1$, and we let

$$(\varphi, v) \in C^\infty(J, T\mathcal{D}^q(\mathbb{S}^1)) \tag{5.1}$$

be the unique solution of the Cauchy problem (2.3), emanating from

$$(id_{\mathbb{S}^1}, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$$

and defined on the *maximal time interval* $J = (t^-, t^+)$. The corresponding solution $u = v \circ \varphi^{-1}$ of the Euler equation (2.6) is a path

$$u \in C^0(J, H^q(\mathbb{S}^1)) \cap C^1(J, H^{q-1}(\mathbb{S}^1)), \tag{5.2}$$

because

$$(\varphi, v) \mapsto v \circ \varphi^{-1}, \qquad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \to H^{q-1}(\mathbb{S}^1),$$

is C^1 for $q > 3/2$ (see [10, Corollary B.6]). Moreover, since A is of order r, the *momentum* $m(t) := Au(t)$ is defined as a path

$$m \in C^0(J, H^{q-r}(\mathbb{S}^1)) \cap C^1(J, H^{q-r-1}(\mathbb{S}^1)). \tag{5.3}$$

It satisfies the *Euler-Poincaré equation*

$$m_t = -m_x u - 2m u_x \quad \text{in} \quad C(J, L^2(\mathbb{S}^1)). \tag{5.4}$$

We will prove that the geodesic $(\varphi(t), v(t))$ is defined for all time, as soon as u_x is bounded below, independently of a particular choice of the inertia operator A, provided that $r \geq 2$.

Remark 3. Global solutions in $H^q(\mathbb{S}^1)$ ($q > 3/2$) of the Camassa–Holm equation, which corresponds to the special case where the inertia operator $A = 1 - D^2$, have been studied in [22]. It was established there, that $u(t)$ is defined on $[0, \infty)$ provided $\|u\|_{C^1}$ is bounded [22, Theorem 2.3]. A similar argument was used in [20] to establish existence of solutions of the Euler equation for the inertia operator $A = (1 - D^2)^k$, $k \geq 1$, for which $m(t)$ does not blow up in L^2.

The main result of this section is the *a priori* estimate contained in the following result.

Theorem 5.1. *Let $r \geq 2$ and $q \geq r + 1$ be given and let*

$$u \in C^0(J, H^q(\mathbb{S}^1)) \cap C^1(J, H^{q-1}(\mathbb{S}^1))$$

be the solution of (2.4) *with initial data $u_0 \in H^q(\mathbb{S}^1)$ on J. Let I be some* bounded *subinterval of J and suppose that*

$$\inf_{t \in I} \left(\min_{x \in \mathbb{S}^1} \{u_x(t, x)\} \right) > -\infty.$$

Then $\|u\|_{H^q}$ is bounded on I.

The approach used here is inspired by that of Taylor [25] and relies on *Friedrichs mollifiers* (see Appendix A). It requires also the following commutator estimate due to Kato and Ponce [16] (see also [24]).

Lemma 5.2. *Let $s > 0$ and $\Lambda^s := \mathbf{op}\left((1 + k^2)^{s/2}\right)$. If $u, v \in H^s(\mathbb{S}^1)$, then*

$$\|\Lambda^s(uv) - u\Lambda^s(v)\|_{L^2} \lesssim \|u_x\|_\infty \left\|\Lambda^{s-1}v\right\|_{L^2} + \|\Lambda^s u\|_{L^2} \|v\|_\infty \tag{5.5}$$

Proof of Theorem 5.1. (1) Let $m(t) = Au(t)$ for $t \in J$. Invoking (5.3) and the fact that $q - r - 1 \geq 0$, we conclude that the curve $[t \mapsto m(t)]$ belongs to $C^1(J, L^2(\mathbb{S}^1))$. Thus the Euler-Poincaré equation (5.4) implies that

$$\frac{d}{dt} \|m\|_{L^2}^2 = -2 \langle m, m_x u + 2mu_x \rangle_{L^2} \quad \text{on} \quad J.$$

Consequently, we get

$$\frac{d}{dt} \|m\|_{L^2}^2 \leq -3 \min_{x \in \mathbb{S}^1} \{u_x(t, x)\} \|m\|_{L^2}^2 ,$$

and by virtue of Gronwall's lemma, we conclude that $\|m\|_{L^2}$ is bounded on I. Recalling that A^{-1} is a bounded operator from $L^2(\mathbb{S}^1)$ to $H^r(\mathbb{S}^1)$, we see that $\|u\|_{H^r}$ is bounded on I. This applies, in particular, to $\|u\|_{H^2}$, because we assumed that $r \geq 2$.

(2) Our next goal is to derive an H^1 *a priori* estimate for m. Since the curve $[t \mapsto m(t)]$ belongs merely to $C^1(J, H^{q-r-1}(\mathbb{S}^1))$ and $q - r - 1$ may be smaller than 1, we need to replace it by the curve $t \mapsto J_\varepsilon m(t)$, where J_ε is a *Friedrichs' mollifier* with respect to the spatial variable in $\mathbb{S}^1$, cf. Appendix A. We note that $J_\varepsilon m \in C^1(J, C^\infty(\mathbb{S}^1))$. For this regularized curve $J_\varepsilon m$, we are going now to show that

$$\frac{d}{dt} \|J_\varepsilon m\|_{H^1}^2 \lesssim \|u\|_{H^2} \|m\|_{H^1}^2 , \tag{5.6}$$

for $\varepsilon \in (0,1]$. To do so, note that

$$\frac{d}{dt}\left\|J_\varepsilon m\right\|_{H^1}^2 = -2\int (J_\varepsilon m)(J_\varepsilon m_x u) - 4\int (J_\varepsilon m)(J_\varepsilon m u_x) \\ - 4\int (J_\varepsilon m_x)(J_\varepsilon m u_{xx}) - 6\int (J_\varepsilon m_x)(J_\varepsilon m_x u_x) - 2\int (J_\varepsilon m_x)(J_\varepsilon m_{xx} u).$$

Using Cauchy–Schwarz' inequality and Lemma A.2, the first four terms of the right hand-side can easily be bounded by $\left\|u\right\|_{H^2}\left\|m\right\|_{H^1}^2$, up to a positive constant independent of ε. The last term in the right hand-side can be rewritten as

$$\int (J_\varepsilon m_x)(u J_\varepsilon m_{xx}) + \int (J_\varepsilon m_x)([J_\varepsilon, uD]m_x).$$

An integration by parts shows that the first term is bounded by $\left\|u_x\right\|_\infty \left\|m\right\|_{H^1}^2$. By Cauchy-Schwarz' inequality and Lemma A.3, the same is true for the second term.

(3) Suppose now that $3/2 < \sigma \le q - r$. We are going to show that

$$\frac{d}{dt}\left\|J_\varepsilon m\right\|_{H^\sigma}^2 \lesssim \left\|u\right\|_{H^{\sigma+1}}\left\|m\right\|_{H^\sigma}^2, \tag{5.7}$$

for $\varepsilon \in (0,1]$. We have

$$\frac{d}{dt}\left\|J_\varepsilon m(t)\right\|_{H^\sigma}^2 = -2\left\langle \Lambda^\sigma J_\varepsilon m, \Lambda^\sigma J_\varepsilon(m_x u)\right\rangle_{L^2} - 4\left\langle \Lambda^\sigma J_\varepsilon m, \Lambda^\sigma J_\varepsilon(m u_x)\right\rangle_{L^2}.$$

Applying Cauchy-Schwarz' inequality, we first get

$$\left\langle \Lambda^\sigma J_\varepsilon m, \Lambda^\sigma J_\varepsilon(m u_x)\right\rangle_{L^2} \le \left\|J_\varepsilon m\right\|_{H^\sigma}\left\|J_\varepsilon(m u_x)\right\|_{H^\sigma}$$

and, by virtue of (A.2), we have

$$\left\|J_\varepsilon m\right\|_{H^\sigma}\left\|J_\varepsilon(m u_x)\right\|_{H^\sigma} \lesssim \left\|m\right\|_{H^\sigma}\left\|m u_x\right\|_{H^\sigma} \lesssim \left\|u\right\|_{H^{\sigma+1}}\left\|m\right\|_{H^\sigma}^2,$$

uniformly in ε (because $H^\sigma(\mathbb{S}^1)$ is a multiplicative algebra as soon as $\sigma > 1/2$). Observing that Λ^σ and J_ε commute (see Appendix A), we have

$$\left\langle \Lambda^\sigma J_\varepsilon m, \Lambda^\sigma J_\varepsilon(m_x u)\right\rangle_{L^2} = \int J_\varepsilon\left(u\Lambda^\sigma m_x\right) J_\varepsilon \Lambda^\sigma m \\ + \int J_\varepsilon\left([\Lambda^\sigma, u]m_x\right) J_\varepsilon \Lambda^\sigma m. \tag{5.8}$$

By virtue of Cauchy–Schwarz' inequality, (A.2) and the Kato–Ponce estimate (Lemma 5.2), the second term in the right hand-side of (5.8) is bounded (up to a constant independent of ε) by

$$\left\|u\right\|_{H^\sigma}\left\|m\right\|_{H^\sigma}^2,$$

because $\left\|m_x\right\|_\infty \lesssim \left\|m\right\|_{H^\sigma}$ for $\sigma > 3/2$. Introducing the operator $L := uD$, the first term in the right hand-side of (5.8) can be written as

$$\int (J_\varepsilon L\Lambda^\sigma m)(J_\varepsilon\Lambda^\sigma m) = \int (LJ_\varepsilon\Lambda^\sigma m)(J_\varepsilon\Lambda^\sigma m) + \int ([J_\varepsilon, L]\Lambda^\sigma m)(J_\varepsilon\Lambda^\sigma m).$$

We have first

$$\int (LJ_\varepsilon\Lambda^\sigma m)(J_\varepsilon\Lambda^\sigma m) = \frac{1}{2}\int \{(L+L^*)J_\varepsilon\Lambda^\sigma m\}(J_\varepsilon\Lambda^\sigma m).$$

But, since $L + L^* = -u_x I$, we get

$$\int \{(L + L^*)J_\varepsilon \Lambda^\sigma m\}(J_\varepsilon \Lambda^\sigma m) \lesssim \|u_x\|_\infty \|m\|_{H^\sigma}^2 .$$

Now, using Cauchy–Schwarz' inequality and Lemma A.3, we have

$$\int ([J_\varepsilon, L]\Lambda^\sigma m)(J_\varepsilon \Lambda^\sigma m) \lesssim \|u_x\|_\infty \|m\|_{H^\sigma}^2 .$$

Combining these estimates, we obtain finally

$$\frac{d}{dt} \|J_\varepsilon m(t)\|_{H^\sigma}^2 \lesssim \|u\|_{H^{\sigma+1}} \|m\|_{H^\sigma}^2 .$$

(4) If either $\sigma = 1$ or $\sigma > 3/2$, we integrate (5.6) or (5.7), respectively, over $[0, t]$ to get

$$\|J_\varepsilon m(t)\|_{H^\sigma}^2 \leq \|J_\varepsilon m(0)\|_{H^\sigma}^2 + C \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^{\sigma+1}} \int_0^t \|m(\tau)\|_{H^\sigma}^2 \, d\tau, \quad t \in J,$$

for some positive constant C (independent of ε). Again, letting $\varepsilon \to 0$ and invoking (A.1) in combination with Gronwall's lemma, we conclude that $\|m(t)\|_{H^\sigma}$ is bounded on I, as soon as $\|u(t)\|_{H^{\sigma+1}}$ is. Therefore, using an inductive argument, we deduce that $\|u(t)\|_{H^q}$ is bounded on I. This completes the proof. □

We next derive estimates on the flow map induced by time-dependent vector fields. These results are independent of the geodesic flow (2.3). Therefore we formulate them in some generality. Note that on a general Banach manifold, the flow of a continuous vector field may not exist [7]. However, in the particular case we consider here, we have the following result.

Proposition 5.3 (Ebin-Marsden, [8])**.** *Let $q > 5/2$ be given and let $u \in C^0\left(I, H^q(\mathbb{S}^1)\right)$ be a time dependent H^q vector field. Then its flow $t \to \varphi(t)$ is a C^1 curve in $\mathcal{D}^q(\mathbb{S}^1)$.*

Lemma 5.4. *Let $u \in C^0\left(J, H^q(\mathbb{S}^1)\right)$ be a time dependent vector field with $q > 3/2$. Assume that its associated flow φ exists and that $\varphi \in C^1(J, \mathcal{D}^q(\mathbb{S}^1))$. If $\|u_x\|_\infty$ is bounded on any bounded subinterval of J, then $\|\varphi_x\|_\infty$ and $\|1/\varphi_x\|_\infty$ are bounded on any bounded subinterval of J.*

Proof. Let

$$\alpha(t) = \max_{x\in\mathbb{S}^1} \varphi_x(t)(x), \quad \text{and} \quad \beta(t) = \max_{x\in\mathbb{S}^1} 1/\varphi_x(t)(x).$$

Note that α and β are continuous functions. Let I denote any bounded subinterval of J, and set

$$K = \sup_{t\in I} \|u_x(t)\|_\infty .$$

From equation $\varphi_t = u \circ \varphi$, we deduce that

$$\varphi_{tx} = (u_x \circ \varphi)\varphi_x, \quad \text{and} \quad (1/\varphi_x)_t = -(u_x \circ \varphi)/\varphi_x,$$

and therefore, we get

$$\alpha(t) \leq \alpha(0) + K \int_0^t \alpha(s)\, ds \quad \text{and} \quad \beta(t) \leq \beta(0) + K \int_0^t \beta(s)\, ds.$$

Thus the conclusion follows from Gronwall's lemma. □

Lemma 5.5. *Let $u \in C^0(J, H^q(\mathbb{S}^1))$ with $q > 3/2$ be a time-dependent vector field and assume that its associated flow φ exists with $\varphi \in C^1(J, \mathcal{D}^q(\mathbb{S}^1))$. If $\|u\|_{H^q}$ is bounded on any bounded subinterval of J, then $\|\varphi_x\|_{H^{q-1}}$ is bounded on any bounded subinterval of J.*

Proof. Let I denote any bounded subinterval of J. For $0 \le \rho \le q-1$, we have

$$\frac{d}{dt}\|\varphi_x\|_{H^\rho}^2 = 2\left\langle (u\circ\varphi)_x, \varphi_x\right\rangle_{H^\rho} \lesssim \|(u\circ\varphi)_x\|_{H^\rho}\|\varphi_x\|_{H^\rho}.$$

(i) Suppose first that $1/2 < \rho \le 1$. Invoking (4.1), we get

$$\begin{aligned}\|(u\circ\varphi)_x\|_{H^\rho} &\lesssim \|u_x\circ\varphi\|_{H^\rho}\|\varphi_x\|_{H^\rho} \\ &\lesssim C_\rho^1\left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}\right)\|u\|_{H^q}\|\varphi_x\|_{H^\rho}.\end{aligned}$$

Therefore, using the fact that $\|\varphi_x\|_\infty$ and $\|1/\varphi_x\|_\infty$ are bounded on I by virtue of Lemma 5.4, we conclude by Gronwall's lemma that $\|\varphi_x\|_{H^\rho}$ is bounded on I, for $0 \le \rho \le 1$.

(ii) Suppose now that $1 \le \rho \le 2$. Invoking (4.3), we get

$$\begin{aligned}\|(u\circ\varphi)_x\|_{H^\rho} &\lesssim \|u\circ\varphi\|_{H^{\rho+1}} \\ &\lesssim C_{\rho+1}^3\left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}\right)\|\varphi_x\|_{H^\rho}\|u\|_{H^q}.\end{aligned}$$

and we conclude again by Gronwall's lemma that $\|\varphi_x\|_{H^\rho}$ is bounded on I, for $0 \le \rho \le 2$.

(iii) Suppose finally that $\rho \ge 3$. Invoking (4.4), we get

$$\begin{aligned}\|(u\circ\varphi)_x\|_{H^\rho} &\lesssim \|u\circ\varphi\|_{H^{\rho+1}} \\ &\lesssim C_{\rho+1}^4\left(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{\rho-1}}\right)\|\varphi_x\|_{H^\rho}\|u\|_{H^q}.\end{aligned}$$

and we conclude by an induction argument on ρ that $\|\varphi_x\|_{H^\rho}$ is bounded on I for $0 \le \rho \le q-1$. This completes the proof. □

Theorem 5.6. *Let $r \ge 2$ and $q \ge r+1$. Assume that conditions 2.2 are satisfied and let*

$$(\varphi, v) \in C^\infty((t^-, t^+), T\mathcal{D}^q(\mathbb{S}^1))$$

denote the non-extendable solution of the geodesic flow (2.3), emanating from

$$(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1).$$

If the Eulerian velocity $u = v\circ\varphi^{-1}$ satisfies the estimate

$$\inf_{t\in[0,t^+)}\left(\min_{x\in\mathbb{S}^1}\{u_x(t,x)\}\right) > -\infty, \tag{5.9}$$

then $t^+ = \infty$. A similar statement holds for t^-.

Proof. Assume that $t^+ < \infty$ and that estimate (5.9) holds. In view of Theorem 5.1 we conclude that $\|u\|_{H^q}$ is bounded on $[0, t^+)$. By Lemma 5.4 we get furthermore that $\|\varphi_x\|_\infty$, and $\|1/\varphi_x\|_\infty$ are bounded on $[0, t^+)$ and by Lemma 5.5 we know that $\|\varphi_x\|_{H^{q-1}}$ is bounded on $[0, t^+)$. We obtain therefore that $\|v\|_{H^q}$ is bounded on $[0, t^+)$, by virtue of Lemma 4.2. Therefore, we deduce that

$$d_q(\varphi_0, \varphi(t)) + \|v(t)\|_{H^q}$$

is bounded on $[0, t^+)$. But this contradicts Corollary 4.3 which shows that

$$\lim_{t \uparrow t^+} [d(\varphi_0, \varphi(t)) + \|v(t)\|_{H^q}] = +\infty.$$

as soon as $t^+ < +\infty$. □

Theorem 1.1 follows from Theorem 5.6 and Lemma 2.5 in combination with Sobolev's embedding Theorem.

Remark 4. The same conclusion holds for the weak Riemannian metric induced by any inertia operator A of order $r > 3$ and satisfying presupposition 2.2, because then the norm

$$\|u\|_A := \langle Au, u \rangle_{L^2}$$

is equivalent to the $H^{r/2}$-norm.

Appendix A. Friedrichs mollifiers

Friedrichs mollifiers were introduced by Kurt Otto Friedrichs in [13]. We briefly recall the construction for periodic functions (see [18] for more details). Let ρ be a nonnegative, even, smooth bump function of total weight 1 and supported in $(-1/2, 1/2)$. We set

$$\rho_\epsilon(x) := \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right),$$

and define the *Friedrichs' mollifer* J_ε as the operator

$$J_\varepsilon u = \rho_\epsilon * u,$$

where $*$ denotes the convolution. Note that if $u \in L^2(\mathbb{S}^1)$, then $J_\varepsilon u \in \mathrm{C}^\infty(\mathbb{S}^1)$ and that J_ε is a bounded operator from $L^2(\mathbb{S}^1)$ to $H^q(\mathbb{S}^1)$ for any $q \geq 0$.

The operator J_ε is a Fourier multiplier. Thus it commutes with any other Fourier multiplier, in particular with the spatial derivative D. It commutes of course also with temporal derivative ∂_t for functions depending on $(t, x) \in \mathbb{R} \times \mathbb{S}^1$. Note also that J_ε is symmetric with respect to the L^2 scalar product. The main properties of J_ε that have been used in this paper are the following.

Lemma A.1. *Given $q \geq 0$ and $u \in H^q(\mathbb{S}^1)$, then*

$$\|J_\varepsilon u - u\|_{H^q} \to 0, \quad \text{as } \varepsilon \to 0. \tag{A.1}$$

Lemma A.1 is a classical result. Its proof can be found in [1, Lemma 3.15]), for instance.

Lemma A.2. *We have*

$$\|J_\varepsilon u\|_{H^q} \lesssim \|u\|_{H^q}, \quad \forall u \in H^q(\mathbb{S}^1), \tag{A.2}$$

uniformly *in $\varepsilon \in (0, 1]$ and $q \geq 0$.*

The proof of Lemma A.2 is a consequence of the following special case of *Young's inequality* ([23, Theorem 2.2, Chapter 1])

$$\|f * u\|_{L^2} \lesssim \|f\|_{L^1} \|u\|_{L^2}, \tag{A.3}$$

and the fact that Λ^q and J_ε commute.

Finally, we have been using the following commutator estimate on $[J_\varepsilon, uD]$.

Lemma A.3. *Let $u \in C^1(\mathbb{S}^1)$ and $m \in L^2(\mathbb{S}^1)$. Then*

$$\|J_\varepsilon(um_x) - uJ_\varepsilon(m_x)\|_{L^2} \lesssim \|u_x\|_\infty \|m\|_{L^2},$$

uniformly in $\varepsilon \in (0,1]$.

Proof. Let $u \in C^1(\mathbb{S}^1)$. Note first that the linear operator

$$K_\varepsilon(m) := J_\varepsilon(um_x) - uJ_\varepsilon(m_x),$$

defined on $\mathrm{C}^\infty(\mathbb{S}^1)$, is an integral operator with kernel

$$k_\varepsilon(x,y) = \frac{\partial}{\partial y}\left\{(u(x)-u(y))\rho_\varepsilon(x-y)\right\}.$$

We have therefore

$$K_\varepsilon(m) = -\rho_\varepsilon * (u_x m) - \int_{\mathbb{S}^1} \rho_\varepsilon'(x-y)[u(x)-u(y)]m(y)\,dy. \tag{A.4}$$

By virtue of Young's inequality (A.3), the L^2-norm of the first term of the right hand-side of (A.4) is bounded (up to some positive constant independent of ε) by

$$\|\rho_\varepsilon\|_{L^1} \|u_x m\|_{L^2} \le \|u_x\|_\infty \|m\|_{L^2},$$

because $\|\rho_\varepsilon\|_{L^1} = 1$. The L^2 norm of the second term of the right hand-side of (A.4) is bounded by

$$(\varepsilon \|u_x\|_\infty) \left\|\rho_\varepsilon' * m\right\|_{L^2},$$

because the support of ρ_ε is contained in $[-\varepsilon/2, \varepsilon/2]$. Using again Young's inequality (A.3), we get then

$$\left\|\rho_\varepsilon' * m\right\|_{L^2} \lesssim \left\|\rho_\varepsilon'\right\|_{L^1} \|m\|_{L^2} \lesssim \frac{1}{\varepsilon}\|m\|_{L^2},$$

because $\|\rho_\varepsilon'\|_{L^1} = \mathcal{O}(1/\varepsilon)$. This concludes the proof. □

References

[1] R. A. Adams. *Sobolev Spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] M. Bauer, M. Bruveris, P. Harms, and P. W. Michor. Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. *Ann. Global Anal. Geom.*, 44(1):5–21, 2013.

[3] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[4] A. Constantin and J. Escher. On the blow-up rate and the blow-up set of breaking waves for a shallow water equation. *Math. Z.*, 233(1):75–91, 2000.

[5] A. Constantin and B. Kolev. Geodesic flow on the diffeomorphism group of the circle. *Comment. Math. Helv.*, 78(4):787–804, 2003.

[6] P. Constantin, P. D. Lax, and A. Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38(6):715–724, 1985.

[7] J. Dieudonné. Deux exemples singuliers d'équations différentielles. *Acta Sci. Math. Szeged*, 12(Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars B):38–40, 1950.

[8] D. G. Ebin and J. E. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.

[9] J. Escher and B. Kolev. The Degasperis-Procesi equation as a non-metric Euler equation. *Math. Z.*, 269(3-4):1137–1153, 2011.

[10] J. Escher and B. Kolev. Right-invariant Sobolev metrics H^s on the diffeomorphisms group of the circle. *ArXiv e-prints*, feb 2012. To appear in The Journal of Geometric Mechanics.

[11] J. Escher, B. Kolev, and M. Wunsch. The geometry of a vorticity model equation. *Commun. Pure Appl. Anal.*, 11(4):1407–1419, Jul 2012.

[12] L. P. Euler. Du mouvement de rotation des corps solides autour d'un axe variable. *Mémoires de l'académie des sciences de Berlin*, 14:154–193, 1765.

[13] K. O. Friedrichs. The identity of weak and strong extensions of differential operators. *Trans. Amer. Math. Soc.*, 55:132–151, 1944.

[14] L. Guieu and C. Roger. *L'algèbre et le groupe de Virasoro*. Les Publications CRM, Montreal, QC, 2007. Aspects géométriques et algébriques, généralisations. [Geometric and algebraic aspects, generalizations], With an appendix by Vlad Sergiescu.

[15] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)*, 7(1):65–222, 1982.

[16] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891 – 907, 1988.

[17] S. Kouranbaeva. The Camassa-Holm equation as a geodesic flow on the diffeomorphism group. *J. Math. Phys.*, 40(2):857–868, 1999.

[18] S. G. Krantz. *Partial Differential Equations and Complex Analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. Lecture notes prepared by Estela A. Gavosto and Marco M. Peloso.

[19] S. Lang. *Fundamentals of Differential Geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[20] R. McLachlan and X. Zhang. Well-posedness of modified Camassa-Holm equations. *J. Differential Equations*, 246(8):3241 – 3259, 2009.

[21] P. Michor and D. Mumford. On Euler's equation and 'EPDiff'. *The Journal of Geometric Mechanics*, 5(3):319 – 344, sep 2013.

[22] G. Misiołek. Classical solutions of the periodic Camassa-Holm equation. *Geom. Funct. Anal.*, 12(5):1080–1104, 2002.

[23] B. E. Petersen. *Introduction to the Fourier Transform & Pseudodifferential Operators*, volume 19 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1983.

[24] M. Taylor. Commutator estimates. *Proc. Amer. Math. Soc.*, 131(5):1501–1507 (electronic), 2003.

[25] M. E. Taylor. *Pseudodifferential Operators and Nonlinear PDE*, volume 100 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1991.

[26] A. Trouvé and L. Younes. Local geometry of deformable templates. *SIAM J. Math. Anal.*, 37(1):17–59 (electronic), 2005.

[27] M. Wunsch. On the geodesic flow on the group of diffeomorphisms of the circle with a fractional Sobolev right-invariant metric. *J. Nonlinear Math. Phys.*, 17(1):7–11, 2010.

Institute for Applied Mathematics, University of Hanover, D-30167 Hanover, Germany

E-mail address: escher@ifam.uni-hannover.de

Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France

E-mail address: boris.kolev@math.cnrs.fr