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Orbits in a stochastic Goodwin-Lotka-Volterra model

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Abstract

This paper examines the cycling behavior of a deterministic and a stochastic version of the economical interpretation of the Lotka-Volterra model, the Goodwin model. We provide a characterization of orbits in the deterministic, highly non-linear version of the model. We then study the cycling behavior for a stochastic version, where a Brownian noise is introduced via an heterogeneous productivity factor. Fundamental properties of the system are provided. We start with a perturbation analysis for small range of volatility and prove that the system locally behaves in average as the deterministic counterpart. We then prove with original stochastic Lyapunov techniques that the system produces cycles around an equilibrium point in finite time for general volatility levels. We also conjecture a limit cycle phenomenon via numerical simulations.

Keywords: Lotka-Volterra model, Goodwin model, Business cycles, Lyapunov techniques, Stochastic differential equation

1. Introduction

The Lotka-Volterra equation is at the heart of population dynamics, but also possesses a famous economical interpretation. Introduced by Richard Goodwin in 1967 [4], the model bears several differences with its biological counterpart. Firstly, the model results from mathematical derivation of economical assumptions and its justification as a pertinent model relies on the latter. Secondly and consequently, the state variables of the planar oscillator represent two economical ratios: the workers wage share of output x and the employment rate y of a closed capitalist economy. As recently noticed in [2], the model shall be modified to ensure the viability of this interpretation:

$$\begin{cases} dx_t/dt = x_t (\Phi(y_t) - \alpha) \\ dy_t/dt = y_t (\kappa(x_t) - \gamma) \end{cases} \quad (G)$$

Here, Φ and κ are highly non-linear functions with vertical asymptotes at $y = 1$ and $x = 1$ respectively. It is proved in [2] that the system stays in the interior of the unit square if it starts inside it under this condition, producing closed orbits around a non-hyperbolic equilibrium point. The purpose of Section 2 is to fully characterize these orbits. The last difference with the biological counterpart is that the aim of (G) is to study the fluctuations of the production level P_t of the economy as an auxiliary variable, and the period of real business cycles. On this matter, Goodwin himself conceded that the model is “starkly schematized and hence quite unrealistic” [4]. Indeed, it hardly connects with irregular observed trajectories, see [5] for an econometric study. Bartlett [1] admitted an equivalent statement for the biological interpretation of the system.

But surprisingly, to our knowledge, there is no reference of addition of stochastic components to the Goodwin model, whereas Bartlett [1] proposed a stochastic Lotka-Volterra equation as an improvement of the deterministic

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system. After this seminal work, stochastic Lotka-Volterra equations have been extensively studied, with focus on random factors affecting the environment and properties of biological systems, see [10, 11, 12, 14, 17] for recent examples. The goal of the present paper is to fill the gap in Economics. To do so, we take into account the three previous remarks. We derive the stochastic model from economical assumptions and an assumption on the heterogeneity of the productivity rate α among workers. The productivity of the labor class is theoretically one of the parameters that should affect real business cycles, see [3] for example, so that it seems a reasonable starting point. Keeping in mind the second distinction of Goodwin interpretation of the prey-predator equation, we study the system and its fundamental properties with functions Φ and κ forbidding values for the state process (x_t, y_t) .

The last particularity of the economical angle motivates the main result of the present paper. Previous references contribute to the mathematical study of such models by analyzing asymptotic distributions and moments. For the present stochastically perturbed system, we prove that the system produces orbits. We first study the model for small values of volatility by perturbation methods. This analysis shows that within first order approximation, the stochastic version preserves the characteristics of the deterministic one, in the sense of expectation. We thus keep the cycling behavior for local times. We then relax the small volatility assumption. We prove under mild assumptions that the system makes orbits, in some new sense, around a non stable equilibrium point in finite time, by means of stochastic Lyapunov techniques [9]. We additionally provide numerical results to conjecture the existence of an asymptotic stochastic limit cycle.

The structure of the paper follows the above outline. Section 2 introduces the deterministic Goodwin model and full description of its orbits. Section 3 introduces the stochastic model and its fundamental properties. In Section 4, we prove the cycling behavior of the new system with small volatility. In Section 5, we show via recurrent domain techniques that any solution to the stochastic system produces orbits in finite time. Examples and illustrations are provided along the paper.

2. Goodwin interpretation, orbits and periods

The Goodwin model [4] describes with Eq. (G) a closed capitalist economy in a dynamical fashion. In this model, total yearly output P_t comes from a Leontief production function with full utilization of invested capital K_t (normalized by a constant ν):

$$P_t = \min(K_t/\nu; a_t y_t N_t) = K_t/\nu = a_t y_t N_t ,$$

with the adjusted employment rate y_t , the productivity of workers $a_t := a_0 \exp(\alpha t)$ and the size of the labor class given by $N_t := N_0 \exp(\beta t)$. A second statement concerns the evolution of capital, which evolves by investing according to an investment function κ minus a depreciation rate:

$$dK_t/dt = (\kappa(x_t) - \delta)K_t .$$

The function κ gives as a fraction of capital the amount put in investment by the capitalists class. It is an increasing function of profits, and thus a decreasing function of the wage share x_t . It is originally given by Say's law, i.e., everything left from production is reinvested:

$$\kappa(x) := \frac{1-x}{\nu} \quad \text{for } x \in [0, +\infty), \quad (1)$$

with ν the capital-to-output ratio introduced above. We will relax this assumption later. The last statement concerns the introduction of Φ in the bargaining power of real wage $w_t := a_t x_t$ by imposing dynamics $dw_t = \Phi(y_t)w_t dt$. The function Φ is economically supported by the Phillips curve and is an increasing function on its domain. Stating $\gamma := \alpha + \beta + \delta$ allows to retrieve (G) for $(x_t, y_t) = (w_t/a_t, K_t/(\nu a_t N_t))$.

Assumption 1. We assume the following:

- (i) $\Phi \in C^2([0, 1])$ is increasing convex on $[0, 1]$. Additionally, $\Phi(0) \in (-\infty, \alpha)$ and $\lim_{y \rightarrow 1^-} \Phi(y) = +\infty$.
- (ii) $\kappa \in C^2([0, +\infty))$ is decreasing. Additionally, $\kappa(0) \in (\gamma, +\infty)$ and $\lim_{x \rightarrow +\infty} \kappa(x) = -\infty$.

Remark 1. The modification proposed by Desai and al [2] is kept for the function Φ but not for κ , keeping the domain \mathbb{R}_+ . We have both economical and mathematical reasons. If investment comes from production outcome only, Say's law (1) ensures stock-flow consistency, an important criterion for economical models. Otherwise, external sources of incomes, as proposed in [7], can justify a total wage greater than output, i.e., $x_t \geq 1$ is possible. Finally, symmetry of x and y is put aside to study the two cases of bounded and unbounded variables. This asymmetry will be emphasized in the subsequent assumptions, and the extension to another domain follows easily.

Under Assumption 1, the Goodwin-Lotka-Volterra model is well-known for its cycling behavior around a non-hyperbolic equilibrium point $(\hat{x}, \hat{y}) := (\kappa^{-1}(\gamma), \Phi^{-1}(\alpha))$ for any starting point (x_0, y_0) in the domain $D := (0, +\infty) \times (0, 1)$. The closed orbits are described by the set of points $(x, y) \in D$ verifying $V(x, y) = V(x_0, y_0)$, with

$$V(x, y) := V_1(x) + V_2(y) := \int_{\hat{x}}^x \frac{\kappa(\hat{x}) - \kappa(s)}{s} ds + \int_{\hat{y}}^y \frac{\Phi(s) - \Phi(\hat{y})}{s} ds$$

being a constant of motion. The function reaches its minimum 0 at point (\hat{x}, \hat{y}) , so that V is a Lyapunov function for (G). The other equilibrium point of the system $(0, 0)$ is a saddle point and not economically relevant: it is eluded in all the paper. It is straightforward that V converges to infinity for (x, y) converging to the boundary of D . The domain of interest immediately follows.

Proposition 1 (Desail and al.[2]). *If $(x_t, y_t)_{t \geq 0}$ is a solution of system (G) starting at $(x_0, y_0) \in D := (0, +\infty) \times (0, 1)$, then $(x_t, y_t) \in D$ for all $t \geq 0$.*

The function V characterizes orbits and allows to study the period of cycles around the center (\hat{x}, \hat{y}) . The following theorem generalizes the approach of [6].

Theorem 1. *Let $(x_t, y_t)_{t \geq 0}$ be a solution to (G) such that $(x_0, y_0) \in D$. Let $V_0 := V(x_0, y_0)$. Then $(x_T, y_T) = (x_0, y_0)$ for T defined by*

$$T(V_0) := \int_{\log(\underline{x})}^{\log(\bar{x})} \frac{1}{F_1^{-1} \circ G(z)} - \frac{1}{F_2^{-1} \circ G(z)} dz \quad (2)$$

where $\underline{x} < \bar{x}$ are the two solutions to equation $V_1(x) = V_0$ and G, F_1 and F_2 are defined as the function $G(z) := V_0 - V_1(e^z)$, $F_1(u) := V_2 \circ \Phi^{-1}(u^+ + \alpha)$ and $F_2(u) := V_2 \circ \Phi^{-1}(-u^- + \alpha)$.

Proof The function V being a constant of motion for system (G), the graph of $\{(x, y) \in D : V(x, y) = V_0\}$ is a closed contour line included in D for all $(x_0, y_0) \neq (\hat{x}, \hat{y})$. Therefore, its orthogonal projection on the x -axis is given by a closed interval I , with boundaries given by $V_1(x) = V_0$, so that $I = [\underline{x}, \bar{x}]$. System (G) being homogeneous, we fix $x_0 = \underline{x}$ without loss of generality and keep $V(x_0, y_0) = V_0$. This implies $V_2(y_0) = 0$, i.e., $y_0 = \hat{y}$. Let $T_1 := \inf\{t \geq 0 : x_t = \bar{x}\}$. For $t \in [0, T_1]$, $(x_t, y_t) \in [\underline{x}, \bar{x}] \times [\hat{y}, \bar{y}]$, with \bar{y} such that $V_2(\bar{y}) = V_0$. Let $z(t) := \log(x_t)$ for $t \geq 0$. It follows the equation $dz = (\Phi(y) - \alpha)dt$ so that $y = \Phi^{-1}(dz/dt + \alpha)$. By differentiating dz/dt , we can write a second order ODE

$$\frac{d^2 z}{dt^2} = \Phi'(y) \frac{dy}{dt} = \Phi' \left(\Phi^{-1} \left(\frac{dz}{dt} + \alpha \right) \right) \Phi^{-1} \left(\frac{dz}{dt} + \alpha \right) [\kappa(e^z) - \gamma]$$

which can be formulated, with additional variable $u := \Phi(y) - \alpha$ and function $\Psi := \Phi' \circ \Phi^{-1} \times \Phi^{-1}$, as

$$\begin{cases} dz/dt = u \\ du/dt = \Psi(u + \alpha) [\kappa(e^z) - \gamma] \end{cases} \quad (3)$$

Since $z_t \in [\underline{z}, \bar{z}] := [\log(\underline{x}), \log(\bar{x})]$ and $u_t \in [0, \Phi(\bar{y}) - \alpha]$ for $t \in [0, T_1]$, we separate variables of Eq. (3) and integrate to obtain two new equal quantities:

$$F(u) := \int_0^u \frac{s}{\Psi(s + \alpha)} ds = \int_{\underline{z}}^{\bar{z}} [\kappa(e^s) - \gamma] ds =: G(z). \quad (4)$$

The functions F verifies $F(0) = G(\bar{z}) = 0$, is increasing on $[0, \Phi(\bar{y}) - \alpha]$ and decreasing on $[\Phi(\underline{y}) - \alpha, 0]$ with $\underline{y} < \hat{y}$ so that $V_2(\underline{y}) = 0$. Coming back to $y = \Phi^{-1}(u + \alpha)$ we get

$$F(u) = \int_0^u \frac{s}{\Phi'(\Phi^{-1}(s + \alpha))\Phi^{-1}(s + \alpha)} ds = \int_{\hat{y}}^{\Phi^{-1}(u + \alpha)} \frac{\Phi(s) - \Phi(\hat{y})}{s} ds = V_2(\Phi^{-1}(u + \alpha)),$$

implying that $F(u) \in [0, V(\underline{x}, \hat{y})]$ for $u \in [\Phi(\underline{y}) + \alpha, \Phi(\bar{y}) + \alpha]$. Altogether, we can construct two monotonous restrictions of F :

$$F_1(u) := V_2(\Phi^{-1}(u^+ + \alpha)) \text{ and } F_2(u) := V_2(\Phi^{-1}(-u^- + \alpha)).$$

Notice that if $t \in [0, T_1]$, then $u_t := \Phi(y_t) + \alpha \in [0, \Phi(\bar{y}) + \alpha]$. Thus, $F_1(u_t)$ is a strictly increasing function of t taking its values in $[0, V(\underline{x}, \hat{y})]$. Getting back to $x = e^z$ for G , we have for $z \geq \underline{z}$

$$G(z) := \int_{\underline{x}}^{\hat{x} \wedge e^z} \frac{\kappa(s) - \gamma}{s} ds + \int_{\hat{x} \wedge e^z}^{e^z} \frac{\kappa(s) - \gamma}{s} ds = V_0 - V_1(e^z)$$

Since $\text{sign}(\kappa(x) - \gamma) = \text{sign}(\hat{x} - x)$ we have $\max_{z \in [\underline{z}, \bar{z}]} G(z) = G(\log(\hat{x})) = V_1(\underline{x}) = V(\underline{x}, \hat{y})$, while minimums are given by $G(\bar{z}) = G(\underline{z}) = 0$. This sums up with $G([\underline{z}, \bar{z}]) \subset [0, V_0]$, so we can write on this interval $F_1^{-1} \circ G(z) = u = dz/dt$ which finally gives

$$T_1 = \int_{\underline{z}}^{\bar{z}} \frac{dz}{F_1^{-1}(G(z))}.$$

We apply the same method for the other half orbit, taking $(x_0, y_0) = (\bar{x}, \hat{y})$ and $T_2 := \inf\{t \geq 0 : x_t = \underline{x}\}$, to reach the other half of expression (2). \square

Remark 2. If we linearize (G) around (\hat{x}, \hat{y}) , we can verify that the solution to the resulting homogeneous linear system is a linear combination of sines and cosines of $(-\hat{x}\Phi'(\hat{y})\hat{y}\kappa'(\hat{x})t)$. It follows that

$$\lim_{V_0 \rightarrow 0} T(V_0) = \frac{2\pi}{\sqrt{-\hat{x}\Phi'(\hat{y})\hat{y}\kappa'(\hat{x})}}.$$

Example 1. We assume that investment follows Say's law (1) and Philips curve is provided by [7]:

$$\kappa(x) := \frac{1-x}{\nu} \text{ and } \Phi(y) := \frac{\phi_1}{(1-y)^2} + \phi_0.$$

Assumption 1 holds here and the equilibrium point is given by $(\hat{x}, \hat{y}) = (1 - \nu\gamma, 1 - \sqrt{\phi_1/(\alpha - \phi_0)})$. The Lyapunov function V is given by $V(x, y) = V_1(x) + V_2(y)$ with

$$V_1(x) = \frac{1}{\nu} \left(x - \hat{x} \left(\log\left(\frac{x}{\hat{x}}\right) + 1 \right) \right) \quad (5)$$

and

$$V_2(y) = \phi_1 \left(\log\left(\frac{1-\hat{y}}{1-y}\right) + \left(\frac{\hat{y}}{1-\hat{y}}\right) \log\left(\frac{\hat{y}}{y}\right) + \frac{1}{y-y^2} - \frac{1}{\hat{y}-\hat{y}^2} \right). \quad (6)$$

Although function T given by Theorem 1 is not explicit here, numerical computations allow to approximate it with a linear function, see first part of Fig. 1. As put in Remark 2 above, the period does not converge to 0 with the orbits concentrating to (\hat{x}, \hat{y}) . A phase portrait with values of T is provided in second part of Fig. 1.

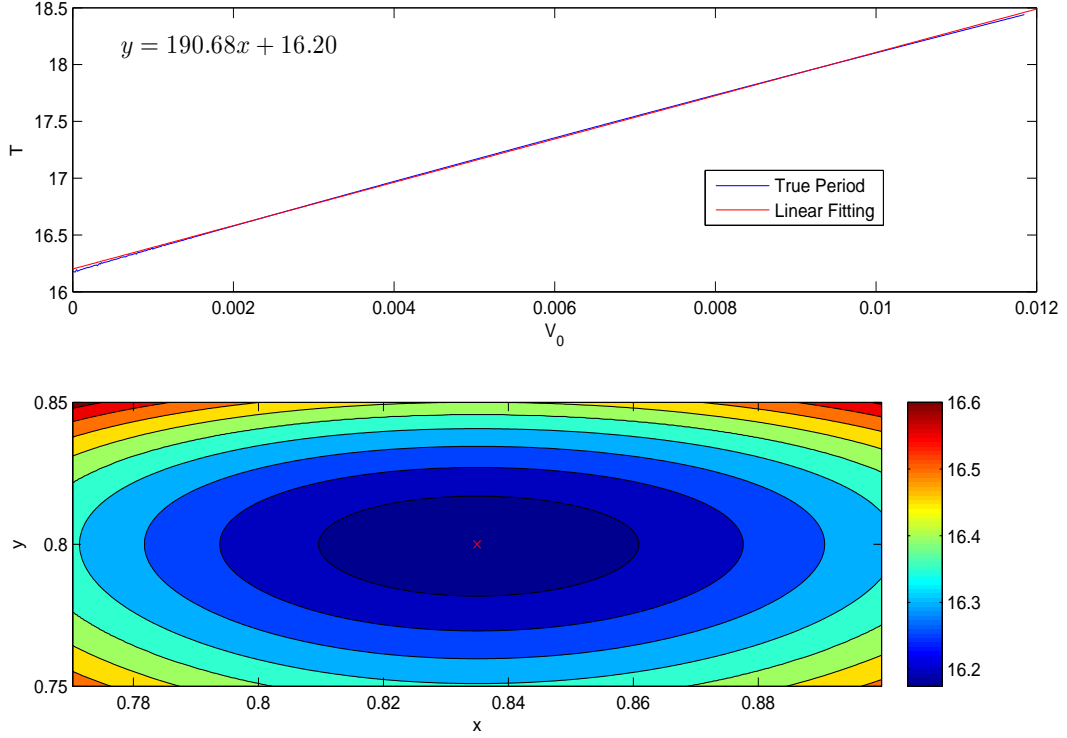


Figure 1: Up: values of T as a function of V . Down: Contour lines with values of T in a subset of D . Parameters set at $(\alpha, \gamma, \nu, \phi_0, \phi_1) = (0.025, 0.055, 3, 0.040064, 0.000064)$. Equilibrium point at $(\hat{x}, \hat{y}) = (0.8350, 0.80)$.

3. Goodwin model with stochastic productivity

Consider a labor force with constant but heterogeneous growth rate of productivity per capita. One can imagine the repartition of a_t among an infinite population to follow an atom-less probability distribution on \mathbb{R} . In the Goodwin model at any given time, only a fraction y of the labor force is employed. Assume now that, for exogenous reasons, the working subset of the labor class changes continuously and unsteadily among it. One can support this assumption by the permanent flow of workers coming in and out of the labor class. However, this random change of the subset of workers is conditioned by y : the higher the fraction of workers y is, the less the variations of this fraction are possible: for $y = 1$, productivity remains constant. Now assume that the productivity growth rate has an average α over the whole population. According to this vision, we propose that productivity follows dynamics

$$da_t := a_t d\alpha_t = a_t (\alpha dt - \sigma(y_t) dW_t) \quad (7)$$

where $\sigma(y)$ is the volatility parameter, a non-negative decreasing function of y verifying $\sigma(1) = 0$. Here, W is a one dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the filtration is $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} = \sigma(W_s, 0 \leq s \leq t)$ completed with \mathbb{P} -null sets. We proceed to the same derivation of the model as in Section 2. Using Itô formula we obtain the following stochastic differential equation:

$$\begin{cases} dx_t = x_t ((\Phi(y_t) - \alpha + \sigma^2(y_t))dt + \sigma(y_t)dW_t) \\ dy_t = y_t ((\kappa(x_t) - \gamma + \sigma^2(y_t))dt + \sigma(y_t)dW_t) \end{cases} \quad (SG)$$

We propose to study a specific case of this system by assuming a unique root to the deterministic part of (SG).

Assumption 2. *There is a unique $(\tilde{x}, \tilde{y}) \in D$ verifying $\Phi(\tilde{y}) - \alpha + \sigma^2(\tilde{y}) = 0$ and $\tilde{x} := \kappa^{-1}(\gamma - \sigma^2(\tilde{y}))$.*

We introduce the Dynkin operator associated to (SG). For $\varphi \in C^{1,2,2}(\mathbb{R}_+ \times (0, +\infty) \times (0, 1))$, we define

$$\mathcal{L}\varphi(t, x, y) := \left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}x(\Phi(y) - \alpha + \sigma^2(y)) + \frac{\partial\varphi}{\partial y}y(\kappa(x) - \gamma + \sigma^2(y)) + \frac{\sigma^2(y)}{2} \left(\frac{\partial^2\varphi}{\partial x^2}x^2 + \frac{\partial^2\varphi}{\partial y^2}y^2 + 2\frac{\partial^2\varphi}{\partial x\partial y}xy \right) \right)(t, x, y)$$

For a stochastic differential equation to have a unique global solution for any given initial value, functions Φ and κ are generally required to satisfy linear growth and local Lipschitz conditions, see [9]. This clearly cannot stand here according to Assumption 1. We thus study the system under the following general condition.

Assumption 3. *There exists $K > 0$ and $k > 0$ such that, for all $(x, y) \in D$,*

$$(i) \quad -x\kappa'(x) - \kappa(x) \leq KV_1(x) + k$$

$$(ii) \quad \sigma^2(y)\Phi'(y) \leq KV_2(y) + k .$$

Remark 3. Notice that in line with Theorem 3.6 in [9], these conditions appear in the proof as *the best possible conditions*, without being formally necessary. Assumption 3.(i) holds for κ having a polynomial growth, suiting a large class of functions. The bound (ii) concerns both Φ and σ to ensure that $y_t \leq 1$ \mathbb{P} -a.s.

Theorem 2. *There exists a regular solution $(x_t, y_t)_{t \geq 0}$ to Eq. (SG) starting at any point $(x_0, y_0) \in D$. Moreover, the solution is unique up to \mathbb{P} -null sets, has the Markov property and remains in D \mathbb{P} -a.s.*

Proof It suffices to show that conditions of Theorem 3.4 and Corollary 3.1 in [9] are satisfied, which we reproduce in Appendix A for convenience. First, the locally Lipschitz growth and sub-linearity conditions of the coefficients of the system on every compact subset included in D follow from Assumption 1. We are only left to check conditions (A.2) and (A.3) in Theorem 6 of Appendix A. Applying the Dynkin operator to the Lyapunov function V we get

$$\begin{aligned} \mathcal{L}V(x, y) &= [\kappa(\hat{x}) - \kappa(x)](\Phi(y) - \alpha + \sigma^2(y)) + [\Phi(y) - \Phi(\hat{y})](\kappa(x) - \gamma + \sigma^2(y)) \\ &\quad + ([\kappa(x) - \kappa(\hat{x}) - x\kappa'(x)] + [\Phi(\hat{y}) - \Phi(y) + y\Phi'(y)])\sigma^2(y)/2 . \end{aligned}$$

Since $\alpha = \Phi(\hat{y})$ and $\gamma = \kappa(\hat{x})$, we obtain

$$\mathcal{L}V(x, y) = ([\kappa(\hat{x}) - \kappa(x) - x\kappa'(x)] + [\Phi(y) - \Phi(\hat{y}) + y\Phi'(y)])\sigma^2(y)/2 . \quad (8)$$

Applying Assumption 3, we obtain $\mathcal{L}V(x, y) \leq \max(\sigma^2(0)/2; 2K)V(x, y) + 2k$ for $K, k > 0$ given in the same assumption, and then condition (A.2) holds for V . From separation of variables in V ,

$$\inf_{x \in [0, +\infty)} V(x, y) = V_2(y) + \inf_{x \in [0, +\infty)} V_1(x) = V_2(y)$$

which tends to infinity when y goes to either 1 or 0. We also have $\inf_{y \in (0, 1)} V(x, y)$ going to infinity as x goes to 0 or $+\infty$. Condition (A.3) is then satisfied, which allows to apply Theorem 6. \square

Example 2. We continue the study of Example 1, with now (SG) and volatility function $\sigma(y) := \sigma_0(1 - y)$ with $\sigma_0 > 0$. According to Assumption 2, $(1 - \tilde{y})^2$ is the root of a quadratic polynomial

$$\sigma_0^2(1 - \tilde{y})^4 - (\alpha + \phi_0)(1 - \tilde{y})^2 + \phi_1 = 0 . \quad (9)$$

By assuming that $\phi_1 \leq (\alpha + \phi_0)/2$ and $\sigma_0 \leq \max\{(\alpha + \phi_0)/(2\sqrt{\phi_1}), (\alpha + \phi_0 - \phi_1)\}$, we have a sufficient condition for having a unique root in $(0, 1)$ to Eq. (9). Assumption 2 thus holds under that restriction. Assumption 3 also holds. Indeed for all $y \in (0, 1)$, following Eq. (6),

$$\sigma^2(y)\Phi'(y) = \frac{2\sigma_0^2\phi_1}{(1-y)} \leq 2\sigma_0^2 \left(V_2(y) - \phi_1 \left(\frac{1}{1-\hat{y}} \left(\hat{y} \log(\hat{y}) - \frac{1}{\hat{y}} \right) + \log(1-\hat{y}) \right) \right)$$

and following Eq. (5) along with the sub-linearity of the log function,

$$-\kappa(x) - x\kappa'(x) = \frac{2x-1}{v} \leq \frac{2}{1-\hat{x}} (V_1(x) + \hat{x} - \hat{x} \log(\hat{x})) . \quad (10)$$

Remark 4. It is a trivial property that if (x_t, y_t) follows (SG),

$$\mathbb{P} \left[\liminf_t x_t = x \right] + \mathbb{P} \left[\liminf_t y_t = y \right] = 0$$

for any (x, y) in D , as the drift part or the volatility coefficient is non null on D . It also holds for the boundaries $\{0\} \times (0, 1)$ and $(0, +\infty) \times \{0, 1\}$. For the point $(0, 0)$, we observe that $\mathcal{L}V(0, 0) < -\varepsilon < 0$ for some small $\varepsilon > 0$. By continuity of $\mathcal{L}V$, it holds also in a region $[0, \varepsilon] \times [0, \varepsilon]$. Since $\lim_{(x,y) \downarrow (0,0)} V(x, y) = +\infty$, the property holds for the closure of D .

It is possible to picture a solution to (SG) as a trajectory continuously jumping from an orbit of (G) to another. Along this idea, V provides an estimate on trajectories.

Theorem 3. *Let $(x_t, y_t)_{t \geq 0}$ be a regular solution to Eq. (SG) with initial condition $(x_0, y_0) \in D$. Let $V_0 := V(x_0, y_0)$ and $\rho \leq V_0$. Let*

$$\tau_\rho := \inf\{t > 0 : |V(x_t, y_t) - V_0| \geq \rho\}.$$

Then for all $\mu > 0$

$$\mathbb{P} \left[\tau_\rho > \Theta(\rho) \right] \geq \left(1 - \frac{I(V_0, \rho)}{\mu^2} \right) \quad (11)$$

with

$$\Theta(\rho) := \frac{2 \left(\mu^2 + \mu \sqrt{\mu^2 + 2\rho R(V_0, \rho)} + \rho R(V_0, \rho) \right)}{(R(V_0, \rho)\sigma)^2}$$

for two constants $R(V_0, \rho)$ and $I(V_0, \rho)$ depending only on V_0 and ρ .

Proof Let us define the compact set $D(V_0, \rho) := \{(x, y) \in D : |V(x, y) - V_0| \leq \rho\}$ and constants

$$R(V_0, \rho) := \max_{D(V_0, \rho)} \left\{ \sigma^2(y) (\kappa(\hat{x}) - \kappa(x) - x\kappa'(x) + y\Phi'(y) + \Phi(y) - \Phi(\hat{y})) \right\}$$

and

$$I(V_0, \rho) := \max_{D(V_0, \rho)} \left\{ \sigma^2(y) (\kappa(\hat{x}) - \kappa(x) + \Phi(y) - \Phi(\hat{y}))^2 \right\} .$$

Fix $\mu > 0$. Now we define the \mathcal{F}_{τ_ρ} -measurable set $A_\mu = \left\{ \omega \in \Omega : \sup_{0 < t \leq \tau_\rho} |M_t(\omega)| \leq \mu \right\}$ where $(M_t)_{t \geq 0}$ is a martingale defined by

$$M_t = \frac{1}{\sqrt{t}} \int_0^t \sigma(y_s) (\kappa(\hat{x}) - \kappa(x_s) + \Phi(y_s) - \Phi(\hat{y})) dW(s) .$$

for $t > 0$ and by $M_t = 0$ for $t = 0$. Notice that M is not right-continuous at $t = 0$ but still verifies $\mathbb{E} \left[M_t^2 \right] \leq I(V_0, \rho)$ for all $0 < t \leq \tau_\rho$. By replacing M_t by its càdlàg representation, the property still holds. Doob's martingale inequality can then be applied:

$$\mathbb{P} \left[A_\mu \right] \geq \left(1 - \frac{I(V_0, \rho)}{\mu^2} \right) .$$

At last, using Itô's formula, we have from Eq. (8):

$$\begin{aligned} |V(x_t, y_t) - V_0| &\leq \frac{1}{2} \int_0^t \sigma^2(y_s) \left| \kappa(\hat{x}) - \kappa(x_s) - x_s \kappa'(x_s) + y_s \Phi'(y_s) + \Phi(y_s) - \Phi(\hat{y}) \right| ds \\ &\quad + \left| \int_0^t \sigma(y_s) (\kappa(\hat{x}) - \kappa(x_s) + \Phi(y_s) - \Phi(\hat{y})) dW_s \right| \end{aligned}$$

so that for \mathbb{P} -a.e. $(\omega, t) \in A_\mu \times [0, \tau_\rho(\omega)]$, $|V(x_t, y_t) - V_0| \leq \frac{1}{2} R(V_0, \rho)t + \mu \sqrt{t} =: S(t)$ \mathbb{P} -a.s. Also, $|V(x_t, y_t) - V_0| \leq \rho$ on that set. Put in another way, $\tau_\rho > S^{-1}(\rho) =: \Theta(\rho)$ on A_μ , so that according to Bayes rule,

$$\mathbb{P}[\tau_\rho > \Theta(\rho)] \geq \mathbb{P}[\tau_\rho > \Theta(\rho) | A_\mu] \mathbb{P}[A_\mu] \geq \mathbb{P}[A_\mu] \geq \left(1 - \frac{I(V_0, \rho)}{\mu^2}\right) \quad \square$$

Remark 5. Since V explodes at the boundary ∂D , I and R are bounded for all $V_0 \geq 0$ and $\rho \leq V_0$. Notice that an estimate of V provides indirectly an estimate on T , although not explicit here.

Example 3. We build on Example 2. Note that, according to Assumptions 1 and 3 the vertical asymptote at $y = 1$ implies that $\sigma^2(y)\Phi(y) \leq K_0 V_2(y) + k_0$ for some $K_0, k_0 \in \mathbb{R}_+^2$. In specification of Example 2, $K_0 = 0$ and $k_0 = \sigma_0^2(\phi_1 + \phi_0)$. We are thus able to claim the existence of $K, k \in \mathbb{R}_+^2$ such that $|R(V_0, \rho)| \leq K(V_0 + \rho) + k$, for the general form of Φ . Applying to Theorem 3, we obtain

$$R(V_0, \rho) := \max_{D(V_0, \rho)} \left\{ \sigma_0^2 (1-y)^2 \left(\frac{2x - \hat{x}}{v} + \frac{2\phi_1 y}{(1-y)^3} + \frac{\phi_1}{(1-y)^2} - \frac{\phi_1}{(1-\hat{y})^2} \right) \right\}.$$

Using Eq. (10), this estimate becomes

$$|R(V_0, \rho)| \leq K(V_0 + \rho) + k \text{ with } K := \frac{2\sigma_0^2}{1 - \hat{x}},$$

and k an explicitly calculable constant. Following the same procedure with Eq. (10), with the same K ,

$$I(V_0, \rho) \leq K(V_0 + \rho)^2 + k', \quad k' \geq 0.$$

Now choosing $\mu = (\rho - \frac{1}{2}(K(V_0 + \rho) + k)\theta) / \sqrt{\theta}$ for some $\theta \geq 0$, so that $\Theta(\rho) = \theta$ we obtain

$$\mathbb{P}[\tau_\rho > \theta] \geq \left(1 - \frac{(K(V_0 + \rho)^2 + k')\theta}{\left(\frac{1}{2}(K(V_0 + \rho) + k)\theta - \rho\right)^2} \right).$$

4. Small volatility approximation

A first approach to a stochastic Lotka-Volterra model is to assume that the white noise is sufficiently small to consider an approximate auxiliary system where the volatility part is considered as a perturbation. See [10] for a related study with a very similar model. Here, we study (SG) and replace $\sigma(y)$ by $\varepsilon\sigma(y)$ for a small $\varepsilon > 0$. We then look for solutions of (SG) of the form

$$\begin{cases} x_t = x_0(t) + \varepsilon x_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \\ y_t = y_0(t) + \varepsilon y_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \end{cases}. \quad (12)$$

with two sets of solution $G_t := [x_0(t), y_0(t)]^\top$ and $P_t := [x_\varepsilon(t), y_\varepsilon(t)]^\top$. By using the Fundamental Theorem of Perturbation [15] with Taylor expansion of Φ and κ , it follows that G_t is a solution to (G), and P_t a solution to the perturbation dynamics

$$dP_t = A_t \cdot P_t \cdot dt + b_t \cdot dW_t \quad (13)$$

where “ \cdot ” and “ \circ ” denote the matrix product and the component-wise product respectively. Terms A and b are periodic functions of period $T \circ V(x_0(0), y_0(0))$ given by Theorem 1:

$$A_t := \begin{bmatrix} \Phi(y_0(t)) - \alpha & x_0(t)\Phi'(y_0(t)) \\ y_0(t)\kappa'(x_0(t)) & \kappa(x_0(t)) - \gamma \end{bmatrix} \quad \text{and} \quad b_t := \begin{bmatrix} x_0(t)\sigma(y_0(t)) \\ y_0(t)\sigma(y_0(t)) \end{bmatrix}.$$

As usual in this type of analysis, initial conditions are given by $(x_0(0), y_0(0)) \in D \setminus (\hat{x}, \hat{y})$ and $P_0 = [0, 0]^\top$. For convenience, we will introduce the following notations: $\Xi_t := \kappa(x_0(t)) - \gamma$ and $\Upsilon_t := \Phi(y_0(t)) - \alpha$.

Theorem 4. *Let us define*

$$G_t := \begin{bmatrix} x_0(t) & 0 \\ 0 & y_0(t) \end{bmatrix}$$

where $(x_0(t), y_0(t))$ solves (G) with initial condition $(x_0(0), y_0(0)) \neq (\hat{x}, \hat{y})$. Consider $(P_t)_{t \geq 0}$ a solution to Eq. (13). Then for $t \geq 0$,

$$P_t = G_t \cdot \Psi_t \cdot \left[G_0^{-1} \cdot P_0 + \int_0^t \sigma(y_0(s)) \cdot \Psi_s^{-1} \cdot \mathbf{1} \cdot dW_s \right] \quad (14)$$

with $\mathbf{1} = [1, 1]^\top$ and

$$\Psi_t := \begin{bmatrix} \Upsilon_t/\Upsilon_0 - \Xi_0\Gamma_t^x & \Upsilon_0\Gamma_t^x \\ \Xi_0\Gamma_t^y & \Xi_t/\Xi_0 - \Upsilon_0\Gamma_t^y \end{bmatrix}$$

with

$$[\Gamma_t^x, \Gamma_t^y] := \left[\Upsilon_t \int_0^t \frac{y_0(s)\Phi'(y_0(s))}{\Upsilon_s^2} ds, \Xi_t \int_0^t \frac{x_0(s)\kappa'(x_0(s))}{\Xi_s^2} ds \right].$$

Proof For $(x_0(0), y_0(0)) \in D \setminus (\hat{x}, \hat{y})$, we define $Z_t := [x_\varepsilon(t)/x_0(t), y_\varepsilon(t)/y_0(t)]^\top$ which is solution to $dZ_t = A'_t \cdot Z_t \cdot dt + b'_t \cdot dW_t$ with

$$A'_t := \begin{bmatrix} 0 & y_0(t)\Phi'(y_0(t)) \\ x_0(t)\kappa'(x_0(t)) & 0 \end{bmatrix} \quad \text{and} \quad b'_t := \sigma(y_0(t))\mathbf{1}.$$

The above elements are periodic of period $T := T \circ V(x_0(0), y_0(0))$. Consider now for $X_t \in \mathbb{R}^2 \times \mathbb{R}^k$, $k \in \mathbb{N}$ the following deterministic dynamics:

$$dX_t = A'_t \cdot X_t \cdot dt, \quad t \geq 0. \quad (15)$$

Define now $[Z_1(t)|Z_2(t)]$ a fundamental matrix of (15) with $Z_1(t) := [\Upsilon_t, \Xi_t]^\top$ and $Z_2(t) := [x_2(t), y_2(t)]^\top$. Indeed, Z_1 is a particular solution to (15). Abel's formula, together with the fact that $\text{tr}(A'_t) = 0$, implies that for any $t \geq 0$

$$\Upsilon_t y_2(t) - x_2(t) \Xi_t = \Upsilon_0 y_2(0) - x_2(0) \Xi_0 =: c,$$

which can be plugged in (15) to obtain a characterization for Z_2 :

$$\begin{cases} \Upsilon_t dx_2(t) = y_0(t)\Phi'(y_0(t))(c + x_2(t)\Xi_t) dt \\ \Xi_t dy_2(t) = x_0(t)\kappa'(x_0(t))(c + y_2(t)\Upsilon_t) dt \end{cases}.$$

Rearranging, we obtain the following system:

$$\begin{cases} d(x_2(t)/\Upsilon_t) = cy_0(t)\Phi'(y_0(t))/\Upsilon_t^2 dt \\ d(y_2(t)/\Xi_t) = cx_0(t)\kappa'(x_0(t))/\Xi_t^2 dt \end{cases}$$

for time t when $\Upsilon_t \neq 0$ and $\Xi_t \neq 0$, i.e., when $c \neq 0$. This allows us to characterize Z_2 and a general solution X of Eq. (15) under the form

$$X_t = X_0 \cdot I_2 \cdot \begin{bmatrix} \Upsilon_t/\Upsilon_0 \\ \Xi_t/\Xi_0 \end{bmatrix} + c \cdot \begin{bmatrix} \Gamma_t^x \\ \Gamma_t^y \end{bmatrix}, \quad c \in \mathbb{R},$$

with I_2 the identity matrix of $\mathbb{M}_{2 \times 2}$. From that, we can identify the state-transition matrix Ψ_t for Eq. (15). Since Ψ_t is a solution to Eq. (15),

$$d\Psi_t^{-1} = -\Psi_t^{-1} \cdot A'_t \cdot dt \quad \text{and} \quad d(\Psi_t^{-1} \cdot Z_t) = \Psi_t^{-1} \cdot b'_t \cdot dW_t.$$

Taking into account that $\Psi_0 = I_2$, we have the following solution for Z_t :

$$Z_t = \Psi_t \cdot \left[Z_0 + \int_0^t \Psi_s^{-1} \cdot b'_s \cdot dW_s \right], \quad t \geq 0,$$

which leads us to the desired result (14) by definition of Z . \square

Coming back to the perturbation analysis, Theorem 4 has the following direct consequence.

Corollary 1. *Any solution to Eq. (14) verifying $P_0 = [0, 0]^\top$ is a martingale with covariance matrix*

$$\text{Var}[P_t] = \int_0^t V(s, t) \cdot V^\top(s, t) \cdot \sigma^2(y_0(s)) \cdot ds \quad (16)$$

with $V(s, t) := G_t \cdot \Psi_t \cdot \Psi_s^{-1} \cdot \mathbf{1}$. Moreover $\text{Var}[P_{nT}] = \mathcal{O}(n^3)$ for $n \in \mathbb{N}$.

Proof The martingale property follows from solution (14) and the variance (16) follows from Itô isometry. Notice that with integration by part, we obtain

$$\Xi_t \Gamma_t^x + \Upsilon_t \Gamma_t^y = \frac{\Xi_t \Upsilon_t}{\Xi_0 \Upsilon_0} - 1$$

so that $\det \Psi_t = 1$. This allows to express straightforwardly Ψ_t^{-1} :

$$\Psi_t^{-1} = \begin{bmatrix} \Xi_t / \Xi_0 - \Upsilon_0 \Gamma_t^y & -\Upsilon_0 \Gamma_t^x \\ -\Xi_0 \Gamma_t^y & \Upsilon_t / \Upsilon_0 - \Xi_0 \Gamma_t^x \end{bmatrix}.$$

We then resort to Floquet theorem to assert that there exists a T -periodic matrix Q_t and a constant matrix R such that $\Psi_t = Q_t \cdot e^{Rt}$, which implies that

$$\Psi_{nT+t} = Q_{nT+t} \cdot e^{R(nT+t)} = Q_t \cdot e^{Rt} e^{RnT} = \Psi_t \cdot \Psi_T^n. \quad (17)$$

According to the periodicity of G_t , we find that $\Xi_0 \Gamma_T^x = -\Upsilon_0 \Gamma_T^y =: \bar{\Gamma}$ which allows to express the monodromy matrix and its inverse

$$\Psi_T = \begin{bmatrix} 1 - \bar{\Gamma} & \bar{\Gamma} \Upsilon_0 / \Xi_0 \\ -\bar{\Gamma} \Xi_0 / \Upsilon_0 & 1 + \bar{\Gamma} \end{bmatrix} \quad \text{and} \quad \Psi^{-1}(T) = \begin{bmatrix} 1 + \bar{\Gamma} & \bar{\Gamma} \Upsilon_0 \Xi_0 \\ \bar{\Gamma} \Xi_0 / \Upsilon_0 & 1 - \bar{\Gamma} \end{bmatrix}.$$

Define

$$M_t = \sigma^2(y_0(t)) \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

From Eq. (17), we obtain for $n \in \mathbb{N}$:

$$\begin{aligned} \text{Var}[P_{nT}] &= G_0 \cdot \int_0^{nT} e^{R(nT-s)} \cdot Q_s^{-1} \cdot M_s \cdot Q_s^\top \cdot e^{R^\top(nT-s)} \cdot ds \cdot G_0 \\ &= G_0 \cdot \sum_{k=0}^{n-1} \left[\int_{kT}^{(k+1)T} e^{R(nT-s)} \cdot Q_s^{-1} \cdot M_s \cdot Q_s^\top \cdot e^{R^\top(nT-s)} \cdot ds \right] \cdot G_0 \\ &= G_0 \sum_{m=1}^n \left[e^{R, mT} \left(\int_0^T e^{-R \cdot s} \cdot Q_s^{-1} \cdot M_s \cdot Q_s^\top \cdot e^{-R^\top \cdot s} \cdot ds \right) \cdot e^{R^\top \cdot mT} \right] \cdot G_0 \\ &= G_0 \cdot V \cdot \sum_{m=1}^n \left\{ \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \cdot \Delta \cdot \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \right\} \cdot V^\top \cdot G_0 \end{aligned}$$

where $\Delta = V^{-1} \cdot \left(\int_0^T e^{-R \cdot s} \cdot Q_s^{-1} \cdot M_s \cdot Q_s^T \cdot e^{-R^T \cdot s} \cdot ds \right) \cdot V^T$ is constant. The variance then grows with order 3 with respect to n . \square

It can be said that the system with small perturbation has orbits in the sense that it follows the deterministic equation (G) with an additional noise. The noise depends on the value of G_t and is locally centered in expectation onto the point G_t . This behavior is local in time according to Corollary 1. However, one can just translate this vision in time by taking another value for G_0 . The cycling behavior of a solution to (SG) with small volatility around the point $(\hat{x}, \hat{y}) \approx (\tilde{x}, \tilde{y})$ is obtained by repeating indefinitely the approach. We provide a numerical illustration with Fig. 2. The approximations have been analyzed through numerical integration with values of $\sigma_0 = \varepsilon\sigma(0)$ ranging from 0.001 to 0.025. Analytical expressions come from Example 2.

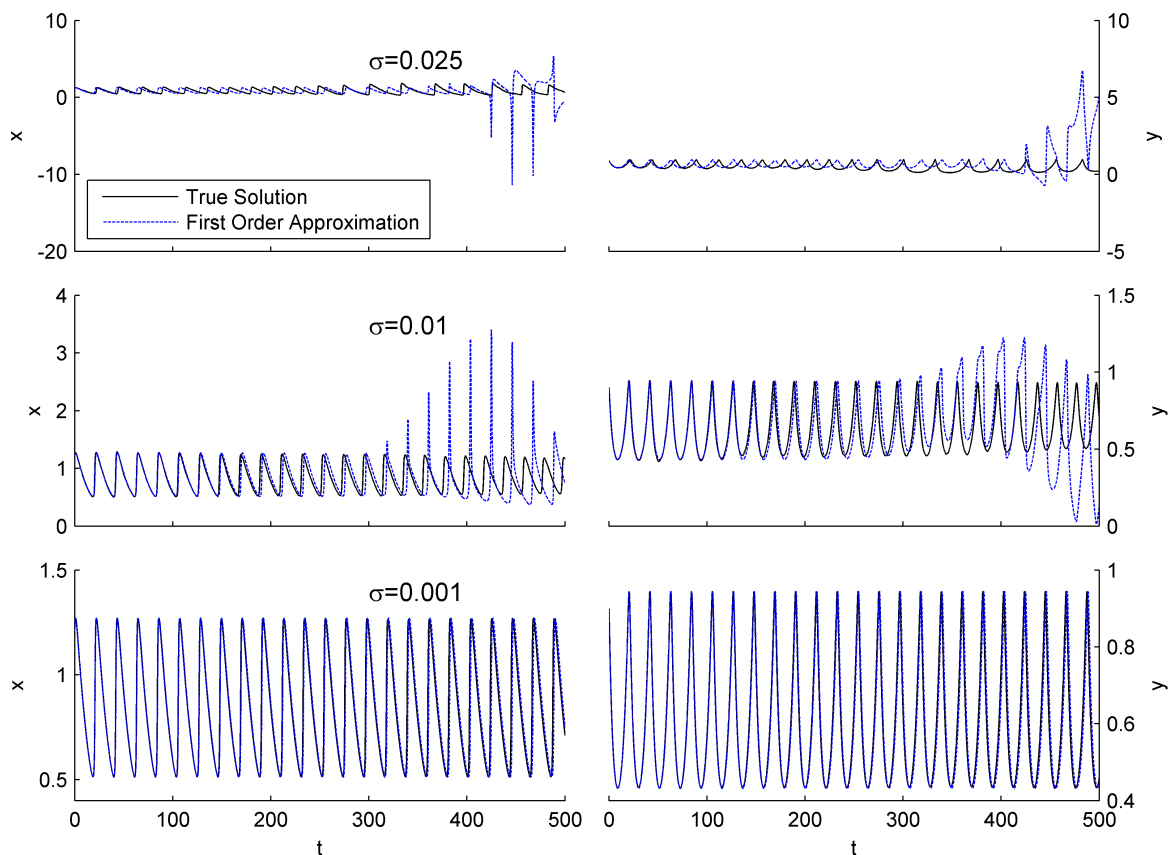


Figure 2: Left column: subsample path of $x(t)$ (left column) and $y(t)$ (right column) of solutions of both Eq. (G) (solid black line) and Eq. (12) (dashed blue line) for different values of σ_0 .

5. Stochastic orbits with recurrent domains

5.1. Definition and result

To our knowledge, there is no formal definition of a cycling behavior for a stochastic oscillator described by (SG). We provide here a temporary and convenient definition for the concept in this framework. Recall $(\tilde{x}, \tilde{y}) \in D$, and consider a regular solution $(x_t, y_t)_{t \geq 0}$ of (SG) starting at an arbitrary point $(x_{t_0}, y_{t_0}) \in D \setminus (\tilde{x}, \tilde{y})$. Let us define the

stochastic angle $(\rho(\omega, t))_{t \geq t_0}$ between $[x_t(\omega) - \tilde{x}, y_t(\omega) - \tilde{y}]^\top$ and $[x_{t_0} - \tilde{x}, y_{t_0} - \tilde{y}]^\top$. A *stochastic orbit* from t_0 around (\tilde{x}, \tilde{y}) for $\omega \in \Omega$ is a subsample path $(x_t(\omega), y_t(\omega))_{t \in [t_0, t_0 + S(\omega)]}$ such that $\rho(\omega, t_0 + S(\omega)) = \rho(\omega, t_0) - 2\pi$. The random quantity S is called the *period of the stochastic orbit*. We omit the case where $(x_s(\omega), y_s(\omega)) = (\tilde{x}, \tilde{y})$ for some $s \in [t_0, t_0 + S(\omega)]$ in order to properly define stochastic orbits which cycle clockwise on the plan (x, y) , see Remark 6 hereafter.

Theorem 5. *For any regular solution $(x_t, y_t)_{t \geq 0}$ to (SG) starting at $(x_0, y_0) \in D \setminus (\tilde{x}, \tilde{y})$, the period S of the stochastic orbit from 0 is finite $\mathbb{P} - a.s.$*

Subsection 5.2 is devoted to the proof of Theorem 5. The idea is to define subregions $(R_i)_i$ of the domain D , illustrated by Fig. 3, and prove that the process exits from them in finite time by the appropriate frontier. Altogether this describes fluctuations of (x, y) in the most general way. In preamble to subsection 5.2 we consider $\theta := (\theta_t)_{t \geq 0}$ defined by $\theta_t := y_t/x_t$, which is \mathbb{F} -adapted and a finite variation process with dynamics

$$d\theta_t = \theta_t (\kappa(x) - \gamma + \alpha - \Phi(y)) dt = \theta_t (\Phi(f(x)) - \Phi(y)) dt ,$$

where f is defined by $f(x) := \Phi^{-1}(\alpha - \gamma + \kappa(x))$ and is a concave decreasing function on \mathbb{R}_+ . Define $\tilde{\theta} := \tilde{y}/\tilde{x}$. We now divide the existence region $D = (0, +\infty) \times (0, 1)$ in 8 sets $(R_i)_{i=1, \dots, 8}$ such that $\bigcap_{i=1}^8 R_i = (\tilde{y}, \tilde{x})$ and $\bigcup_{i=1}^8 R_i = D$.

$$\left\{ \begin{array}{l} R_1 := \{(x, y) \in D : y \geq \tilde{y} \text{ and } \theta_t \leq \tilde{\theta}\} \\ R_2 := \{(x, y) \in D : f(x) \leq y \leq \tilde{y}\} \\ R_3 := \{(x, y) \in D : y \leq f(x) \text{ and } x \geq \tilde{x}\} \\ R_4 := \{(x, y) \in D : x \leq \tilde{x} \text{ and } \theta \leq \tilde{\theta}\} \\ R_5 := \{(x, y) \in D : y \leq \tilde{y} \text{ and } \theta_t \geq \tilde{\theta}\} \\ R_6 := \{(x, y) \in D : \tilde{y} \leq y \leq f(x)\} \\ R_7 := \{(x, y) \in D : y \geq f(x) \text{ and } x_t \leq \tilde{x}\} \\ R_8 := \{(x, y) \in D : y \geq f(x) \text{ and } x_t \leq \tilde{x}\} \end{array} \right.$$

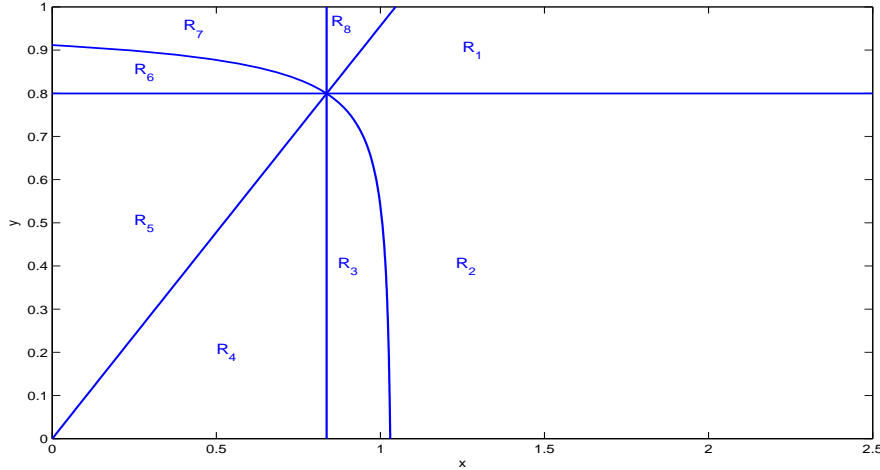


Figure 3: Covering of $D := (0, +\infty) \times (0, 1)$ by $(R_i)_{i=1, \dots, 8}$. Since $f(0) < 1$ and $\lim_{y \uparrow 1} \Phi(y) = +\infty$, the graph illustrates the general case.

Remark 6. Notice that $\tilde{y} = f(\tilde{x})$ and $\hat{y} = f(\hat{x})$. We also emphasize that $\sigma(y)$ is bounded from below and $d\theta_t \neq 0$ for $y \neq f(x)$. We can thus fairly assume along Remark 4 that

$$\mathbb{P}[(x_t, y_t) = (\tilde{x}, \tilde{y}), \text{ for some } t > 0] = 0.$$

Accordingly, we will work as if (\tilde{x}, \tilde{y}) is never reached, implying that (x_t, y_t) can only leave a region R_i to one of its neighboring regions R_j , where $j \in \{\text{mod}(i \pm 1, 8)\}$.

Our proof, although unwieldy, allows us to describe precisely the possible trajectories of the stochastic system. If the result of Theorem 2 seems trivial, an interesting consequence follows.

Proposition 2. *Consider the function \mathcal{S} from $(0, \tilde{y})$ to $(0, \tilde{y})$ such that for $y \in (0, \tilde{y})$, $\mathcal{S}(y) = \mathbb{E}[y_S]$, where (x_t, y_t) is a solution to (SG) verifying $(x_0, y_0) = (y/\tilde{\theta}, y)$, and S is the period of the stochastic orbit of (x_t, y_t) from 0. Then \mathcal{S} has a fixed point.*

Proof If we introduce a modification of V :

$$\tilde{V}(x, y) := \int_{\tilde{x}}^x \frac{\kappa(\tilde{x}) - \kappa(s)}{s} ds + \int_{\tilde{y}}^y \frac{\Phi(s) - \Phi(\tilde{y})}{s} ds,$$

we can see that $\mathcal{L}\tilde{V}(\tilde{x}, \tilde{y}) = \sigma^2(\tilde{y})[\tilde{y}\Phi'(\tilde{y}) - \tilde{x}\kappa'(\tilde{x})]/2 > 0$, so that according to Theorem 6, (x_t, y_t) always leaves a neighborhood of (\tilde{x}, \tilde{y}) in finite time. For any y close enough to \tilde{y} , we have that $\mathcal{S}(y) \leq y$. Now for y close to 0, we already noticed in Remark 4 that $\mathcal{L}V(0, 0) < \varepsilon$ in some neighborhood $(0, \varepsilon]^2$. Therefore the same reasoning applies and $\mathcal{S}(y) \geq y$ for y close to 0. The function \mathcal{S} is continuous, the conclusion follows. \square

5.2. Proof of Theorem 5

According to Theorem 2 any regular solution of (SG) is a Markov process. We then repeatedly change the initial condition of the system, as equivalent of a time translation. We consider a solution (x_t, y_t) of (SG) such that $(x_0, y_0) = (x, y)$ for $(x, y) \in D$ and the stopping times

$$\tau_i(x, y) := \inf\{t \geq 0 : (x_t, y_t) \in R_i\}, \quad i = 1, \dots, 8.$$

To ease the reading of the proof of Theorem 5 which follows from the following Propositions 3 to 12, we divide it in four quadrants around (\tilde{x}, \tilde{y}) . We first prove that the process cycles, even in infinite time.

Proposition 3. *If $(x_0, y_0) \in R_1$, then $\mathbb{P}[\tau_8(x_0, y_0) \leq \tau_7(x_0, y_0)] = 0$. If $(x_0, y_0) \in R_5$, then $\mathbb{P}[\tau_4(x_0, y_0) \leq \tau_3(x_0, y_0)] = 0$.*

Proof This is a direct consequence of the absence of Brownian motion in θ . Take $(x_0, y_0) \in R_1$. Then on $[0, \tau_3(x_0, y_0)]$, the process θ is non increasing \mathbb{P} -a.s., meaning that R_8 cannot be reached without first crossing region R_7 . The other side is identical. \square

Remark 7. Proposition 3 holds even if $\tau_i = +\infty$, for any i involved. Notice also that Proposition 3 and Remark 6 imply that if $(x_0, y_0) \in R_1 \cup R_5$, then $\tau_i(x_0, y_0) \leq \tau_j(x_0, y_0)$ \mathbb{P} -a.s. for $j \in \{\text{mod}(i + 1, 8)\}$.

5.2.1. Eastern quadrant

We ought to prove that for $(x_0, y_0) \in R_1$, the process reaches R_3 in finite time \mathbb{P} -a.s. In the following propositions, we use results of sections 3.7 and 3.8 in [9] on recurrent domains, where the main theorem is reproduced in Appendix B.

Proposition 4. *If $(x_0, y_0) \in R_1$ then $\mathbb{P}[\tau_2(x_0, y_0) < +\infty] = 1$.*

Proof We apply Theorem 7 in Appendix B with function $h(y) := \sqrt{y}$, for which we have $h(y) \geq h(\tilde{y}) > 0$ and, for every $(x, y) \in R_1$,

$$\mathcal{L}h(y) = \frac{h(y)}{2} \left(\kappa(x) - \gamma + \frac{3}{4}\sigma^2(y) \right) \leq -\frac{\sigma^2(\tilde{y})h(y)}{8} \leq -\frac{\sigma^2(\tilde{y})h(\tilde{y})}{8} < 0.$$

The theorem stipulates that (x_t, y_t) leaves R_1 in finite time \mathbb{P} -a.s. which is only possible via R_2 according to Proposition 3 and Theorem 2. \square

Proposition 5. *If $(x_0, y_0) \in R_2 \cup R_3$ then $\mathbb{P}[\tau_1(x_0, y_0) \wedge \tau_4(x_0, y_0) < +\infty] = 1$.*

Proof The proof is similar to the one of Proposition 4 by using $h(x) = \sqrt{x}$. Theorem 2 ensures that the boundary of D is not reached in finite time. \square

Proposition 6. *If $(x_0, y_0) \in R_1 \cup R_2$ then $\mathbb{P}[\tau_3(x_0, y_0) < +\infty] = 1$.*

Proof 1. Let $(u_n)_{n \geq 0}$ be a sequence of stopping times defined by $u_0 = 0$ and

$$u_n := \inf\{t \geq u_{n-1} : y_t = \tilde{y} \text{ or } (x_t, y_t) \in R_3\}, \quad n \geq 1.$$

By construction if $(x_{u_n}, y_{u_n}) \in R_3$ for some $n \geq 1$, then $u_k = u_n$ for all $k > n$. Following Propositions 3, 4 and 5, $u_n < +\infty$ \mathbb{P} -a.s. for all $n \geq 1$, and $\{\tau_3(x, y) = +\infty\} \subset \bigcap_{n \geq 1} \{y_{u_n} = \tilde{y}\}$. We prove in step 2 that this implies

$$\lim_{t \rightarrow \infty} \theta_t(\omega) = 0, \text{ for } \mathbb{P} \text{ - a.e. } \omega \in \{\tau_3(x, y) = +\infty\}. \quad (18)$$

Providing that (18) holds we immediately get $\mathbb{P}[\tau_3(x_0, y_0) = +\infty] \leq \mathbb{P}[\lim_n x_{u_n} = +\infty] = 0$.

2. If $\omega \in \{\tau_3(x, y) = +\infty\}$, then for all $n \geq 1$, $y_{u_n} = \tilde{y}$ and according to Proposition 5, (x_t, y_t) does not converge to the set $R_2 \cap R_3$. Since θ_t is a positive decreasing process for $(x_t, y_t) \in R_1 \cup R_2$, Doob's martingale convergence theorem implies that θ_t converges path-wise in $L^\infty([0, \tilde{\theta}))$ on D . Assume now that $\theta_t(\omega)$ does not converge with t to 0 on $E \subset \{\tau_3(x_0, y_0) = +\infty\}$. Then for any $\varepsilon > 0$, and for \mathbb{P} -a.e. $\omega \in E$

$$\lim_t \int_0^t \mathbb{1}_{\{\kappa(x_s(\omega)) - \gamma + \alpha - \Phi(y_s(\omega)) < -\varepsilon\}} ds = C_\varepsilon(\omega) < +\infty. \quad (19)$$

It the integral in (19) explodes to $+\infty$ for some $\varepsilon > 0$ on some \mathbb{P} -non null set $F \subset E$, then for \mathbb{P} -a.e. $\omega \in F$,

$$\mathcal{L} \log \theta_t(\omega) = (\kappa(x_t(\omega)) - \gamma + \alpha - \Phi(y_t(\omega))) < -\varepsilon \mathbb{1}_{\{\kappa(x_t(\omega)) - \gamma + \alpha - \Phi(y_t(\omega)) < -\varepsilon\}}$$

and $\lim_{t \uparrow \infty} \log \theta_t(\omega) = -\infty$ for \mathbb{P} -a.e. $\omega \in F$, implying that θ_t converges to 0 on F , a contradiction with $F \subset E$. Eq. (19) holding on E , we consider the random time $t_{\varepsilon, n}$, being the first time such that

$$\int_0^{t_{\varepsilon, n}} \mathbb{1}_{\{\kappa(x_s) - \gamma + \alpha - \Phi(y_s) < -\varepsilon\}} ds \geq C_\varepsilon - \frac{1}{n}. \quad (20)$$

and k_n the smallest index such that $u_{k_n} \geq t_{\varepsilon, n}$. Note that $t_{\varepsilon, n}$ is not a \mathbb{F} -stopping time and k_n is not \mathbb{F} -adapted since they depend on C_ε which is \mathcal{F}_∞ -measurable. Eq. (20) implies that there exists a random time $s_n \in (u_{k_n}, u_{k_n} + 1/n)$ such that $-\varepsilon < \kappa(x_{s_n}) - \gamma + \alpha - \Phi(y_{s_n}) < 0$, otherwise we would have a contradiction of (19) on a subset of E :

$$\int_0^{u_{k_n} + 1/n} \mathbb{1}_{\{\kappa(x_s) - \gamma + \alpha - \Phi(y_s) < -\varepsilon\}} ds \geq C_\varepsilon.$$

This implies that $\lim_n (s_n - u_{k_n})(\omega) = 0$ for \mathbb{P} -a.e. $\omega \in E$, and $(y_t)_{t \geq 0}$ being a continuous process

$$\lim_n y_{s_n}(\omega) = \tilde{y}, \quad \text{for } \mathbb{P} \text{ - a.e. } \omega \in D \subset \{\tau_3(x_0, y_0) = +\infty\}.$$

This is impossible for $\varepsilon > 0$ small enough since θ_t is strictly decreasing and thus E is a \mathbb{P} -null set. Eq. (18) holds. \square

5.2.2. Southern quadrant

We show that starting from $R_2 \cap R_3$, (x_t, y_t) reaches R_5 in finite time \mathbb{P} - a.s.

Proposition 7. *If $(x_0, y_0) \in R_2 \cap R_3$ then $\mathbb{P}[\tau_4(x_0, y_0) < +\infty] = 1$.*

Proof 1. We consider $R_t := R(x_t, y_t) = (y_t - \tilde{y})/(x_t - \tilde{x})$, and aim to prove that the process $F_t := F \circ R(x_t, y_t) = \tan(\tan^{-1}(R(x_t, y_t)) + \tan^{-1}(c))$ is a super-martingale on $R_1 \cup R_2 \cup R_3$, for $c \in (0, \tilde{\theta}^{-1})$. Notice that it is bounded in $R_1 \cup R_2 \cup R_3$. Applying Itô to R first gives

$$dR_t = \frac{dy_t}{x_t - \tilde{x}} - \frac{y_t - \tilde{y}}{(x_t - \tilde{x})^2} dx_t + \frac{\sigma^2(y_t)}{(x_t - \tilde{x})^2} \left[\frac{y_t - \tilde{y}}{x_t - \tilde{x}} x_t^2 - x_t y_t \right] dt.$$

Then, noticing that $F_t = (R_t + c)(1 - R_t c)$, we obtain

$$dF_t = \frac{1 + c^2}{(1 - R_t c)^2} \left(dR_t + \frac{c}{1 - R_t c} d\langle R \rangle_t \right).$$

It is clear that $-(y - \tilde{y})(\Phi(y) - \alpha + \sigma^2(y)) \leq 0$ for all $y \in [0, 1]$. Now notice that for $(x, y) \in R_1$, we have $(x - \tilde{x})(\kappa(x) - \gamma + \sigma^2(y)) < 0$ so that

$$\begin{aligned} \frac{(x - \tilde{x})^4 (1 - Rc)^2}{\sigma^2(y) (1 + c^2)} \mathcal{L}F &\leq (y - \tilde{y})(x - \tilde{x})x^2 - xy(x - \tilde{x})^2 + \frac{1}{\tilde{y}/\tilde{x} - R} (\tilde{y}x - y\tilde{x})^2 \\ &= (x - \tilde{x}) \left[x^2(y - \tilde{y}) - xy(x - \tilde{x}) + \tilde{x}(\tilde{y}x - y\tilde{x}) \right] \\ &= (x - \tilde{x})^2 x \tilde{x} \left[\frac{y}{x} - \frac{\tilde{y}}{\tilde{x}} \right] < 0. \end{aligned}$$

Now on $R_2 \cup R_3$, $\hat{x} < \tilde{x}$ implies that $(\kappa(x) - \gamma) < 0$, so that

$$\begin{aligned} \frac{(x - \tilde{x})^4 (1 - Rc)^2}{\sigma^2(y) (1 + c^2)} \mathcal{L}F &\leq (y - \tilde{y})(x - \tilde{x})x^2 - y\tilde{x}(x - \tilde{x})^2 + \frac{1}{\tilde{y}/\tilde{x} - R} (\tilde{y}x - y\tilde{x})^2 \\ &= (x - \tilde{x}) \left[x^2(y - \tilde{y}) - y\tilde{x}(x - \tilde{x}) + \tilde{x}(\tilde{y}x - y\tilde{x}) \right] \\ &= (x - \tilde{x})^2 x (y - \tilde{y}) < 0. \end{aligned}$$

Denoting $\tau_{1,4} := \tau_1(x_0, y_0) \wedge \tau_4(x_0, y_0)$, we conclude that $F_{t \wedge \tau_{1,4}}$ is a super-martingale for $t \geq 0$. Using optional sampling theorem, assisted by Proposition 5, $\tau_{1,4} < +\infty$ \mathbb{P} - a.s. and

$$F_0 \geq \mathbb{E}[F_{\tau_{1,4}}] = \frac{1}{c} \mathbb{P}[\tau_4(x, y) < \tau_1(x, y)] + c \mathbb{P}[\tau_1(x_0, y_0) < \tau_4(x_0, y_0)]$$

Since $M := \max\{F \circ R(x, y) : (x, y) \in R_2 \cap R_3\} < c$ then

$$\mathbb{P}[\tau_4(x_0, y_0) < \tau_1(x_0, y_0)] \geq \frac{c(c - M)}{c^2 + 1} > 0 \quad \forall (x_0, y_0) \in R_2 \cap R_3.$$

2. According to Proposition 5, $\tau_{1,4} < +\infty$ \mathbb{P} - a.s. for any $(x_0, y_0) \in R_2 \cap R_3$, and according to Proposition 6, $\tau_3(x_0, y_0) < +\infty$ \mathbb{P} - a.s. for all $(x_0, y_0) \in R_1$. Taking $(x_0, y_0) \in R_2 \cap R_3$, we define the sequence $(\tau_{1,4}^n, \tau_3^n)_{n \geq 0}$ with $\tau_3^0 = 0$ and

$$\begin{cases} \tau_{1,4}^n := \inf\{t \geq \tau_3^n : (x_t, y_t) \in R_1 \cup R_4\} \\ \tau_3^{n+1} := \inf\{t \geq \tau_{1,4}^n : (x_t, y_t) \in (R_2 \cap R_3) \cup R_4\} \end{cases}, \text{ for all } n \geq 1.$$

We then have $\{\tau_4(x_0, y_0) = +\infty\} \subset \cap_{n \geq 1} \{x_{\tau_{1,4}^n} > \tilde{x}\}$ for any $(x_0, y_0) \in R_2 \cap R_3$. The sequence $(\{x_{\tau_{1,4}^n} > \tilde{x}\})_{n \geq 1}$ is decreasing in the sense of inclusion, so that

$$\mathbb{P}[\tau_4(x, y) = +\infty] = \lim_n \mathbb{P}[x_{\tau_{1,4}^n} > \tilde{x}]. \quad (21)$$

Using Baye's rule,

$$\mathbb{P} \left[x_{\tau_{1,4}^n} > \tilde{x} \right] \leq \prod_{k=1}^n \mathbb{P} \left[x_{\tau_{1,4}^k} > \tilde{x} | x_{\tau_{1,4}^{k-1}} > \tilde{x} \right] \leq \prod_{k=1}^n \mathbb{P} \left[x_{\tau_{1,4}^k} > \tilde{x} | x_{\tau_3^k} > \tilde{x} \right].$$

Using step 1 of the present proof and the Markov property of (x_t, y_t) ,

$$\mathbb{P} \left[x_{\tau_{1,4}^n} > \tilde{x} \right] \leq \prod_{k=1}^n \mathbb{P} \left[\tau_1(x_{\tau_3^k}, y_{\tau_3^k}) < \tau_4(x_{\tau_3^k}, y_{\tau_3^k}) \right] \leq \prod_{k=1}^n \left(1 - \frac{c(c-M)}{c^2+1} \right).$$

Plugging this inequality into (21) concludes the proof. \square

Remark 8. Notice that by choosing c properly in the above proof, it is possible to be arbitrarily close to R_5 in finite time. The device is used later in Proposition 11.

Proposition 8. *If $(x_0, y_0) \in R_3 \cap R_4$ then $\mathbb{P}[\tau_5(x_0, y_0) < +\infty] = 1$.*

Proof 1. We claim that $\tau_{2,5} := \tau_2(x_0, y_0) \wedge \tau_5(x_0, y_0) < +\infty$ \mathbb{P} -a.s. Consider the process $h_t := \sqrt{x_{t \wedge \tau_{2,5}}}$, which is a positive super martingale on $R_2 \cup R_3 \cup R_4$:

$$\mathcal{L}h_t = \frac{h_t}{2} \left(\Phi(y_t) - \alpha + \frac{\sigma^2(y_t)}{2} \right) \leq -\frac{\sigma^2(y_t)h_t}{4}. \quad (22)$$

According to Doob's martingale convergence theorem, h_t converges point-wise with t . Let $\varepsilon > 0$ and define $R_\varepsilon := \bigcup_{i=2}^4 R_i \cap \{x \geq \varepsilon\}$. Then $h(x_t) \geq \sqrt{\varepsilon}$ on R_ε , and similarly to Proposition 5, we use Theorem 7 with $\mathcal{L}h_t \leq -\sqrt{\varepsilon}\sigma^2(\tilde{y})/4 < 0$ to assert that (x_t, y_t) leaves R_ε in finite time. This being true for any $\varepsilon > 0$, h_t can only converge to 0 for \mathbb{P} -a.e. $\omega \in \Omega$ such that $\tau_{2,5}(\omega) = +\infty$. In regard of the region R_5 , this is only possible if $\lim_t y_t(\omega) = 0$ also, implying that $(x_t(\omega), y_t(\omega))$ converges to $(0, 0)$. Recalling Remark 4, it is improbable and $\tau_{2,5} < +\infty$ \mathbb{P} -a.s.

2. By denoting $\tau_4^0 = 0$, we then define the sequence $(\tau_{2,5}^n, \tau_4^n)_{n \geq 0}$ by

$$\begin{cases} \tau_4^n := \inf\{t \geq \tau_{2,5}^{n-1} : x_t = \tilde{x} \text{ or } (x_t, y_t) \in R_5\} \\ \tau_{2,5}^n := \inf\{t \geq \tau_4^n : (x_t, y_t) \in R_2 \cup R_5\} \end{cases}, \text{ for all } n \geq 1.$$

If $(x_{\tau_{2,5}^0}, y_{\tau_{2,5}^0}) \in R_2$, then, according to Proposition 7, the process reaches back R_4 in finite time. Using step 1, we have that $\mathbb{P}[\tau_4^n < +\infty] = \mathbb{P}[\tau_{2,5}^n < +\infty] = 1$. By construction and Proposition 7, for $n \geq 1$

$$\{(x_{\tau_{2,5}^n}, y_{\tau_{2,5}^n}) \in R_2\} \subset \{x_{\tau_4^n} = \tilde{x}\} = \{(x_{\tau_{2,5}^{n-1}}, y_{\tau_{2,5}^{n-1}}) \in R_2\} = \{x_{\tau_{2,5}^{n-1}} > \tilde{x}\}. \quad (23)$$

Therefore, $\{\tau_5(x_0, y_0) = +\infty\} = \bigcap_{n \geq 0} \{(x_{\tau_{2,5}^n}, y_{\tau_{2,5}^n}) \in R_2\}$ and the sequence of sets $\left(\{(x_{\tau_{2,5}^n}, y_{\tau_{2,5}^n}) \in R_2\} \right)_{n \geq 0}$ is decreasing in the sense of inclusion. Altogether we get

$$\mathbb{P}[\tau_5(x_0, y_0) = +\infty] = \lim_n \mathbb{P} \left[x_{\tau_{2,5}^n} > \tilde{x} \right]. \quad (24)$$

Now using Bayes formula and Eq. (23), we finally obtain for every $n \geq 1$

$$\mathbb{P} \left[x_{\tau_{2,5}^n} > \tilde{x} \right] \leq \prod_{k=1}^n \mathbb{P} \left[x_{\tau_{2,5}^k} > \tilde{x} | x_{\tau_{2,5}^{k-1}} > \tilde{x} \right] = \prod_{k=1}^n \mathbb{P} \left[x_{\tau_{2,5}^k} > \tilde{x} | x_{\tau_4^k} = \tilde{x} \right] \quad (25)$$

Putting (24) and (25) together, $\mathbb{P}[\tau_5(x_0, y_0) = +\infty] > 0$ implies that

$$\lim_n \mathbb{P} \left[x_{\tau_{2,5}^n} > \tilde{x} | x_{\tau_4^n} = \tilde{x} \right] = 1. \quad (26)$$

3. Let $h(x, t) := \sqrt{x} \exp(\frac{1}{8}\sigma^2(\tilde{y})t)$. According to Eq. (22) of step 1, the process $h(x_t, t)$ is a super-martingale on $[\tau_4^n, \tau_{2,5}^n]$. Fixing $t > 0$ and applying optional sampling theorem, we obtain

$$\mathbb{E} \left[h(x_{t \wedge \tau_{2,5}^n}, t \wedge \tau_{2,5}^n) - h(x_{t \wedge \tau_4^n}, t \wedge \tau_4^n) \middle| x_{t \wedge \tau_4^n} = \tilde{x} \right] \leq 0 .$$

Since $\max(\tau_4^n, \tau_{2,5}^n) < +\infty$ \mathbb{P} -a.s., we apply Fatou's lemma and obtain

$$\mathbb{E} \left[\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n] \right) \sqrt{x_{\tau_{2,5}^n}} (\mathbb{1}_{\{x_{\tau_{2,5}^n} < \tilde{x}\}} + \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}}) \middle| x_{\tau_4^n} = \tilde{x} \right] \leq \sqrt{\tilde{x}} . \quad (27)$$

Since $\sqrt{x_{\tau_{2,5}^n}} \mathbb{1}_{\{x_{\tau_{2,5}^n} < \tilde{x}\}} \geq 0$ and $\sqrt{x_{\tau_{2,5}^n}} \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}} \geq \sqrt{\tilde{x}} \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}}$ for all $n \geq 1$, Eq. (27) implies

$$\mathbb{E} \left[\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n] \right) \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}} \middle| x_{\tau_4^n} = \tilde{x} \right] \leq 1 ,$$

leading to

$$\mathbb{E} \left[\left(\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n] \right) - 1 \right) \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}} \middle| x_{\tau_4^n} = \tilde{x} \right] \leq 1 - \mathbb{P} [x_{\tau_{2,5}^n} > \tilde{x} | x_{\tau_4^n} = \tilde{x}] . \quad (28)$$

If $x_{\tau_4^n} = \tilde{x}$ then $y_{\tau_4^n} < \tilde{y}$ and by continuity $\{\tau_{2,5}^n > \tau_4^n\} \supset \{x_{\tau_4^n} = \tilde{x}\}$, implying

$$\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n(\omega) - \tau_4^n(\omega)] \right) > 1, \text{ for } \mathbb{P} - \text{a.e. } \omega \in \{x_{\tau_4^n} = \tilde{x}\} . \quad (29)$$

Let's assume that $\mathbb{P} [\tau_5(x_0, y_0) = +\infty] > 0$, so that (26) holds. According to Eq. (28), we get

$$0 \leq \mathbb{E} \left[\left(\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n] \right) - 1 \right) \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}} \middle| x_{\tau_4^n} = \tilde{x} \right] \rightarrow 0 \text{ as } n \rightarrow \infty .$$

Markov inequality then leads to the following convergence for any $\varepsilon > 0$:

$$\lim_n \mathbb{P} \left[\left(\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n] \right) - 1 \right) \mathbb{1}_{\{x_{\tau_{2,5}^n} > \tilde{x}\}} > \varepsilon \middle| x_{\tau_4^n} = \tilde{x} \right] = 0 .$$

Now Bayes rules with Eq. (29) provides

$$\begin{aligned} & \mathbb{P} \left[\left(\exp \left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n] \right) - 1 \right) > \varepsilon \middle| x_{\tau_{2,5}^n} > \tilde{x}, x_{\tau_4^n} = \tilde{x} \right] \mathbb{P} [x_{\tau_{2,5}^n} > \tilde{x} | x_{\tau_4^n} = \tilde{x}] \\ & = \mathbb{P} [(\tau_{2,5}^n - \tau_4^n) > 8 \ln(1 + \varepsilon) / \sigma^2(\tilde{y}) \middle| x_{\tau_{2,5}^n} > \tilde{x}, x_{\tau_4^n} = \tilde{x}] \mathbb{P} [x_{\tau_{2,5}^n} > \tilde{x} | x_{\tau_4^n} = \tilde{x}] \end{aligned}$$

which leads to the following convergence, for any $\varepsilon > 0$:

$$\mathbb{P} [(\tau_{2,5}^n - \tau_4^n) > \varepsilon \middle| x_{\tau_{2,5}^n} > \tilde{x}, x_{\tau_4^n} = \tilde{x}] \rightarrow 0 \text{ as } n \rightarrow \infty .$$

From step 2, $\{\tau_5(x_0, y_0) = +\infty\} = \bigcap_{n \geq 0} (\{x_{\tau_4^n} = \tilde{x}\} \cap \{x_{\tau_{2,5}^n} > \tilde{x}\})$. Therefore on this set, the continuous mapping theorem asserts that (x, y) at consecutive stopping times converge in probability. By continuity, this implies $\lim_n y_{\tau_4^n}(\omega) = \tilde{y}$ and $\lim_n x_{\tau_{2,5}^n}(\omega) = \tilde{x}$ for \mathbb{P} -a.e. $\omega \in \{\tau_5(x_0, y_0) = +\infty\}$. By the Markov property of (x_t, y_t) ,

$$\lim_{t \rightarrow \infty} (x_t, y_t)(\omega) = (\tilde{x}, \tilde{y}) \text{ for } \mathbb{P} - \text{a.e. } \omega \in \{\tau_5(x_0, y_0) = +\infty\} .$$

Recalling Remark 4, we conclude that $\mathbb{P} [\tau_5(x_0, y_0) = +\infty] = 0$. \square

5.2.3. Western Quadrant

Proposition 9. *If $(x_0, y_0) \in R_5 \cup R_6$ then $\mathbb{P}[\tau_7(x_0, y_0) < +\infty] = 1$.*

Proof 1. Consider $R_\varepsilon := R_5 \cup R_6 \cap \{x \leq \hat{x} - \varepsilon\}$ for arbitrarily fixed $\varepsilon > 0$. Assume that $(x_0, y_0) \in R_\varepsilon$. Denoting $h_t := 1/y_t \geq 0$ and recalling the definition of f ,

$$\mathcal{L}h_t = -h_t(\kappa(x_t) - \gamma) < -(\kappa(\hat{x} - \varepsilon) - \gamma)/f(0) < 0 \text{ on } R_\varepsilon .$$

Theorem 7 then states that (x_t, y_t) exits R_ε in finite time almost surely. Since that θ is non-decreasing on this set, and recalling Theorem 6, it is only possible via R_7 and $\mathbb{P}[\tau_7(x_0, y_0) < +\infty] = 1$ This holds for any $\varepsilon > 0$.

2. Assume now that $(x_0, y_0) \in (R_5 \cup R_6) \setminus R_\varepsilon$. According to step 1, $\{\tau_7(x_0, y_0) = +\infty\} \subset \{x_t \geq \hat{x}, \forall t \geq 0\}$ and thus $\{\tau_7(x_0, y_0) = +\infty\} \subset \{\theta_t \leq f(\hat{x})/\hat{x}, \forall t \geq 0\}$. Because θ_t is non decreasing, θ_t converges to $\theta_0 \in L^0([\tilde{\theta}, f(\hat{\omega})/\hat{\omega}])$ on $\{\tau_7(x_0, y_0) = +\infty\}$ according to Doob's martingale convergence theorem. This implies that (x_t, y_t) converges with t to $R_6 \cap R_7$ on $\{\tau_7(x_0, y_0) = +\infty\}$. Along Remark 4, since $\sigma(y) > \sigma(f(0))$, this convergence is improbable and $\mathbb{P}[\tau_7(x_0, y_0) = +\infty] = 0$. \square

5.2.4. Northern quadrant

Finally we prove that if $(x_0, y_0) \in R_7$, then the process reaches R_1 in finite time \mathbb{P} -a.s. One can notice that proofs are very similar to those of Subsections 5.2.1 and 5.2.2.

Proposition 10. *If $(x_0, y_0) \in R_6 \cup R_7$ then $\mathbb{P}[\tau_5 \wedge \tau_8(x_0, y_0) < +\infty] = 1$.*

Proof Define the sequence of regions $\{B_n\}_{n \in \mathbb{N}}$ through $B_n = R_6 \cup R_7 \cap \{y < 1 - k/n\} \cap \{x > k/n\}$ where $k > 0$ is sufficiently small to have $(x_0, y_0) \in B_1$. Applying Itô to $h_t := \sqrt{\hat{x} - x_t}$, we find that for all $(x, y) \in B_n$

$$\mathcal{L}h_t = -\frac{1}{2h_t} \left[x \left[\Phi(y) - \alpha + \sigma^2(y) \right] + \frac{1}{4} \frac{x^2}{\hat{x} - x} \sigma^2(y) \right] \leq -\frac{1}{8} \frac{x^2}{(\hat{x} - x)^{3/2}} \sigma^2(y)$$

while $\mathcal{L}h_t \leq 0$ in $R_6 \cup R_7$. Doob's supermartingale convergence theorem implies the existence of the point-wise limit $h_\infty(\omega) := \lim_t h_{t \wedge \tau_{5,8}}(\omega)$ for \mathbb{P} -a.e. $\omega \in \Omega$, where we use the notation $\tau_{5,8} := \tau_5(x_0, y_0) \wedge \tau_8(x_0, y_0)$. In addition, Theorem 7 guarantees that every set B_n is exited in finite time \mathbb{P} -a.s. Consequently if $\omega \in \{\tau_{5,8} = +\infty\}$, we have that either $x_t(\omega) \rightarrow 0$ or $y_t(\omega) \rightarrow 1$. Either way, according to Remark 4, we have a contradiction. \square

Proposition 11. *If $(x_0, y_0) \in R_6 \cap R_7$ then $\mathbb{P}[\tau_8(x_0, y_0) < +\infty] = 1$.*

Proof The proof is identical to the one of Proposition 7, with small modifications. Here $R_t := (y_t - \tilde{y})/(x_t - \tilde{x})$, and

$$F_t := F \circ R(x_t, y_t) := \tan\left(\tan^{-1}(R_t) + \tan^{-1}(c)\right)$$

is a supermartingale on $R_5 \cup R_6 \cup R_7$ if we chose $c \in (0, (\tilde{\theta} + M/m)^{-1})$ where (m, M) are two positive constants given by $m := \min_{[\tilde{x}, \hat{x}] \times [\tilde{y}, \hat{y}]} x \hat{x} \sigma^2(y)$ and

$$M := \max_{[\tilde{x}, \hat{x}] \times [\tilde{y}, \hat{y}]} y(x - \hat{x}) [\kappa(x) - \gamma] - x(y - \tilde{y}) [\Phi(y) - \alpha] .$$

The justification is the following. The domain $S_c := D \setminus \{\tilde{\theta} \leq \frac{y-\tilde{y}}{x-\tilde{x}} \leq 1/c\}$ contains the area of interest $R_5 \cup R_6 \cup R_7$. Using Proposition 7, we can prove that F_t is a supermartingale on $S_c \setminus [\hat{x}, \tilde{x}] \times [\tilde{y}, \hat{y}]$. On $[\hat{x}, \tilde{x}] \times [\tilde{y}, \hat{y}]$,

$$\begin{aligned}
(x - \tilde{x})^2 \frac{(1 - R_t c)^2}{1 + c^2} \mathcal{L}F_t &\leq y(x - \tilde{x})[\kappa(x) - \gamma] - x(y - \tilde{y})[\Phi(y) - \alpha] \\
&\quad + \sigma^2(y) [y(x - \tilde{x}) - x(y - \tilde{y}) + \tilde{x}(y - x(1/c - \tilde{x} + \tilde{x}))] \\
&\leq y(x - \tilde{x})[\kappa(x) - \gamma] - x(y - \tilde{y})[\Phi(y) - \alpha] - x\tilde{x}\sigma^2(y)(1/c - \tilde{\theta}) \\
&\leq M - m(1/c - \tilde{\theta}) \leq 0 .
\end{aligned}$$

We then reproduce step 2 of the designated proof, using Propositions 9 and 10 above instead of Propositions 4 and 5. \square

Proposition 12. *If $(x_0, y_0) \in R_7 \cap R_8$ then $\mathbb{P}[\tau_1(x_0, y_0) < +\infty] = 1$.*

Proof We follow Proposition 8 with the minor following modifications.

1 We consider $\tau_{1,6} := \tau_1(x_0, y_0) \wedge \tau_6(x_0, y_0)$ the exit time of $R_7 \cup R_8$. The process $h_t := x_t^{-2}$ verifies

$$\mathcal{L}h_t = -2h_t \left(\Phi(y_t) - \alpha + \frac{3}{2}\sigma^2(y_t) \right) < -\varepsilon h_t < -\varepsilon h(\tilde{\theta}) < 0$$

for some $\varepsilon > 0$. Indeed $\Phi(y) - \alpha + \sigma^2(y) \geq 0$ and is null only if $y = \tilde{y}$, whereas $\sigma^2(y) = 0$ only if $y = 1$. Applying Theorem 7 to $R_7 \cup R_8$, $\tau_{1,6} < +\infty$ \mathbb{P} -a.s.

2. If $(x_{\tau_{1,6}}, y_{\tau_{1,6}}) \in R_6$, then the process reaches R_8 in finite time \mathbb{P} -a.s. according to Proposition 11. We define the sequence $(\tau_{1,6}^n, \tau_8^n)_{n \geq 0}$ with $\tau_8^0 := 0$ and

$$\begin{cases} \tau_{1,6}^n := \inf\{t \geq \tau_8^n : (x_t, y_t) \in R_6 \cup R_1\} \\ \tau_8^{n+1} := \inf\{t > \tau_{1,6}^n : (x_t, y_t) \in (R_7 \cap R_8) \cup R_1\} \end{cases} , \text{ for all } n \geq 0 .$$

Proceeding as in step 2 Proposition 8, we obtain that $\mathbb{P}[\tau_1(x_0, y_0) = +\infty] > 0$ implies that

$$\lim_n \mathbb{P}[x_{\tau_{1,6}^n} < \tilde{x} | x_{\tau_8^n} = \tilde{x}] = 1 . \quad (30)$$

3. Define $m := \inf\{2(\Phi(y) - \alpha) + 3\sigma^2(y) : y \in [\tilde{y}, 1)\}$, which is strictly positive according to step 1. Consider the new process $h_t := \exp(-mt)x_t^2$. It is a positive submartingale on $[0, \tau_{1,6}^0]$, and similarly to step 3 of Proposition 8, we can obtain

$$\tilde{x}^2 \leq \mathbb{E} \left[x_{\tau_{1,6}^n}^2 e^{-m(\tau_{1,6}^n - \tau_8^n)} | x_{\tau_8^n} = \tilde{x} \right] \leq \tilde{x}^2 \mathbb{E} \left[e^{-m(\tau_{1,6}^n - \tau_8^n)} \mathbb{1}_{\{x_{\tau_{1,6}^n} < \tilde{x}\}} | x_{\tau_8^n} = \tilde{x} \right] + \tilde{\theta}^{-2} \mathbb{E} \left[e^{-m(\tau_{1,6}^n - \tau_8^n)} \mathbb{1}_{\{x_{\tau_{1,6}^n} \geq \tilde{x}\}} | x_{\tau_8^n} = \tilde{x} \right] .$$

Assuming Eq. (30), we have

$$0 \leq \mathbb{E} \left[\left(1 - e^{-m(\tau_{1,6}^n - \tau_8^n)}\right) \mathbb{1}_{\{x_{\tau_{1,6}^n} < \tilde{x}\}} | x_{\tau_8^n} = \tilde{x} \right] \leq (1/\tilde{\lambda} - 1)(1 - \mathbb{P}[x_{\tau_{1,6}^n} < \tilde{x} | x_{\tau_8^n} = \tilde{x}]) \xrightarrow{n} 0 .$$

We then proceed exactly as in step 3 of Proposition 8 to finish the proof. \square

5.3. Numerical insights

The result of Theorem 5 is a straightly observable phenomenon with simulations. However, apart from specific subregions of D as R_1 or R_5 where Corollary 3.2 in [9] can provide an estimate for the expectation of the exit time, a bound for the expected period $\mathbb{E}[S]$ seems out of reach. In the present section, we use the parameters of Examples 1,2

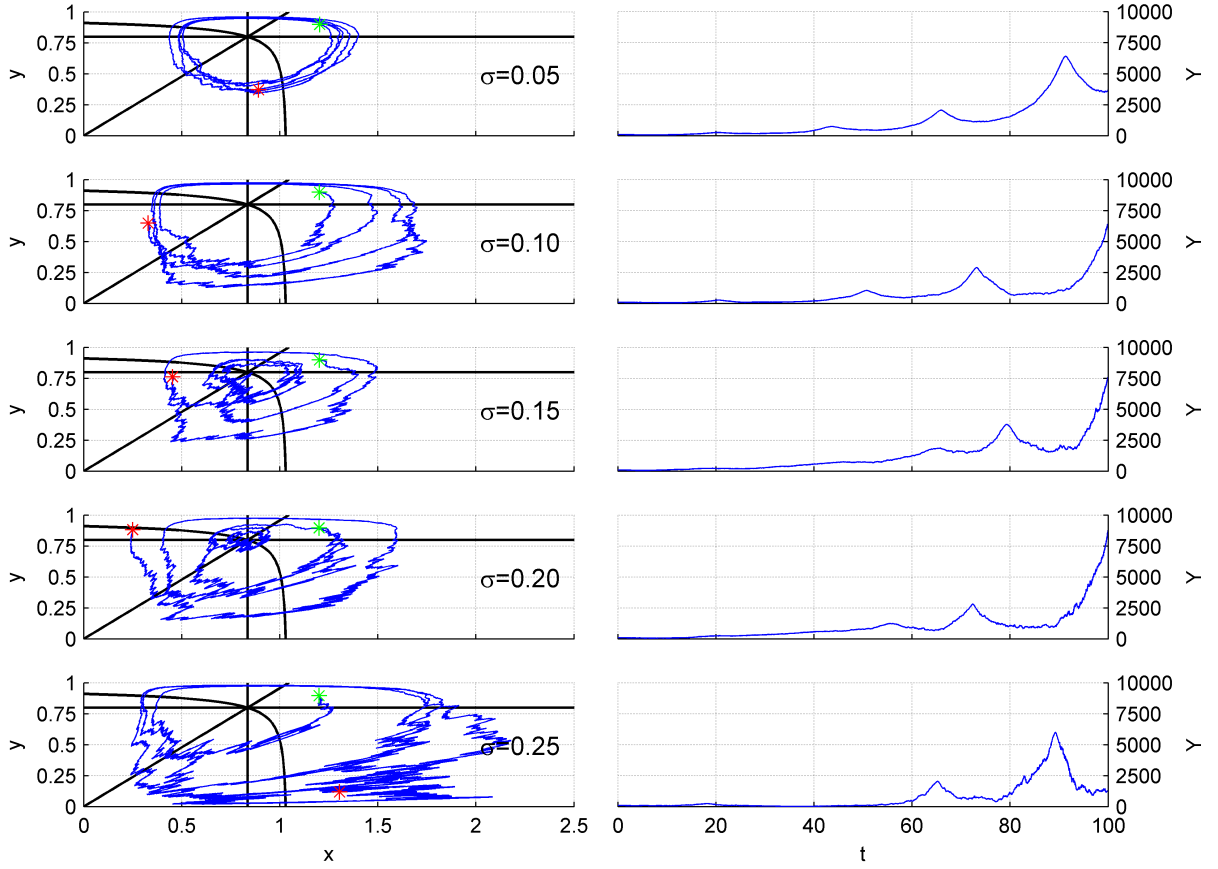


Figure 4: Left column : phase diagram (x, y) of subsample paths of trajectories for (SG) with different values of volatility σ_0 , starting from the green star and stopping at the red start. Right column: evolution of output P_t over time for the subsample path.

and 3. The system has been simulated using XPPAUT with a fourth order Runge-Kutta scheme for the deterministic part, and an Euler scheme for the Brownian part.

Fig. 4 illustrates the effect of the volatility level $\sigma_0 := \sigma(0)$ on trajectories of the system, as for the economical quantity $P_t := a_t y_t N_t$. One can observe that with σ_0 , the oscillating behavior on the subregion below $y = \tilde{\theta}x$, i.e., $\cup_{i=1}^4 R_i$, is amplified. We can mostly observe high variations of x with low variations of y .

The second phenomenon we ought to analyze is the period of the system and Proposition 2. According to the previous subsection, we start with $(x_0, y_0) \in R_1 \cap R_8$ and formulate S as the time the process crosses the line $y = \hat{\theta}x$ for the second time. This is equivalent to take $(x_0, y_0) \in R_4 \cap R_5$. Resorting to numerical methods, we have simulated the system 2000 times for 100 different starting points in $R_1 \cap R_8$ and recorded the position at the time when this line is crossed the second time, that is the positions after a full loop. Fig. 5 contains such examination for an array of values of σ_0 . The expected time $\mathbb{E}[S]$ to complete a full-loop is also illustrated. As observed, there seems to be a stable attractive fixed point to $y_0 \mapsto \mathbb{E}[y_S]$ for sufficiently large values of σ_0 . This implies that after many loops, the expectation converges, and so does $\mathbb{E}[S]$ with the number of loops around (\bar{x}, \bar{y}) . This confirms Proposition 2.

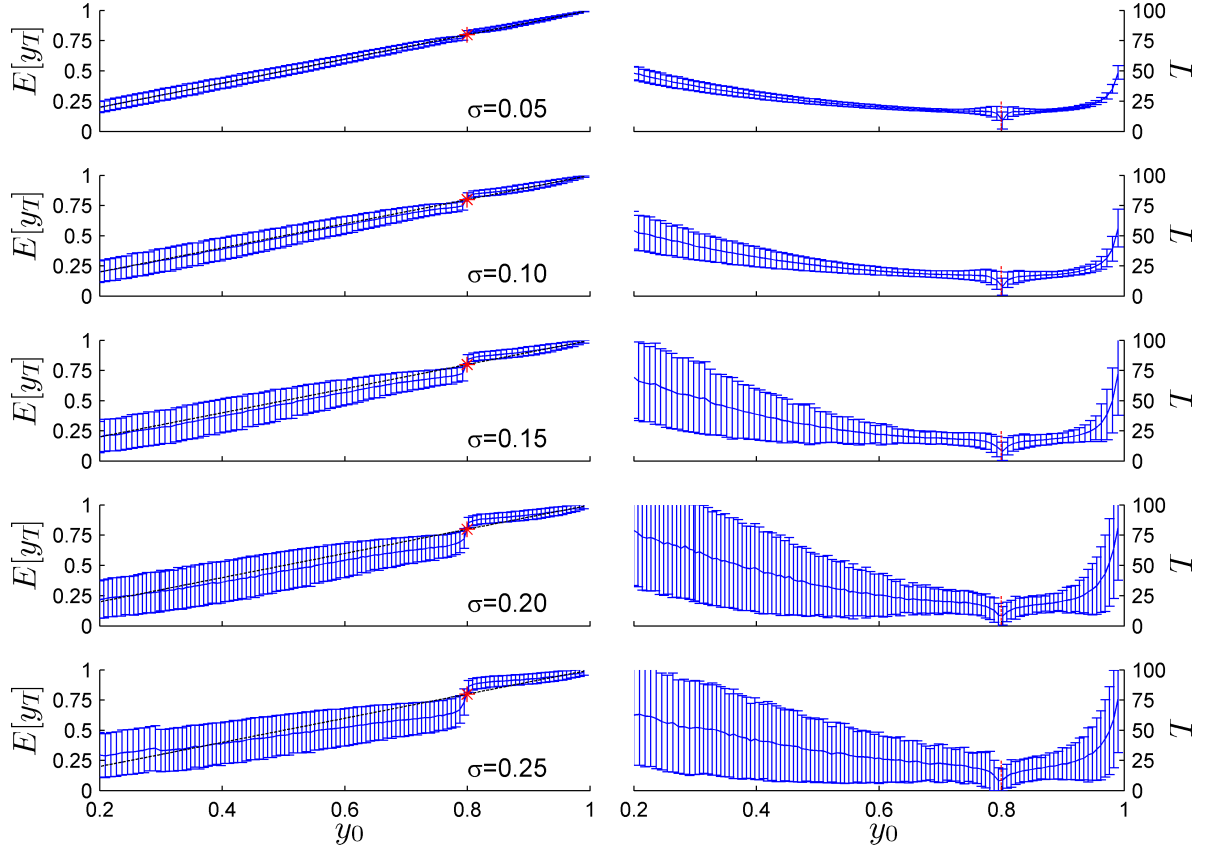


Figure 5: Expected values of employment y after one full loop y_T (left), and expected elapsed time T (right). Computation performed in MATLAB, with 2000 simulations for every value single one of the 100 initial values taken along the line $y = \hat{\theta}x$.

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Appendix A. Existence and regularity of a solution to a SDE

The following theorem is an exhaustive version of Corollary 3.1 p.76 with elements of Theorem 3.4 p.71 and Theorem 3.5 p.75 in [9].

Theorem 6. *Let (D_n) be an increasing sequence of open sets whose closure are contained in D and such that $\bigcup_n D_n = D \subset \mathbb{R}^k$. We define the stochastic differential equation for elements $X \in \mathbb{R}^k$:*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t . \quad (\text{A.1})$$

Assume that on each set D_n , μ and σ are Lipschitz and sub-linear. Assume that there exists a function $V(X_t, t) \in C^{2,1}(D \times \mathbb{R}_+)$ such that

$$\mathcal{L}V(X_t, t) \leq KV(X_t, t) + k \quad \text{on } D \times \mathbb{R}_+ \quad (\text{A.2})$$

for some $K, k \in \mathbb{R}_+$, and

$$\liminf_n \inf_{D \setminus D_n} V(c, t) = +\infty. \quad (\text{A.3})$$

Then Eq. (A.1) possesses a unique \mathbb{P} – a.s. continuous regular solution for any $X_0 \in L^0(D, \mathcal{F}_0)$ and it verifies also $\mathbb{P}[X_t \in D] = 1$ for all $t \geq 0$.

Appendix B. Recurrent domains

The following theorem is Theorem 3.9 in [9] and is repeatedly used in Section 5. We adapt the original theorem to our notations.

Theorem 7. Let $(x_t, y_t)_{t \geq 0}$ be a regular process in D , with $(x_0, y_0) \in U$, for some $U \subset D$. Assume that there exists a non-negative function $V(t, x, y) \in C^{1,2,2}(\mathbb{R}_+ \times U)$ verifying

$$\mathcal{L}V(s, x, y) \leq -f(s)$$

where $f(s) \geq 0$ and $\lim_t \int_0^t f(s) ds = +\infty$. Then (x_t, y_t) leaves the region U in finite time \mathbb{P} – a.s.

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