



HAL
open science

Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

Hung Nguyen Quoc, Laurent Veron

► **To cite this version:**

Hung Nguyen Quoc, Laurent Veron. Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption. 2014. hal-00851381v4

HAL Id: hal-00851381

<https://hal.science/hal-00851381v4>

Preprint submitted on 10 Oct 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

Nguyen Quoc Hung*

Laurent Véron†

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We obtain sufficient conditions, expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities, for the existence of large solutions to equations (1) $-\Delta_p u + e^u - 1 = 0$ or (2) $-\Delta_p u + u^q = 0$ in a bounded domain Ω when $q > p - 1 > 0$. We apply our results to equations (3) $-\Delta_p u + a |\nabla u|^q + bu^s = 0$, (4) $\Delta_p u + u^{-\gamma} = 0$ with $1 < p \leq 2$, $1 \leq q \leq p$, $a > 0$, $b > 0$ and $q > p - 1$, $s \geq p - 1$, $\gamma > 0$.

2010 Mathematics Subject Classification. 31C15, 35J92, 35F21, 35B44.

Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) and $1 < p \leq N$. We denote $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. In this paper we study some questions relative to the existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \tag{1.1}$$

where g is a continuous nondecreasing function vanishing at 0, and most often $g(u)$ is either $\operatorname{sign}(u)(e^{|u|} - 1)$ or $|u|^{q-1}u$ with $q > p - 1$. A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data ϕ

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega, \\ u - \phi &\in W_0^{1,p}(\Omega), u \in W_{\text{loc}}^{1,p}(\Omega) \cap L^\infty(\Omega), \end{aligned} \tag{1.2}$$

admits a solution $u \in C(\overline{\Omega})$, it is clear that problem (1.1) admits a solution provided problem $-\Delta_p u + g(u) = 0$ in Ω having a maximal solution, see [14, Chapter 5]. It is known that a

*E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr

†E-mail address: Laurent.Veron@lmpt.univ-tours.fr

necessary and sufficient condition for the solvability of problem (1.2) in case $g(u) \equiv 0$ is the *Wiener criterion*, due to Wiener [22] when $p = 2$ and Maz'ya [15], Kilpelainen and Malý [7] when $p \neq 2$, in general case is proved by Malý and Ziemer [12]. This condition is

$$\int_0^1 \left(\frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$. The existence of a maximal solution is guaranteed for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition [19]

$$\int_a^\infty \frac{dt}{\sqrt[p]{G(t)}} < \infty \quad \text{where } G(t) = \int_0^t g(s) ds \quad (1.4)$$

for some $a > 0$. This is an extension of the Keller-Osserman condition [8], [16], which is the above relation when $p = 2$. If for $R > \text{diam}(\Omega)$ there exists a function v which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\}, \\ v &= 0 && \text{on } \partial B_R, \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution u of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of $\partial\Omega$. Indeed, $x \mapsto v(x - y)$ is a solution of (1.6) in Ω for all $y \in \partial\Omega$, thus $u(x) \geq v(x - y)$ for any $x \in \Omega, y \in \partial\Omega$. It follows $\lim_{\rho(x) \rightarrow 0} u(x) = \infty$ since $\lim_{z \rightarrow 0} v(z) = \infty$.

Remark that the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron prove in [20] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad \text{with } p < N, \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p - 1 < q < \frac{N(p-1)}{N-p}$ with $p < N$, Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [9] show that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned}$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega,$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$, $N \geq 3$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. We refer

to [14] for other related results. Concerning the exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = \text{sign}(u)(e^{|u|} - 1)$ and $g(u) = |u|^{q-1}u$, $q > p - 1$. For $1 < p \leq N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\}$$

where the B_j are balls and $h^{N-p}(B_r) = r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 2$ and $1 < p \leq N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.9)$$

then there exists $u \in C^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^u - 1 &= 0 && \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty. \end{aligned} \quad (1.10)$$

Clearly, when $p = N$, we have $\mathcal{H}_1^{N-p}(\{x_0\}) = 1$ for all $x_0 \in \mathbb{R}^N$ thus, (1.9) is true for any open domain Ω .

We also obtain a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{\alpha,s}$ Bessel capacity in \mathbb{R}^N associated to the Besov space $B^{\alpha,s}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 2$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.11)$$

then, for any $p-1 < q < \frac{pq_1}{N}$ there exists $u \in C^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + u^q &= 0 && \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty. \end{aligned} \quad (1.12)$$

We can see that condition (1.9) implies (1.11). In view of Labutin's theorem this previous result is not optimal in the case $p = 2$, since the involved capacity is C_{2,q'_1} with q'_1 and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned}$$

with $q_1 > q$.

At end we apply the previous theorem to quasilinear viscous Hamilton-Jacobi equations:

$$\begin{aligned} -\Delta_p u + a|\nabla u|^q + b|u|^{s-1}u &= 0 \quad \text{in } \Omega, \\ u &\in C^1(\Omega), \quad \lim_{\rho(x) \rightarrow 0} u(x) = \infty. \end{aligned} \quad (1.13)$$

For $q_1 > p - 1$ and $1 < p \leq 2$, if equation (1.12) admits a solution with $q = q_1$, then for any $a > 0, b > 0$ and $q \in (p - 1, \frac{pq_1}{q_1+1})$, $s \in [p - 1, q_1)$ there exists a positive solution to (1.13). Conversely, if for some $a, b > 0$, $s > p - 1$ there exists a solution to equation (1.13) with $1 < q = p \leq 2$, then for any $q_1 > p - 1$, $1 \leq q_1 \leq p$, $s_1 \geq p - 1$, $a_1, b_1 > 0$ there exists a positive solution to equation (1.13) with parameters q_1, s_1, a_1, b_1 replacing q, s, a, b . Moreover, we also prove that the previous statement holds if for some $\gamma > 0$ there exists $u \in C(\overline{\Omega}) \cap C^1(\Omega)$, $u > 0$ in Ω satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We would like to remark that the case $p = 2$ was studied in [10]. In particular, if the boundary of Ω is smooth then (1.13) has a solution with $s = 1$ and $1 < q \leq 2, a > 0, b > 0$.

2 Morrey classes and Wolff potential estimates

In this section we assume that Ω is a bounded open subset of \mathbb{R}^N and $1 < p < N$. We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (1.1) belongs to $C_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 2.1 *A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that*

$$\int_{\Omega \cap B_r(x)} |f| dy \leq K r^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N.$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{\mathcal{M}^s(\Omega)}$. Clearly $L^s(\Omega) \subset \mathcal{M}^s(\Omega)$.

Definition 2.2 *Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}_+^b(\Omega)$, the set of nonnegative and bounded Radon measures in Ω . We define the (R -truncated) Wolff potential of μ by*

$$\mathbf{W}_{1,p}^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N,$$

and the (R -truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N,$$

where the measure is extended by 0 in Ω^c .

We recall a result proved in [6] (see also [2, Theorem 2.4]).

Theorem 2.3 Let μ be a nonnegative Radon measure in \mathbb{R}^N . There exist positive constants C_1, C_2 depending on N, p such that

$$\int_{2B} \exp(C_1 \mathbf{W}_{1,p}^R[\chi_B \mu]) dx \leq C_2 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$.

For $k \geq 0$, we set $T_k(u) = \text{sign}(u) \min\{k, |u|\}$.

Definition 2.4 Assume $f \in L_{loc}^1(\Omega)$. We say that a measurable function u defined in Ω is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.1)$$

if, for any $k > 0$, $T_k(u) \in W_{loc}^{1,p}(\Omega)$, $|\nabla u|^{p-1} \in L_{loc}^1(\Omega)$ and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f$ is a positive distribution in Ω .

The following result is proved in [12, Theorem 4.35].

Theorem 2.5 If $f \in \mathcal{M}_{\frac{N}{p-\epsilon}}(\Omega)$ for some $\epsilon \in (0, p)$, u is a nonnegative renormalized supersolution of (2.1) and set $\mu := -\Delta_p u + f$. Then there holds

$$u(x) + \|f\|_{\mathcal{M}_{\frac{N}{p-\epsilon}}(\Omega)}^{\frac{1}{p-1}} \geq C \mathbf{W}_{1,p}^{\frac{\epsilon}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega,$$

for some C depending only on $N, p, \epsilon, \text{diam}(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u + f = \mu \quad \text{in } \Omega, \quad (2.2)$$

where $f \in L^1(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$, we have

Corollary 2.6 Let $f \in \mathcal{M}_{\frac{N}{p-\epsilon}}(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. If u is a renormalized solution to (2.2) and $\inf_{\Omega} u > -\infty$ then there exists a positive constant C depending only on $N, p, \epsilon, \text{diam}(\Omega)$ such that

$$u(x) + \|f\|_{\mathcal{M}_{\frac{N}{p-\epsilon}}(\Omega)}^{\frac{1}{p-1}} \geq \inf_{\Omega} u + C \mathbf{W}_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega.$$

The next result, proved in [2, Theorem 1.1, 1.2], is an important tool for the proof of Theorems 1 and 2. Before presenting we introduce the notation.

Definition 2.7 Let $s > 1$ and $\alpha > 0$. We denote by $C_{\alpha,s}(E)$ the Bessel capacity of Borel set $E \subset \mathbb{R}^N$,

$$C_{\alpha,s}(E) = \inf\{\|\phi\|_{L^s(\mathbb{R}^N)}^s : \phi \in L_+^s(\mathbb{R}^N), G_{\alpha} * \phi \geq \chi_E\}$$

where χ_E is the characteristic function of E and G_{α} the Bessel kernel of order α .

We say that a measure μ in Ω is absolutely continuous with respect to the capacity $C_{\alpha,s}$ in Ω if

$$\text{for all } E \subset \Omega, E \text{ Borel}, C_{\alpha,s}(E) = 0 \Rightarrow |\mu|(E) = 0.$$

Theorem 2.8 Let $\mu \in \mathfrak{M}_+^b(\Omega)$ and $q > p - 1$.

a. If μ is absolutely continuous with respect to the capacity $C_{p, \frac{q}{q+1-p}}$ in Ω , then there exists a nonnegative renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + u^q &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which satisfies

$$u(x) \leq C \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu](x) \quad \forall x \in \Omega \quad (2.3)$$

where C is a positive constant depending on p and N .

b. If $\exp(\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu]) \in L^1(\Omega)$ where C is the previous constant, then there exists a nonnegative renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which satisfies (2.3).

3 Estimates from below

If G is any domain in \mathbb{R}^N with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7) there always exists a maximal solution to (1.6) in G . It is constructed as the limit, when $n \rightarrow \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 && \text{in } G_n \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty && \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 && \text{if } G_n \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where $\{G_n\}_n$ is a sequence of smooth domains such that $G_n \subset \overline{G_n} \subset G_{n+1}$ for all n , $\{\partial G_n\}_n$ is a bounded and $\bigcup_{n=1}^{\infty} G_n = G$ and $\rho_n(x) := \text{dist}(x, \partial G_n)$. Our main estimates are the following.

Theorem 3.1 Let $K \subset B_{1/4} \setminus \{0\}$ be a compact set and let $U_j \in C^1(K^c)$, $j = 1, 2$, be the maximal solutions of

$$-\Delta_p u + e^u - 1 = 0 \quad \text{in } K^c \quad (3.2)$$

for U_1 and

$$-\Delta_p u + u^q = 0 \quad \text{in } K^c \quad (3.3)$$

for U_2 , where $p - 1 < q < \frac{pq_1}{N}$. Then there exist constants C_k , $k = 1, 2, 3, 4$, depending on N , p and q such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

Proof. 1. For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [18, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\text{supp}(\mu_j) \subset K \cap S_j$, $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$c_1^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c_1 \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j,$$

for some $c_1 = c_1(N, p)$.

Now, we will show that for $\varepsilon = \varepsilon(N, p) > 0$ small enough, there holds,

$$A := \int_{B_1} \exp\left(\varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x)\right) dx \leq c_2, \quad (3.6)$$

where c_2 does not depend on J .

Indeed, define $\mu_j \equiv 0$ for all $j \geq J+1$ and $j \leq 2$. We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp\left(\varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x)\right) dx.$$

Since for any j

$$\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \leq c(p) \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] + c(p) \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] + c(p) \sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k],$$

with $c(p) = \max\{1, 5^{\frac{2-p}{p-1}}\}$ and $\exp(\sum_{i=1}^5 a_i) \leq \sum_{i=1}^5 \exp(5a_i)$ for all a_i . Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_3 \varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x)\right) dx + \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_3 \varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x)\right) dx \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1,3\}}^{j+1} \int_{S_j} \exp(c_3 \varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)) dx := A_1 + A_2 + A_3, \text{ with } c_3 = 5c(p). \end{aligned}$$

Estimate of A_3 : We apply Theorem 2.3 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp(c_3 \varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)) dx \leq c_4 r_{k-1}^N$$

with $c_3 \varepsilon \in (0, C_1]$, the constant C_1 is in Theorem 2.3. In particular,

$$\int_{S_j} \exp(c_3 \varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)) dx \leq c_4 r_{k-1}^N \quad \text{for } k = j-1, j, j+1,$$

which implies

$$A_3 \leq c_5 \sum_{j=1}^{+\infty} r_j^N = c_5 < \infty. \quad (3.7)$$

Estimate of A_1 : Since $\sum_{k \geq j+2} \mu_k(B_t(x)) = 0$ for all $x \in S_j, t \in (0, r_{j+1})$. Thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp \left(c_3 \varepsilon \int_{r_{j+1}}^1 \left(\frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right) dx \\ &\leq \sum_{j=1}^{\infty} \exp \left(c_3 \varepsilon \frac{p-1}{N-p} \left(\sum_{k \geq j+2} \mu_k(S_k) \right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \right) |S_j|. \end{aligned}$$

Note that $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k \geq j+2} \mu_k(S_k) \right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left(\sum_{k \geq j+2} r_{k-1}^{N-p} \right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k \geq 0} r_k^{N-p} \right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}}.$$

Therefore

$$A_1 \leq \exp \left(c_3 \varepsilon \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} \right) |B_1| = c_6. \quad (3.8)$$

Estimate of A_2 : for $x \in S_j$,

$$\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Since $r_i < t < r_{i-1}$, $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$, thus

$$\begin{aligned} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(S_k)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \sum_{i=1}^{j-1} \left(\sum_{k=i-1}^{j-2} r_{k-1}^{N-p} \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \leq c_7 j, \text{ with } c_7 = \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp(c_3 c_7 \varepsilon j) dx = \sum_{j=1}^{\infty} r_j^N \exp(c_3 c_7 \varepsilon j) |S_1| \\ &= \sum_{j=1}^{\infty} \exp((c_3 c_7 \varepsilon - N \log(2)) j) |S_1| \leq c_8 \quad \text{for } \varepsilon \leq N \log(2) / (2c_3 c_7). \end{aligned} \quad (3.9)$$

Consequently, from (3.8), (3.9) and (3.7), we obtain $A \leq c_2 := c_6 + c_8 + c_5$ for $\varepsilon = \varepsilon(N, p)$ small enough. This implies

$$\left\| \exp \left(\frac{p}{2N} \varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1)} \leq c_9 \left(\int_{B_1} \exp \left(\varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right) dx \right)^{\frac{p}{2N}} \leq c_{10}, \quad (3.10)$$

where the constant c_{10} does not depend on J . Set $B = B_{\frac{1}{4}}$. For $\varepsilon_0 = (\frac{p\varepsilon}{2NC})^{1/(p-1)}$, where C is the constant in (2.3), by Theorem 2.8 and estimate (3.10), there exists a nonnegative renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \varepsilon_0 \sum_{j=3}^J \mu_j && \text{in } B, \\ u &= 0 && \text{in } \partial B, \end{aligned}$$

satisfying (2.3) with $\mu = \varepsilon_0 \sum_{j=3}^J \mu_j$. Thus, from Corollary 2.6 and estimate (3.10), we have

$$u(0) \geq -c_{11} + c_{12} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

Therefore

$$\begin{aligned} u(0) &\geq -c_{11} + c_{12} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{11} + c_{12} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= -c_{11} + c_{12} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{11} + c_{13} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &= -c_{11} + c_{13} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}. \end{aligned}$$

From the inequality

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i,$$

we deduce that

$$\begin{aligned} u(0) &\geq -c_{11} + c_{13} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{11} + c_{13} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{14} + c_{15} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}. \end{aligned}$$

Since U_1 is the maximal solution in K^c , u satisfies the same equation in $B \setminus K$ and $U_1 \geq u = 0$ on ∂B , it follows that U_1 dominates u in $B \setminus K$. Then $U_1(0) \geq u(0)$ and we obtain (3.4).

2. By [1, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\text{supp}(\mu_j) \subset K \cap S_j$ and

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{\frac{q_1}{p-1}} dx = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

By Jensen's inequality, we have for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k \right)^s \leq \sum_{k=0}^{\infty} \theta_{k,s} a_k^s$$

where $\theta_{k,s}$ has the following expression with $\theta > 0$,

$$\theta_{k,s} = \begin{cases} 1 & \text{if } s \in (0, 1], \\ \left(\frac{\theta+1}{\theta}\right)^{s-1} (\theta+1)^{k(s-1)} & \text{if } s > 1. \end{cases}$$

Thus,

$$\begin{aligned} \int_{B_1} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} dx &\leq \int_{B_1} \left(\sum_{k=3}^J \theta_{k, \frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right)^{q_1} dx \\ &\leq \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1} (\mathbf{W}_{1,p}^1[\mu_k](x))^{q_1} dx \\ &\leq c_{16} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_{\mathbb{R}^N} (G_p * \mu_k(x))^{\frac{q_1}{p-1}} dx \\ &= c_{16} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\ &\leq c_{17} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k} \left(N - \frac{pq_1}{q_1-p+1} \right) \\ &\leq c_{18}, \end{aligned}$$

for θ small enough. Here the third inequality follows from [2, Theorem 2.3] and the constant c_{18} does not depend on J . Hence,

$$\left\| \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1)} \leq c_{19} \left\| \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1)}^q \leq c_{20}, \quad (3.11)$$

where c_{20} is independent of J . Take $B = B_{\frac{1}{4}}$. Since $\sum_{j=3}^J \mu_j$ is absolutely continuous with respect to the capacity $C_{p, \frac{q}{q+1-p}}$ in B , thus by Theorem 2.8, there exists a nonnegative renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + u^q &= \sum_{j=3}^J \mu_j && \text{in } B, \\ u &= 0 && \text{on } \partial B. \end{aligned}$$

satisfying (2.3) with $\mu = \sum_{j=3}^J \mu_j$. Thus, from Corollary 2.6 and estimate (3.11), we have

$$u(0) \geq -c_{21} + c_{22} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{23} + c_{24} \int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have $U_2(0) \geq u(0)$. Therefore, we obtain(3.5). ■

4 Proof of the main results

First, we prove theorem 1 in the case case $p = N$. To do this we consider the function

$$x \mapsto U(x) = U(|x|) = \log \left(\frac{N-1}{2^{N+1}} \frac{1}{R^N} \left(\frac{R}{|x|} + 1 \right) \right) \quad \text{in } B_R(0) \setminus \{0\}.$$

One has

$$U'(|x|) = \frac{1}{R+|x|} - \frac{1}{|x|} \quad \text{and} \quad U''(|x|) = -\frac{1}{(R+|x|)^2} + \frac{1}{|x|^2},$$

thus, for any $0 < |x| < R$,

$$\begin{aligned} -\Delta_N U + e^U - 1 &= -(N-1)|U'(|x|)|^{N-2} \left(U''(|x|) + \frac{1}{|x|} U'(|x|) \right) + e^U - 1 \\ &= -\frac{(N-1)R^{N-1}}{(R+|x|)^N |x|^{N-1}} + \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left(\frac{R}{|x|} + 1 \right) - 1 \\ &\leq -\frac{(N-1)R^{N-1}}{(2R)^N |x|^{N-1}} + \frac{N-1}{2^{N+1}} \frac{1}{R^N} \frac{2R}{|x|} \\ &\leq -1. \end{aligned}$$

Hence, if $u \in C^1(\Omega)$ is the maximal solution of

$$-\Delta_N u + e^u - 1 = 0 \quad \text{in } \Omega$$

and $R = 2\text{diam}(\Omega)$, then $u(x) \geq U(|x-y|)$ for any $x \in \Omega$ and $y \in \partial\Omega$. Therefore, u is a large solution and satisfies

$$u(x) \geq \log \left(\frac{N-1}{2^{N+1}} \frac{1}{R^N} \left(\frac{R}{\rho(x)} + 1 \right) \right) \quad \forall x \in \Omega.$$

Now, we prove Theorem 1 in the case $p < N$ and Theorem 2. Let $u, v \in C^1(\Omega)$ be the maximal solutions of

$$\begin{aligned} (i) \quad & -\Delta_p u + e^u - 1 = 0 && \text{in } \Omega, \\ (ii) \quad & -\Delta_p v + v^q = 0 && \text{in } \Omega. \end{aligned}$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B}_{1/4}(z_0)$. Let $U_1, U_2 \in C^1(K^c)$ be the maximal solutions of (3.2) and (3.3) respectively. We have $u \geq U_1$ and $v \geq U_2$ in Ω . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -c_1 + c_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -c_1 + c_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -c_1 + c_2 \int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -c_1 + c_3 \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned}$$

We deduce

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -c_1 + c_3 \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Similarly, we also obtain

$$\inf_{B_\delta \cap \Omega} v \geq -c_4 + c_5 \int_\delta^{1/2} \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Therefore, u and v satisfy (1.10) and (1.12) respectively. This completes the proof.

5 Large solutions of quasilinear Hamilton-Jacobi equations

Let Ω be a bounded open subset of \mathbb{R}^N with $N \geq 2$. In this section we use our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + b u^s &= 0 \quad \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \tag{5.1}$$

where $a > 0, b > 0$ and $1 \leq q < p \leq 2, q > p - 1, s \geq p - 1$.

First we have the result of existence solutions to equation (5.1).

Proposition 5.1 *Let $a > 0, b > 0$ and $q > p - 1, s \geq p - 1, 1 \leq q \leq p$ and $1 < p \leq 2$. There exists a maximal nonnegative solution $u \in C^1(\Omega)$ to equation*

$$-\Delta_p u + a |\nabla u|^q + b u^s = 0 \quad \text{in } \Omega, \tag{5.2}$$

which satisfies

$$u(x) \leq c(N, p, s) b^{-\frac{1}{s-p+1}} d(x, \partial\Omega)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega, \tag{5.3}$$

if $s > p - 1$,

$$u(x) \leq c(N, p, q) \left(a^{-\frac{1}{q-p+1}} d(x, \partial\Omega)^{-\frac{p-q}{q-p+1}} + a^{-\frac{1}{q-p+1}} b^{-\frac{1}{p-1}} d(x, \partial\Omega)^{-\frac{q}{(p-1)(q-p+1)}} \right) \quad \forall x \in \Omega, \quad (5.4)$$

if $p-1 < q < p$ and $s = p-1$, and

$$u(x) \leq c(N, p) a^{-1} b^{-\frac{1}{p-1}} d(x, \partial\Omega)^{-\frac{p}{p-1}} \quad \forall x \in \Omega, \quad (5.5)$$

if $q = p$ and $s = p-1$.

Proof. Case $s = p-1$ and $p-1 < q < p$. We consider

$$U_1(x) = U_1(|x|) = c_1 \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{p-q}{q-p+1}} + c_2 \in C^1(B_R(0)).$$

with $p' = \frac{p}{p-1}$ and $c_1, c_2 > 0$. We have

$$\begin{aligned} U_1'(|x|) &= \frac{c_1(p-q)}{q-p+1} \frac{|x|^{p'-1}}{R^{p'-1}} \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{1}{q-p+1}}, \\ U_1''(|x|) &= \frac{c_1(p-q)(p'-1)}{q-p+1} \frac{|x|^{p'-2}}{R^{p'-1}} \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{1}{q-p+1}} \\ &\quad + \frac{c_1(p-q)}{(q-p+1)^2} \left(\frac{|x|^{p'-1}}{R^{p'-1}} \right)^2 \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{1}{q-p+1}-1} \end{aligned}$$

and

$$A = -\Delta_p U_1 + a |\nabla U_1|^q + b U_1^{p-1} \geq -\Delta_p U_1 + a |\nabla U_1|^q + b c_2^{p-1}.$$

Thus, for all $x \in B_R(0)$

$$\begin{aligned} A &\geq -(p-1) |U_1'(|x|)|^{p-2} U_1''(|x|) - \frac{N-1}{|x|} |U_1'(|x|)|^{p-2} U_1'(|x|) + a |U_1'(|x|)|^q + b c_1^{p-1} \\ &= \left(\frac{c_1(p-q)(p'-1)}{q-p+1} \right)^{p-1} \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{q}{q-p+1}} \left\{ -(p-1) \frac{p'-1}{p'} \left(1 - \left(\frac{|x|}{R} \right)^{p'} \right) \right. \\ &\quad \left. - \frac{1}{q-p+1} \left(\frac{|x|}{R} \right)^{p'} - \frac{N-1}{p'} \left(\frac{|x|}{R} \right)^{p'} \left(1 - \left(\frac{|x|}{R} \right)^{p'} \right) \right. \\ &\quad \left. + a \left(\frac{c_1(p-q)}{q-p+1} \right)^{q-p+1} \left(\frac{|x|}{R} \right)^{\frac{q}{q-p+1}} \right\} + b c_2^{p-1} \\ &\geq \left(\frac{c_1(p-q)(p'-1)}{q-p+1} \right)^{p-1} \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{q}{q-p+1}} \\ &\quad \times \left\{ -\frac{N(p-1)}{p} - \frac{1}{q-p+1} + a \left(\frac{c_1(p-q)}{q-p+1} \right)^{q-p+1} \left(\frac{|x|}{R} \right)^{\frac{q}{q-p+1}} \right\} + b c_2^{p-1}. \end{aligned}$$

Clearly, one can find $c_1 = c_2(N, p, q)a^{-\frac{1}{q-p+1}} > 0$ and $c_3 = c_3(N, p, q) > 0$ such that

$$A \geq -c_3 a^{-\frac{p-1}{q-p+1}} R^{-\frac{q}{q-p+1}} + b c_2^{p-1}.$$

Choosing $c_2 = c_3^{\frac{1}{p-1}} a^{-\frac{1}{q-p+1}} b^{-\frac{1}{p-1}} R^{-\frac{q}{(p-1)(q-p+1)}}$, we get

$$-\Delta_p U_1 + a|\nabla U_1|^q + bU_1^{p-1} \geq 0 \quad \text{in } B_R(0). \quad (5.6)$$

Likewise, we can verify that the function U_2 below

$$U_2(x) = c_4 a^{-1} \log \left(\frac{R^{p'}}{R^{p'} - |x|^{p'}} \right) + c_4 a^{-1} b^{-\frac{1}{p-1}} R^{-\frac{p}{p-1}}$$

belongs to $C_+^1(B_R(0))$ and satisfies

$$-\Delta_p U_2 + a|\nabla U_2|^p + bU_2^{p-1} \geq 0 \quad \text{in } B_R(0). \quad (5.7)$$

While, if $s > p - 1$,

$$U_3(x) = c_5 b^{-\frac{1}{s-p+1}} \left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{-\frac{p}{s-p+1}}$$

belongs to $C^1(B_R(0))$ and verifies

$$-\Delta_p U_3 + bU_3^s \geq 0 \quad \text{in } B_R(0), \quad (5.8)$$

for some positive constants $c_4 = c_4(N, p, q)$, $c_5 = c_5(N, p, s)$ and $\beta = \beta(N, p, q) > 1$.

We emphasize the fact that with the condition $1 < p \leq 2$ and $q \geq 1$, equation (5.2) satisfies a comparison principle, see [17, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains Ω_n satisfying $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. For each $n, k \in \mathbb{N}^*$, there

exist nonnegative solution $u_{n,k} = u \in W_k^{1,p}(\Omega_n) := W_0^{1,p}(\Omega_n) + k$ of equation (5.2) in Ω_n .

Since $-\Delta_p u_{k,n} \leq 0$ in Ω_n , so using the maximum principle we get $u_{n,k} \leq k$ in Ω_n for all n . Thus, by standard regularity (see [4] and [11]), $u_{n,k} \in C^{1,\alpha}(\overline{\Omega}_n)$ for some $\alpha \in (0, 1)$. It follows from the comparison principle and (5.6)-(5.8), that

$$u_{n,k} \leq u_{n,k+1} \quad \text{in } \Omega_n$$

and (5.3)-(5.5) are satisfied with $u_{n,k}$ and Ω_n in place of u and Ω respectively. From this, we derive uniform local bounds for $\{u_{n,k}\}_k$, and by standard interior regularity (see [4]) we obtain uniform local bounds for $\{u_{n,k}\}_k$ in $C_{loc}^{1,\eta}(\Omega_n)$. It implies that the sequence $\{u_{n,k}\}_k$ is pre-compact in C^1 . Therefore, up to a subsequence, $u_{n,k} \rightarrow u_n$ in $C^1(\Omega_n)$. Hence, we can verify that u_n is a solution of (5.2) and satisfies (5.3)-(5.5) with u_n and Ω_n replacing u and Ω and $u_n(x) \rightarrow \infty$ as $d(x, \Omega_n) \rightarrow 0$.

Next, since $u_{n,k} \geq u_{n+1,k}$ in Ω_n there holds $u_n \geq u_{n+1}$ in Ω_n . In particular, $\{u_n\}$ is uniformly locally bounded in Ω . Arguing as above, we obtain $u_n \rightarrow u$ in $C^1(\Omega)$, thus u is a solution of (5.2) in Ω and satisfies (5.3)-(5.5). Clearly, u is the maximal solution of (5.2). \blacksquare

Theorem 5.2 *Let $q_1 > p - 1$ and $1 < p \leq 2$. Assume that equation (1.12) admits a solution with $q = q_1$. Then for any $a > 0, b > 0$ and $q \in (p - 1, \frac{pq_1}{q_1 + 1})$, $s \in [p - 1, q_1)$ equation (5.2) has a large solution satisfying (5.3) and (5.4).*

Proof. Assume that equation (1.12) admits a solution v with $q = q_1$ and set $v = \beta w^\sigma$ with $\beta > 0, \sigma \in (0, 1)$, then $w > 0$ and

$$-\Delta_p w + (-\sigma + 1)(p - 1) \frac{|\nabla w|^p}{w} + \beta^{q_1 - p + 1} \sigma^{-p + 1} w^{\sigma(q_1 - p + 1) + p - 1} = 0 \text{ in } \Omega.$$

If we impose $\max\{\frac{s - p + 1}{q_1 - p + 1}, (\frac{q}{p - q} - p + 1) \frac{1}{q_1 - p + 1}\} < \sigma < 1$, we can see that

$$(-\sigma + 1)(p - 1) \frac{|\nabla w|^p}{w} + \beta^{q_1 - p + 1} \sigma^{-p + 1} w^{\sigma(q_1 - p + 1) + p - 1} \geq a |\nabla w|^q + b w^s \text{ in } \{x : w(x) \geq M\},$$

where a positive constant M depends on p, q_1, q, s, a, b . Therefore

$$-\Delta_p w + a |\nabla w|^q + b w^s \leq 0 \text{ in } \{x : w(x) \geq M\}.$$

Now we take an open subset Ω' of Ω with $\overline{\Omega'} \subset \Omega$ such that the set $\{x : w(x) \geq M\}$ contains $\Omega \setminus \overline{\Omega'}$. So w is a subsolution of $-\Delta_p u + a |\nabla u|^q + b u^s = 0$ in $\Omega \setminus \overline{\Omega'}$ and the same property holds with $w_\varepsilon := \varepsilon w$ for any $\varepsilon \in (0, 1)$. Let u be as in Proposition 5.1. Set $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$ and $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$. Thus $w_\varepsilon < u$ on $\partial\Omega'$ with $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$. Hence, from the construction of u in the proof of Proposition 5.1 and the comparison principle, we obtain $w_\varepsilon \leq u$ in $\Omega \setminus \overline{\Omega'}$. This implies the result. \blacksquare

Remark 5.3 From the proof of above Theorem, we can show that under the assumption as in Proposition 5.1, equation (5.2) has a large solution in Ω if and only if equation (5.2) has a large solution in $\Omega \setminus K$ for some a compact set $K \subset \Omega$ with smooth boundary.

Now we deal with (5.1) in the case $q = p$.

Theorem 5.4 Assume that equation (5.2) has a large solution in Ω for some $a, b > 0, s > p - 1$ and $q = p > 1$. Then for any $a_1, b_1 > 0$ and $q_1 > p - 1, s_1 \geq p - 1, 1 \leq q_1 \leq p \leq 2$, equation (5.2) also has a large solution u in Ω with parameters a_1, b_1, q_1, s_1 in place of a, b, q, s respectively, and it satisfies (5.3)-(5.5).

Proof. For $\sigma > 0$ we set $u = v^\sigma$ thus

$$-\Delta_p v - (\sigma - 1)(p - 1) \frac{|\nabla v|^p}{v} + a \sigma v^{\sigma - 1} |\nabla v|^p + b \sigma^{-p + 1} v^{(s - p + 1)\sigma + p - 1} = 0.$$

Choose $\sigma = \frac{s_1 - p + 1}{s - p + 1} + 2$, it is easy to see that

$$-\Delta_p v + a_1 |\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \text{ in } \{x : v(x) \geq M\},$$

for some a positive constant M only depending on $p, s, a, b, a_1, b_1, q_1, s_1$. Similarly as in the proof of Theorem 5.2, we get the result as desired. \blacksquare

Remark 5.5 If we set $u = e^v$ then v satisfies

$$-\Delta_p v + b e^{(s - p + 1)v} = |\nabla v|^p (p - 1 - a e^v) \text{ in } \Omega.$$

From this, we can construct a large solution of

$$-\Delta_p u + b e^{(s - p + 1)u} = 0 \text{ in } \Omega \setminus K,$$

for any a compact set $K \subset \Omega$ with smooth boundary such that $v \geq \ln\left(\frac{p-1}{a}\right)$ in $\Omega \setminus K$. In case $p = 2$, It would be interesting to see what Wiener type criterion is implied by the existence as such a large solution. We conjecture that this condition must be

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = \infty \quad \forall x \in \partial\Omega.$$

We now consider the function

$$U_4(x) = c \left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0.$$

As in the proof of proposition 5.1, it is easy to check that there exist positive constants β large enough and c small enough so that inequality $\Delta_p U_4 + U_4^{-\gamma} \geq 0$ holds.

From this, we get the existence of minimal solution to equation

$$\Delta_p u + u^{-\gamma} = 0 \quad \text{in } \Omega. \quad (5.9)$$

Proposition 5.6 *Assume $\gamma > 0$. Then there exists a minimal solution $u \in C^1(\Omega)$ to equation (5.9) and it satisfies $u(x) \geq Cd(x, \partial\Omega)^{\frac{p}{\gamma+p-1}}$ in Ω .*

We can verify that if the boundary of Ω is satisfied (1.3), then above minimal solution u belongs to $C(\overline{\Omega})$, vanishes on $\partial\Omega$ and it is therefore a solution to the quenching problem

$$\begin{aligned} \Delta_p u + u^{-\gamma} &= 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \partial\Omega. \end{aligned} \quad (5.10)$$

Theorem 5.7 *Let $\gamma > 0$. Assume that there exists a solution $u \in C(\overline{\Omega})$ to problem (5.10). Then, for any $a, b > 0$ and $q > p - 1, s \geq p - 1, 1 \leq q \leq p \leq 2$, equation (5.2) admits a large solution in Ω and it satisfies (5.3)-(5.5).*

Proof. We set $u = e^{-\frac{a}{p-1}v}$, then v is a large solution of

$$-\Delta_p v + a|\nabla v|^p + \left(\frac{p-1}{a}\right)^{p-1} e^{\frac{a}{p-1}(\gamma+p-1)v} = 0 \quad \text{in } \Omega.$$

So

$$-\Delta_p v + a|\nabla v|^q + bv^s \leq 0 \quad \text{in } \{x : v(x) \geq M\},$$

for some a positive constant M only depending on p, q, s, a, b, γ . Similarly to the proof of Theorem 5.2, we get the result as desired. \blacksquare

References

- [1] D. R. Adams, L. I. Hedberg: *Function spaces and potential theory*. Grundlehren der Mathematischen Wissenschaften **314**, Springer-Verlag, Berlin, 1996. xii+366 pp.
- [2] M.F. Bidaut-Véron, H. Nguyen Quoc, L. Véron: *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. **102**, 315-337 (2014).
- [3] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa **28**, 741-808 (1999).

- [4] E. DiBenedetto: *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis **7**, 827-850 (1983).
- [5] A. Friedman, L. Véron: *Singular Solutions of Some Quasilinear Elliptic Equations*, Arch. Rat. Mech. Anal. **96**, 259-287 (1986).
- [6] P. Honzik, B. Jaye: *On the good- λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180 (2012).
- [7] T. Kilpelainen, J. Malý: *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).
- [8] J. B. Keller: *On solutions of $\Delta u = f(u)$* , Comm. Pure Appl. Math. **10**, 503-510 (1957).
- [9] D. Labutin: *Wiener regularity for large solutions of nonlinear equations*, Ark. Mat. **41**, 307-39 (2003).
- [10] J.M. Lasry and P.L. Lions: *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints: 1. The Model problem*, Math. Ann. **283**, 583-630(1989).
- [11] G.M. Lieberman: *Boundary regularity for solution of degenerate elliptic equations* , Nonlinear Analysis **12**, 1203-1219 (1988).
- [12] J. Maly, W.P. Ziemer: *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, A.M.S (1997).
- [13] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, Series in Nonlinear Analysis and Applications **21**, De Gruyter, Berlin/Boston (2013).
- [14] M. Marcus, L. Véron: *Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets*, J. Math. Pures Appl. **91**, 256-295 (2009).
- [15] V. Maz'ya: *On the continuity at a boundary point of solutions of quasilinear equations*, Vestnik Leningrad Univ. Math. **3**, 225-242 (1976).
- [16] R. Osserman: *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math. **7**, 1641-1647 (1957)
- [17] P. Pucci, J. Serrin: *The Maximum Principle*, Progress in Nonlinear Differential Equations and Their Applications, 2007.
- [18] B. O. Turesson: *Nonlinear Potential Theory and Sobolev Spaces*, Lecture Notes in Mathematics **1736**, Springer-Verlag Berlin, Heidelberg (2000).
- [19] J. L. Vazquez: *An a priori interior estimate for the solution of a nonlinear problem representing weak diffusion*, Nonlinear Anal. T. M. A. **5**, 95-103 (1981).
- [20] J. L. Vazquez, L. Véron: *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. **33**, 129-144 (1980).
- [21] L. Véron: *On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data*, Math. Z. **273**, 1-17 (2013).
- [22] N. Wiener: *The Dirichlet problem*, J. Math. Phys. **3**, 127-146 (1924).
- [23] W. Ziemer: *Weakly Differentiable Functions*, Graduate Texts in Mathematics **120**, Springer-Verlag, Berlin (1989).