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# Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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## Abstract

We obtain sufficient conditions, expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities, for the existence of large solutions to equations (1)  $-\Delta_p u + e^u - 1 = 0$  or (2)  $-\Delta_p u + u^q = 0$  in a bounded domain  $\Omega$  when  $q > p - 1 > 0$ . We apply our results to equations (3)  $-\Delta_p u + a|\nabla u|^q + bu^s = 0$ , (4)  $\Delta_p u + u^{-\gamma} = 0$  with  $1 < p \leq 2$ ,  $1 \leq q \leq p$ ,  $a > 0$ ,  $b > 0$  and  $q > p - 1$ ,  $s \geq p - 1$ ,  $\gamma > 0$ .

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Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $1 < p \leq N$ . We denote  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ . In this paper we study some questions relative to the existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \tag{1.1}$$

where  $g$  is a continuous nondecreasing function vanishing at 0, and most often  $g(u)$  is either  $\operatorname{sign}(u)(e^{|u|} - 1)$  or  $|u|^{q-1}u$  with  $q > p - 1$ . A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data  $\phi$

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega, \\ u - \phi &\in W_0^{1,p}(\Omega), u \in W_{\text{loc}}^{1,p}(\Omega) \cap L^\infty(\Omega), \end{aligned} \tag{1.2}$$

admits a solution  $u \in C(\overline{\Omega})$ , it is clear that problem (1.1) admits a solution provided problem  $-\Delta_p u + g(u) = 0$  in  $\Omega$  having a maximal solution, see [14, Chapter 5]. It is known that a

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necessary and sufficient condition for the solvability of problem (1.2) in case  $g(u) \equiv 0$  is the *Wiener criterion*, due to Wiener [22] when  $p = 2$  and Maz'ya [15], Kilpelainen and Malý [7] when  $p \neq 2$ , in general case is proved by Malý and Ziemer [12]. This condition is

$$\int_0^1 \left( \frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where  $C_{1,p}$  denotes the capacity associated to the space  $W^{1,p}(\mathbb{R}^N)$ . The existence of a maximal solution is guaranteed for a large class of nondecreasing nonlinearities  $g$  satisfying the Vazquez condition [19]

$$\int_a^\infty \frac{dt}{\sqrt[p]{G(t)}} < \infty \quad \text{where } G(t) = \int_0^t g(s) ds \quad (1.4)$$

for some  $a > 0$ . This is an extension of the Keller-Osserman condition [8], [16], which is the above relation when  $p = 2$ . If for  $R > \text{diam}(\Omega)$  there exists a function  $v$  which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\}, \\ v &= 0 && \text{on } \partial B_R, \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution  $u$  of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of  $\partial\Omega$ . Indeed,  $x \mapsto v(x - y)$  is a solution of (1.6) in  $\Omega$  for all  $y \in \partial\Omega$ , thus  $u(x) \geq v(x - y)$  for any  $x \in \Omega, y \in \partial\Omega$ . It follows  $\lim_{\rho(x) \rightarrow 0} u(x) = \infty$  since  $\lim_{z \rightarrow 0} v(z) = \infty$ .

Remark that the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of  $g$  is too strong since Vazquez and Véron prove in [20] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad \text{with } p < N, \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if  $p - 1 < q < \frac{N(p-1)}{N-p}$  with  $p < N$ , Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [9] show that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned}$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega,$$

where  $C_{2,q'}$  is the capacity associated to the Sobolev space  $W^{2,q'}(\mathbb{R}^N)$  and  $q' = q/(q-1)$ ,  $N \geq 3$ . Notice that this condition is always satisfied if  $q$  is subcritical, i.e.  $q < N/(N-2)$ . We refer

to [14] for other related results. Concerning the exponential case of problem (1.1) nothing is known, even in the case  $p = 2$ , besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases  $g(u) = \text{sign}(u)(e^{|u|} - 1)$  and  $g(u) = |u|^{q-1}u$ ,  $q > p - 1$ . For  $1 < p \leq N$ , we denote by  $\mathcal{H}_1^{N-p}(E)$  the Hausdorff capacity of a set  $E$  defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\}$$

where the  $B_j$  are balls and  $h^{N-p}(B_r) = r^{N-p}$ . Our main result concerning the exponential case is the following

**Theorem 1.** *Let  $N \geq 2$  and  $1 < p \leq N$ . If*

$$\int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.9)$$

*then there exists  $u \in C^1(\Omega)$  satisfying*

$$\begin{aligned} -\Delta_p u + e^u - 1 &= 0 && \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty. \end{aligned} \quad (1.10)$$

Clearly, when  $p = N$ , we have  $\mathcal{H}_1^{N-p}(\{x_0\}) = 1$  for all  $x_0 \in \mathbb{R}^N$  thus, (1.9) is true for any open domain  $\Omega$ .

We also obtain a sufficient condition for the existence of a large solution in the power case expressed in terms of some  $C_{\alpha,s}$  Bessel capacity in  $\mathbb{R}^N$  associated to the Besov space  $B^{\alpha,s}(\mathbb{R}^N)$ .

**Theorem 2.** *Let  $N \geq 2$ ,  $1 < p < N$  and  $q_1 > \frac{N(p-1)}{N-p}$ . If*

$$\int_0^1 \left( \frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.11)$$

*then, for any  $p-1 < q < \frac{pq_1}{N}$  there exists  $u \in C^1(\Omega)$  satisfying*

$$\begin{aligned} -\Delta_p u + u^q &= 0 && \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty. \end{aligned} \quad (1.12)$$

We can see that condition (1.9) implies (1.11). In view of Labutin's theorem this previous result is not optimal in the case  $p = 2$ , since the involved capacity is  $C_{2,q'_1}$  with  $q'_1$  and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned}$$

with  $q_1 > q$ .

At end we apply the previous theorem to quasilinear viscous Hamilton-Jacobi equations:

$$\begin{aligned} -\Delta_p u + a|\nabla u|^q + b|u|^{s-1}u &= 0 & \text{in } \Omega, \\ u &\in C^1(\Omega), \lim_{\rho(x) \rightarrow 0} u(x) = \infty. \end{aligned} \quad (1.13)$$

For  $q_1 > p - 1$  and  $1 < p \leq 2$ , if equation (1.12) admits a solution with  $q = q_1$ , then for any  $a > 0, b > 0$  and  $q \in (p - 1, \frac{pq_1}{q_1+1})$ ,  $s \in [p - 1, q_1)$  there exists a positive solution to (1.13). Conversely, if for some  $a, b > 0$ ,  $s > p - 1$  there exists a solution to equation (1.13) with  $1 < q = p \leq 2$ , then for any  $q_1 > p - 1$ ,  $1 \leq q_1 \leq p$ ,  $s_1 \geq p - 1$ ,  $a_1, b_1 > 0$  there exists a positive solution to equation (1.13) with parameters  $q_1, s_1, a_1, b_1$  replacing  $q, s, a, b$ . Moreover, we also prove that the previous statement holds if for some  $\gamma > 0$  there exists  $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ ,  $u > 0$  in  $\Omega$  satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

We would like to remark that the case  $p = 2$  was studied in [10]. In particular, if the boundary of  $\Omega$  is smooth then (1.13) has a solution with  $s = 1$  and  $1 < q \leq 2, a > 0, b > 0$ .

## 2 Morrey classes and Wolff potential estimates

In this section we assume that  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$  and  $1 < p < N$ . We also denote by  $B_r(x)$  the open ball of center  $x$  and radius  $r$  and  $B_r = B_r(0)$ . We also recall that a solution of (1.1) belongs to  $C_{loc}^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ , and is more regular (depending on  $g$ ) on the set  $\{x \in \Omega : |\nabla u(x)| \neq 0\}$ .

**Definition 2.1** *A function  $f \in L^1(\Omega)$  belongs to the Morrey space  $\mathcal{M}^s(\Omega)$ ,  $1 \leq s \leq \infty$ , if there is a constant  $K$  such that*

$$\int_{\Omega \cap B_r(x)} |f| dy \leq K r^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N.$$

*The norm is defined as the smallest constant  $K$  that satisfies this inequality; it is denoted by  $\|f\|_{\mathcal{M}^s(\Omega)}$ . Clearly  $L^s(\Omega) \subset \mathcal{M}^s(\Omega)$ .*

**Definition 2.2** *Let  $R \in (0, \infty]$  and  $\mu \in \mathfrak{M}_+^b(\Omega)$ , the set of nonnegative and bounded Radon measures in  $\Omega$ . We define the ( $R$ -truncated) Wolff potential of  $\mu$  by*

$$\mathbf{W}_{1,p}^R[\mu](x) = \int_0^R \left( \frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N,$$

*and the ( $R$ -truncated) fractional maximal potential of  $\mu$  by*

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N,$$

*where the measure is extended by 0 in  $\Omega^c$ .*

We recall a result proved in [6] (see also [2, Theorem 2.4]).

**Theorem 2.3** Let  $\mu$  be a nonnegative Radon measure in  $\mathbb{R}^N$ . There exist positive constants  $C_1, C_2$  depending on  $N, p$  such that

$$\int_{2B} \exp(C_1 \mathbf{W}_{1,p}^R[\chi_B \mu]) dx \leq C_2 r^N,$$

for all  $B = B_r(x_0) \subset \mathbb{R}^N$ ,  $2B = B_{2r}(x_0)$ ,  $R > 0$  such that  $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$ .

For  $k \geq 0$ , we set  $T_k(u) = \text{sign}(u) \min\{k, |u|\}$ .

**Definition 2.4** Assume  $f \in L^1_{loc}(\Omega)$ . We say that a measurable function  $u$  defined in  $\Omega$  is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.1)$$

if, for any  $k > 0$ ,  $T_k(u) \in W^1_{loc}(\Omega)$ ,  $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$  and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0$$

for all  $\varphi \in W^{1,p}(\Omega)$  with compact support in  $\Omega$  and such that  $0 \leq \varphi \leq k - T_k(u)$ , and if  $-\Delta_p u + f$  is a positive distribution in  $\Omega$ .

The following result is proved in [12, Theorem 4.35].

**Theorem 2.5** If  $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$  for some  $\epsilon \in (0, p)$ ,  $u$  is a nonnegative renormalized supersolution of (2.1) and set  $\mu := -\Delta_p u + f$ . Then there holds

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)}^{\frac{1}{p-1}} \geq C \mathbf{W}_{1,p}^{\frac{\epsilon}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega,$$

for some  $C$  depending only on  $N, p, \epsilon, \text{diam}(\Omega)$ .

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u + f = \mu \quad \text{in } \Omega, \quad (2.2)$$

where  $f \in L^1(\Omega)$  and  $\mu \in \mathfrak{M}_+^b(\Omega)$ , we have

**Corollary 2.6** Let  $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$  and  $\mu \in \mathfrak{M}_+^b(\Omega)$ . If  $u$  is a renormalized solution to (2.2) and  $\inf_{\Omega} u > -\infty$  then there exists a positive constant  $C$  depending only on  $N, p, \epsilon, \text{diam}(\Omega)$  such that

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)}^{\frac{1}{p-1}} \geq \inf_{\Omega} u + C \mathbf{W}_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega.$$

The next result, proved in [2, Theorem 1.1, 1.2], is an important tool for the proof of Theorems 1 and 2. Before presenting we introduce the notation.

**Definition 2.7** Let  $s > 1$  and  $\alpha > 0$ . We denote by  $C_{\alpha,s}(E)$  the Bessel capacity of Borel set  $E \subset \mathbb{R}^N$ ,

$$C_{\alpha,s}(E) = \inf\{\|\phi\|_{L^s(\mathbb{R}^N)}^s : \phi \in L^s_+(\mathbb{R}^N), G_\alpha * \phi \geq \chi_E\}$$

where  $\chi_E$  is the characteristic function of  $E$  and  $G_\alpha$  the Bessel kernel of order  $\alpha$ .

We say that a measure  $\mu$  in  $\Omega$  is absolutely continuous with respect to the capacity  $C_{\alpha,s}$  in  $\Omega$  if

$$\text{for all } E \subset \Omega, E \text{ Borel}, C_{\alpha,s}(E) = 0 \Rightarrow |\mu|(E) = 0.$$

**Theorem 2.8** Let  $\mu \in \mathfrak{M}_+^b(\Omega)$  and  $q > p - 1$ .

- a. If  $\mu$  is absolutely continuous with respect to the capacity  $C_{p, \frac{q}{q+1-p}}$  in  $\Omega$ , then there exists a nonnegative renormalized solution  $u$  to equation

$$\begin{aligned} -\Delta_p u + u^q &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which satisfies

$$u(x) \leq C \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu](x) \quad \forall x \in \Omega \quad (2.3)$$

where  $C$  is a positive constant depending on  $p$  and  $N$ .

- b. If  $\exp(C \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu]) \in L^1(\Omega)$  where  $C$  is the previous constant, then there exists a nonnegative renormalized solution  $u$  to equation

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which satisfies (2.3).

### 3 Estimates from below

If  $G$  is any domain in  $\mathbb{R}^N$  with a compact boundary and  $g$  is nondecreasing,  $g(0) = g^{-1}(0) = 0$  and satisfies (1.7) there always exists a maximal solution to (1.6) in  $G$ . It is constructed as the limit, when  $n \rightarrow \infty$ , of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 && \text{in } G_n \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty && \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 && \text{if } G_n \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where  $\{G_n\}_n$  is a sequence of smooth domains such that  $G_n \subset \overline{G}_n \subset G_{n+1}$  for all  $n$ ,  $\{\partial G_n\}_n$  is a bounded and  $\bigcup_{n=1}^{\infty} G_n = G$  and  $\rho_n(x) := \text{dist}(x, \partial G_n)$ . Our main estimates are the following.

**Theorem 3.1** Let  $K \subset B_{1/4} \setminus \{0\}$  be a compact set and let  $U_j \in C^1(K^c)$ ,  $j = 1, 2$ , be the maximal solutions of

$$-\Delta_p u + e^u - 1 = 0 \quad \text{in } K^c \quad (3.2)$$

for  $U_1$  and

$$-\Delta_p u + u^q = 0 \quad \text{in } K^c \quad (3.3)$$

for  $U_2$ , where  $p - 1 < q < \frac{pq_1}{N}$ . Then there exist constants  $C_k$ ,  $k = 1, 2, 3, 4$ , depending on  $N$ ,  $p$  and  $q$  such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left( \frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

**Proof.** 1. For  $j \in \mathbb{Z}$  define  $r_j = 2^{-j}$  and  $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$ ,  $B_j = B_{r_j}$ . Fix a positive integer  $J$  such that  $K \subset \{x : r_J \leq |x| < 1/8\}$ . Consider the sets  $K \cap S_j$  for  $j = 3, \dots, J$ . By [18, Theorem 3.4.27], there exists  $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$  such that  $\text{supp}(\mu_j) \subset K \cap S_j$ ,  $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \leq 1$  and

$$c_1^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c_1 \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j,$$

for some  $c_1 = c_1(N, p)$ .

Now, we will show that for  $\varepsilon = \varepsilon(N, p) > 0$  small enough, there holds,

$$A := \int_{B_1} \exp\left(\varepsilon \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] (x)\right) dx \leq c_2, \quad (3.6)$$

where  $c_2$  does not depend on  $J$ .

Indeed, define  $\mu_j \equiv 0$  for all  $j \geq J+1$  and  $j \leq 2$ . We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp\left(\varepsilon \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] (x)\right) dx.$$

Since for any  $j$

$$\mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \leq c(p) \mathbf{W}_{1,p}^1 \left[ \sum_{k \geq j+2} \mu_k \right] + c(p) \mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] + c(p) \sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k],$$

with  $c(p) = \max\{1, 5^{\frac{2-p}{p-1}}\}$  and  $\exp\left(\sum_{i=1}^5 a_i\right) \leq \sum_{i=1}^5 \exp(5a_i)$  for all  $a_i$ . Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_3 \varepsilon \mathbf{W}_{1,p}^1 \left[ \sum_{k \geq j+2} \mu_k \right] (x)\right) dx + \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_3 \varepsilon \mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] (x)\right) dx \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1,3\}}^{j+1} \int_{S_j} \exp(c_3 \varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)) dx := A_1 + A_2 + A_3, \text{ with } c_3 = 5c(p). \end{aligned}$$

*Estimate of  $A_3$ :* We apply Theorem 2.3 for  $\mu = \mu_k$  and  $B = B_{k-1}$ ,

$$\int_{2B_{k-1}} \exp(c_3 \varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)) dx \leq c_4 r_{k-1}^N$$

with  $c_3 \varepsilon \in (0, C_1]$ , the constant  $C_1$  is in Theorem 2.3. In particular,

$$\int_{S_j} \exp(c_3 \varepsilon \mathbf{W}_{1,p}^1[\mu_k](x)) dx \leq c_4 r_{k-1}^N \quad \text{for } k = j-1, j, j+1,$$

which implies

$$A_3 \leq c_5 \sum_{j=1}^{+\infty} r_j^N = c_5 < \infty. \quad (3.7)$$

Estimate of  $A_1$ : Since  $\sum_{k \geq j+2} \mu_k(B_t(x)) = 0$  for all  $x \in S_j, t \in (0, r_{j+1})$ . Thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp \left( c_3 \varepsilon \int_{r_{j+1}}^1 \left( \frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right) dx \\ &\leq \sum_{j=1}^{\infty} \exp \left( c_3 \varepsilon \frac{p-1}{N-p} \left( \sum_{k \geq j+2} \mu_k(S_k) \right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \right) |S_j|. \end{aligned}$$

Note that  $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$ , which leads to

$$\left( \sum_{k \geq j+2} \mu_k(S_k) \right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left( \sum_{k \geq j+2} r_{k-1}^{N-p} \right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left( \sum_{k \geq 0} r_k^{N-p} \right)^{\frac{1}{p-1}} = \left( \frac{1}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}}.$$

Therefore

$$A_1 \leq \exp \left( c_3 \varepsilon \frac{p-1}{N-p} \left( \frac{1}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} \right) |B_1| = c_6. \quad (3.8)$$

Estimate of  $A_2$ : for  $x \in S_j$ ,

$$\mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left( \frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left( \frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Since  $r_i < t < r_{i-1}$ ,  $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$ , thus

$$\begin{aligned} \mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left( \frac{\sum_{k=i-1}^{j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left( \frac{\sum_{k=i-1}^{j-2} \mu_k(S_k)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \sum_{i=1}^{j-1} \left( \sum_{k=i-1}^{j-2} r_{k-1}^{N-p} \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \leq c_7 j, \text{ with } c_7 = \left( \frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp(c_3 c_7 \varepsilon j) dx = \sum_{j=1}^{\infty} r_j^N \exp(c_3 c_7 \varepsilon j) |S_1| \\ &= \sum_{j=1}^{\infty} \exp((c_3 c_7 \varepsilon - N \log(2)) j) |S_1| \leq c_8 \quad \text{for } \varepsilon \leq N \log(2) / (2c_3 c_7). \end{aligned} \quad (3.9)$$

Consequently, from (3.8), (3.9) and (3.7), we obtain  $A \leq c_2 := c_6 + c_8 + c_5$  for  $\varepsilon = \varepsilon(N, p)$  small enough. This implies

$$\left\| \exp \left( \frac{p}{2N} \varepsilon \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1)} \leq c_9 \left( \int_{B_1} \exp \left( \varepsilon \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] (x) \right) dx \right)^{\frac{p}{2N}} \leq c_{10}, \quad (3.10)$$

where the constant  $c_{10}$  does not depend on  $J$ . Set  $B = B_{\frac{1}{4}}$ . For  $\varepsilon_0 = (\frac{p\varepsilon}{2NC})^{1/(p-1)}$ , where  $C$  is the constant in (2.3), by Theorem 2.8 and estimate (3.10), there exists a nonnegative renormalized solution  $u$  to equation

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \varepsilon_0 \sum_{j=3}^J \mu_j && \text{in } B, \\ u &= 0 && \text{in } \partial B, \end{aligned}$$

satisfying (2.3) with  $\mu = \varepsilon_0 \sum_{j=3}^J \mu_j$ . Thus, from Corollary 2.6 and estimate (3.10), we have

$$u(0) \geq -c_{11} + c_{12} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[ \sum_{j=3}^J \mu_j \right] (0).$$

Therefore

$$\begin{aligned} u(0) &\geq -c_{11} + c_{12} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left( \frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{11} + c_{12} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left( \frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= -c_{11} + c_{12} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left( \frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{11} + c_{13} \sum_{i=2}^{J-2} \left( \mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &= -c_{11} + c_{13} \sum_{i=4}^{\infty} \left( \mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}. \end{aligned}$$

From the inequality

$$\left( \mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left( \mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left( \mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i,$$

we deduce that

$$\begin{aligned} u(0) &\geq -c_{11} + c_{13} \sum_{i=4}^{\infty} \left( \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left( \mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left( \mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{11} + c_{13} \left( \frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left( \mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{14} + c_{15} \int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_t)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}. \end{aligned}$$

Since  $U_1$  is the maximal solution in  $K^c$ ,  $u$  satisfies the same equation in  $B \setminus K$  and  $U_1 \geq u = 0$  on  $\partial B$ , it follows that  $U_1$  dominates  $u$  in  $B \setminus K$ . Then  $U_1(0) \geq u(0)$  and we obtain (3.4).

2. By [1, Theorem 2.5.3], there exists  $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$  such that  $\text{supp}(\mu_j) \subset K \cap S_j$  and

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{\frac{q_1}{p-1}} dx = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

By Jensen's inequality, we have for any  $a_k \geq 0$ ,

$$\left( \sum_{k=0}^{\infty} a_k \right)^s \leq \sum_{k=0}^{\infty} \theta_{k,s} a_k^s$$

where  $\theta_{k,s}$  has the following expression with  $\theta > 0$ ,

$$\theta_{k,s} = \begin{cases} 1 & \text{if } s \in (0, 1], \\ \left(\frac{\theta+1}{\theta}\right)^{s-1} (\theta+1)^{k(s-1)} & \text{if } s > 1. \end{cases}$$

Thus,

$$\begin{aligned} \int_{B_1} \left( \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} dx &\leq \int_{B_1} \left( \sum_{k=3}^J \theta_{k, \frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right)^{q_1} dx \\ &\leq \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1} (\mathbf{W}_{1,p}^1[\mu_k](x))^{q_1} dx \\ &\leq c_{16} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_{\mathbb{R}^N} (G_p * \mu_k(x))^{\frac{q_1}{p-1}} dx \\ &= c_{16} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\ &\leq c_{17} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k} \left( N - \frac{pq_1}{q_1-p+1} \right) \\ &\leq c_{18}, \end{aligned}$$

for  $\theta$  small enough. Here the third inequality follows from [2, Theorem 2.3] and the constant  $c_{18}$  does not depend on  $J$ . Hence,

$$\left\| \left( \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1)} \leq c_{19} \left\| \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1)}^q \leq c_{20}, \quad (3.11)$$

where  $c_{20}$  is independent of  $J$ . Take  $B = B_{\frac{1}{4}}$ . Since  $\sum_{j=3}^J \mu_j$  is absolutely continuous with respect to the capacity  $C_{p, \frac{q}{q+1-p}}$  in  $B$ , thus by Theorem 2.8, there exists a nonnegative renormalized solution  $u$  to equation

$$\begin{aligned} -\Delta_p u + u^q &= \sum_{j=3}^J \mu_j && \text{in } B, \\ u &= 0 && \text{on } \partial B. \end{aligned}$$

satisfying (2.3) with  $\mu = \sum_{j=3}^J \mu_j$ . Thus, from Corollary 2.6 and estimate (3.11), we have

$$u(0) \geq -c_{21} + c_{22} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[ \sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{23} + c_{24} \int_0^1 \left( \frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have  $U_2(0) \geq u(0)$ . Therefore, we obtain(3.5). ■

## 4 Proof of the main results

First, we prove theorem 1 in the case case  $p = N$ . To do this we consider the function

$$x \mapsto U(x) = U(|x|) = \log \left( \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left( \frac{R}{|x|} + 1 \right) \right) \quad \text{in } B_R(0) \setminus \{0\}.$$

One has

$$U'(|x|) = \frac{1}{R+|x|} - \frac{1}{|x|} \quad \text{and} \quad U''(|x|) = -\frac{1}{(R+|x|)^2} + \frac{1}{|x|^2},$$

thus, for any  $0 < |x| < R$ ,

$$\begin{aligned} -\Delta_N U + e^U - 1 &= -(N-1)|U'(|x|)|^{N-2} \left( U''(|x|) + \frac{1}{|x|} U'(|x|) \right) + e^U - 1 \\ &= -\frac{(N-1)R^{N-1}}{(R+|x|)^N |x|^{N-1}} + \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left( \frac{R}{|x|} + 1 \right) - 1 \\ &\leq -\frac{(N-1)R^{N-1}}{(2R)^N |x|^{N-1}} + \frac{N-1}{2^{N+1}} \frac{1}{R^N} \frac{2R}{|x|} \\ &\leq -1. \end{aligned}$$

Hence, if  $u \in C^1(\Omega)$  is the maximal solution of

$$-\Delta_N u + e^u - 1 = 0 \quad \text{in } \Omega$$

and  $R = 2\text{diam}(\Omega)$ , then  $u(x) \geq U(|x-y|)$  for any  $x \in \Omega$  and  $y \in \partial\Omega$ . Therefore,  $u$  is a large solution and satisfies

$$u(x) \geq \log \left( \frac{N-1}{2^{N+1}} \frac{1}{R^N} \left( \frac{R}{\rho(x)} + 1 \right) \right) \quad \forall x \in \Omega.$$

Now, we prove Theorem 1 in the case  $p < N$  and Theorem 2. Let  $u, v \in C^1(\Omega)$  be the maximal solutions of

$$\begin{aligned} (i) \quad & -\Delta_p u + e^u - 1 = 0 && \text{in } \Omega, \\ (ii) \quad & -\Delta_p v + v^q = 0 && \text{in } \Omega. \end{aligned}$$

Fix  $x_0 \in \partial\Omega$ . We can assume that  $x_0 = 0$ . Let  $\delta \in (0, 1/12)$ . For  $z_0 \in \overline{B_\delta} \cap \Omega$ . Set  $K = \Omega^c \cap \overline{B_{1/4}}(z_0)$ . Let  $U_1, U_2 \in C^1(K^c)$  be the maximal solutions of (3.2) and (3.3) respectively. We have  $u \geq U_1$  and  $v \geq U_2$  in  $\Omega$ . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -c_1 + c_2 \int_\delta^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -c_1 + c_2 \int_\delta^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -c_1 + c_2 \int_{2\delta}^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -c_1 + c_3 \int_\delta^{1/2} \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned}$$

We deduce

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -c_1 + c_3 \int_\delta^{1/2} \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Similarly, we also obtain

$$\inf_{B_\delta \cap \Omega} v \geq -c_4 + c_5 \int_\delta^{1/2} \left( \frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Therefore,  $u$  and  $v$  satisfy (1.10) and (1.12) respectively. This completes the proof.

## 5 Large solutions of quasilinear Hamilton-Jacobi equations

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$  with  $N \geq 2$ . In this section we use our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + b u^s &= 0 \quad \text{in } \Omega, \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \tag{5.1}$$

where  $a > 0, b > 0$  and  $1 \leq q < p \leq 2, q > p - 1, s \geq p - 1$ .

First we have the result of existence solutions to equation (5.1).

**Proposition 5.1** *Let  $a > 0, b > 0$  and  $q > p - 1, s \geq p - 1, 1 \leq q \leq p$  and  $1 < p \leq 2$ . There exists a maximal nonnegative solution  $u \in C^1(\Omega)$  to equation*

$$-\Delta_p u + a |\nabla u|^q + b u^s = 0 \quad \text{in } \Omega, \tag{5.2}$$

which satisfies

$$u(x) \leq c(N, p, s) b^{-\frac{1}{s-p+1}} d(x, \partial\Omega)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega, \tag{5.3}$$

if  $s > p - 1$ ,

$$u(x) \leq c(N, p, q) \left( a^{-\frac{1}{q-p+1}} d(x, \partial\Omega)^{-\frac{p-q}{q-p+1}} + a^{-\frac{1}{q-p+1}} b^{-\frac{1}{p-1}} d(x, \partial\Omega)^{-\frac{q}{(p-1)(q-p+1)}} \right) \quad \forall x \in \Omega, \quad (5.4)$$

if  $p-1 < q < p$  and  $s = p-1$ , and

$$u(x) \leq c(N, p) a^{-1} b^{-\frac{1}{p-1}} d(x, \partial\Omega)^{-\frac{p}{p-1}} \quad \forall x \in \Omega, \quad (5.5)$$

if  $q = p$  and  $s = p-1$ .

**Proof.** Case  $s = p-1$  and  $p-1 < q < p$ . We consider

$$U_1(x) = U_1(|x|) = c_1 \left( \frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{p-q}{q-p+1}} + c_2 \in C^1(B_R(0)).$$

with  $p' = \frac{p}{p-1}$  and  $c_1, c_2 > 0$ . We have

$$\begin{aligned} U_1'(|x|) &= \frac{c_1(p-q)}{q-p+1} \frac{|x|^{p'-1}}{R^{p'-1}} \left( \frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{1}{q-p+1}}, \\ U_1''(|x|) &= \frac{c_1(p-q)(p'-1)}{q-p+1} \frac{|x|^{p'-2}}{R^{p'-1}} \left( \frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{1}{q-p+1}} \\ &\quad + \frac{c_1(p-q)}{(q-p+1)^2} \left( \frac{|x|^{p'-1}}{R^{p'-1}} \right)^2 \left( \frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{1}{q-p+1}-1} \end{aligned}$$

and

$$A = -\Delta_p U_1 + a |\nabla U_1|^q + b U_1^{p-1} \geq -\Delta_p U_1 + a |\nabla U_1|^q + b c_2^{p-1}.$$

Thus, for all  $x \in B_R(0)$

$$\begin{aligned} A &\geq -(p-1) |U_1'(|x|)|^{p-2} U_1''(|x|) - \frac{N-1}{|x|} |U_1'(|x|)|^{p-2} U_1'(|x|) + a |U_1'(|x|)|^q + b c_1^{p-1} \\ &= \left( \frac{c_1(p-q)(p'-1)}{q-p+1} \right)^{p-1} \left( \frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{q}{q-p+1}} \left\{ -(p-1) \frac{p'-1}{p'} \left( 1 - \left( \frac{|x|}{R} \right)^{p'} \right) \right. \\ &\quad \left. - \frac{1}{q-p+1} \left( \frac{|x|}{R} \right)^{p'} - \frac{N-1}{p'} \left( \frac{|x|}{R} \right)^{p'} \left( 1 - \left( \frac{|x|}{R} \right)^{p'} \right) \right. \\ &\quad \left. + a \left( \frac{c_1(p-q)}{q-p+1} \right)^{q-p+1} \left( \frac{|x|}{R} \right)^{\frac{q}{q-p+1}} \right\} + b c_2^{p-1} \\ &\geq \left( \frac{c_1(p-q)(p'-1)}{q-p+1} \right)^{p-1} \left( \frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{q}{q-p+1}} \\ &\quad \times \left\{ -\frac{N(p-1)}{p} - \frac{1}{q-p+1} + a \left( \frac{c_1(p-q)}{q-p+1} \right)^{q-p+1} \left( \frac{|x|}{R} \right)^{\frac{q}{q-p+1}} \right\} + b c_2^{p-1}. \end{aligned}$$

Clearly, one can find  $c_1 = c_2(N, p, q)a^{-\frac{1}{q-p+1}} > 0$  and  $c_3 = c_3(N, p, q) > 0$  such that

$$A \geq -c_3 a^{-\frac{p-1}{q-p+1}} R^{-\frac{q}{q-p+1}} + b c_2^{p-1}.$$

Choosing  $c_2 = c_3^{\frac{1}{p-1}} a^{-\frac{1}{q-p+1}} b^{-\frac{1}{p-1}} R^{-\frac{q}{(p-1)(q-p+1)}}$ , we get

$$-\Delta_p U_1 + a|\nabla U_1|^q + bU_1^{p-1} \geq 0 \quad \text{in } B_R(0). \quad (5.6)$$

Likewise, we can verify that the function  $U_2$  below

$$U_2(x) = c_4 a^{-1} \log \left( \frac{R^{p'}}{R^{p'} - |x|^{p'}} \right) + c_4 a^{-1} b^{-\frac{1}{p-1}} R^{-\frac{p}{p-1}}$$

belongs to  $C_+^1(B_R(0))$  and satisfies

$$-\Delta_p U_2 + a|\nabla U_2|^p + bU_2^{p-1} \geq 0 \quad \text{in } B_R(0). \quad (5.7)$$

While, if  $s > p - 1$ ,

$$U_3(x) = c_5 b^{-\frac{1}{s-p+1}} \left( \frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{-\frac{p}{s-p+1}}$$

belongs to  $C^1(B_R(0))$  and verifies

$$-\Delta_p U_3 + bU_3^s \geq 0 \quad \text{in } B_R(0), \quad (5.8)$$

for some positive constants  $c_4 = c_4(N, p, q)$ ,  $c_5 = c_5(N, p, s)$  and  $\beta = \beta(N, p, q) > 1$ .

We emphasize the fact that with the condition  $1 < p \leq 2$  and  $q \geq 1$ , equation (5.2) satisfies a comparison principle, see [17, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains  $\Omega_n$  satisfying  $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$  for all  $n$  and  $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ . For each  $n, k \in \mathbb{N}^*$ , there

exist nonnegative solution  $u_{n,k} = u \in W_k^{1,p}(\Omega_n) := W_0^{1,p}(\Omega_n) + k$  of equation (5.2) in  $\Omega_n$ .

Since  $-\Delta_p u_{k,n} \leq 0$  in  $\Omega_n$ , so using the maximum principle we get  $u_{n,k} \leq k$  in  $\Omega_n$  for all  $n$ . Thus, by standard regularity (see [4] and [11]),  $u_{n,k} \in C^{1,\alpha}(\overline{\Omega}_n)$  for some  $\alpha \in (0, 1)$ . It follows from the comparison principle and (5.6)-(5.8), that

$$u_{n,k} \leq u_{n,k+1} \quad \text{in } \Omega_n$$

and (5.3)-(5.5) are satisfied with  $u_{n,k}$  and  $\Omega_n$  in place of  $u$  and  $\Omega$  respectively. From this, we derive uniform local bounds for  $\{u_{n,k}\}_k$ , and by standard interior regularity (see [4]) we obtain uniform local bounds for  $\{u_{n,k}\}_k$  in  $C_{loc}^{1,\eta}(\Omega_n)$ . It implies that the sequence  $\{u_{n,k}\}_k$  is pre-compact in  $C^1$ . Therefore, up to a subsequence,  $u_{n,k} \rightarrow u_n$  in  $C^1(\Omega_n)$ . Hence, we can verify that  $u_n$  is a solution of (5.2) and satisfies (5.3)-(5.5) with  $u_n$  and  $\Omega_n$  replacing  $u$  and  $\Omega$  and  $u_n(x) \rightarrow \infty$  as  $d(x, \Omega_n) \rightarrow 0$ .

Next, since  $u_{n,k} \geq u_{n+1,k}$  in  $\Omega_n$  there holds  $u_n \geq u_{n+1}$  in  $\Omega_n$ . In particular,  $\{u_n\}$  is uniformly locally bounded in  $\Omega$ . Arguing as above, we obtain  $u_n \rightarrow u$  in  $C^1(\Omega)$ , thus  $u$  is a solution of (5.2) in  $\Omega$  and satisfies (5.3)-(5.5). Clearly,  $u$  is the maximal solution of (5.2).  $\blacksquare$

**Theorem 5.2** *Let  $q_1 > p - 1$  and  $1 < p \leq 2$ . Assume that equation (1.12) admits a solution with  $q = q_1$ . Then for any  $a > 0, b > 0$  and  $q \in (p - 1, \frac{pq_1}{q_1+1})$ ,  $s \in [p - 1, q_1)$  equation (5.2) has a large solution satisfying (5.3) and (5.4).*

**Proof.** Assume that equation (1.12) admits a solution  $v$  with  $q = q_1$  and set  $v = \beta w^\sigma$  with  $\beta > 0, \sigma \in (0, 1)$ , then  $w > 0$  and

$$-\Delta_p w + (-\sigma + 1)(p-1) \frac{|\nabla w|^p}{w} + \beta^{q_1-p+1} \sigma^{-p+1} w^{\sigma(q_1-p+1)+p-1} = 0 \text{ in } \Omega.$$

If we impose  $\max\{\frac{s-p+1}{q_1-p+1}, (\frac{q}{p-q} - p + 1) \frac{1}{q_1-p+1}\} < \sigma < 1$ , we can see that

$$(-\sigma + 1)(p-1) \frac{|\nabla w|^p}{w} + \beta^{q_1-p+1} \sigma^{-p+1} w^{\sigma(q_1-p+1)+p-1} \geq a|\nabla w|^q + bw^s \text{ in } \{x : w(x) \geq M\},$$

where a positive constant  $M$  depends on  $p, q_1, q, s, a, b$ . Therefore

$$-\Delta_p w + a|\nabla w|^q + bw^s \leq 0 \text{ in } \{x : w(x) \geq M\}.$$

Now we take an open subset  $\Omega'$  of  $\Omega$  with  $\overline{\Omega'} \subset \Omega$  such that the set  $\{x : w(x) \geq M\}$  contains  $\Omega \setminus \overline{\Omega'}$ . So  $w$  is a subsolution of  $-\Delta_p u + a|\nabla u|^q + bu^s = 0$  in  $\Omega \setminus \overline{\Omega'}$  and the same property holds with  $w_\varepsilon := \varepsilon w$  for any  $\varepsilon \in (0, 1)$ . Let  $u$  be as in Proposition 5.1. Set  $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$  and  $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$ . Thus  $w_\varepsilon < u$  on  $\partial\Omega'$  with  $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$ . Hence, from the construction of  $u$  in the proof of Proposition 5.1 and the comparison principle, we obtain  $w_\varepsilon \leq u$  in  $\Omega \setminus \overline{\Omega'}$ . This implies the result.  $\blacksquare$

**Remark 5.3** From the proof of above Theorem, we can show that under the assumption as in Proposition 5.1, equation (5.2) has a large solution in  $\Omega$  if and only if equation (5.2) has a large solution in  $\Omega \setminus K$  for some a compact set  $K \subset \Omega$  with smooth boundary.

Now we deal with (5.1) in the case  $q = p$ .

**Theorem 5.4** Assume that equation (5.2) has a large solution in  $\Omega$  for some  $a, b > 0, s > p-1$  and  $q = p > 1$ . Then for any  $a_1, b_1 > 0$  and  $q_1 > p-1, s_1 \geq p-1, 1 \leq q_1 \leq p \leq 2$ , equation (5.2) also has a large solution  $u$  in  $\Omega$  with parameters  $a_1, b_1, q_1, s_1$  in place of  $a, b, q, s$  respectively, and it satisfies (5.3)-(5.5).

**Proof.** For  $\sigma > 0$  we set  $u = v^\sigma$  thus

$$-\Delta_p v - (\sigma - 1)(p-1) \frac{|\nabla v|^p}{v} + a\sigma v^{\sigma-1} |\nabla v|^p + b\sigma^{-p+1} v^{(s-p+1)\sigma+p-1} = 0.$$

Choose  $\sigma = \frac{s_1-p+1}{s-p+1} + 2$ , it is easy to see that

$$-\Delta_p v + a_1 |\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \text{ in } \{x : v(x) \geq M\},$$

for some a positive constant  $M$  only depending on  $p, s, a, b, a_1, b_1, q_1, s_1$ . Similarly as in the proof of Theorem 5.2, we get the result as desired.  $\blacksquare$

**Remark 5.5** If we set  $u = e^v$  then  $v$  satisfies

$$-\Delta_p v + be^{(s-p+1)v} = |\nabla v|^p (p-1 - ae^v) \text{ in } \Omega.$$

From this, we can construct a large solution of

$$-\Delta_p u + be^{(s-p+1)u} = 0 \text{ in } \Omega \setminus K,$$

for any a compact set  $K \subset \Omega$  with smooth boundary such that  $v \geq \ln\left(\frac{p-1}{a}\right)$  in  $\Omega \setminus K$ . In case  $p = 2$ , It would be interesting to see what Wiener type criterion is implied by the existence as such a large solution. We conjecture that this condition must be

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = \infty \quad \forall x \in \partial\Omega.$$

We now consider the function

$$U_4(x) = c \left( \frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0.$$

As in the proof of proposition 5.1, it is easy to check that there exist positive constants  $\beta$  large enough and  $c$  small enough so that inequality  $\Delta_p U_4 + U_4^{-\gamma} \geq 0$  holds.

From this, we get the existence of minimal solution to equation

$$\Delta_p u + u^{-\gamma} = 0 \quad \text{in } \Omega. \quad (5.9)$$

**Proposition 5.6** *Assume  $\gamma > 0$ . Then there exists a minimal solution  $u \in C^1(\Omega)$  to equation (5.9) and it satisfies  $u(x) \geq Cd(x, \partial\Omega)^{\frac{p}{\gamma+p-1}}$  in  $\Omega$ .*

We can verify that if the boundary of  $\Omega$  is satisfied (1.3), then above minimal solution  $u$  belongs to  $C(\overline{\Omega})$ , vanishes on  $\partial\Omega$  and it is therefore a solution to the quenching problem

$$\begin{aligned} \Delta_p u + u^{-\gamma} &= 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \partial\Omega. \end{aligned} \quad (5.10)$$

**Theorem 5.7** *Let  $\gamma > 0$ . Assume that there exists a solution  $u \in C(\overline{\Omega})$  to problem (5.10). Then, for any  $a, b > 0$  and  $q > p - 1, s \geq p - 1, 1 \leq q \leq p \leq 2$ , equation (5.2) admits a large solution in  $\Omega$  and it satisfies (5.3)-(5.5).*

**Proof.** We set  $u = e^{-\frac{a}{p-1}v}$ , then  $v$  is a large solution of

$$-\Delta_p v + a|\nabla v|^p + \left(\frac{p-1}{a}\right)^{p-1} e^{\frac{a}{p-1}(\gamma+p-1)v} = 0 \quad \text{in } \Omega.$$

So

$$-\Delta_p v + a|\nabla v|^q + bv^s \leq 0 \quad \text{in } \{x : v(x) \geq M\},$$

for some a positive constant  $M$  only depending on  $p, q, s, a, b, \gamma$ . Similarly to the proof of Theorem 5.2, we get the result as desired.  $\blacksquare$

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