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# Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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## Abstract

We obtain sufficient conditions expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities for the existence of large solutions to equations (1)  $-\Delta_p u + e^{\lambda u} + \beta = 0$  or (2)  $-\Delta_p u + \lambda|u|^{q-1}u + \beta = 0$  in a bounded domain  $\Omega$  when  $q > p - 1 > 0, \lambda > 0$  and  $\beta \in \mathbb{R}$ . We apply our results to equations (3)  $-\Delta_p u + a|\nabla u|^q + bu^s = 0$ , (4)  $\Delta_p u + u^{-\gamma} = 0$  with  $1 < p \leq 2, 1 \leq q \leq p, a > 0, b \geq 0$  and  $(q - p + 1) + b(s - p + 1) > 0, \gamma > 0$ .

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*Key words:* quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \geq 3$ ) and  $1 < p \leq N$ . We consider the question of existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \tag{1.1}$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $\rho(x) = \operatorname{dist}(x, \partial\Omega)$  and  $g$  is a continuous nondecreasing function vanishing at 0; most often  $g(u)$  is either  $e^{\lambda u} + \beta$  or  $\lambda|u|^{q-1}u + \beta$  with  $q > p - 1, \lambda > 0$  and  $\beta \in \mathbb{R}$ . A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data  $\phi$

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ u &= \phi && \text{in } \partial\Omega \end{aligned} \tag{1.2}$$

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admits a solution, it is clear that problem (1.1) admits a solution. It is known that a necessary and sufficient condition for such a result is the so called Wiener criterion (for  $p = 2$  see [21]), for  $p \neq 2$  see [13], [7])

$$\int_0^1 \left( \frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where  $C_{1,p}$  denotes the capacity associated to the space  $W^{1,p}(\mathbb{R}^N)$ . We remark that the Lipschitz and the Reifenberg flat domains are satisfied (1.3). The existence of a large solution is guaranteed for a large class of nondecreasing nonlinearities  $g$  satisfying the Vazquez condition [18]

$$\int_a^\infty \frac{dt}{G^{\frac{1}{p}}(t)} < \infty \quad G(t) = \int_0^t g(s)ds, \quad (1.4)$$

a variant of the Keller-Osserman estimate [8], [15], which is the above relation when  $p = 2$ . If for  $R > \text{diam}(\Omega)$  there exists a function  $v$  which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\} \\ v &= 0 && \text{on } \partial B_R \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of  $\partial\Omega$ , provided (1.3) is satisfied. However the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of  $g$  is too strong since Vazquez and Véron proved [19] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if  $p-1 < q < \frac{N(p-1)}{N-p}$ , Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [9] proved that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.9)$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.10)$$

where  $C_{2,q'}$  is the capacity associated to the Sobolev space  $W^{2,q'}(\mathbb{R}^N)$  and  $q' = q/(q-1)$ . Notice that this condition is always satisfied if  $q$  is subcritical, i.e.  $q < N/(N-2)$ . Concerning the

exponential case of problem (1.1) nothing is known, even in the case  $p = 2$ , besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases  $g(u) = e^{\lambda u} + \beta$  and  $g(u) = \lambda |u|^{q-1} u + \beta$ ,  $q > p - 1, \lambda > 0, \beta \in \mathbb{R}$ . For  $1 < p < N$ , we denote by  $\mathcal{H}_1^{N-p}(E)$  the Hausdorff capacity of a set  $E$  defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\} \quad (1.11)$$

where the  $B_j$  are balls and  $h^{N-p}(B_r) = c_N r^{N-p}$ . Our main result concerning the exponential case is the following

**Theorem 1.** *Let  $N \geq 3$  and  $1 < p < N$ . If*

$$\int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.12)$$

then for any  $\lambda > 0$  and  $\beta \in \mathbb{R}$  there exists  $u \in C_{loc}^1(\Omega)$  satisfying

$$\begin{aligned} -\Delta_p u + e^{\lambda u} + \beta &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.13)$$

As a consequence we obtained a sufficient condition for the existence of a large solution in the power case expressed in terms of some  $C_{s,r}$  Bessel capacity in  $\mathbb{R}^N$  associated to the Besov space  $B^{s,r}(\mathbb{R}^N)$ .

**Theorem 2.** *Let  $N \geq 3$ ,  $1 < p < N$  and  $q_1 > \frac{N(p-1)}{N-p}$ . If*

$$\int_0^1 \left( \frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.14)$$

then, for any  $p-1 < q < \frac{pq_1}{N}$ ,  $\lambda > 0$  and  $\beta \in \mathbb{R}$  there exists  $u \in C_{loc}^1(\Omega)$  satisfying

$$\begin{aligned} -\Delta_p u + \lambda u^q + \beta &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.15)$$

In view of Labutin's theorem this last result is not optimal in the case  $p = 2$ , since the involved capacity is  $C_{2,q'_1}$  with  $q'_1$  and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.16)$$

with  $q_1 > q$ .

At end we apply the previous theorems to quasilinear viscous Hamilton-Jacobi equations:

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + b |u|^{s-1} u &= 0 && \text{in } \Omega \\ u \in C_{loc}^1(\Omega), \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.17)$$

We prove that if  $1 < p \leq 2$ ,  $1 \leq q < p$ ,  $a > 0$ ,  $b \geq 0$ ,  $(q - p + 1) + b(s - p + 1) > 0$  and (1.12) holds, there exists a positive solution to (1.17). Conversely, if for some  $a, b > 0$ ,  $s > p - 1$  there exists a solution to equation (1.17) with  $q = p$ , then for any  $\max\{p - 1, 1\} \leq q_1 \leq p$ ,  $s_1 \geq p - 1$ ,  $a_1, b_1 \geq 0$ ,  $a_1(q_1 - p + 1) + b_1(s_1 - p + 1) > 0$  there exists a positive solution to equation (1.17) with  $q_1, s_1, a_1, b_1$  replacing  $q, s, a, b$ , here and we add  $1 < p \leq 2$  if  $a_1 > 0$ . Moreover, we also prove that previous claim holds if for some  $\gamma > 0$  there exists  $u \in C(\overline{\Omega})$ ,  $u > 0$  in  $\Omega$  satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (1.18)$$

## 2 Morrey classes and Wolff potential estimates

In this section  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ . We also denote by  $B_r(x)$  the open ball of center  $x$  and radius  $r$  and  $B_r = B_r(0)$ . We also recall that a solution of (1.1) belongs to  $C_{loc}^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ , and is more regular (depending on  $g$ ) on the set  $\{x \in \Omega : |\nabla u(x)| \neq 0\}$ .

**Definition 2.1** 1- A function  $f \in L^1(\Omega)$  belongs to the Morrey space  $\mathcal{M}^s(\Omega)$ ,  $1 \leq s \leq \infty$ , if there is a constant  $K$  such that

$$\int_{\Omega \cap B_r(x)} |f| dy \leq Kr^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N \quad (2.1)$$

The norm is defined as the smallest constant  $K$  that satisfies this inequality; it is denoted by  $\|f\|_{\mathcal{M}^s(\Omega)}$ .

2- A function  $f \in L^1(\Omega)$  belongs to the weak  $L^s$ -space  $M^s(\Omega)$ ,  $1 \leq s \leq \infty$ , if there is a constant  $K$  such that

$$\int_E |f| dy \leq K|E|^{\frac{1}{s}} \quad \forall E \subset \Omega, E \text{ Borel.} \quad (2.2)$$

The quasi-norm is defined as the smallest constant  $K$  that satisfies this inequality; it is denoted by  $\|f\|_{M^s(\Omega)}$

Clearly  $L^p(\Omega) \subset M^p(\Omega) \subset \mathcal{M}^p(\Omega)$ .

**Definition 2.2** Let  $R \in (0, \infty]$  and  $\mu \in \mathfrak{M}_+(\Omega)$ , the set of positive Radon measures in  $\Omega$ . If  $\alpha > 0$  and  $1 < p < \alpha^{-1}N$ , we define the ( $R$ -truncated) Wolff potential of  $\mu$  by

$$\mathbf{W}_p^R[\mu](x) = \int_0^R \left( \frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N, \quad (2.3)$$

and, for  $1 < p < N$ , the ( $R$ -truncated) fractional maximal potential of  $\mu$  by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N, \quad (2.4)$$

where the measure is extended by 0 in  $\Omega^c$ .

For  $k \geq 0$ , we set  $T_k(u) = \text{sign}(u) \min\{k, |u|\}$ .

**Definition 2.3** Assume  $f \in L^1_{loc}(\Omega)$ . We say that a Borel function  $u$  defined in  $\Omega$  is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.5)$$

if for any  $k > 0$ ,  $T_k(u) \in W^{1,p}_{loc}(\Omega)$ ,  $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$  and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0 \quad (2.6)$$

for all  $\varphi \in W^{1,p}(\Omega)$  with compact support in  $\Omega$  and such that  $0 \leq \varphi \leq k - T_k(u)$ , and if  $-\Delta_p u + f := \mu$  is a positive distribution in  $\Omega$ .

The following result is proved in [14, Theorem 4.35].

**Theorem 2.4** Let  $\Omega$  be an open bounded domain in  $\mathbb{R}^N$ . If  $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$  for some  $\varepsilon \in (0, p)$ ,  $u$  is a nonnegative renormalized supersolution of (2.5) and set  $\mu := -\Delta_p u + f$ . Then there holds

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_1 W_{1,p}^{\frac{\varepsilon}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega, \quad (2.7)$$

for some  $c_1$  depending only on  $N, p, \varepsilon, \text{diam}(\Omega)$ .

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u = f + \mu \quad \text{in } \Omega \quad (2.8)$$

where  $f \in L^1_{loc}(\Omega)$  and  $\mu$  is a Radon measure, we have

**Corollary 2.5** Let  $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$  and  $\mu \in \mathfrak{M}^b_+(\Omega)$ , the set of positive and bounded Radon measures in  $\Omega$ . If  $u$  is a renormalized solution to (2.8) and  $\inf_{\Omega} u > -\infty$  then there exists a positive constant  $c_2$  depending only on  $N, p, \varepsilon, \text{diam}(\Omega)$  such that

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq \inf_{\Omega} u + c_2 W_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega. \quad (2.9)$$

We now recall [2, Theorem 3.8].

**Theorem 2.6** Let  $\mu \in \mathfrak{M}^b(\Omega)$ . There exists a positive constant  $c_3$  such that if  $u$  is a renormalized solution to  $-\Delta_p u = \mu$  in  $\Omega$  and  $u = 0$  on  $\partial\Omega$ , then for any  $x \in \Omega$

$$|u(x)| \leq c_3 W_{1,p}^{2\text{diam}(\Omega)}[|\mu|](x). \quad (2.10)$$

**Definition 2.7** Let  $s > 0$  and  $0 < \alpha s < N$ . We denote by  $L^{\alpha,s}(\mathbb{R}^N)$  the Besov space the space functions  $\phi = G_{\alpha} * f$  for  $f \in L^s(\mathbb{R}^N)$  and we set  $\|\phi\|_{L^{\alpha,s}(\mathbb{R}^N)} = \|f\|_{L^s(\mathbb{R}^N)}$ , where  $G_{\alpha}$  is Bessel kernel of order  $\alpha$ . The dual space of  $L^{s,q}(\mathbb{R}^N)$  is the space  $L^{-s,q'}(\mathbb{R}^N)$  and it is naturally endowed with the dual norm. We also denote by  $C_{\alpha,s}(E)$  the Bessel capacity of Borel set  $E \subset \mathbb{R}^N$  defined by

$$C_{\alpha,s}(E) = \inf\{\|\phi\|_{L^s(\mathbb{R}^N)}^s : \phi \in L^s_+(\mathbb{R}^N), G_{\alpha} * \phi \geq \chi_E\}$$

where  $\chi_E$  is the characteristic function of  $E$ .

From Corollary 2.5, Theorem 2.6 and [2, Theorem 2.3] we can verify the following result.

**Theorem 2.8** Let  $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$  and  $\mu \in \mathfrak{M}_+^b(\Omega)$ . Assume that  $u$  is a nonnegative renormalized solution to equation (2.8). If  $\mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$  for some  $q > p - 1$ , then  $u \in L^q(\Omega)$  and  $\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} = \|G_p * \mu\|_{L^{\frac{q}{q-1}}(\mathbb{R}^N)}$ ,

$$\|u\|_{L^q(\Omega)} \leq C \left( \|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \right) \quad (2.11)$$

for some a positive constant  $C$  depending only on  $N, p, q, \varepsilon, \text{diam}(\Omega)$ .

Conversely, if  $u \in L^q(\Omega)$ , then for any compact set  $K \subset \Omega$ , there exists a positive constant  $C_K$  depending only on  $N, p, q, \varepsilon, \text{diam}(\Omega)$  and  $\text{dist}(K, \partial\Omega)$  such that  $\chi_K \mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$  and

$$\|\chi_K \mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} \leq C_K \left( \|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \right). \quad (2.12)$$

In particular, for any Borel set  $E \subset \Omega$ ,

$$C_{p, \frac{q}{q+1-p}}(E) = 0 \implies \mu(E) = 0. \quad (2.13)$$

The next statement is proved in [2, Theorem 2.4], and in [6] for a variant.

**Theorem 2.9** Let  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ . There exist positive constants  $c_4, c_5$  such that

$$\int_{2B} \exp(c_4 W_{1,p}^R[\mu_B]) \leq c_5 r^N,$$

for all  $B = B_r(x_0) \subset \mathbb{R}^N$ ,  $2B = B_{2r}(x_0)$ ,  $R > 0$  such that  $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$ .

### 3 Estimates from below

If  $G$  is any domain in  $\mathbb{R}^N$  with a compact boundary and  $g$  is nondecreasing,  $g(0) = g^{-1}(0) = 0$  and satisfies (1.7), there always exists a maximal solution to (1.3) in  $G$ . It is constructed as the limit, when  $n \rightarrow \infty$ , of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 && \text{in } G_n \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty && \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 && \text{if } G_n \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where  $\{G_n\}_n$  is a sequence of smooth domains such that  $G_n \subset \overline{G_n} \subset G_{n+1}$  for all  $n$ ,  $\{\partial G_n\}_n$  is a bounded and  $\bigcup_{n=1}^{\infty} G_n = G$  and  $\rho_n(x) := \text{dist}(x, \partial G_n)$ . Our main estimates are the following.

**Theorem 3.1** Let  $K \subset B_{1/4} \setminus \{0\}$  be a compact set and let  $U_j \in C_{loc}^1(K^c)$ ,  $j = 1, 2$ , be the maximal solutions of

$$-\Delta_p u + e^{\lambda u} + \beta = 0 \quad \text{in } K^c \quad (3.2)$$

for  $U_1$  and

$$-\Delta_p u + \lambda u^q + \beta = 0 \quad \text{in } K^c \quad (3.3)$$

for  $U_2$ , where  $p-1 < q < \frac{pq_1}{N}$ . Then there exist constants  $C_k$ ,  $k=1,2,3,4$ , depending on  $N, p$  and  $q$  such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left( \frac{C_{p, \frac{q_1}{1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

*Proof. Step 1.* For  $j \in \mathbb{Z}$  define  $r_j = 2^{-j}$  and  $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$ ,  $B_j = B_{r_j}$ . Fix a positive integer  $J$  such that  $K \subset \{x : r_J \leq |x| < 1/8\}$ . Consider the sets  $K \cap S_j$  for  $j = 3, \dots, J$ . By [17, Theorem 3.4.27], there exists  $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$  such that  $\text{supp}(\mu_j) \subset K \cap S_j$ ,  $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \leq 1$  and

$$c^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j$$

Now, we will show that for  $\varepsilon$  small enough, there holds,

$$A := \int_{B_1} \exp \left( \frac{2N}{p} \tilde{c}_3 \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \varepsilon \mu_k \right] (x) \right) \leq C, \quad (3.6)$$

where  $\tilde{c}_3 = \max\{1, 2^{\frac{2-p}{p-1}}\} \lambda c_3$ ,  $c_3$  is the constant in Theorem 2.6, and  $C$  does not depend on  $J$ . Indeed, define  $\mu_j \equiv 0$  for all  $j \geq J+1$  and  $j \leq 2$ . We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp \left( \frac{2N}{p} \tilde{c}_3 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] (x) \right).$$

Since for any  $j$

$$\mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \leq c_p \mathbf{W}_{1,p}^1 \left[ \sum_{k \geq j+2} \mu_k \right] + c_p \mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] + c_p \sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k]$$

with  $c_p = \max\{1, 5^{\frac{2-p}{p-1}}\}$  and  $\exp(\sum_{i=1}^5 a_i) \leq \sum_{i=1}^5 \exp(5a_i)$  for all  $a_i$ . Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp \left( c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[ \sum_{k \geq j+2} \mu_k \right] (x) \right) + \sum_{j=1}^{\infty} \int_{S_j} \exp \left( c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] (x) \right) \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1,3\}}^{j+1} \int_{S_j} \exp \left( c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right) := A_1 + A_2 + A_3, \text{ with } c_6 = 5c_p \frac{2N}{p} \tilde{c}_3. \end{aligned}$$

*Estimate of  $A_3$ :* We apply Theorem 2.9 for  $\mu = \mu_k$  and  $B = B_{k-1}$ ,

$$\int_{2B_{k-1}} \exp \left( c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right) \leq c_5 r_{k-1}^N$$



with  $c_6\varepsilon^{\frac{1}{p-1}} \in (0, c_4]$ , the constant  $c_4$  is in Theorem 2.9. In particular,

$$\int_{S_j} \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) \leq c_5 r_{k-1}^N \quad k = j-1, j, j+1,$$

which implies

$$A_3 \leq c_7 \sum_{j=1}^{+\infty} r_j^N = c_8 < \infty. \quad (3.7)$$

*Estimate of  $A_1$ :* Since  $\sum_{k \geq j+2} \mu_k(B_t(x)) = 0$  for all  $x \in S_j, t \in (0, r_{j+1})$ . Thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \int_{r_{j+1}}^1 \left(\frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right) \\ &\leq \sum_{j=1}^{\infty} \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right) |S_j|. \end{aligned}$$

Note that  $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$ , which leads to

$$\left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left(\sum_{k \geq j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k \geq 0} r_k^{N-p}\right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$A_1 \leq \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) |B_1| = c_9. \quad (3.8)$$

*Estimate of  $A_2$ :* for  $x \in S_j$ ,

$$\mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Since  $r_i < t < r_{i-1}$ ,  $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$ , thus

$$\begin{aligned} \mathbf{W}_{1,p}^1 \left[ \sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(S_k)}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \sum_{i=1}^{j-1} \left(\sum_{k=i-1}^{j-2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \leq c_{10} j, \quad \text{with } c_{10} = \left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} c_{10} j\right) = \sum_{j=1}^{\infty} r_j^N \exp\left(c_6 c_{10} \varepsilon^{\frac{1}{p-1}} j\right) |S_1| \\
&= \sum_{j=1}^{\infty} \exp\left(\left(c_6 c_{10} \varepsilon^{\frac{1}{p-1}} - N \log(2)\right) j\right) |S_1| = c_{11} \quad \text{for } \varepsilon \text{ small enough.} \quad (3.9)
\end{aligned}$$

Consequently,  $A \leq C := c_9 + c_{11} + c_8$  for  $\varepsilon$  small enough. This implies

$$\left\| \exp\left(\tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k\right]\right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1(0))} \leq c_{12} \left( \int_{B_1(0)} \exp\left(\frac{2N}{p} \tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k\right](x)\right) \right)^{\frac{p}{2N}} \leq c_{13} \quad (3.10)$$

where the constant  $c_{13}$  does not depend on  $J$ . Set  $B = B_{\frac{1}{4}}$ . For  $\varepsilon$  small enough, it follows from [2], (3.6) and Theorem 2.6, that there exists a renormalized solution  $u$  to equation

$$\begin{aligned}
-\Delta_p u + e^{\lambda u} + \beta &= \varepsilon \sum_{j=3}^J \mu_j && \text{in } B \\
u &= 0 && \text{in } \partial B.
\end{aligned} \quad (3.11)$$

satisfied (2.10) with  $\mu = -\beta + 1 + \varepsilon \sum_{j=3}^J \mu_j$  and  $u \geq \tilde{u}$  in  $B$  where  $\tilde{u}$  is a unique solution to equation (3.11) which the right hand side equals zero. By standard regularity theory,  $u \in C_{loc}^{1,\alpha}(B \setminus K)$  for some  $\alpha \in (0, 1)$ . Thus, from Corollary 2.5 and estimate (3.10), we have  $\exp(\lambda|u|) \in \mathcal{M}^{\frac{2N}{p}}(B)$ ,

$$u(0) \geq -c_{14} + c_{15} W_{1,p}^{\frac{1}{4}} \left[ \sum_{j=3}^J \mu_j \right] (0).$$

Therefore

$$\begin{aligned}
u(0) &\geq -c_{14} + c_{15} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left( \frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{14} + c_{15} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left( \frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= -c_{14} + c_{15} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left( \frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{14} + c_{16} \sum_{i=2}^{J-2} \left( \mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&= -c_{14} + c_{17} \sum_{i=4}^{\infty} \left( \mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}.
\end{aligned}$$

From the inequality

$$\left( \mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{N-p}{p-1}})} \left( \mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left( \mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i.$$

We deduce that

$$\begin{aligned}
u(0) &\geq -c_{14} + c_{17} \sum_{i=4}^{\infty} \left( \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left( \mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left( \mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{14} + c_{17} \left( \frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left( \mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{18} + c_{19} \int_0^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_t)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\end{aligned}$$

Since  $U_1$  is the maximal solution in  $K^c$ ,  $u$  satisfies the same equation in  $B \setminus K$  and  $U_1 \geq u = 0$  on  $\partial B$ , it follows that  $U_1$  dominates  $u$  in  $B \setminus K$ . Then  $U_1(0) \geq u(0)$  and we derive (3.4).

*Step 2.* Fix a positive integer  $J$  such that  $K \subset \{x : r_J \leq |x| < 1/8\}$ . Consider the sets  $K \cap S_j$  for  $j = 3, \dots, J$ . By [1, Theorem 2.5.3], there exists  $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$  such that

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{q_1} = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

We have, for any  $a_k \geq 0$ ,

$$\left( \sum_{k=0}^{\infty} a_k \right)^r \leq \sum_{k=0}^{\infty} \theta_{k,r} a_k^r$$

where  $\theta_{k,r}$  has the following expression with  $\theta > 0$ ,

$$\theta_{k,r} = \begin{cases} 1 & \text{if } r \in (0, 1], \\ \left( \frac{\theta+1}{\theta} \right)^{r-1} (\theta+1)^{kr} & \text{if } r > 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int_{B_1(0)} \left( \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} &\leq \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1(0)} \left( \mathbf{W}_{1,p}^1[\mu_k](x) \right)^{q_1} \\
&\leq c_{20} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} \int_{\mathbb{R}^N} (G_p * \mu_k(x))^{\frac{q_1}{p-1}} \\
&= c_{20} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\
&\leq c_{21} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k \left( N - \frac{pq_1}{q_1-p+1} \right)} \\
&\leq c_{22} \text{ for } \theta \text{ small enough,}
\end{aligned}$$

Here the second inequality follows from [2, Theorem 2.3] and the constant  $c_{22}$  does not depend on  $J$ . Hence,

$$\left\| \left( \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1(0))} \leq c_{23} \left\| \mathbf{W}_{1,p}^1 \left[ \sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1(0))}^q \leq c_{24} \quad (3.12)$$

where  $c_{23}$  is independent of  $J$ . Take  $B = B_{\frac{1}{4}}$ . By [2], (3.12), Corollary 2.5 and Theorem 2.6, there exists a renormalized solution  $u$  to equation

$$\begin{aligned} -\Delta_p u + \lambda|u|^{q-1}u + \beta &= \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{on } \partial B. \end{aligned} \quad (3.13)$$

It belongs to  $C_{loc}^{1,\alpha}(B \setminus K)$  for some  $\alpha \in (0, 1)$  and

$$u(0) \geq -c_{25} + c_{26} W_{1,p}^{\frac{1}{4}} \left[ \sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{27} + c_{28} \int_0^1 \left( \frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have  $U_2(0) \geq u(0)$ . Therefore, we get (3.5).  $\square$

## 4 Proof of the main results

Here we only prove Theorem 1. Let  $u \in C_{loc}^1(\Omega)$  be the maximal solution of

$$-\Delta_p u + e^{\lambda u} + \beta = 0 \quad \text{in } \Omega \quad (4.1)$$

Fix  $x_0 \in \partial\Omega$ . We can assume that  $x_0 = 0$ . Let  $\delta \in (0, 1/12)$ . For  $z_0 \in \overline{B_\delta} \cap \Omega$ . Set  $K = \Omega^c \cap \overline{B_{1/4}}(z_0)$ . Let  $U_1 \in C^1(K^c)$  be the maximal solution of (3.2). We have  $u \geq U_1$  in  $\Omega$ . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2 \int_\delta^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -C_1 + C_2 \int_{2\delta}^1 \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2' \int_\delta^{1/2} \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned}$$

We deduce

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -C_1 + C_2' \int_\delta^{1/2} \left( \frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Therefore,  $u$  is satisfied (1.13).  $\square$

## 5 Large solutions of quasilinear Hamilton-Jacobi equations

In this section we used our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + bu^s &= 0 & \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \quad (5.1)$$

where  $a > 0, b \geq 0$  and  $1 < p \leq 2, 1 \leq q < p, s \geq p - 1$  such that  $(q - p + 1) + b(s - p + 1) > 0$ . First we have the result of existence solutions to equation (5.1).

**Proposition 5.1** *Let  $a > 0, b \geq 0$  and  $s \geq p - 1, 1 \leq q \leq p, (q - p + 1) + b(s - p + 1) > 0$  and  $1 < p \leq 2$ . There exists a maximal solution  $u \in C_{loc}^1(\Omega)$  to equation*

$$\begin{aligned} -\Delta_p u + a |\nabla u|^p + bu^s &= 0 & \text{in } \Omega \\ 0 < u &\in C_{loc}^1(\Omega) \end{aligned} \quad (5.2)$$

satisfied

$$u(x) \leq c(N, p, s) b^{-\frac{1}{s-p+1}} d(x, \partial\Omega)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega \quad (5.3)$$

if  $b(s - p + 1) > 0$  and

$$u(x) \leq c(N, p, q) a^{-\frac{1}{q-p+1}} d(x, \partial\Omega)^{-\frac{p-q}{q-p+1}} \quad \forall x \in \Omega \quad (5.4)$$

if  $p - 1 < q < p$  and

$$u(x) \leq c(N, p) a^{-1} \log(2 \text{diam}(\Omega) d(x, \partial\Omega)^{-1}) \quad \forall x \in \Omega \quad (5.5)$$

if  $q = p$ .

**Proof.** Direct calculations show that the function

$$U_1(x) = c_1 a^{-\frac{1}{q-p+1}} \left( \frac{R^{\beta_1} - |x|^{\beta_1}}{\beta_1 R^{\beta_1-1}} \right)^{-\frac{p-q}{q-p+1}} \in C^1(B_R(0))$$

with  $a > 0$  and  $p - 1 < q < p$  satisfies

$$-\Delta_p U_1 + a |\nabla U_1|^q \geq 0 \quad \text{in } B_R(0), \quad (5.6)$$

the function

$$U_2(x) = c_2 a^{-1} \log \left( \frac{\beta_2 R^{\beta_2}}{R^{\beta_2} - |x|^{\beta_2}} \right) \in C_+^1(B_R(0))$$

with  $a > 0$  satisfies

$$-\Delta_p U_2 + a |\nabla U_2|^p \geq 0 \quad \text{in } B_R(0) \quad (5.7)$$

and the function

$$U_3(x) = c_3 b^{-\frac{1}{s-p+1}} \left( \frac{R^{\beta_3} - |x|^{\beta_3}}{\beta_3 R^{\beta_3-1}} \right)^{-\frac{p}{s-p+1}} \in C^1(B_R(0))$$

with  $b > 0$  and  $s > p - 1$  satisfies

$$-\Delta_p U_3 + b U_3^s \geq 0 \quad \text{in } B_R(0) \quad (5.8)$$

for some positive constants  $c_1 = c_1(N, p, q)$ ,  $c_2 = c_2(N, p)$ ,  $c_3 = c_3(N, p, s)$  and  $\beta_1 = \beta_1(N, p, q) \geq 1$ ,  $\beta_2 = \beta_2(N, p) \geq 1$ ,  $\beta_3 = \beta_3(N, p, s) \geq 1$ .

We emphasize the condition  $1 < p \leq 2$  and  $q \geq 1$  so that the equation (5.2) admits the Comparison Principle, see [16, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains  $\Omega_n$  satisfied  $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$  for all  $n$  and  $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ . For each  $n, k \in \mathbb{N}^*$ , there exist nonnegative

solution  $u_{n,k} = u \in W_k^{1,p}(\Omega_n) = W_0^{1,p}(\Omega_n) + k$  of equation (5.2) in  $\Omega_n$ .

Since  $-\Delta_p u_{k,n} \leq 0$  in  $\Omega_n$ , so using the Maximum Principle we get  $u_{n,k} \leq k$  in  $\Omega_n$  for all  $n$ . Thus, by standard regularity (see [4] and [11]),  $u_{n,k} \in C^{1,\alpha}(\overline{\Omega}_n)$  for some  $\alpha \in (0, 1)$ . After, using the comparison Principle and (5.6)-(5.8), we obtain that  $u_{n,k} \leq u_{n,k+1}$  in  $\Omega_n$  and (5.3)-(5.5) are satisfied with  $u_{n,k}, \Omega_n$  replacing  $u, \Omega$  respectively. From this, we lead to uniform local bounds for  $\{u_{n,k}\}_k$ . Thus, by standard interior regularity (see [4]) we obtain uniform local bounds for  $\{u_{n,k}\}_k$  in  $C_{loc}^{1,\eta}(\Omega_n)$ . It implies that  $\{u_{n,k}\}_k$  is pre-compact in  $C^1$  and hence up to subsequence  $u_{n,k} \rightarrow u_n$  in  $C_{loc}^1(\Omega_n)$ . Hence, we can verify that  $u_n$  is a solution of (5.2) and satisfies (5.3)-(5.5) with  $u_n, \Omega_n$  replacing  $u, \Omega$  and  $u_n(x) \rightarrow \infty$  as  $d(x, \Omega_n) \rightarrow 0$ .

Next, since  $u_{n,k} \geq u_{n+1,k}$  in  $\Omega_n$  thus  $u_n \geq u_{n+1}$  in  $\Omega_n$ . In particular,  $\{u_n\}$  is uniform local bounded in  $\Omega$ . We can argue as above, to obtain  $u_n \rightarrow u$  in  $C_{loc}^1(\Omega)$ ,  $u$  is a solution of (5.2) in  $\Omega$  and satisfies (5.3)-(5.5). Clearly,  $u$  is a maximal solution of (5.2).  $\blacksquare$

**Theorem 5.2** *Assume that the equation (1.13) admits a solution for some  $\lambda > 0$  and  $\beta \in \mathbb{R}$ . Then for any  $a, b \geq 0$  and  $q, s \geq p - 1$ ,  $1 \leq q < p$ ,  $a(q - p + 1) + b(s - p + 1) > 0$  and we will add  $1 < p \leq 2$  if  $a > 0$ , the equation (5.2) has a large solution and satisfied (5.3),(5.4).*

**Proof.** Assume that the equation (1.13) admits a solution  $v$  for some  $\lambda > 0$  and  $\beta \in \mathbb{R}$ . Here we only need to prove for case  $a(q - p + 1) > 0$ . We set  $v = \sigma \ln w$  with  $\sigma > 0$ , then  $w > 0$  and

$$-\Delta_p w + (p-1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha\lambda+p-1} + \beta \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega.$$

Choose  $\sigma = \frac{s-p+2}{\lambda}$ , we can see that

$$(p-1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha\lambda+p-1} + \beta \alpha^{1-p} w^{p-1} \geq a |\nabla w|^q + b w^s \quad \text{in } \{x : w(x) \geq M\},$$

where a positive constant  $M$  depends on  $p, q, s, \lambda, \beta, a, b$ . Therefore

$$-\Delta_p w + a |\nabla w|^q + b w^s \leq 0 \quad \text{in } \{x : w(x) \geq M\}.$$

Now we take an open subset  $\Omega'$  of  $\Omega$  with  $\overline{\Omega'} \subset \Omega$  such that the set  $\{x : w(x) \geq M\}$  contains in  $\Omega \setminus \overline{\Omega'}$ . So  $w$  is a subsolution of  $-\Delta_p u + a |\nabla u|^q + b u^s = 0$  in  $\Omega \setminus \overline{\Omega'}$  and  $w_\varepsilon := \varepsilon w$  is too for any  $\varepsilon \in (0, 1)$ . Let  $u$  be as in Proposition 5.1. Set  $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$  and  $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$ . So, we have  $w_\varepsilon < u$  on  $\partial\Omega'$  with  $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$ . Hence, from the construction of  $u$  in the proof of Proposition 5.1 and the Comparison Principle we assert  $w_\varepsilon \leq u$  in  $\Omega \setminus \overline{\Omega'}$ . Therefore, the Theorem follows.  $\blacksquare$

**Remark 5.3** *From the proof of above Theorem, we can show that under the assumption as in Proposition 5.1, the equation (5.2) has a large solution in  $\Omega$  if and only if the equation (5.2) has a large solution in  $\Omega \setminus K$  for some a compact set  $K \subset \Omega$  with smooth boundary.*

Now we concern (5.1) in case  $q = p$ .

**Theorem 5.4** Assume that the equation (5.2) has a large solution in  $\Omega$  for some  $a, b > 0$ ,  $s > p - 1$  and  $q = p$ . Then for any  $a_1, b_1 \geq 0$  and  $q_1, s_1 \geq p - 1$ ,  $1 \leq q_1 \leq p$ ,  $a_1(q_1 - p + 1) + b_1(s_1 - p + 1) > 0$ , the equation (5.2) also has a large solution in  $\Omega$  with  $a_1, b_1, q_1, s_1$  replacing  $a, b, q, s$  respectively, and satisfied (5.3)-(5.5). Here if  $a_2 > 0$ , we add the condition  $1 < p \leq 2$ .

*Proof.* For  $\sigma > 0$  we set  $u = v^\sigma$  thus

$$-\Delta_p v - (\sigma - 1)(p - 1) \frac{|\nabla v|^p}{v} + a\sigma v^{\sigma-1} |\nabla v|^p + b\sigma^{-p+1} v^{(s-p+1)\sigma+p-1} = 0.$$

Choose  $\sigma = \frac{s_1-p+1}{s-p+1} + 2$ , it is easy to see that

$$-\Delta_p v + a_1 |\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \quad \text{in } \{x : v(x) \geq M\},$$

for some a positive constant  $M$  only depending on  $p, s, a, b, a_1, b_1, q_1, s_1$ .

Similar to the proof of Theorem 5.2, we get the result as desired.  $\square$

**Remark 5.5** If we set  $u = e^v$  then  $v$  satisfies

$$-\Delta_p v + b e^{(s-p+1)v} = |\nabla v|^p (p - 1 - a e^v) \quad \text{in } \Omega. \quad (5.9)$$

From this, we can construct a large solution of

$$-\Delta_p u + b e^{(s-p+1)u} = 0 \quad \text{in } \Omega \setminus K. \quad (5.10)$$

for any a compact set  $K \subset \Omega$  with smooth boundary such that  $v \geq \ln\left(\frac{p-1}{a}\right)$  in  $\Omega \setminus K$ . In case  $p = 2$ , It would be interesting to see what Wiener type criterion the existence as such a large solution implies. We conjecture that this condition is

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c) dr}{r^{N-2}} \frac{dr}{r} = \infty \quad \forall x \in \partial\Omega. \quad (5.11)$$

We now consider the function

$$U_4(x) = c \left( \frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0 \quad (5.12)$$

It is easy to see that  $\Delta_p U_4 + U_4^{-\gamma} \geq 0$  for some positive constants  $\beta$  large and  $c$  small enough. From this, we get the existence of minimal solution to equation  $\Delta_p u + u^{-\gamma} = 0$  in  $\Omega$ .

**Proposition 5.6** For  $\gamma > 0$ , there exists a minimal solution  $u \in C_{loc}^1(\Omega)$  to equation  $\Delta_p u + u^{-\gamma} = 0$  in  $\Omega$  and satisfied  $u(x) \geq Cd(x, \Omega)^{\frac{p}{\gamma+p-1}}$  in  $\Omega$ .

We can verify that if the boundary of  $\Omega$  is satisfied (1.3), then above minimal solution  $u$  is a singular solution this means  $u \in C(\overline{\Omega})$  and  $u = 0$  on  $\partial\Omega$ .

**Theorem 5.7** Let  $\gamma > 0$ . Assume that there exists a singular solution to equation  $\Delta_p u + u^{-\gamma} = 0$  in  $\Omega$ . Then for any  $a, b \geq 0$  and  $q, s \geq p - 1$ ,  $1 \leq q \leq p$ ,  $a(q - p + 1) + b(s - p + 1) > 0$ , the equation (5.2) has a large solution in  $\Omega$  and satisfied (5.3)-(5.5). Here if  $a > 0$ , we add the condition  $1 < p \leq 2$ .

*Proof.* We set  $u = e^{-\frac{a}{p-1}v}$ , then  $v$  is a large solution of

$$-\Delta_p v + a |\nabla v|^p + \left(\frac{p-1}{a}\right)^{p-1} e^{\frac{a}{p-1}(\gamma+p-1)v} = 0 \quad \text{in } \Omega \quad (5.13)$$

So

$$-\Delta_p v + a |\nabla v|^q + b v^s \leq 0 \quad \text{in } \{x : v(x) \geq M\}$$

for some a positive constant  $M$  only depending on  $p, q, s, a, b, \gamma$ .

Similar to the proof of Theorem 5.2, we get the result as desired.  $\square$

## References

- [1] D. R. Adams, L. I. Hedberg, *Function spaces and potential theory*. Grundlehren der Mathematischen Wissenschaften **314**, Springer-Verlag, Berlin, 1996. xii+366 pp.
- [2] M.F. Bidaut-Véron, H. Nguyen Quoc, L. Véron: *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. (to appear).
- [3] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa **28**, 741-808 (1999).
- [4] E. DiBenedetto,  *$C^{1+\alpha}$  local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis 7, 827-850(1983).
- [5] A. Friedman, L. Véron: *Singular Solutions of Some Quasilinear Elliptic Equations*, Arch. Rat. Mech. Anal. **96**, 259-287 (1986).
- [6] P. Honzik, B. Jaye: *On the good- $\lambda$  inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180 (2012).
- [7] T. Kilpelainen, J. Malý: *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).
- [8] J. B. Keller: *On solutions of  $\Delta u = f(u)$* , Comm. Pure Appl. Math. **10**, 503-510 (1957).
- [9] D. Labutin: *Wiener regularity for large solutions of nonlinear equations*, Ark. Mat. **41**, no. 2, 307-39 (2003).
- [10] J.M. Lasry and P.L. Lions: *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints: 1. The Model problem*, Math. Ann. 283, 583-630(1989).
- [11] G.M. Lieberman *Boundary regularity for solution of degenerate elliptic equations* , Nonlinear Analysis 12, 1203-1219(1988).
- [12] M. Marcus, L. Véron: *Maximal solutions for  $-\Delta u + u^q = 0$  in open and finely open sets*, J. Math. Pures Appl. **91**, 256295 (2009).
- [13] V. Maz'ya: *On the continuity at a boundary point of solutions of quasilinear equations*, Vestnik Leningrad Univ. Math. **3**, 225-242 (1976).
- [14] J. Maly, W.P. Ziemer: *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, A.M.S (1997).
- [15] R. Osserman: *On the inequality  $\Delta u \geq f(u)$* , Pacific J. Math. **7**, 1641-1647 (1957)
- [16] P. Pucci, J. Serrin, *The Maximum Principle*, Progress in Nonlinear Differential Equations and Their Applications, 2007.
- [17] B. O. Turesson: *Nonlinear Potential Theory and Sobolev Spaces*, Lecture Notes in Mathematics **1736**, Springer-Verlag Berlin, Heidelberg (2000).
- [18] J. L. Vazquez: *An a priori interior estimate for the solution of a nonlinear problem representing weak diffusion*, Nonlinear Anal. T. M. A. **5**, 95103 (1981).
- [19] J. L. Vazquez, L. Véron: *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. **33**, 129-144 (1980).
- [20] L. Véron: *On the equation  $-\Delta u + e^u - 1 = 0$  with measures as boundary data*, Math. Z. **273** 1-2, 1-17 (2013).
- [21] N. Wiener: *The Dirichlet problem*, J. Math. Phys. **3**, 127-146 (1924).
- [22] W. Ziemer: *Weakly Differentiable Functions*, Graduate Texts in Mathematics **120**, Springer-Verlag, Berlin (1989).