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Hung Nguyen Quoc, Laurent Veron. Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption. 2013. hal-00851381v2

HAL Id: hal-00851381

<https://hal.science/hal-00851381v2>

Preprint submitted on 1 Sep 2013 (v2), last revised 10 Oct 2014 (v4)

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Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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Abstract

We obtain sufficient conditions expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities for the existence of large solutions to equations (1) $-\Delta_p u + e^{\lambda u} + \beta = 0$ or (2) $-\Delta_p u + \lambda|u|^{q-1}u + \beta = 0$ in a bounded domain Ω when $q > p - 1 > 0, \lambda > 0$ and $\beta \in \mathbb{R}$. We apply our results to equations (3) $-\Delta_p u + a|\nabla u|^q + bu^s = 0$, (4) $\Delta_p u + u^{-\gamma} = 0$ with $1 < p \leq 2, 1 \leq q \leq p, a > 0, b \geq 0$ and $(q - p + 1) + b(s - p + 1) > 0, \gamma > 0$.

2010 *Mathematics Subject Classification.* 31C15, 35J92, 35F21, 35B44.

Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 3$) and $1 < p \leq N$. We consider the question of existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty && \end{aligned} \quad (1.1)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and g is a continuous nondecreasing function vanishing at 0; most often $g(u)$ is either $e^{\lambda u} + \beta$ or $\lambda|u|^{q-1}u + \beta$ with $q > p - 1, \lambda > 0$ and $\beta \in \mathbb{R}$. A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data ϕ

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ u &= \phi && \text{in } \partial\Omega \end{aligned} \quad (1.2)$$

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admits a solution, it is clear that problem (1.1) admits a solution. It is known that a necessary and sufficient condition for such a result is the so called Wiener criterion (for $p = 2$ see [21]), for $p \neq 2$ see [13], [7])

$$\int_0^1 \left(\frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$. We remark that the Lipschitz and the Reifenberg flat domains are satisfied (1.3). The existence of a large solution is guaranteed for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition [18]

$$\int_a^\infty \frac{dt}{G^{\frac{1}{p}}(t)} < \infty \quad G(t) = \int_0^t g(s)ds, \quad (1.4)$$

a variant of the Keller-Osserman estimate [8], [15], which is the above relation when $p = 2$. If for $R > \text{diam}(\Omega)$ there exists a function v which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\} \\ v &= 0 && \text{on } \partial B_R \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of $\partial\Omega$, provided (1.3) is satisfied. However the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron proved [19] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p-1 < q < \frac{N(p-1)}{N-p}$, Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [9] proved that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.9)$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.10)$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. Concerning the

exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = e^{\lambda u} + \beta$ and $g(u) = \lambda |u|^{q-1} u + \beta$, $q > p - 1, \lambda > 0, \beta \in \mathbb{R}$. For $1 < p < N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\} \quad (1.11)$$

where the B_j are balls and $h^{N-p}(B_r) = c_N r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 3$ and $1 < p < N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.12)$$

then for any $\lambda > 0$ and $\beta \in \mathbb{R}$ there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^{\lambda u} + \beta &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.13)$$

As a consequence we obtained a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{s,r}$ Bessel capacity in \mathbb{R}^N associated to the Besov space $B^{s,r}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 3$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.14)$$

then, for any $p-1 < q < \frac{pq_1}{N}$, $\lambda > 0$ and $\beta \in \mathbb{R}$ there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + \lambda u^q + \beta &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.15)$$

In view of Labutin's theorem this last result is not optimal in the case $p = 2$, since the involved capacity is C_{2,q'_1} with q'_1 and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.16)$$

with $q_1 > q$.

At end we apply the previous theorems to quasilinear viscous Hamilton-Jacobi equations:

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + b |u|^{s-1} u &= 0 && \text{in } \Omega \\ u \in C_{loc}^1(\Omega), \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.17)$$

We prove that if $1 < p \leq 2$, $1 \leq q < p$, $a > 0$, $b \geq 0$, $(q - p + 1) + b(s - p + 1) > 0$ and (1.12) holds, there exists a positive solution to (1.17). Conversely, if for some $a, b > 0$, $s > p - 1$ there exists a solution to equation (1.17) with $q = p$, then for any $\max\{p - 1, 1\} \leq q_1 \leq p$, $s_1 \geq p - 1$, $a_1, b_1 \geq 0$, $a_1(q_1 - p + 1) + b_1(s_1 - p + 1) > 0$ there exists a positive solution to equation (1.17) with q_1, s_1, a_1, b_1 replacing q, s, a, b , here and we add $1 < p \leq 2$ if $a_1 > 0$. Moreover, we also prove that previous claim holds if for some $\gamma > 0$ there exists $u \in C(\overline{\Omega})$, $u > 0$ in Ω satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (1.18)$$

2 Morrey classes and Wolff potential estimates

In this section Ω is a bounded domain in \mathbb{R}^N . We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (1.1) belongs to $C_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 2.1 1- A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_{\Omega \cap B_r(x)} |f| dy \leq Kr^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N \quad (2.1)$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{\mathcal{M}^s(\Omega)}$.

2- A function $f \in L^1(\Omega)$ belongs to the weak L^s -space $M^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_E |f| dy \leq K|E|^{\frac{1}{s}} \quad \forall E \subset \Omega, E \text{ Borel.} \quad (2.2)$$

The quasi-norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{M^s(\Omega)}$

Clearly $L^p(\Omega) \subset M^p(\Omega) \subset \mathcal{M}^p(\Omega)$.

Definition 2.2 Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}_+(\Omega)$, the set of positive Radon measures in Ω . If $\alpha > 0$ and $1 < p < \alpha^{-1}N$, we define the (R -truncated) Wolff potential of μ by

$$\mathbf{W}_p^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N, \quad (2.3)$$

and, for $1 < p < N$, the (R -truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N, \quad (2.4)$$

where the measure is extended by 0 in Ω^c .

For $k \geq 0$, we set $T_k(u) = \text{sign}(u) \min\{k, |u|\}$.

Definition 2.3 Assume $f \in L^1_{loc}(\Omega)$. We say that a Borel function u defined in Ω is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.5)$$

if for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0 \quad (2.6)$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f := \mu$ is a positive distribution in Ω .

The following result is proved in [14, Theorem 4.35].

Theorem 2.4 Let Ω be an open bounded domain in \mathbb{R}^N . If $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ for some $\varepsilon \in (0, p)$, u is a nonnegative renormalized supersolution of (2.5) and set $\mu := -\Delta_p u + f$. Then there holds

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_1 W_{1,p}^{\frac{\varepsilon}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega, \quad (2.7)$$

for some c_1 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u = f + \mu \quad \text{in } \Omega \quad (2.8)$$

where $f \in L^1_{loc}(\Omega)$ and μ is a Radon measure, we have

Corollary 2.5 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of positive and bounded Radon measures in Ω . If u is a renormalized solution to (2.8) and $\inf_{\Omega} u > -\infty$ then there exists a positive constant c_2 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$ such that

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq \inf_{\Omega} u + c_2 W_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega. \quad (2.9)$$

We now recall [2, Theorem 3.8].

Theorem 2.6 Let $\mu \in \mathfrak{M}^b(\Omega)$. There exists a positive constant c_3 such that if u is a renormalized solution to $-\Delta_p u = \mu$ in Ω and $u = 0$ on $\partial\Omega$, then for any $x \in \Omega$

$$|u(x)| \leq c_3 W_{1,p}^{2\text{diam}(\Omega)}[|\mu|](x). \quad (2.10)$$

Definition 2.7 Let $s > 0$ and $0 < \alpha s < N$. We denote by $L^{\alpha,s}(\mathbb{R}^N)$ the Besov space the space functions $\phi = G_{\alpha} * f$ for $f \in L^s(\mathbb{R}^N)$ and we set $\|\phi\|_{L^{\alpha,s}(\mathbb{R}^N)} = \|f\|_{L^s(\mathbb{R}^N)}$, where G_{α} is Bessel kernel of order α . The dual space of $L^{s,q}(\mathbb{R}^N)$ is the space $L^{-s,q'}(\mathbb{R}^N)$ and it is naturally endowed with the dual norm. We also denote by $C_{\alpha,s}(E)$ the Bessel capacity of Borel set $E \subset \mathbb{R}^N$ defined by

$$C_{\alpha,s}(E) = \inf\{\|\phi\|_{L^s(\mathbb{R}^N)}^s : \phi \in L^s_+(\mathbb{R}^N), G_{\alpha} * \phi \geq \chi_E\}$$

where χ_E is the characteristic function of E .

From Corollary 2.5, Theorem 2.6 and [2, Theorem 2.3] we can verify the following result.

Theorem 2.8 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. Assume that u is a nonnegative renormalized solution to equation (2.8). If $\mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$ for some $q > p - 1$, then $u \in L^q(\Omega)$ and $\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} = \|G_p * \mu\|_{L^{\frac{q}{q-1}}(\mathbb{R}^N)}$,

$$\|u\|_{L^q(\Omega)} \leq C \left(\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \right) \quad (2.11)$$

for some a positive constant C depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$.

Conversely, if $u \in L^q(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C_K depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$ and $\text{dist}(K, \partial\Omega)$ such that $\chi_K \mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$ and

$$\|\chi_K \mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} \leq C_K \left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \right). \quad (2.12)$$

In particular, for any Borel set $E \subset \Omega$,

$$C_{p, \frac{q}{q+1-p}}(E) = 0 \implies \mu(E) = 0. \quad (2.13)$$

The next statement is proved in [2, Theorem 2.4], and in [6] for a variant.

Theorem 2.9 Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. There exist positive constants c_4, c_5 such that

$$\int_{2B} \exp(c_4 W_{1,p}^R[\mu_B]) \leq c_5 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$.

3 Estimates from below

If G is any domain in \mathbb{R}^N with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7), there always exists a maximal solution to (1.3) in G . It is constructed as the limit, when $n \rightarrow \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 && \text{in } G_n \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty && \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 && \text{if } G_n \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where $\{G_n\}_n$ is a sequence of smooth domains such that $G_n \subset \overline{G_n} \subset G_{n+1}$ for all n , $\{\partial G_n\}_n$ is a bounded and $\bigcup_{n=1}^{\infty} G_n = G$ and $\rho_n(x) := \text{dist}(x, \partial G_n)$. Our main estimates are the following.

Theorem 3.1 Let $K \subset B_{1/4} \setminus \{0\}$ be a compact set and let $U_j \in C_{loc}^1(K^c)$, $j = 1, 2$, be the maximal solutions of

$$-\Delta_p u + e^{\lambda u} + \beta = 0 \quad \text{in } K^c \quad (3.2)$$

for U_1 and

$$-\Delta_p u + \lambda u^q + \beta = 0 \quad \text{in } K^c \quad (3.3)$$

for U_2 , where $p-1 < q < \frac{pq_1}{N}$. Then there exist constants C_k , $k = 1, 2, 3, 4$, depending on N , p and q such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{C_{p, \frac{q_1}{1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

Proof. Step 1. For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [17, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\text{supp}(\mu_j) \subset K \cap S_j$, $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$c^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j$$

Now, we will show that for ε small enough, there holds,

$$A := \int_{B_1} \exp \left(\frac{2N}{p} \tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x) \right) \leq C, \quad (3.6)$$

where $\tilde{c}_3 = \max\{1, 2^{\frac{2-p}{p-1}}\} \lambda c_3$, c_3 is the constant in Theorem 2.6, and C does not depend on J . Indeed, define $\mu_j \equiv 0$ for all $j \geq J+1$ and $j \leq 2$. We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp \left(\frac{2N}{p} \tilde{c}_3 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right).$$

Since for any j

$$\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \leq c_p \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] + c_p \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] + c_p \sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k]$$

with $c_p = \max\{1, 5^{\frac{2-p}{p-1}}\}$ and $\exp(\sum_{i=1}^5 a_i) \leq \sum_{i=1}^5 \exp(5a_i)$ for all a_i . Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x) \right) + \sum_{j=1}^{\infty} \int_{S_j} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) \right) \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1,3\}}^{j+1} \int_{S_j} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right) := A_1 + A_2 + A_3, \text{ with } c_6 = 5c_p \frac{2N}{p} \tilde{c}_3. \end{aligned}$$

Estimate of A_3 : We apply Theorem 2.9 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right) \leq c_5 r_{k-1}^N$$

with $c_6\varepsilon^{\frac{1}{p-1}} \in (0, c_4]$, the constant c_4 is in Theorem 2.9. In particular,

$$\int_{S_j} \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) \leq c_5 r_{k-1}^N \quad k = j-1, j, j+1,$$

which implies

$$A_3 \leq c_7 \sum_{j=1}^{+\infty} r_j^N = c_8 < \infty. \quad (3.7)$$

Estimate of A_1 : Since $\sum_{k \geq j+2} \mu_k(B_t(x)) = 0$ for all $x \in S_j, t \in (0, r_{j+1})$. Thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \int_{r_{j+1}}^1 \left(\frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right) \\ &\leq \sum_{j=1}^{\infty} \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right) |S_j|. \end{aligned}$$

Note that $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left(\sum_{k \geq j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k \geq 0} r_k^{N-p}\right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$A_1 \leq \exp\left(c_6\varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) |B_1| = c_9. \quad (3.8)$$

Estimate of A_2 : for $x \in S_j$,

$$\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Since $r_i < t < r_{i-1}$, $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$, thus

$$\begin{aligned} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(S_k)}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \sum_{i=1}^{j-1} \left(\sum_{k=i-1}^{j-2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \leq c_{10}j, \quad \text{with } c_{10} = \left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} c_{10} j\right) = \sum_{j=1}^{\infty} r_j^N \exp\left(c_6 c_{10} \varepsilon^{\frac{1}{p-1}} j\right) |S_1| \\
&= \sum_{j=1}^{\infty} \exp\left(\left(c_6 c_{10} \varepsilon^{\frac{1}{p-1}} - N \log(2)\right) j\right) |S_1| = c_{11} \quad \text{for } \varepsilon \text{ small enough.} \quad (3.9)
\end{aligned}$$

Consequently, $A \leq C := c_9 + c_{11} + c_8$ for ε small enough. This implies

$$\left\| \exp\left(\tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k\right]\right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1(0))} \leq c_{12} \left(\int_{B_1(0)} \exp\left(\frac{2N}{p} \tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k\right](x)\right) \right)^{\frac{p}{2N}} \leq c_{13} \quad (3.10)$$

where the constant c_{13} does not depend on J . Set $B = B_{\frac{1}{4}}$. For ε small enough, it follows from [2], (3.6) and Theorem 2.6, that there exists a renormalized solution u to equation

$$\begin{aligned}
-\Delta_p u + e^{\lambda u} + \beta &= \varepsilon \sum_{j=3}^J \mu_j && \text{in } B \\
u &= 0 && \text{in } \partial B.
\end{aligned} \quad (3.11)$$

satisfied (2.10) with $\mu = -\beta + 1 + \varepsilon \sum_{j=3}^J \mu_j$ and $u \geq \tilde{u}$ in B where \tilde{u} is a unique solution to equation (3.11) which the right hand side equals zero. By standard regularity theory, $u \in C_{loc}^{1,\alpha}(B \setminus K)$ for some $\alpha \in (0, 1)$. Thus, from Corollary 2.5 and estimate (3.10), we have $\exp(\lambda|u|) \in \mathcal{M}^{\frac{2N}{p}}(B)$,

$$u(0) \geq -c_{14} + c_{15} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

Therefore

$$\begin{aligned}
u(0) &\geq -c_{14} + c_{15} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{14} + c_{15} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= -c_{14} + c_{15} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{14} + c_{16} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&= -c_{14} + c_{17} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}.
\end{aligned}$$

From the inequality

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i.$$

We deduce that

$$\begin{aligned}
u(0) &\geq -c_{14} + c_{17} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{14} + c_{17} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{18} + c_{19} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\end{aligned}$$

Since U_1 is the maximal solution in K^c , u satisfies the same equation in $B \setminus K$ and $U_1 \geq u = 0$ on ∂B , it follows that U_1 dominates u in $B \setminus K$. Then $U_1(0) \geq u(0)$ and we derive (3.4).

Step 2. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [1, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{q_1} = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

We have, for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k \right)^r \leq \sum_{k=0}^{\infty} \theta_{k,r} a_k^r$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,r} = \begin{cases} 1 & \text{if } r \in (0, 1], \\ \left(\frac{\theta+1}{\theta} \right)^{r-1} (\theta+1)^{kr} & \text{if } r > 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int_{B_1(0)} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} &\leq \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1(0)} \left(\mathbf{W}_{1,p}^1[\mu_k](x) \right)^{q_1} \\
&\leq c_{20} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} \int_{\mathbb{R}^N} (G_p * \mu_k(x))^{\frac{q_1}{p-1}} \\
&= c_{20} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\
&\leq c_{21} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k \left(N - \frac{pq_1}{q_1-p+1} \right)} \\
&\leq c_{22} \text{ for } \theta \text{ small enough,}
\end{aligned}$$

Here the second inequality follows from [2, Theorem 2.3] and the constant c_{22} does not depend on J . Hence,

$$\left\| \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1(0))} \leq c_{23} \left\| \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1(0))}^q \leq c_{24} \quad (3.12)$$

where c_{23} is independent of J . Take $B = B_{\frac{1}{4}}$. By [2], (3.12), Corollary 2.5 and Theorem 2.6, there exists a renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + \lambda|u|^{q-1}u + \beta &= \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{on } \partial B. \end{aligned} \quad (3.13)$$

It belongs to $C_{loc}^{1,\alpha}(B \setminus K)$ for some $\alpha \in (0, 1)$ and

$$u(0) \geq -c_{25} + c_{26} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{27} + c_{28} \int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have $U_2(0) \geq u(0)$. Therefore, we get (3.5). \square

4 Proof of the main results

Here we only prove Theorem 1. Let $u \in C_{loc}^1(\Omega)$ be the maximal solution of

$$-\Delta_p u + e^{\lambda u} + \beta = 0 \quad \text{in } \Omega \quad (4.1)$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B_\delta} \cap \Omega$. Set $K = \Omega^c \cap \overline{B_{1/4}}(z_0)$. Let $U_1 \in C^1(K^c)$ be the maximal solution of (3.2). We have $u \geq U_1$ in Ω . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned}$$

We deduce

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Therefore, u is satisfied (1.13). \square

5 Large solutions of quasilinear Hamilton-Jacobi equations

In this section we used our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + bu^s &= 0 & \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \quad (5.1)$$

where $a > 0, b \geq 0$ and $1 < p \leq 2, 1 \leq q < p, s \geq p - 1$ such that $(q - p + 1) + b(s - p + 1) > 0$. First we have the result of existence solutions to equation (5.1).

Proposition 5.1 *Let $a > 0, b \geq 0$ and $s \geq p - 1, 1 \leq q \leq p, (q - p + 1) + b(s - p + 1) > 0$ and $1 < p \leq 2$. There exists a maximal solution $u \in C_{loc}^1(\Omega)$ to equation*

$$\begin{aligned} -\Delta_p u + a |\nabla u|^p + bu^s &= 0 & \text{in } \Omega \\ 0 < u &\in C_{loc}^1(\Omega) \end{aligned} \quad (5.2)$$

satisfied

$$u(x) \leq c(N, p, s) b^{-\frac{1}{s-p+1}} d(x, \partial\Omega)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega \quad (5.3)$$

if $b(s - p + 1) > 0$ and

$$u(x) \leq c(N, p, q) a^{-\frac{1}{q-p+1}} d(x, \partial\Omega)^{-\frac{p-q}{q-p+1}} \quad \forall x \in \Omega \quad (5.4)$$

if $p - 1 < q < p$ and

$$u(x) \leq c(N, p) a^{-1} \log(2 \text{diam}(\Omega) d(x, \partial\Omega)^{-1}) \quad \forall x \in \Omega \quad (5.5)$$

if $q = p$.

Proof. Direct calculations show that the function

$$U_1(x) = c_1 a^{-\frac{1}{q-p+1}} \left(\frac{R^{\beta_1} - |x|^{\beta_1}}{\beta_1 R^{\beta_1-1}} \right)^{-\frac{p-q}{q-p+1}} \in C^1(B_R(0))$$

with $a > 0$ and $p - 1 < q < p$ satisfies

$$-\Delta_p U_1 + a |\nabla U_1|^q \geq 0 \quad \text{in } B_R(0), \quad (5.6)$$

the function

$$U_2(x) = c_2 a^{-1} \log \left(\frac{\beta_2 R^{\beta_2}}{R^{\beta_2} - |x|^{\beta_2}} \right) \in C_+^1(B_R(0))$$

with $a > 0$ satisfies

$$-\Delta_p U_2 + a |\nabla U_2|^p \geq 0 \quad \text{in } B_R(0) \quad (5.7)$$

and the function

$$U_3(x) = c_3 b^{-\frac{1}{s-p+1}} \left(\frac{R^{\beta_3} - |x|^{\beta_3}}{\beta_3 R^{\beta_3-1}} \right)^{-\frac{p}{s-p+1}} \in C^1(B_R(0))$$

with $b > 0$ and $s > p - 1$ satisfies

$$-\Delta_p U_3 + b U_3^s \geq 0 \quad \text{in } B_R(0) \quad (5.8)$$

for some positive constants $c_1 = c_1(N, p, q)$, $c_2 = c_2(N, p)$, $c_3 = c_3(N, p, s)$ and $\beta_1 = \beta_1(N, p, q) \geq 1$, $\beta_2 = \beta_2(N, p) \geq 1$, $\beta_3 = \beta_3(N, p, s) \geq 1$.

We emphasize the condition $1 < p \leq 2$ and $q \geq 1$ so that the equation (5.2) admits the Comparison Principle, see [16, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains Ω_n satisfied $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. For each $n, k \in \mathbb{N}^*$, there exist nonnegative

solution $u_{n,k} = u \in W_k^{1,p}(\Omega_n) = W_0^{1,p}(\Omega_n) + k$ of equation (5.2) in Ω_n .

Since $-\Delta_p u_{k,n} \leq 0$ in Ω_n , so using the Maximum Principle we get $u_{n,k} \leq k$ in Ω_n for all n . Thus, by standard regularity (see [4] and [11]), $u_{n,k} \in C^{1,\alpha}(\overline{\Omega}_n)$ for some $\alpha \in (0, 1)$. After, using the comparison Principle and (5.6)-(5.8), we obtain that $u_{n,k} \leq u_{n,k+1}$ in Ω_n and (5.3)-(5.5) are satisfied with $u_{n,k}, \Omega_n$ replacing u, Ω respectively. From this, we lead to uniform local bounds for $\{u_{n,k}\}_k$. Thus, by standard interior regularity (see [4]) we obtain uniform local bounds for $\{u_{n,k}\}_k$ in $C_{loc}^{1,\eta}(\Omega_n)$. It implies that $\{u_{n,k}\}_k$ is pre-compact in C^1 and hence up to subsequence $u_{n,k} \rightarrow u_n$ in $C_{loc}^1(\Omega_n)$. Hence, we can verify that u_n is a solution of (5.2) and satisfies (5.3)-(5.5) with u_n, Ω_n replacing u, Ω and $u_n(x) \rightarrow \infty$ as $d(x, \Omega_n) \rightarrow 0$.

Next, since $u_{n,k} \geq u_{n+1,k}$ in Ω_n thus $u_n \geq u_{n+1}$ in Ω_n . In particular, $\{u_n\}$ is uniform local bounded in Ω . We can argue as above, to obtain $u_n \rightarrow u$ in $C_{loc}^1(\Omega)$, u is a solution of (5.2) in Ω and satisfies (5.3)-(5.5). Clearly, u is a maximal solution of (5.2). \blacksquare

Lemma 5.2 *The maximal solution of (4.1) is a large solution with $\lambda = 1$ and $\beta = -1$ if and only if for any $a > 0$ and $b < b_a := \theta_1 a^{1-p}$ the maximal solution of*

$$-\Delta_p v + e^{av} + b = 0 \quad \text{in } \Omega, \quad (5.9)$$

is a large solution, where θ_1 is a positive constant depending on N, p and Ω .

Proof. Since monotonicity and Vazquez' condition (1.3) hold, it is sufficient to exhibit a large subsolution (i.e. tending to infinity on the boundary) in order to conclude on the existence of a large solution to (5.9).

Assume $u := u_{1,-1}$ is a large solution of (4.1), then for any $\Lambda \geq 1$

$$-\Delta_p u_{1,-1} + e^{u_{1,-1}} - \Lambda = 1 - \Lambda \leq 0 \quad \text{in } \Omega,$$

thus $u_{1,-1}$ is a subsolution of the corresponding solution and there exists a larger solution which is necessarily a large solution $u_{1,-\Lambda}$ of

$$-\Delta_p u + e^u - \Lambda = 0 \quad \text{in } \Omega. \quad (5.10)$$

Set $\min\{u_{1,-1}(x) : x \in \Omega\} = \theta > 0$. then, for any $c \in (0, 1)$ and $d \geq 0$ there holds

$$e^{u_{1,-1}} - 1 \geq m_\theta e^{cu_{1,-1}} \geq m_\theta e^{cu_{1,-1}} - d \quad \text{on } [\theta, \infty)$$

with $m_\theta = e^{(1-c)\theta} - e^{-c\theta}$. This implies that $-\Delta_p u_{1,-1} + m_\theta e^{cu_{1,-1}} - d \leq 0$, therefore $v := u_{1,-1} + c^{-1} \ln m_\theta$ satisfies $-\Delta_p v + e^{cv} - d \leq 0$. Therefore there exists a large solution $u_{c,-d}$ to

$$-\Delta_p u + e^{cu} - d = 0 \quad \text{in } \Omega. \quad (5.11)$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, set $u_{c,-d} = \alpha w + \beta$, then $-\Delta_p w + \alpha^{1-p} e^{\beta c} e^{\alpha c w} - d \alpha^{1-p} = 0$. If we take $\beta = \frac{p-1}{c} \ln \alpha$, then

$$-\Delta_p w + e^{\alpha c w} - d \alpha^{1-p} = 0 \quad \text{in } \Omega. \quad (5.12)$$

Since $\alpha > 0$ and $d \geq 0$ are arbitrary, we see that for any $a > 0$ and $b \geq 0$, there exists a large solution $u = u_{a,-b}$ to

$$-\Delta_p u + e^{au} - b = 0 \quad \text{in } \Omega. \quad (5.13)$$

We can notice that since $u_{a,0} = a^{-1}u_{1,0} + (1-p)a^{-1} \ln a$, the minimum $\theta := \theta_a$ of $u_{a,0}$ satisfies $\theta_a = a^{-1}\theta_1 + (1-p)a^{-1} \ln a$. For $\epsilon > 0$, there holds

$$e^{au_{a,0}} - \epsilon e^{au_{a,0}} = (1-\epsilon)e^{au_{a,0}} \geq (1-\epsilon)\theta_1 a^{1-p}.$$

Therefore $-\Delta_p u_{a,0} + \epsilon e^{au_{a,0}} + (1-\epsilon)\theta_1 a^{1-p} \leq 0$. Thus $v = u_{a,0} - a^{-1} \ln \epsilon$ satisfies

$$-\Delta_p v + e^{av} + (1-\epsilon)\theta_1 a^{1-p} \leq 0$$

which implies that there exists a large solution to the corresponding equation. Since ϵ is arbitrary, it follows that for any $b < b_a := \theta_1 a^{1-p}$, there exists a large solution $u = u_{a,b}$ to (5.9).

Remark: the constant θ only depends on N, p and $\text{diam}(\Omega)$. In fact, we can see that $\theta \geq \min_{B_{\text{diam}(\Omega)}(0)} U$, where U is a unique solution of $-\Delta_p U + e^U - 1 = 0$ in $B_{\text{diam}(\Omega)}(0)$ and $U(x) \rightarrow \infty$ as $|x| \uparrow \text{diam}(\Omega)$ \square

Theorem 5.3 *Assume $p-1 < q < p$ and (1.12) holds. Then there exists $b^* = b^*(p, q, N, \Omega) > 0$ such that for any $b \in (-\infty, b^*)$, problem (5.1) admits a solution.*

Proof. If (1.12) holds, for any $a > 0$ and $b < b_a$, there exists a large solution u to (5.9). We set $u = \alpha \ln w$ with $\alpha > 0$, then

$$-\Delta_p w + (p-1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha a + p-1} + b \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega. \quad (5.14)$$

By Hölder's inequality

$$(p-1) \frac{|\nabla w|^p}{w} \geq |\nabla w|^q - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}},$$

therefore

$$-\Delta_p w + |\nabla w|^q + \alpha^{1-p} w^{\alpha a + p-1} + b \alpha^{1-p} w^{p-1} - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}} \leq 0.$$

Since $q > p-1$, $\frac{q}{p-q} > p-1$. We choose α and a such that

$$\alpha a + p - 1 = \frac{q}{p-q} \quad \text{and} \quad \alpha^{1-p} = \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}}.$$

Therefore w satisfies

$$-\Delta_p w + |\nabla w|^q + b \alpha^{1-p} w^{p-1} \leq 0.$$

This implies that there exists a large solution to (5.1).

Remark: According to [16], we need $1 < p \leq 2$, $q \geq 1$ and $b \geq 0$ in order that the Comparison Principle is applied to the equation $-\Delta_p u + a|\nabla u|^q + bu^s = 0$. \square

Theorem 5.4 *Assume that the equation (1.13) admits a solution for some $\lambda > 0$ and $\beta \in \mathbb{R}$. Then for any $a, b \geq 0$ and $q, s \geq p - 1$, $1 \leq q < p$, $a(q - p + 1) + b(s - p + 1) > 0$ and we will add $1 < p \leq 2$ if $a > 0$, the equation (5.2) has a large solution and satisfied (5.3),(5.4).*

Proof. Assume that the equation (1.13) admits a solution v for some $\lambda > 0$ and $\beta \in \mathbb{R}$. Here we only need to prove for case $a(q - p + 1) > 0$. We set $v = \sigma \ln w$ with $\sigma > 0$, then $w > 0$ and

$$-\Delta_p w + (p - 1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha\lambda+p-1} + \beta \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega.$$

Choose $\sigma = \frac{s-p+2}{\lambda}$, we can see that

$$(p - 1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha\lambda+p-1} + \beta \alpha^{1-p} w^{p-1} \geq a |\nabla w|^q + b w^s \quad \text{in } \{x : w(x) \geq M\},$$

where a positive constant M depends on $p, q, s, \lambda, \beta, a, b$. Therefore

$$-\Delta_p w + a |\nabla w|^q + b w^s \leq 0 \quad \text{in } \{x : w(x) \geq M\}.$$

Now we take an open subset Ω' of Ω with $\overline{\Omega'} \subset \Omega$ such that the set $\{x : w(x) \geq M\}$ contains in $\Omega \setminus \overline{\Omega'}$. So w is a subsolution of $-\Delta_p u + a |\nabla u|^q + b u^s = 0$ in $\Omega \setminus \overline{\Omega'}$ and $w_\varepsilon := \varepsilon w$ is too for any $\varepsilon \in (0, 1)$. Let u be as in Proposition 5.1. Set $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$ and $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$. So, we have $w_\varepsilon < u$ on $\partial\Omega'$ with $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$. Hence, from the construction of u in the proof of Proposition 5.1 and the Comparison Principle we assert $w_\varepsilon \leq u$ in $\Omega \setminus \overline{\Omega'}$. Therefore, the Theorem follows. \blacksquare

Remark 5.5 *From the proof of above Theorem, we can show that under the assumption as in Proposition 5.1, the equation (5.2) has a large solution in Ω if and only if the equation (5.2) has a large solution in $\Omega \setminus K$ for some a compact set $K \subset \Omega$ with smooth boundary.*

Now we concern (5.1) in case $q = p$.

Theorem 5.6 *Assume that the equation (5.2) has a large solution in Ω for some $a, b > 0$, $s > p - 1$ and $q = p$. Then for any $a_1, b_1 \geq 0$ and $q_1, s_1 \geq p - 1$, $1 \leq q_1 \leq p$, $a_1(q_1 - p + 1) + b_1(s_1 - p + 1) > 0$, the equation (5.2) also has a large solution in Ω with a_1, b_1, q_1, s_1 replacing a, b, q, s respectively, and satisfied (5.3)-(5.5). Here if $a_2 > 0$, we add the condition $1 < p \leq 2$.*

Proof. For $\sigma > 0$ we set $u = v^\sigma$ thus

$$-\Delta_p v - (\sigma - 1)(p - 1) \frac{|\nabla v|^p}{v} + a \sigma v^{\sigma-1} |\nabla v|^p + b \sigma^{-p+1} v^{(s-p+1)\sigma+p-1} = 0.$$

Choose $\sigma = \frac{s_1-p+1}{s-p+1} + 2$, it is easy to see that

$$-\Delta_p v + a_1 |\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \quad \text{in } \{x : v(x) \geq M\},$$

for some a positive constant M only depending on $p, s, a, b, a_1, b_1, q_1, s_1$.

Similar to the proof of Theorem 5.4, we get the result as desired. \square

Remark 5.7 *If we set $u = e^v$ then v satisfies*

$$-\Delta_p v + b e^{(s-p+1)v} = |\nabla v|^p (p - 1 - a e^v) \quad \text{in } \Omega. \quad (5.15)$$

From this, we can construct a large solution of

$$-\Delta_p u + b e^{(s-p+1)u} = 0 \quad \text{in } \Omega \setminus K. \quad (5.16)$$

for any a compact set $K \subset \Omega$ with smooth boundary such that $v \geq \ln\left(\frac{p-1}{a}\right)$ in $\Omega \setminus K$. In case $p = 2$, It would be interesting to see what Wiener type criterion the existence as such a large solution implies. We conjecture that this condition is

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c) dr}{r^{N-2}} = \infty \quad \forall x \in \partial\Omega. \quad (5.17)$$

We now consider the function

$$U_4(x) = c \left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0 \quad (5.18)$$

It is easy to see that $\Delta_p U_4 + U_4^{-\gamma} \geq 0$ for some positive constants β large and c small enough. From this, we get the existence of minimal solution to equation $\Delta_p u + u^{-\gamma} = 0$ in Ω .

Proposition 5.8 For $\gamma > 0$, there exists a minimal solution $u \in C_{loc}^1(\Omega)$ to equation $\Delta_p u + u^{-\gamma} = 0$ in Ω and satisfied $u(x) \geq Cd(x, \Omega)^{\frac{p}{\gamma+p-1}}$ in Ω .

We can verify that if the boundary of Ω is satisfied (1.3), then above minimal solution u is a singular solution this means $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

Theorem 5.9 Let $\gamma > 0$. Assume that there exists a singular solution to equation $\Delta_p u + u^{-\gamma} = 0$ in Ω . Then for any $a, b \geq 0$ and $q, s \geq p-1$, $1 \leq q \leq p$, $a(q-p+1) + b(s-p+1) > 0$, the equation (5.2) has a large solution in Ω and satisfied (5.3)-(5.5). Here if $a > 0$, we add the condition $1 < p \leq 2$.

Proof. We set $u = e^{-\frac{a}{p-1}v}$, then v is a large solution of

$$-\Delta_p v + a |\nabla v|^p + \left(\frac{p-1}{a}\right)^{p-1} e^{\frac{a}{p-1}(\gamma+p-1)v} = 0 \quad \text{in } \Omega \quad (5.19)$$

So

$$-\Delta_p v + a |\nabla v|^q + bv^s \leq 0 \quad \text{in } \{x : v(x) \geq M\}$$

for some a positive constant M only depending on p, q, s, a, b, γ .

Similar to the proof of Theorem 5.4, we get the result as desired. \square

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