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Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

Nguyen Quoc Hung*

Laurent Véron†

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We obtain sufficient conditions expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities for the existence of large solutions to equations (1) $-\Delta_p u + e^{\lambda u} + \beta = 0$ or (2) $-\Delta_p u + \lambda|u|^{q-1}u + \beta = 0$ in a bounded domain Ω when $q > p - 1 > 0, \lambda > 0$ and $\beta \in \mathbb{R}$. We apply our results to equations (3) $-\Delta_p u + a|\nabla u|^q + bu^s = 0$, (4) $\Delta_p u + u^{-\gamma} = 0$ with $1 < p \leq 2, 1 \leq q \leq p, a > 0, b \geq 0$ and $(q - p + 1) + b(s - p + 1) > 0, \gamma > 0$.

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Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 3$) and $1 < p \leq N$. We consider the question of existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty && \end{aligned} \quad (1.1)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and g is a continuous nondecreasing function vanishing at 0; most often $g(u)$ is either $e^{\lambda u} + \beta$ or $\lambda|u|^{q-1}u + \beta$ with $q > p - 1, \lambda > 0$ and $\beta \in \mathbb{R}$. A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data ϕ

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ u &= \phi && \text{in } \partial\Omega \end{aligned} \quad (1.2)$$

*E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr

†E-mail address: Laurent.Veron@lmpt.univ-tours.fr

admits a solution, it is clear that problem (1.1) admits a solution. It is known that a necessary and sufficient condition for such a result is the so called Wiener criterion (for $p = 2$ see [21]), for $p \neq 2$ see [13], [7])

$$\int_0^1 \left(\frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$. We remark that the Lipschitz and the Reifenberg flat domains are satisfied (1.3). The existence of a large solution is guaranteed for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition [18]

$$\int_a^\infty \frac{dt}{G^{\frac{1}{p}}(t)} < \infty \quad G(t) = \int_0^t g(s) ds, \quad (1.4)$$

a variant of the Keller-Osserman estimate [8], [15], which is the above relation when $p = 2$. If for $R > \text{diam}(\Omega)$ there exists a function v which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\} \\ v &= 0 && \text{on } \partial B_R \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of $\partial\Omega$, provided (1.3) is satisfied. However the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron proved [19] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p-1 < q < \frac{N(p-1)}{N-p}$, Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [9] proved that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.9)$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.10)$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. Concerning the

exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = e^{\lambda u} + \beta$ and $g(u) = \lambda |u|^{q-1} u + \beta$, $q > p - 1, \lambda > 0, \beta \in \mathbb{R}$. For $1 < p < N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\} \quad (1.11)$$

where the B_j are balls and $h^{N-p}(B_r) = c_N r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 3$ and $1 < p < N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.12)$$

then for any $\lambda > 0$ and $\beta \in \mathbb{R}$ there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^{\lambda u} + \beta &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.13)$$

As a consequence we obtained a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{s,r}$ Bessel capacity in \mathbb{R}^N associated to the Besov space $B^{s,r}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 3$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.14)$$

then, for any $p-1 < q < \frac{pq_1}{N}$, $\lambda > 0$ and $\beta \in \mathbb{R}$ there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + \lambda u^q + \beta &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.15)$$

In view of Labutin's theorem this last result is not optimal in the case $p = 2$, since the involved capacity is C_{2,q'_1} with q'_1 and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.16)$$

with $q_1 > q$.

At end we apply the previous theorems to quasilinear viscous Hamilton-Jacobi equations:

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + b |u|^{s-1} u &= 0 && \text{in } \Omega \\ u \in C_{loc}^1(\Omega), \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.17)$$

We prove that if $1 < p \leq 2$, $1 \leq q < p$, $a > 0$, $b \geq 0$, $(q - p + 1) + b(s - p + 1) > 0$ and (1.12) holds, there exists a positive solution to (1.17). Conversely, if for some $a, b > 0$, $s > p - 1$ there exists a solution to equation (1.17) with $q = p$, then for any $\max\{p - 1, 1\} \leq q_1 \leq p$, $s_1 \geq p - 1$, $a_1, b_1 \geq 0$, $a_1(q_1 - p + 1) + b_1(s_1 - p + 1) > 0$ there exists a positive solution to equation (1.17) with q_1, s_1, a_1, b_1 replacing q, s, a, b , here and we add $1 < p \leq 2$ if $a_1 > 0$. Moreover, we also prove that previous claim holds if for some $\gamma > 0$ there exists $u \in C(\overline{\Omega})$, $u > 0$ in Ω satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (1.18)$$

2 Morrey classes and Wolff potential estimates

In this section Ω is a bounded domain in \mathbb{R}^N . We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (1.1) belongs to $C_{loc}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 2.1 1- A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_{\Omega \cap B_r(x)} |f| dy \leq Kr^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N \quad (2.1)$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{\mathcal{M}^s(\Omega)}$.

2- A function $f \in L^1(\Omega)$ belongs to the weak L^s -space $M^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_E |f| dy \leq K|E|^{\frac{1}{s}} \quad \forall E \subset \Omega, E \text{ Borel}. \quad (2.2)$$

The quasi-norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{M^s(\Omega)}$

Clearly $L^p(\Omega) \subset M^p(\Omega) \subset \mathcal{M}^p(\Omega)$.

Definition 2.2 Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}_+(\Omega)$, the set of positive Radon measures in Ω . If $\alpha > 0$ and $1 < p < \alpha^{-1}N$, we define the (R -truncated) Wolff potential of μ by

$$\mathbf{W}_p^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N, \quad (2.3)$$

and, for $1 < p < N$, the (R -truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N, \quad (2.4)$$

where the measure is extended by 0 in Ω^c .

For $k \geq 0$, we set $T_k(u) = \text{sign}(u) \min\{k, |u|\}$.

Definition 2.3 Assume $f \in L^1_{loc}(\Omega)$. We say that a Borel function u defined in Ω is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.5)$$

if for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0 \quad (2.6)$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f := \mu$ is a positive distribution in Ω .

The following result is proved in [14, Theorem 4.35].

Theorem 2.4 Let Ω be an open bounded domain in \mathbb{R}^N . If $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ for some $\varepsilon \in (0, p)$, u is a nonnegative renormalized supersolution of (2.5) and set $\mu := -\Delta_p u + f$. Then there holds

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_1 W_{1,p}^{\frac{\varepsilon}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega, \quad (2.7)$$

for some c_1 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u = f + \mu \quad \text{in } \Omega \quad (2.8)$$

where $f \in L^1_{loc}(\Omega)$ and μ is a Radon measure, we have

Corollary 2.5 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of positive and bounded Radon measures in Ω . If u is a renormalized solution to (2.8) and $\inf_{\Omega} u > -\infty$ then there exists a positive constant c_2 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$ such that

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq \inf_{\Omega} u + c_2 W_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega. \quad (2.9)$$

We now recall [2, Theorem 3.8].

Theorem 2.6 Let $\mu \in \mathfrak{M}^b(\Omega)$. There exists a positive constant c_3 such that if u is a renormalized solution to $-\Delta_p u = \mu$ in Ω and $u = 0$ on $\partial\Omega$, then for any $x \in \Omega$

$$|u(x)| \leq c_3 W_{1,p}^{2\text{diam}(\Omega)}[|\mu|](x). \quad (2.10)$$

Definition 2.7 Let $s > 0$ and $0 < \alpha s < N$. We denote by $L^{\alpha,s}(\mathbb{R}^N)$ the Besov space the space functions $\phi = G_{\alpha} * f$ for $f \in L^s(\mathbb{R}^N)$ and we set $\|\phi\|_{L^{\alpha,s}(\mathbb{R}^N)} = \|f\|_{L^s(\mathbb{R}^N)}$, where G_{α} is Bessel kernel of order α . The dual space of $L^{s,q}(\mathbb{R}^N)$ is the space $L^{-s,q'}(\mathbb{R}^N)$ and it is naturally endowed with the dual norm. We also denote by $C_{\alpha,s}(E)$ the Bessel capacity of Borel set $E \subset \mathbb{R}^N$ defined by

$$C_{\alpha,s}(E) = \inf\{\|\phi\|_{L^s(\mathbb{R}^N)}^s : \phi \in L^s_+(\mathbb{R}^N), G_{\alpha} * \phi \geq \chi_E\}$$

where χ_E is the characteristic function of E .

From Corollary 2.5, Theorem 2.6 and [2, Theorem 2.3] we can verify the following result.

Theorem 2.8 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. Assume that u is a nonnegative renormalized solution to equation (2.8). If $\mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$ for some $q > p - 1$, then $u \in L^q(\Omega)$ and $\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} = \|G_p * \mu\|_{L^{\frac{q}{q-1}}(\mathbb{R}^N)}$,

$$\|u\|_{L^q(\Omega)} \leq C \left(\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \right) \quad (2.11)$$

for some a positive constant C depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$.

Conversely, if $u \in L^q(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C_K depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$ and $\text{dist}(K, \partial\Omega)$ such that $\chi_K \mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$ and

$$\|\chi_K \mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} \leq C_K \left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \right). \quad (2.12)$$

In particular, for any Borel set $E \subset \Omega$,

$$C_{p, \frac{q}{q+1-p}}(E) = 0 \implies \mu(E) = 0. \quad (2.13)$$

The next statement is proved in [2, Theorem 2.4], and in [6] for a variant.

Theorem 2.9 Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. There exist positive constants c_4, c_5 such that

$$\int_{2B} \exp(c_4 W_{1,p}^R[\mu_B]) \leq c_5 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$.

3 Estimates from below

If G is any domain in \mathbb{R}^N with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7), there always exists a maximal solution to (1.3) in G . It is constructed as the limit, when $n \rightarrow \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 && \text{in } G_n \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty && \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 && \text{if } G_n \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where $\{G_n\}_n$ is a sequence of smooth domains such that $G_n \subset \overline{G_n} \subset G_{n+1}$ for all n , $\{\partial G_n\}_n$ is a bounded and $\bigcup_{n=1}^{\infty} G_n = G$ and $\rho_n(x) := \text{dist}(x, \partial G_n)$. Our main estimates are the following.

Theorem 3.1 Let $K \subset B_{1/4} \setminus \{0\}$ be a compact set and let $U_j \in C_{loc}^1(K^c)$, $j = 1, 2$, be the maximal solutions of

$$-\Delta_p u + e^{\lambda u} + \beta = 0 \quad \text{in } K^c \quad (3.2)$$

for U_1 and

$$-\Delta_p u + \lambda u^q + \beta = 0 \quad \text{in } K^c \quad (3.3)$$

for U_2 , where $p-1 < q < \frac{pq_1}{N}$. Then there exist constants C_k , $k = 1, 2, 3, 4$, depending on N , p and q such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{C_{p, \frac{q_1}{1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

Proof. Step 1. For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [17, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\text{supp}(\mu_j) \subset K \cap S_j$, $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$c^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j$$

Now, we will show that for ε small enough, there holds,

$$A := \int_{B_1} \exp \left(\frac{2N}{p} \tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x) \right) \leq C, \quad (3.6)$$

where $\tilde{c}_3 = \max\{1, 2^{\frac{2-p}{p-1}}\} \lambda c_3$, c_3 is the constant in Theorem 2.6, and C does not depend on J . Indeed, define $\mu_j \equiv 0$ for all $j \geq J+1$ and $j \leq 2$. We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp \left(\frac{2N}{p} \tilde{c}_3 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right).$$

Since for any j

$$\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \leq c_p \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] + c_p \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] + c_p \sum_{k=\max\{j-1, 3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k]$$

with $c_p = \max\{1, 5^{\frac{2-p}{p-1}}\}$ and $\exp(\sum_{i=1}^5 a_i) \leq \sum_{i=1}^5 \exp(5a_i)$ for all a_i . Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x) \right) + \sum_{j=1}^{\infty} \int_{S_j} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) \right) \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1, 3\}}^{j+1} \int_{S_j} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right) := A_1 + A_2 + A_3, \text{ with } c_6 = 5c_p \frac{2N}{p} \tilde{c}_3. \end{aligned}$$

Estimate of A_3 : We apply Theorem 2.9 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp \left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x) \right) \leq c_5 r_{k-1}^N$$

with $c_6 \varepsilon^{\frac{1}{p-1}} \in (0, c_4]$, the constant c_4 is in Theorem 2.9. In particular,

$$\int_{S_j} \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) \leq c_5 r_{k-1}^N \quad k = j-1, j, j+1,$$

which implies

$$A_3 \leq c_7 \sum_{j=1}^{+\infty} r_j^N = c_8 < \infty. \quad (3.7)$$

Estimate of A_1 : Since $\sum_{k \geq j+2} \mu_k(B_t(x)) = 0$ for all $x \in S_j, t \in (0, r_{j+1})$. Thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} \int_{r_{j+1}}^1 \left(\frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right) \\ &\leq \sum_{j=1}^{\infty} \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right) |S_j|. \end{aligned}$$

Note that $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left(\sum_{k \geq j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k \geq 0} r_k^{N-p}\right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$A_1 \leq \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) |B_1| = c_9. \quad (3.8)$$

Estimate of A_2 : for $x \in S_j$,

$$\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Since $r_i < t < r_{i-1}$, $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$, thus

$$\begin{aligned} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k=i-1}^{j-2} \mu_k(S_k)}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \sum_{i=1}^{j-1} \left(\sum_{k=i-1}^{j-2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \leq c_{10} j, \quad \text{with } c_{10} = \left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(c_6 \varepsilon^{\frac{1}{p-1}} c_{10} j\right) = \sum_{j=1}^{\infty} r_j^N \exp\left(c_6 c_{10} \varepsilon^{\frac{1}{p-1}} j\right) |S_1| \\
&= \sum_{j=1}^{\infty} \exp\left(\left(c_6 c_{10} \varepsilon^{\frac{1}{p-1}} - N \log(2)\right) j\right) |S_1| = c_{11} \quad \text{for } \varepsilon \text{ small enough.} \quad (3.9)
\end{aligned}$$

Consequently, $A \leq C := c_9 + c_{11} + c_8$ for ε small enough. This implies

$$\left\| \exp\left(\tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right]\right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1(0))} \leq c_{12} \left(\int_{B_1(0)} \exp\left(\frac{2N}{p} \tilde{c}_3 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right](x)\right) \right)^{\frac{p}{2N}} \leq c_{13} \quad (3.10)$$

where the constant c_{13} does not depend on J . Set $B = B_{\frac{1}{4}}$. For ε small enough, it follows from [2], (3.6) and Theorem 2.6, that there exists a renormalized solution u to equation

$$\begin{aligned}
-\Delta_p u + e^{\lambda u} + \beta &= \varepsilon \sum_{j=3}^J \mu_j && \text{in } B \\
u &= 0 && \text{in } \partial B.
\end{aligned} \quad (3.11)$$

satisfied (2.10) with $\mu = -\beta + 1 + \varepsilon \sum_{j=3}^J \mu_j$ and $u \geq \tilde{u}$ in B where \tilde{u} is a unique solution to equation (3.11) which the right hand side equals zero. By standard regularity theory, $u \in C_{loc}^{1,\alpha}(B \setminus K)$ for some $\alpha \in (0, 1)$. Thus, from Corollary 2.5 and estimate (3.10), we have $\exp(\lambda|u|) \in \mathcal{M}^{\frac{2N}{p}}(B)$,

$$u(0) \geq -c_{14} + c_{15} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right](0).$$

Therefore

$$\begin{aligned}
u(0) &\geq -c_{14} + c_{15} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{14} + c_{15} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= -c_{14} + c_{15} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{14} + c_{16} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&= -c_{14} + c_{17} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}.
\end{aligned}$$

From the inequality

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{N-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i.$$

We deduce that

$$\begin{aligned}
u(0) &\geq -c_{14} + c_{17} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{14} + c_{17} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{18} + c_{19} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\end{aligned}$$

Since U_1 is the maximal solution in K^c , u satisfies the same equation in $B \setminus K$ and $U_1 \geq u = 0$ on ∂B , it follows that U_1 dominates u in $B \setminus K$. Then $U_1(0) \geq u(0)$ and we derive (3.4).

Step 2. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [1, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{q_1} = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

We have, for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k \right)^r \leq \sum_{k=0}^{\infty} \theta_{k,r} a_k^r$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,r} = \begin{cases} 1 & \text{if } r \in (0, 1], \\ \left(\frac{\theta+1}{\theta} \right)^{r-1} (\theta+1)^{kr} & \text{if } r > 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int_{B_1(0)} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} &\leq \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1(0)} \left(\mathbf{W}_{1,p}^1[\mu_k](x) \right)^{q_1} \\
&\leq c_{20} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} \int_{\mathbb{R}^N} (G_p * \mu_k(x))^{\frac{q_1}{p-1}} \\
&= c_{20} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\
&\leq c_{21} \sum_{k=3}^J \theta_{k, \frac{q_1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k \left(N - \frac{pq_1}{q_1-p+1} \right)} \\
&\leq c_{22} \text{ for } \theta \text{ small enough,}
\end{aligned}$$

Here the second inequality follows from [2, Theorem 2.3] and the constant c_{22} does not depend on J . Hence,

$$\left\| \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1(0))} \leq c_{23} \left\| \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1(0))}^q \leq c_{24} \quad (3.12)$$

where c_{23} is independent of J . Take $B = B_{\frac{1}{4}}$. By [2], (3.12), Corollary 2.5 and Theorem 2.6, there exists a renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + \lambda|u|^{q-1}u + \beta &= \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{on } \partial B. \end{aligned} \quad (3.13)$$

It belongs to $C_{loc}^{1,\alpha}(B \setminus K)$ for some $\alpha \in (0, 1)$ and

$$u(0) \geq -c_{25} + c_{26} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{27} + c_{28} \int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have $U_2(0) \geq u(0)$. Therefore, we get (3.5). \square

4 Proof of the main results

Here we only prove Theorem 1. Let $u \in C_{loc}^1(\Omega)$ be the maximal solution of

$$-\Delta_p u + e^{\lambda u} + \beta = 0 \quad \text{in } \Omega \quad (4.1)$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B_\delta} \cap \Omega$. Set $K = \Omega^c \cap \overline{B_{1/4}}(z_0)$. Let $U_1 \in C^1(K^c)$ be the maximal solution of (3.2). We have $u \geq U_1$ in Ω . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \end{aligned}$$

We deduce

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

Therefore, u is satisfied (1.13). \square

5 Large solutions of quasilinear Hamilton-Jacobi equations

In this section we used our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + a |\nabla u|^q + bu^s &= 0 & \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \quad (5.1)$$

where $a > 0, b \geq 0$ and $1 < p \leq 2, 1 \leq q < p, s \geq p - 1$ such that $(q - p + 1) + b(s - p + 1) > 0$. First we have the result of existence solutions to equation (5.1).

Proposition 5.1 *Let $a > 0, b \geq 0$ and $s \geq p - 1, 1 \leq q \leq p, (q - p + 1) + b(s - p + 1) > 0$ and $1 < p \leq 2$. There exists a maximal solution $u \in C_{loc}^1(\Omega)$ to equation*

$$\begin{aligned} -\Delta_p u + a |\nabla u|^p + bu^s &= 0 & \text{in } \Omega \\ 0 < u &\in C_{loc}^1(\Omega) \end{aligned} \quad (5.2)$$

satisfied

$$u(x) \leq c(N, p, s) b^{-\frac{1}{s-p+1}} d(x, \partial\Omega)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega \quad (5.3)$$

if $b(s - p + 1) > 0$ and

$$u(x) \leq c(N, p, q) a^{-\frac{1}{q-p+1}} d(x, \partial\Omega)^{-\frac{p-q}{q-p+1}} \quad \forall x \in \Omega \quad (5.4)$$

if $p - 1 < q < p$ and

$$u(x) \leq c(N, p) a^{-1} \log(2 \text{diam}(\Omega) d(x, \partial\Omega)^{-1}) \quad \forall x \in \Omega \quad (5.5)$$

if $q = p$.

Proof. Direct calculations show that the function

$$U_1(x) = c_1 a^{-\frac{1}{q-p+1}} \left(\frac{R^{\beta_1} - |x|^{\beta_1}}{\beta_1 R^{\beta_1-1}} \right)^{-\frac{p-q}{q-p+1}} \in C^1(B_R(0))$$

with $a > 0$ and $p - 1 < q < p$ satisfies

$$-\Delta_p U_1 + a |\nabla U_1|^q \geq 0 \quad \text{in } B_R(0), \quad (5.6)$$

the function

$$U_2(x) = c_2 a^{-1} \log \left(\frac{\beta_2 R^{\beta_2}}{R^{\beta_2} - |x|^{\beta_2}} \right) \in C_+^1(B_R(0))$$

with $a > 0$ satisfies

$$-\Delta_p U_2 + a |\nabla U_2|^p \geq 0 \quad \text{in } B_R(0) \quad (5.7)$$

and the function

$$U_3(x) = c_3 b^{-\frac{1}{s-p+1}} \left(\frac{R^{\beta_3} - |x|^{\beta_3}}{\beta_3 R^{\beta_3-1}} \right)^{-\frac{p}{s-p+1}} \in C^1(B_R(0))$$

with $b > 0$ and $s > p - 1$ satisfies

$$-\Delta_p U_3 + b U_3^s \geq 0 \quad \text{in } B_R(0) \quad (5.8)$$

for some positive constants $c_1 = c_1(N, p, q)$, $c_2 = c_2(N, p)$, $c_3 = c_3(N, p, s)$ and $\beta_1 = \beta_1(N, p, q) \geq 1$, $\beta_2 = \beta_2(N, p) \geq 1$, $\beta_3 = \beta_3(N, p, s) \geq 1$.

We emphasize the condition $1 < p \leq 2$ and $q \geq 1$ so that the equation (5.2) admits the Comparison Principle, see [16, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains Ω_n satisfied $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. For each $n, k \in \mathbb{N}^*$, there exist nonnegative

solution $u_{n,k} = u \in W_k^{1,p}(\Omega_n) = W_0^{1,p}(\Omega_n) + k$ of equation (5.2) in Ω_n .

Since $-\Delta_p u_{k,n} \leq 0$ in Ω_n , so using the Maximum Principle we get $u_{n,k} \leq k$ in Ω_n for all n . Thus, by standard regularity (see [4] and [11]), $u_{n,k} \in C^{1,\alpha}(\overline{\Omega}_n)$ for some $\alpha \in (0, 1)$. After, using the comparison Principle and (5.6)-(5.8), we obtain that $u_{n,k} \leq u_{n,k+1}$ in Ω_n and (5.3)-(5.5) are satisfied with $u_{n,k}, \Omega_n$ replacing u, Ω respectively. From this, we lead to uniform local bounds for $\{u_{n,k}\}_k$. Thus, by standard interior regularity (see [4]) we obtain uniform local bounds for $\{u_{n,k}\}_k$ in $C_{loc}^{1,\eta}(\Omega_n)$. It implies that $\{u_{n,k}\}_k$ is pre-compact in C^1 and hence up to subsequence $u_{n,k} \rightarrow u_n$ in $C_{loc}^1(\Omega_n)$. Hence, we can verify that u_n is a solution of (5.2) and satisfies (5.3)-(5.5) with u_n, Ω_n replacing u, Ω and $u_n(x) \rightarrow \infty$ as $d(x, \Omega_n) \rightarrow 0$.

Next, since $u_{n,k} \geq u_{n+1,k}$ in Ω_n thus $u_n \geq u_{n+1}$ in Ω_n . In particular, $\{u_n\}$ is uniform local bounded in Ω . We can argue as above, to obtain $u_n \rightarrow u$ in $C_{loc}^1(\Omega)$, u is a solution of (5.2) in Ω and satisfies (5.3)-(5.5). Clearly, u is a maximal solution of (5.2). \blacksquare

Lemma 5.2 *The maximal solution of (4.1) is a large solution with $\lambda = 1$ and $\beta = -1$ if and only if for any $a > 0$ and $b < b_a := \theta_1 a^{1-p}$ the maximal solution of*

$$-\Delta_p v + e^{av} + b = 0 \quad \text{in } \Omega, \quad (5.9)$$

is a large solution, where θ_1 is a positive constant depending on N, p and Ω .

Proof. Since monotonicity and Vazquez' condition (1.3) hold, it is sufficient to exhibit a large subsolution (i.e. tending to infinity on the boundary) in order to conclude on the existence of a large solution to (5.9).

Assume $u := u_{1,-1}$ is a large solution of (4.1), then for any $\Lambda \geq 1$

$$-\Delta_p u_{1,-1} + e^{u_{1,-1}} - \Lambda = 1 - \Lambda \leq 0 \quad \text{in } \Omega,$$

thus $u_{1,-1}$ is a subsolution of the corresponding solution and there exists a larger solution which is necessarily a large solution $u_{1,-\Lambda}$ of

$$-\Delta_p u + e^u - \Lambda = 0 \quad \text{in } \Omega. \quad (5.10)$$

Set $\min\{u_{1,-1}(x) : x \in \Omega\} = \theta > 0$. then, for any $c \in (0, 1)$ and $d \geq 0$ there holds

$$e^{u_{1,-1}} - 1 \geq m_\theta e^{cu_{1,-1}} \geq m_\theta e^{cu_{1,-1}} - d \quad \text{on } [\theta, \infty)$$

with $m_\theta = e^{(1-c)\theta} - e^{-c\theta}$. This implies that $-\Delta_p u_{1,-1} + m_\theta e^{cu_{1,-1}} - d \leq 0$, therefore $v := u_{1,-1} + c^{-1} \ln m_\theta$ satisfies $-\Delta_p v + e^{cv} - d \leq 0$. Therefore there exists a large solution $u_{c,-d}$ to

$$-\Delta_p u + e^{cu} - d = 0 \quad \text{in } \Omega. \quad (5.11)$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, set $u_{c,-d} = \alpha w + \beta$, then $-\Delta_p w + \alpha^{1-p} e^{\beta c} e^{\alpha c w} - d \alpha^{1-p} = 0$. If we take $\beta = \frac{p-1}{c} \ln \alpha$, then

$$-\Delta_p w + e^{\alpha c w} - d \alpha^{1-p} = 0 \quad \text{in } \Omega. \quad (5.12)$$

Since $\alpha > 0$ and $d \geq 0$ are arbitrary, we see that for any $a > 0$ and $b \geq 0$, there exists a large solution $u = u_{a,-b}$ to

$$-\Delta_p u + e^{au} - b = 0 \quad \text{in } \Omega. \quad (5.13)$$

We can notice that since $u_{a,0} = a^{-1}u_{1,0} + (1-p)a^{-1} \ln a$, the minimum $\theta := \theta_a$ of $u_{a,0}$ satisfies $\theta_a = a^{-1}\theta_1 + (1-p)a^{-1} \ln a$. For $\epsilon > 0$, there holds

$$e^{au_{a,0}} - \epsilon e^{au_{a,0}} = (1-\epsilon)e^{au_{a,0}} \geq (1-\epsilon)\theta_1 a^{1-p}.$$

Therefore $-\Delta_p u_{a,0} + \epsilon e^{au_{a,0}} + (1-\epsilon)\theta_1 a^{1-p} \leq 0$. Thus $v = u_{a,0} - a^{-1} \ln \epsilon$ satisfies

$$-\Delta_p v + e^{av} + (1-\epsilon)\theta_1 a^{1-p} \leq 0$$

which implies that there exists a large solution to the corresponding equation. Since ϵ is arbitrary, it follows that for any $b < b_a := \theta_1 a^{1-p}$, there exists a large solution $u = u_{a,b}$ to (5.9).

Remark: the constant θ only depends on N, p and $\text{diam}(\Omega)$. In fact, we can see that $\theta \geq \min_{B_{\text{diam}(\Omega)}(0)} U$, where U is a unique solution of $-\Delta_p U + e^U - 1 = 0$ in $B_{\text{diam}(\Omega)}(0)$ and $U(x) \rightarrow \infty$ as $|x| \uparrow \text{diam}(\Omega)$ \square

Theorem 5.3 *Assume $p-1 < q < p$ and (1.12) holds. Then there exists $b^* = b^*(p, q, N, \Omega) > 0$ such that for any $b \in (-\infty, b^*)$, problem (5.1) admits a solution.*

Proof. If (1.12) holds, for any $a > 0$ and $b < b_a$, there exists a large solution u to (5.9). We set $u = \alpha \ln w$ with $\alpha > 0$, then

$$-\Delta_p w + (p-1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha a + p-1} + b \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega. \quad (5.14)$$

By Hölder's inequality

$$(p-1) \frac{|\nabla w|^p}{w} \geq |\nabla w|^q - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}},$$

therefore

$$-\Delta_p w + |\nabla w|^q + \alpha^{1-p} w^{\alpha a + p-1} + b \alpha^{1-p} w^{p-1} - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}} \leq 0.$$

Since $q > p-1$, $\frac{q}{p-q} > p-1$. We choose α and a such that

$$\alpha a + p - 1 = \frac{q}{p-q} \quad \text{and} \quad \alpha^{1-p} = \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}}.$$

Therefore w satisfies

$$-\Delta_p w + |\nabla w|^q + b \alpha^{1-p} w^{p-1} \leq 0.$$

This implies that there exists a large solution to (5.1).

Remark: According to [16], we need $1 < p \leq 2$, $q \geq 1$ and $b \geq 0$ in order that the Comparison Principle is applied to the equation $-\Delta_p u + a|\nabla u|^q + bu^s = 0$. \square

Theorem 5.4 *Assume that the equation (1.13) admits a solution for some $\lambda > 0$ and $\beta \in \mathbb{R}$. Then for any $a, b \geq 0$ and $q, s \geq p - 1$, $1 \leq q < p$, $a(q - p + 1) + b(s - p + 1) > 0$ and we will add $1 < p \leq 2$ if $a > 0$, the equation (5.2) has a large solution and satisfied (5.3),(5.4).*

Proof. Assume that the equation (1.13) admits a solution v for some $\lambda > 0$ and $\beta \in \mathbb{R}$. Here we only need to prove for case $a(q - p + 1) > 0$. We set $v = \sigma \ln w$ with $\sigma > 0$, then $w > 0$ and

$$-\Delta_p w + (p - 1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha\lambda+p-1} + \beta \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega.$$

Choose $\sigma = \frac{s-p+2}{\lambda}$, we can see that

$$(p - 1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha\lambda+p-1} + \beta \alpha^{1-p} w^{p-1} \geq a |\nabla w|^q + b w^s \quad \text{in } \{x : w(x) \geq M\},$$

where a positive constant M depends on $p, q, s, \lambda, \beta, a, b$. Therefore

$$-\Delta_p w + a |\nabla w|^q + b w^s \leq 0 \quad \text{in } \{x : w(x) \geq M\}.$$

Now we take an open subset Ω' of Ω with $\overline{\Omega'} \subset \Omega$ such that the set $\{x : w(x) \geq M\}$ contains in $\Omega \setminus \overline{\Omega'}$. So w is a subsolution of $-\Delta_p u + a |\nabla u|^q + b u^s = 0$ in $\Omega \setminus \overline{\Omega'}$ and $w_\varepsilon := \varepsilon w$ is too for any $\varepsilon \in (0, 1)$. Let u be as in Proposition 5.1. Set $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$ and $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$. So, we have $w_\varepsilon < u$ on $\partial\Omega'$ with $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$. Hence, from the construction of u in the proof of Proposition 5.1 and the Comparison Principle we assert $w_\varepsilon \leq u$ in $\Omega \setminus \overline{\Omega'}$. Therefore, the Theorem follows. \blacksquare

Remark 5.5 *From the proof of above Theorem, we can show that under the assumption as in Proposition 5.1, the equation (5.2) has a large solution in Ω if and only if the equation (5.2) has a large solution in $\Omega \setminus K$ for some a compact set $K \subset \Omega$ with smooth boundary.*

Now we concern (5.1) in case $q = p$.

Theorem 5.6 *Assume that the equation (5.2) has a large solution in Ω for some $a, b > 0$, $s > p - 1$ and $q = p$. Then for any $a_1, b_1 \geq 0$ and $q_1, s_1 \geq p - 1$, $1 \leq q_1 \leq p$, $a_1(q_1 - p + 1) + b_1(s_1 - p + 1) > 0$, the equation (5.2) also has a large solution in Ω with a_1, b_1, q_1, s_1 replacing a, b, q, s respectively, and satisfied (5.3)-(5.5). Here if $a_2 > 0$, we add the condition $1 < p \leq 2$.*

Proof. For $\sigma > 0$ we set $u = v^\sigma$ thus

$$-\Delta_p v - (\sigma - 1)(p - 1) \frac{|\nabla v|^p}{v} + a \sigma v^{\sigma-1} |\nabla v|^p + b \sigma^{-p+1} v^{(s-p+1)\sigma+p-1} = 0.$$

Choose $\sigma = \frac{s_1-p+1}{s-p+1} + 2$, it is easy to see that

$$-\Delta_p v + a_1 |\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \quad \text{in } \{x : v(x) \geq M\},$$

for some a positive constant M only depending on $p, s, a, b, a_1, b_1, q_1, s_1$.

Similar to the proof of Theorem 5.4, we get the result as desired. \square

Remark 5.7 *If we set $u = e^v$ then v satisfies*

$$-\Delta_p v + b e^{(s-p+1)v} = |\nabla v|^p (p - 1 - a e^v) \quad \text{in } \Omega. \quad (5.15)$$

From this, we can construct a large solution of

$$-\Delta_p u + b e^{(s-p+1)u} = 0 \quad \text{in } \Omega \setminus K. \quad (5.16)$$

for any a compact set $K \subset \Omega$ with smooth boundary such that $v \geq \ln\left(\frac{p-1}{a}\right)$ in $\Omega \setminus K$. In case $p = 2$, It would be interesting to see what Wiener type criterion the existence as such a large solution implies. We conjecture that this condition is

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c) dr}{r^{N-2}} = \infty \quad \forall x \in \partial\Omega. \quad (5.17)$$

We now consider the function

$$U_4(x) = c \left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}} \right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0 \quad (5.18)$$

It is easy to see that $\Delta_p U_4 + U_4^{-\gamma} \geq 0$ for some positive constants β large and c small enough. From this, we get the existence of minimal solution to equation $\Delta_p u + u^{-\gamma} = 0$ in Ω .

Proposition 5.8 For $\gamma > 0$, there exists a minimal solution $u \in C_{loc}^1(\Omega)$ to equation $\Delta_p u + u^{-\gamma} = 0$ in Ω and satisfied $u(x) \geq Cd(x, \Omega)^{\frac{p}{\gamma+p-1}}$ in Ω .

We can verify that if the boundary of Ω is satisfied (1.3), then above minimal solution u is a singular solution this means $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

Theorem 5.9 Let $\gamma > 0$. Assume that there exists a singular solution to equation $\Delta_p u + u^{-\gamma} = 0$ in Ω . Then for any $a, b \geq 0$ and $q, s \geq p-1$, $1 \leq q \leq p$, $a(q-p+1) + b(s-p+1) > 0$, the equation (5.2) has a large solution in Ω and satisfied (5.3)-(5.5). Here if $a > 0$, we add the condition $1 < p \leq 2$.

Proof. We set $u = e^{-\frac{a}{p-1}v}$, then v is a large solution of

$$-\Delta_p v + a |\nabla v|^p + \left(\frac{p-1}{a}\right)^{p-1} e^{-\frac{a}{p-1}(\gamma+p-1)v} = 0 \quad \text{in } \Omega \quad (5.19)$$

So

$$-\Delta_p v + a |\nabla v|^q + bv^s \leq 0 \quad \text{in } \{x : v(x) \geq M\}$$

for some a positive constant M only depending on p, q, s, a, b, γ .

Similar to the proof of Theorem 5.4, we get the result as desired. \square

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