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Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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Abstract

We obtain necessary conditions expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities for the existence of large solutions to equations (1) $-\Delta_p u + e^u - 1 = 0$ or (2) $-\Delta_p u + u^q = 0$ in a bounded domain Ω when $q > p - 1 > 0$. We apply our results to equations (3) $-\Delta_p u + |\nabla u|^q + bu^{p-1} = 0$, (4) $-\Delta u + |\nabla u|^2 + u^q = 0$ and $-\Delta u + u^{-q} = 0$ with $q > 0$.

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Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) and $1 < p \leq N$. We consider the question of existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and g is a continuous nondecreasing function vanishing at 0; most often $g(u)$ is either $e^u - 1$ or $|u|^{q-1} u$ with $q > p - 1$. A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data ϕ

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ u &= \phi && \text{in } \partial\Omega \end{aligned} \tag{1.2}$$

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admits a solution, it is clear that problem (1.1) admits a solution. It is known that a necessary and sufficient condition for such a result is the so called Wiener criterion (for $p = 2$ see [17]), for $p \neq 2$ see [10], [6])

$$\int_0^1 \left(\frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$: the existence of a large solution is guaranteed for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition [14]

$$\int_a^\infty \frac{dt}{G^{\frac{1}{p}}(t)} < \infty \quad G(t) = \int_0^t g(s)ds, \quad (1.4)$$

a variant of the Keller-Osserman estimate [7], [12], which is the above relation when $p = 2$. If for $R > \text{diam}(\Omega)$ there exists a function v which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\} \\ v &= 0 && \text{on } \partial B_R \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of $\partial\Omega$, provided (1.4) is satisfied. However the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron proved [15] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p-1 < q < \frac{N(p-1)}{N-p}$, Friedman and Véron [4] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [8] proved that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.9)$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.10)$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. Concerning the exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = e^u - 1$ and $g(u) = |u|^{q-1}u$, $q > p - 1$. For $1 < p < N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\} \quad (1.11)$$

where the B_j are balls and $h^{N-p}(B_r) = c_N r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 2$ and $1 < p < N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.12)$$

then there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^u - 1 &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.13)$$

As a consequence we obtained a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{s,r}$ Bessel capacity in \mathbb{R}^N associated to the Besov space $B^{s,r}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 2$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.14)$$

then, for any $p-1 < q < \frac{pq_1}{N}$, there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + u^q &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.15)$$

In view of Labutin's theorem this last result is not optimal in the case $p = 2$, since the involved capacity is C_{2,q'_1} with q'_1 and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.16)$$

with $q_1 > q$.

At end we apply the previous theorems to quasilinear viscous Hamilton-Jacobi equations. We prove that if $p-1 < q < p$, $b < b^*$ for some $b^* > 0$ depending on p, q, Ω , and (1.12) holds, there exists a solution to

$$\begin{aligned} -\Delta_p u + |\nabla u|^q + bu^{p-1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.17)$$

Conversely, we prove that if for some $q > 1$, there exists a solution to

$$\begin{aligned} -\Delta u + |\nabla u|^2 + |u|^{q-1}u &= 0 & \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \quad (1.18)$$

then necessarily

$$\int_0^1 \frac{C_{2,s}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.19)$$

for all $s > 1$. This condition holds also if for some $p > 0$ there exists $u \in C(\overline{\Omega})$, $u > 0$ in Ω satisfying

$$\begin{aligned} -\Delta u + u^{-p} &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (1.20)$$

2 Morrey classes and Wolff potential estimates

In this section Ω is a bounded domain in \mathbb{R}^N . We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (1.4) belongs to $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 2.1 1- A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_{\Omega \cap B_r(x)} |f| dy \leq K r^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N \quad (2.1)$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{\mathcal{M}^s(\Omega)}$.

2- A function $f \in L^1(\Omega)$ belongs to the weak L^s -space $M^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_E |f| dy \leq K |E|^{\frac{1}{s}} \quad \forall E \subset \Omega, E \text{ Borel}. \quad (2.2)$$

The quasi-norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{M^s(\Omega)}$

Clearly $L^p(\Omega) \subset M^p(\Omega) \subset \mathcal{M}^p(\Omega)$.

Definition 2.2 Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}_+(\Omega)$, the set of positive Radon measures in Ω . If $\alpha > 0$ and $1 < p < \alpha^{-1}N$, we define the (R -truncated) Wolff potential of μ by

$$\mathbf{W}_p^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N, \quad (2.3)$$

and, for $1 < p < N$, the (R -truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N, \quad (2.4)$$

where the measure is extended by 0 in Ω^c .

For $k \geq 0$, we set $T_k(u) = \text{sign}(u) \min\{k, |u|\}$.

Definition 2.3 Assume $f \in L^1_{loc}(\Omega)$. We say that a Borel function u defined in Ω is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.5)$$

if for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0 \quad (\text{resp. } \leq 0) \quad (2.6)$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f := \mu$ is a positive (resp. negative) distribution in Ω .

The following result is proved in [11, Theorem 4.35].

Theorem 2.4 Let Ω be an open bounded domain in \mathbb{R}^N . If $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ for some $\varepsilon \in (0, p)$, u is a nonnegative supersolution of (2.5) and set $\mu := -\Delta_p u + f$. Then there holds

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_1 W_{1,p}^{\frac{r}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega, \quad (2.7)$$

for some c_1 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u = f + \mu \quad \text{in } \Omega \quad (2.8)$$

where $f \in L^1_{loc}(\Omega)$ and μ is a Radon measure, we have

Corollary 2.5 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of positive and bounded Radon measures in Ω . If u is a nonnegative renormalized solution to (2.8), then there exists a positive constant c_2 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$ such that

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_2 W_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega. \quad (2.9)$$

Definition 2.6 For $1 \leq s, q < \infty$, let $L^{s,q}(\Omega)$ denote the Lorentz space endowed with the norm

$$\|f\|_{L^{s,q}} = \left(\int_0^\infty t^{\frac{q}{s}} (f^{**}(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad (2.10)$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(t) dt$$

with $f^*(t) = \inf\{s > 0 : |\{x \in \Omega : |f(x)| > s\}| \leq t\}$. The dual space of $L^{s,q}(\Omega)$ is the space $L^{-s,q'}(\Omega)$ and it is naturally endowed with the dual norm.

The following result is proved in [2].

Theorem 2.7 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. Assume that u is a nonnegative renormalized solution to equation (2.13). If $\mu \in L^{-p, \frac{q}{p-1}}$ for some $q > p - 1$, then $u \in L^q(\Omega)$ and

$$\|u\|_{L^q(\Omega)} \leq C \left(\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} + \|f\|_{\mathcal{M}^{\frac{1}{p-1}}(\Omega)} \right) \quad (2.11)$$

for some a positive constant C depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$.

Conversely, if $u \in L^q(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C_K depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$ and $\text{dist}(K, \partial\Omega)$ such that $\chi_K \mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$ and

$$\|\chi_K \mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} \leq C_K \left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{M}^{\frac{1}{p-1}}(\Omega)} \right). \quad (2.12)$$

In particular, for any Borel set $E \subset \Omega$,

$$C_{p, \frac{q}{q+1-p}}(E) = 0 \implies \mu(E) = 0. \quad (2.13)$$

We recall [2, Theorem 3.8].

Theorem 2.8 There exists a positive constant c_1 such that if u is a renormalized solution to $-\Delta_p u = \mu$ in Ω and $u = 0$ on $\partial\Omega$, then for any $x \in \Omega$

$$|u(x)| \leq c_1 W_{1,p}^{2\text{diam}(\Omega)}[\mu](x).$$

The next statement is proved in [2, Theorem 2.4], and in [5] for a variant.

Theorem 2.9 There exist positive constants c_2, c_3 such that

$$\int_{2B} \exp(c_2 W_{1,p}^R[\mu_B]) \leq c_3 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$.

3 Estimates from below

If G is any domain in \mathbb{R}^N with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7), there always exists a maximal solution to (1.4) in G . It is constructed as the limit, when $n \rightarrow \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 & \text{in } G_n := \{x \in \mathbb{R}^N : \text{dist}(x, G^c) > \frac{1}{n}\} \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 & \text{if } G \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where $\rho_n(x) := \text{dist}(x, \partial\Omega_n)$. Our main estimates are the following.

Theorem 3.1 Let $K \subset B_{1/4} \setminus \{0\}$ be a compact set and let $U_j \in C_{loc}^1(K^c)$, $j = 1, 2$, be the maximal solutions of

$$-\Delta_p u + e^u - 1 = 0 \quad \text{in } K^c \quad (3.2)$$

for U_1 and

$$-\Delta_p u + u^q = 0 \quad \text{in } K^c \quad (3.3)$$

for U_2 , where $p-1 < q < \frac{2q_1}{N}$. Then there exist constants C_k , $k = 1, 2, 3, 4$, depending on N , p and q such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{C_{p, \frac{q_1}{41-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

Proof. Step 1. For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [13, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\text{supp}(\mu_j) \subset K \cap S_j$,

$$c^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j$$

and

$$\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} = 1.$$

Now, we will show that for ε small enough, there holds,

$$\int_{B_1} \exp\left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x)\right) dx \leq C, \quad (3.6)$$

where c_1 is the constant in Theorem 2.8, and C does not depend on J .

Indeed, we have

$$A := \int_{B_1} \exp\left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x)\right) dx = \sum_{j=1}^{\infty} \int_{S_j} \exp\left(\frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x)\right) dx.$$

Since

$$\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \leq \max\{1, 5^{\frac{2-p}{p-1}}\} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x) + \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) + \sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k](x) \right)$$

and

$$\exp\left(\sum_{i=1}^5 a_i\right) \leq \sum_{i=1}^5 \exp(5a_i) \quad \forall a_i.$$

Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x)\right) dx \\ &\quad + \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x)\right) dx \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1,3\}}^{j+1} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

Estimate of A_3 : We apply Theorem 2.9 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \leq c_3 r_k^N$$

with $5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} K \varepsilon^{\frac{1}{p-1}} \in (0, c_2]$. In particular,

$$\int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} K \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \leq 4c_3 r_k^N \quad k = j-1, j, j+1.$$

Which implies

$$A_3 \leq c_4 \sum_{j=1}^{+\infty} 4c_3 r_j^N = c_5 < \infty. \quad (3.7)$$

Estimate of A_1 : Since

$$\sum_{k \geq j+2} \mu_k(B_t(x)) = 0 \quad \forall x \in S_j, t \in (0, r_{j+1}),$$

thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \int_{r_{j+1}}^1 \left(\frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right) dx \\ &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right) dx. \end{aligned}$$

Note that $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left(\sum_{k \geq j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k \geq 0} r_k^{N-p}\right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$\begin{aligned} A_1 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) dx \\ &= \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) |B_1| = c_6 \end{aligned}$$

Estimate of A_2 : for $x \in S_j$,

$$\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Since $r_i < t < r_{i-1}$, $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$, thus

$$\begin{aligned}
\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{i-1 \leq k \leq j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{i-1 \leq k \leq j-2} \mu_k(S_k)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \sum_{i=1}^{j-1} \left(\sum_{i-1 \leq k \leq j-2} \mu_k(S_k) \right)^{\frac{1}{p-1}} \frac{p-1}{N-p} r_i^{-\frac{N-p}{p-1}} \\
&\leq \sum_{i=1}^{j-1} \left(\sum_{i-1 \leq k \leq j-2} r_{k-1}^{N-p} \right)^{\frac{1}{p-1}} \frac{p-1}{N-p} r_i^{-\frac{N-p}{p-1}} \\
&\leq \frac{p-1}{N-p} \sum_{i=1}^{j-1} \left(\sum_{i-1 \leq k \leq j-2} r_{k-1-i}^{N-p} \right)^{\frac{1}{p-1}} \\
&\leq \frac{p-1}{N-p} \sum_{i=1}^{j-1} \left(\sum_{k \geq i-1} r_{k-1-i}^{N-p} \right)^{\frac{1}{p-1}} \\
&= \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} (j-1) \\
&\leq \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp \left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} j \right) dx \\
&= \sum_{j=1}^{\infty} c_7 r_j^N \exp \left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} j \right) \\
&= \sum_{j=1}^{\infty} c_7 \exp \left(\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} - N \log(2) \right) j \right) \\
&= c_8 \quad \text{for } \varepsilon \text{ small enough.}
\end{aligned}$$

Consequently, $A \leq C := c_6 + c_8 + c_5$ for ε small enough. This implies

$$\left\| \exp \left(c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] \right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1(0))} \leq c_{10} \left(\int_{B_1(0)} \exp \left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x) \right) dx \right)^{\frac{p}{2N}} \leq c_{11} \quad (3.8)$$

where the constant c_{11} does not depend on J . Set $B = B_{\frac{1}{4}}$. For ε small enough, it follows from [2], 3.6 and Theorem 2.8, that there exists a renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \varepsilon \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{in } \partial B. \end{aligned} \quad (3.9)$$

By standard regularity theory, $u \in C_{loc}^{1,\alpha}(B \setminus K)$. From Corollary 2.5 and estimate (3.8), we have

$$u(0) \geq -c_{12} + c_{13} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

Therefore

$$\begin{aligned} u(0) &\geq -c_{12} + c_{13} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0) = -c_{12} + c_{13} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\geq -c_{12} + c_{13} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = -c_{12} + c_{13} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\geq -c_{12} + c_{14} \sum_{i=2}^{J-2} (\mu_{i+2}(S_{i+2}))^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} = -c_{12} + c_{14} \sum_{i=2}^{J-2} (\mu_{i+2}(S_{i+2}))^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{12} + c_{15} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} = -c_{12} + c_{16} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}. \end{aligned}$$

Note that

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i.$$

Therefore,

$$\begin{aligned} u(0) &\geq -c_{12} + c_{16} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\ &= -c_{12} + c_{16} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} - \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \right) \\ &= -c_{12} + c_{16} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} r_{i-1}^{-\frac{N-p}{p-1}} - \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \right) \\ &\geq -c_{12} + c_{16} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{17} + c_{18} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}. \end{aligned}$$

Since U_1 is the maximal solution in K^c , u satisfies the same equation in $B \setminus K$ and $U_1 \geq u = 0$ on ∂B , it follows that U_1 dominates u in $B \setminus K$. Then $U_1(0) \geq u(0)$ and we derive (3.4).

Step 2. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [13, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{q_1} dx = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

We have, for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k \right)^r \leq \sum_{k=0}^{\infty} \theta_{k,r} a_k^r$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,r} = \begin{cases} 1 & \text{if } r \in (0, 1] \\ \left(\frac{\theta+1}{\theta}\right)^{r-1} (\theta+1)^{kr} & \text{if } r > 1. \end{cases}$$

Thus,

$$\begin{aligned} \int_{B_1(0)} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} dx &\leq \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1(0)} (\mathbf{W}_{1,p}^1[\mu_k](x))^{q_1} dx \\ &\leq \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_R (\mathbf{W}_{1,p}^1[\mu_k](x))^{q_1} dx \\ &\leq c_{19} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_R (G_p[\mu_k](x))^{q_1} dx \\ &= c_{19} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\ &\leq c_{20} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k} \left(N - \frac{pq_1}{q_1-p+1} \right) \\ &\leq c_{21} \quad \text{for } \theta \text{ small enough,} \end{aligned}$$

where the constant c_{21} does not depend on J . Hence,

$$\left\| \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1(0))} \leq c_{22} \left\| \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1(0))} \leq c_{23} \quad (3.10)$$

where c_{23} is independent of J . Take $B = B_{\frac{1}{4}}$. Note that $\frac{q_1}{q} > \frac{N}{p}$. By [2], 3.10, Corollary 2.5 and Theorem 2.8, there exists a renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + u^q &= \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{on } \partial B. \end{aligned} \quad (3.11)$$

It belongs to $C_{loc}^{1,\alpha}(B \setminus K)$ and

$$u(0) \geq -c_{24} + c_{25} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{26} + c_{27} \int_0^1 \left(\frac{C_{p, \frac{q_1}{4_1 - p + 1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Since U_2 is the maximal solution in U_2 in $B \setminus K$, it dominates the solution u in $B \setminus K$, and thus $U_2(0) \geq u(0)$. Therefore, we get (3.5). \square

4 Proof of the main results

Proof of Theorem 1. Let u be the maximal solution of

$$-\Delta_p u + e^u - 1 = 0 \quad \text{in } \Omega \quad (4.1)$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B}_{1/4}(z_0)$. Let $U_1 \in C^1(K^c)$ be the maximal solution of (3.2). We have $u \geq U_1$ in Ω . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \end{aligned}$$

Therefore,

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

\square

Proof of Theorem 2. Let u be the maximal solution to

$$-\Delta_p u + u^q = 0 \quad \text{in } \Omega. \quad (4.2)$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B}_{1/4}(z_0)$. Let $U_2 \in C^1(K^c)$ be the maximal solution of (3.3). We have $u \geq U_2$

in Ω . By Theorem 3.1,

$$\begin{aligned}
U_1(z_0) &\geq -C_1 + C_2 \int_{\delta}^1 \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -C_1 + C_2 \int_{\delta}^1 \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\
&\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -C_1 + C_2' \int_{\delta}^{1/2} \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.
\end{aligned}$$

Therefore,

$$\inf_{B_{\delta} \cap \Omega} u \geq \inf_{B_{\delta} \cap \Omega} U \geq -C_1 + C_2' \int_{\delta}^{1/2} \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

□

5 Large solutions of quasilinear Hamilton-Jacobi equations

In this section we used our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned}
-\Delta_p u + |\nabla u|^q + bu^{p-1} &= 0 & \text{in } \Omega \\
\lim_{\rho(x) \rightarrow 0} u(x) &= \infty,
\end{aligned} \tag{5.1}$$

where b is a real number and $p-1 < q < p$.

Lemma 5.1 *The maximal solution of (4.1) is a large solution if and only if for any $a > 0$ and $b < b_a := \theta_1 a^{1-p}$ the maximal solution of*

$$-\Delta_p v + e^{av} + b = 0 \quad \text{in } \Omega, \tag{5.2}$$

is a large solution, where θ_1 is a positive constant depending on N, p and Ω .

Proof. Since monotonicity and Vazquez' condition (1.4) hold, it is sufficient to exhibit a large subsolution (i.e. tending to infinity on the boundary) in order to conclude on the existence of a large solution to (5.2).

Assume $u := u_{1,-1}$ is a large solution of (4.1), then for any $\Lambda \geq 1$

$$-\Delta_p u_{1,-1} + e^{u_{1,-1}} - \Lambda = 1 - \Lambda \leq 0 \quad \text{in } \Omega,$$

thus $u_{1,-1}$ is a subsolution of the corresponding solution and there exists a larger solution which is necessarily a large solution $u_{1,-\Lambda}$ of

$$-\Delta_p u + e^u - \Lambda = 0 \quad \text{in } \Omega. \tag{5.3}$$

Set $\min\{u_{1,1}(x) : x \in \Omega\} = \theta > 0$. then, for any $c \in (0, 1)$ and $d \geq 0$ there holds

$$e^{u_{1,-1}} - 1 \geq m_{\theta} e^{cu_{1,-1}} \geq m_{\theta} e^{cu_{1,-1}} - d \quad \text{on } [\theta, \infty)$$

with $m_\theta = e^{(1-c)\theta} - e^{-c\theta}$. This implies that $-\Delta_p u_{1,-1} + m_\theta e^{cu_{1,-1}} - d \leq 0$, therefore $v := u_{1,-1} + c^{-1} \ln m_\theta$ satisfies $-\Delta_p u_{1,-1} + e^{cu_{1,-1}} - d \leq 0$. Therefore there exists a large solution $u_{c,-d}$ to

$$-\Delta_p u + e^{cu} - d = 0 \quad \text{in } \Omega. \quad (5.4)$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, set $u_{c,-d} = \alpha w + \beta$, then $-\Delta_p w + \alpha^{1-p} e^{\beta c} e^{\alpha c w} - d \alpha^{1-p} = 0$. If we take $\beta = \frac{p-1}{c} \ln \alpha$, then

$$-\Delta_p w + e^{\alpha c w} - d \alpha^{1-p} = 0 \quad \text{in } \Omega. \quad (5.5)$$

Since $\alpha > 0$ and $d \geq 0$ are arbitrary, we see that for any $a > 0$ and $b \geq 0$, there exists a large solution $u = u_{a,-b}$ to

$$-\Delta_p u + e^{au} - b = 0 \quad \text{in } \Omega. \quad (5.6)$$

We can notice that since $u_{a,0} = a^{-1} u_{1,0} + (1-p)a^{-1} \ln a$, the minimum $\theta := \theta_a$ of $u_{a,0}$ satisfies $\theta_a = a^{-1} \theta_1 + (1-p)a^{-1} \ln a$. For $\epsilon > 0$, there holds

$$e^{a u_{a,0}} - \epsilon e^{a u_{a,0}} = (1-\epsilon) e^{a u_{a,0}} \geq (1-\epsilon) \theta_1 a^{1-p}.$$

Therefore $-\Delta_p u_{a,0} + \epsilon e^{a u_{a,0}} + (1-\epsilon) \theta_1 a^{1-p} \leq 0$. Thus $v = u_{a,0} - a^{-1} \ln \epsilon$ satisfies

$$-\Delta_p v + e^{av} + (1-\epsilon) \theta_1 a^{1-p} \leq 0$$

which implies that there exists a large solution to the corresponding equation. Since ϵ is arbitrary, it follows that for any $b < b_a := \theta_1 a^{1-p}$, there exists a large solution $u = u_{a,b}$ to (5.2). \square

Theorem 5.2 *Assume $p-1 < q < p$ and (1.12) holds. Then there exists $b^* = b^*(p, q, N, \Omega) > 0$ such that for any $b \in (-\infty, b^*)$, problem (5.1) admits a solution.*

Proof. If (1.12) holds, for any $a > 0$ and $b < b_a$, there exists a large solution u to (5.2). We set $u = \alpha \ln w$ with $\alpha > 0$, then

$$-\Delta_p w + (p-1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha a + p - 1} + b \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega. \quad (5.7)$$

By Hölder's inequality

$$(p-1) \frac{|\nabla w|^p}{w} \geq |\nabla w|^q - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}},$$

therefore

$$-\Delta_p w + |\nabla w|^q + \alpha^{1-p} w^{\alpha a + p - 1} + b \alpha^{1-p} w^{p-1} - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}} \leq 0.$$

Since $q > p-1$, $\frac{q}{p-q} > p-1$. We choose α and a such that

$$\alpha a + p - 1 = \frac{q}{p-q} \quad \text{and} \quad \alpha^{1-p} = \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}}.$$

Therefore w satisfies

$$-\Delta_p w + |\nabla w|^q + b \alpha^{1-p} w^{p-1} \leq 0.$$

This implies that there exists a large solution to (5.1). \square

Theorem 5.3 *Let $q > 1$ and assume that there exists a solution to*

$$\begin{aligned} -\Delta u + |\nabla u|^2 + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty. \end{aligned} \quad (5.8)$$

Then for any $s > 1$ there holds

$$\int_0^1 \frac{C_{2,s}(B_r(x) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = \infty \quad \forall x \in \partial\Omega. \quad (5.9)$$

Proof. For $\delta > 0$ set $\Omega_\delta := \{x \in \Omega : \rho(x) < \delta\}$. There exists $\delta_0 > 0$ such that $u(x) > 1$ in Ω_{δ_0} . For $\sigma > 0$ we set $u = v^\sigma$, therefore

$$-\Delta v - (\sigma - 1) \frac{|\nabla v|^2}{v} + \sigma v^{\sigma-1} |\nabla v|^2 + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0.$$

Since $v > 1$ in Ω_{δ_0} , it follows

$$-\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = \sigma \frac{|\nabla v|^2}{v} \left(1 - \frac{1}{\sigma} - v^\sigma\right) \leq 0 \quad \text{in } \Omega_{\delta_0}.$$

For $0 < \delta < \delta_0$ and $m > \inf\{u^{\frac{1}{\sigma}}(x) : x \in \partial\Omega_{\delta_0}\}$ we denote by $v_{m,\delta}$ the solution of

$$\begin{aligned} -\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} &= 0 && \text{in } \Omega_{\delta,\delta_0} := \{x \in \Omega : \delta < \rho(x) < \delta_0\} \\ v &= m && \text{in } \partial\Omega_{\delta_0} \\ \lim_{\text{dist}(x,\partial\Omega_\delta) \rightarrow 0} v(x) &= \infty. \end{aligned} \quad (5.10)$$

Then $v_\delta \geq u^{\frac{1}{\sigma}}$ and $v_{\delta'} < v_\delta$ if $0 < \delta' < \delta$. Thus $v_m = \lim_{\delta \rightarrow 0} v_{m,\delta}$ satisfies

$$\begin{aligned} -\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} &= 0 && \text{in } \Omega_{\delta_0} \\ v &= m && \text{in } \partial\Omega_{\delta_0} \\ \lim_{\rho(x) \rightarrow 0} v(x) &= \infty. \end{aligned} \quad (5.11)$$

Notice that, since $\partial\Omega_{\delta_0}$ is Lipschitz, the boundary data is preserved in the approximation process. Letting $m \rightarrow \infty$ and using the monotonicity of $\{v_m\}$, it implies that there exists a large solution to

$$-\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0 \quad \text{in } \Omega_{\delta_0}.$$

By Labutin's result it implies in particular (5.9) with $s = \frac{(q-1)\sigma+1}{(q-1)\sigma}$ and $s > 1$ is arbitrary. \square

Remark. If we set $v = e^u$ in (5.8), then v satisfies

$$-\Delta v + e^{(q+1)v} = |\nabla v|^2 (1 - e^v) \quad \text{in } \Omega. \quad (5.12)$$

From this, we can construct a large solution of

$$-\Delta v + e^{(q+1)v} = 0 \quad \text{in } \Omega_{\delta_0} \setminus \Omega_\delta. \quad (5.13)$$

It would be interesting to see what Wiener type criterion the existence of such a large solution implies. We conjecture that this condition is

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c) dr}{r^{N-2}} = \infty \quad \forall x \in \partial\Omega. \quad (5.14)$$

Theorem 5.4 *Assume that for some $p > 0$ there exists a function $u \in C(\overline{\Omega})$ satisfying*

$$\begin{aligned} -\Delta u + u^{-p} &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (5.15)$$

Then for any $s > 1$ (5.9) holds.

Proof. We set $v = e^{-v}$, then v is a large solution of

$$-\Delta v + |\nabla v|^2 + e^{(p+1)v} = 0 \quad \text{in } \Omega \quad (5.16)$$

and we conclude using the preceding theorem (and the remark hereafter). \square

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