



HAL
open science

Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

Hung Nguyen Quoc, Laurent Veron

► **To cite this version:**

Hung Nguyen Quoc, Laurent Veron. Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption. 2013. hal-00851381v1

HAL Id: hal-00851381

<https://hal.science/hal-00851381v1>

Preprint submitted on 13 Aug 2013 (v1), last revised 10 Oct 2014 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

Nguyen Quoc Hung*

Laurent Véron†

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We obtain necessary conditions expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities for the existence of large solutions to equations (1) $-\Delta_p u + e^u - 1 = 0$ or (2) $-\Delta_p u + u^q = 0$ in a bounded domain Ω when $q > p - 1 > 0$. We apply our results to equations (3) $-\Delta_p u + |\nabla u|^q + bu^{p-1} = 0$, (4) $-\Delta u + |\nabla u|^2 + u^q = 0$ and $-\Delta u + u^{-q} = 0$ with $q > 0$.

2010 Mathematics Subject Classification. 31C15, 35J92, 35F21, 35B44.

Key words: quasilinear elliptic equations, Wolff potential, maximal functions, Hausdorff capacities, Bessel capacities.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) and $1 < p \leq N$. We consider the question of existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and g is a continuous nondecreasing function vanishing at 0; most often $g(u)$ is either $e^u - 1$ or $|u|^{q-1} u$ with $q > p - 1$. A solution to problem (1.1) is called a *large solution*. When the domain is regular in the sense that the Dirichlet problem with continuous boundary data ϕ

$$\begin{aligned} -\Delta_p u + g(u) &= 0 && \text{in } \Omega \\ u &= \phi && \text{in } \partial\Omega \end{aligned} \tag{1.2}$$

*E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr

†E-mail address: Laurent.Veron@lmpt.univ-tours.fr

admits a solution, it is clear that problem (1.1) admits a solution. It is known that a necessary and sufficient condition for such a result is the so called Wiener criterion (for $p = 2$ see [17]), for $p \neq 2$ see [10], [6])

$$\int_0^1 \left(\frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.3)$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$: the existence of a large solution is guaranteed for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition [14]

$$\int_a^\infty \frac{dt}{G^{\frac{1}{p}}(t)} < \infty \quad G(t) = \int_0^t g(s)ds, \quad (1.4)$$

a variant of the Keller-Osserman estimate [7], [12], which is the above relation when $p = 2$. If for $R > \text{diam}(\Omega)$ there exists a function v which satisfies

$$\begin{aligned} -\Delta_p v + g(v) &= 0 && \text{in } B_R \setminus \{0\} \\ v &= 0 && \text{on } \partial B_R \\ \lim_{x \rightarrow 0} v(x) &= \infty, \end{aligned} \quad (1.5)$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \quad \text{in } \Omega \quad (1.6)$$

is a large solution, without any assumption on the regularity of $\partial\Omega$, provided (1.4) is satisfied. However the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron proved [15] that if

$$\liminf_{|r| \rightarrow \infty} |r|^{-\frac{N(p-1)}{N-p}} \text{sign}(r)g(r) > 0 \quad (1.7)$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p-1 < q < \frac{N(p-1)}{N-p}$, Friedman and Véron [4] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \quad (1.8)$$

with an isolated singularities. In 2003, Labutin [8] proved that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.9)$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \quad \forall x \in \partial\Omega, \quad (1.10)$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. Concerning the exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = e^u - 1$ and $g(u) = |u|^{q-1}u$, $q > p - 1$. For $1 < p < N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j, \text{diam}(B_j) \leq 1 \right\} \quad (1.11)$$

where the B_j are balls and $h^{N-p}(B_r) = c_N r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 2$ and $1 < p < N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.12)$$

then there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^u - 1 &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.13)$$

As a consequence we obtained a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{s,r}$ Bessel capacity in \mathbb{R}^N associated to the Besov space $B^{s,r}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 2$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.14)$$

then, for any $p-1 < q < \frac{pq_1}{N}$, there exists $u \in C_{loc}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + u^q &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.15)$$

In view of Labutin's theorem this last result is not optimal in the case $p = 2$, since the involved capacity is C_{2,q'_1} with q'_1 and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.16)$$

with $q_1 > q$.

At end we apply the previous theorems to quasilinear viscous Hamilton-Jacobi equations. We prove that if $p-1 < q < p$, $b < b^*$ for some $b^* > 0$ depending on p, q, Ω , and (1.12) holds, there exists a solution to

$$\begin{aligned} -\Delta_p u + |\nabla u|^q + bu^{p-1} &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty \end{aligned} \quad (1.17)$$

Conversely, we prove that if for some $q > 1$, there exists a solution to

$$\begin{aligned} -\Delta u + |\nabla u|^2 + |u|^{q-1}u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty, \end{aligned} \quad (1.18)$$

then necessarily

$$\int_0^1 \frac{C_{2,s}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \quad (1.19)$$

for all $s > 1$. This condition holds also if for some $p > 0$ there exists $u \in C(\overline{\Omega})$, $u > 0$ in Ω satisfying

$$\begin{aligned} -\Delta u + u^{-p} &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (1.20)$$

2 Morrey classes and Wolff potential estimates

In this section Ω is a bounded domain in \mathbb{R}^N . We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (1.4) belongs to $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 2.1 1- A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_{\Omega \cap B_r(x)} |f| dy \leq K r^{\frac{N}{s}} \quad \forall r > 0, \forall x \in \mathbb{R}^N \quad (2.1)$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{\mathcal{M}^s(\Omega)}$.

2- A function $f \in L^1(\Omega)$ belongs to the weak L^s -space $M^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that

$$\int_E |f| dy \leq K |E|^{\frac{1}{s}} \quad \forall E \subset \Omega, E \text{ Borel}. \quad (2.2)$$

The quasi-norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $\|f\|_{M^s(\Omega)}$

Clearly $L^p(\Omega) \subset M^p(\Omega) \subset \mathcal{M}^p(\Omega)$.

Definition 2.2 Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}_+(\Omega)$, the set of positive Radon measures in Ω . If $\alpha > 0$ and $1 < p < \alpha^{-1}N$, we define the (R -truncated) Wolff potential of μ by

$$\mathbf{W}_p^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \forall x \in \mathbb{R}^N, \quad (2.3)$$

and, for $1 < p < N$, the (R -truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0 < t < R} \frac{\mu(B_t(x))}{t^{N-p}} \quad \forall x \in \mathbb{R}^N, \quad (2.4)$$

where the measure is extended by 0 in Ω^c .

For $k \geq 0$, we set $T_k(u) = \text{sign}(u) \min\{k, |u|\}$.

Definition 2.3 Assume $f \in L^1_{loc}(\Omega)$. We say that a Borel function u defined in Ω is a renormalized supersolution of

$$-\Delta_p u + f = 0 \quad \text{in } \Omega \quad (2.5)$$

if for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds

$$\int_{\Omega} (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f \varphi) dx \geq 0 \quad (\text{resp. } \leq 0) \quad (2.6)$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f := \mu$ is a positive (resp. negative) distribution in Ω .

The following result is proved in [11, Theorem 4.35].

Theorem 2.4 Let Ω be an open bounded domain in \mathbb{R}^N . If $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ for some $\varepsilon \in (0, p)$, u is a nonnegative supersolution of (2.5) and set $\mu := -\Delta_p u + f$. Then there holds

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_1 W_{1,p}^{\frac{r}{4}}[\mu](x) \quad \forall x \in \Omega \text{ s.t. } B_r(x) \subset \Omega, \quad (2.7)$$

for some c_1 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u = f + \mu \quad \text{in } \Omega \quad (2.8)$$

where $f \in L^1_{loc}(\Omega)$ and μ is a Radon measure, we have

Corollary 2.5 Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of positive and bounded Radon measures in Ω . If u is a nonnegative renormalized solution to (2.8), then there exists a positive constant c_2 depending only on $N, p, \varepsilon, \text{diam}(\Omega)$ such that

$$u(x) + \|f\|_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}^{\frac{1}{p-1}} \geq c_2 W_{1,p}^{\frac{d(x, \partial\Omega)}{4}}[\mu](x) \quad \forall x \in \Omega. \quad (2.9)$$

Definition 2.6 For $1 \leq s, q < \infty$, let $L^{s,q}(\Omega)$ denote the Lorentz space endowed with the norm

$$\|f\|_{L^{s,q}} = \left(\int_0^\infty t^{\frac{q}{s}} (f^{**}(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad (2.10)$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$$

with $f^*(t) = \inf\{s > 0 : |\{x \in \Omega : |f(x)| > s\}| \leq t\}$. The dual space of $L^{s,q}(\Omega)$ is the space $L^{-s,q'}(\Omega)$ and it is naturally endowed with the dual norm.

The following result is proved in [2].

Theorem 2.7 *Let $f \in \mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. Assume that u is a nonnegative renormalized solution to equation (2.13). If $\mu \in L^{-p, \frac{q}{p-1}}$ for some $q > p - 1$, then $u \in L^q(\Omega)$ and*

$$\|u\|_{L^q(\Omega)} \leq C \left(\|\mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} + \|f\|_{\mathcal{M}^{\frac{1}{p-1}}(\Omega)} \right) \quad (2.11)$$

for some a positive constant C depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$.

Conversely, if $u \in L^q(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C_K depending only on $N, p, q, \varepsilon, \text{diam}(\Omega)$ and $\text{dist}(K, \partial\Omega)$ such that $\chi_K \mu \in L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)$ and

$$\|\chi_K \mu\|_{L^{-p, \frac{q}{p-1}}(\mathbb{R}^N)} \leq C_K \left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{M}^{\frac{1}{p-1}}(\Omega)} \right). \quad (2.12)$$

In particular, for any Borel set $E \subset \Omega$,

$$C_{p, \frac{q}{q+1-p}}(E) = 0 \implies \mu(E) = 0. \quad (2.13)$$

We recall [2, Theorem 3.8].

Theorem 2.8 *There exists a positive constant c_1 such that if u is a renormalized solution to $-\Delta_p u = \mu$ in Ω and $u = 0$ on $\partial\Omega$, then for any $x \in \Omega$*

$$|u(x)| \leq c_1 W_{1,p}^{2\text{diam}(\Omega)}[\mu](x).$$

The next statement is proved in [2, Theorem 2.4], and in [5] for a variant.

Theorem 2.9 *There exist positive constants c_2, c_3 such that*

$$\int_{2B} \exp(c_2 W_{1,p}^R[\mu_B]) \leq c_3 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $\|\mathbf{M}_{p,R}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1$.

3 Estimates from below

If G is any domain in \mathbb{R}^N with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7), there always exists a maximal solution to (1.4) in G . It is constructed as the limit, when $n \rightarrow \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 & \text{in } G_n := \{x \in \mathbb{R}^N : \text{dist}(x, G^c) > \frac{1}{n}\} \\ \lim_{\rho_n(x) \rightarrow 0} u_n(x) &= \infty \\ \lim_{|x| \rightarrow \infty} u_n(x) &= 0 & \text{if } G \text{ is unbounded,} \end{aligned} \quad (3.1)$$

where $\rho_n(x) := \text{dist}(x, \partial\Omega_n)$. Our main estimates are the following.

Theorem 3.1 *Let $K \subset B_{1/4} \setminus \{0\}$ be a compact set and let $U_j \in C_{loc}^1(K^c)$, $j = 1, 2$, be the maximal solutions of*

$$-\Delta_p u + e^u - 1 = 0 \quad \text{in } K^c \quad (3.2)$$

for U_1 and

$$-\Delta_p u + u^q = 0 \quad \text{in } K^c \quad (3.3)$$

for U_2 , where $p-1 < q < \frac{2q_1}{N}$. Then there exist constants C_k , $k = 1, 2, 3, 4$, depending on N , p and q such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (3.4)$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{C_{p, \frac{q_1}{4_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \quad (3.5)$$

Proof. Step 1. For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [13, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\text{supp}(\mu_j) \subset K \cap S_j$,

$$c^{-1} \mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j$$

and

$$\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} = 1.$$

Now, we will show that for ε small enough, there holds,

$$\int_{B_1} \exp\left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x)\right) dx \leq C, \quad (3.6)$$

where c_1 is the constant in Theorem 2.8, and C does not depend on J .

Indeed, we have

$$A := \int_{B_1} \exp\left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x)\right) dx = \sum_{j=1}^{\infty} \int_{S_j} \exp\left(\frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x)\right) dx.$$

Since

$$\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \leq \max\{1, 5^{\frac{2-p}{p-1}}\} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x) + \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) + \sum_{k=\max\{j-1,3\}}^{j+1} \mathbf{W}_{1,p}^1[\mu_k](x) \right)$$

and

$$\exp\left(\sum_{i=1}^5 a_i\right) \leq \sum_{i=1}^5 \exp(5a_i) \quad \forall a_i.$$

Thus,

$$\begin{aligned} A &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \geq j+2} \mu_k \right] (x)\right) dx \\ &\quad + \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x)\right) dx \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=\max\{j-1,3\}}^{j+1} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

Estimate of A_3 : We apply Theorem 2.9 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \leq c_3 r_k^N$$

with $5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} K \varepsilon^{\frac{1}{p-1}} \in (0, c_2]$. In particular,

$$\int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} K \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \leq 4c_3 r_k^N \quad k = j-1, j, j+1.$$

Which implies

$$A_3 \leq c_4 \sum_{j=1}^{+\infty} 4c_3 r_j^N = c_5 < \infty. \quad (3.7)$$

Estimate of A_1 : Since

$$\sum_{k \geq j+2} \mu_k(B_t(x)) = 0 \quad \forall x \in S_j, t \in (0, r_{j+1}),$$

thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \int_{r_{j+1}}^1 \left(\frac{\sum_{k \geq j+2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right) dx \\ &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right) dx. \end{aligned}$$

Note that $\mu_k(S_k) \leq \mu_k(B_{r_{k-1}}(0)) \leq r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k \geq j+2} \mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \leq \left(\sum_{k \geq j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k \geq 0} r_k^{N-p}\right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$\begin{aligned} A_1 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) dx \\ &= \exp\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right) |B_1| = c_6 \end{aligned}$$

Estimate of A_2 : for $x \in S_j$,

$$\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{k \leq j-2} \mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Since $r_i < t < r_{i-1}$, $\sum_{k \leq i-2} \mu_k(B_t(x)) = 0, \forall i = 1, \dots, j-1$, thus

$$\begin{aligned}
\mathbf{W}_{1,p}^1 \left[\sum_{k \leq j-2} \mu_k \right] (x) &= \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{i-1 \leq k \leq j-2} \mu_k(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \sum_{i=1}^{j-1} \int_{r_i}^{r_{i-1}} \left(\frac{\sum_{i-1 \leq k \leq j-2} \mu_k(S_k)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq \sum_{i=1}^{j-1} \left(\sum_{i-1 \leq k \leq j-2} \mu_k(S_k) \right)^{\frac{1}{p-1}} \frac{p-1}{N-p} r_i^{-\frac{N-p}{p-1}} \\
&\leq \sum_{i=1}^{j-1} \left(\sum_{i-1 \leq k \leq j-2} r_{k-1}^{N-p} \right)^{\frac{1}{p-1}} \frac{p-1}{N-p} r_i^{-\frac{N-p}{p-1}} \\
&\leq \frac{p-1}{N-p} \sum_{i=1}^{j-1} \left(\sum_{i-1 \leq k \leq j-2} r_{k-1-i}^{N-p} \right)^{\frac{1}{p-1}} \\
&\leq \frac{p-1}{N-p} \sum_{i=1}^{j-1} \left(\sum_{k \geq i-1} r_{k-1-i}^{N-p} \right)^{\frac{1}{p-1}} \\
&= \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} (j-1) \\
&\leq \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 &\leq \sum_{j=1}^{\infty} \int_{S_j} \exp \left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} j \right) dx \\
&= \sum_{j=1}^{\infty} c_7 r_j^N \exp \left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} j \right) \\
&= \sum_{j=1}^{\infty} c_7 \exp \left(\left(5 \max\{1, 5^{\frac{2-p}{p-1}}\} \frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{4^{N-p}}{1-2^{-(N-p)}} \right)^{\frac{1}{p-1}} - N \log(2) \right) j \right) \\
&= c_8 \quad \text{for } \varepsilon \text{ small enough.}
\end{aligned}$$

Consequently, $A \leq C := c_6 + c_8 + c_5$ for ε small enough. This implies

$$\left\| \exp \left(c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] \right) \right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1(0))} \leq c_{10} \left(\int_{B_1(0)} \exp \left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \varepsilon \mu_k \right] (x) \right) dx \right)^{\frac{p}{2N}} \leq c_{11} \quad (3.8)$$

where the constant c_{11} does not depend on J . Set $B = B_{\frac{1}{4}}$. For ε small enough, it follows from [2], 3.6 and Theorem 2.8, that there exists a renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \varepsilon \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{in } \partial B. \end{aligned} \quad (3.9)$$

By standard regularity theory, $u \in C_{loc}^{1,\alpha}(B \setminus K)$. From Corollary 2.5 and estimate (3.8), we have

$$u(0) \geq -c_{12} + c_{13} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

Therefore

$$\begin{aligned} u(0) &\geq -c_{12} + c_{13} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0) = -c_{12} + c_{13} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^J \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\geq -c_{12} + c_{13} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = -c_{12} + c_{13} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\geq -c_{12} + c_{14} \sum_{i=2}^{J-2} (\mu_{i+2}(S_{i+2}))^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} = -c_{12} + c_{14} \sum_{i=2}^{J-2} (\mu_{i+2}(S_{i+2}))^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{12} + c_{15} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} = -c_{12} + c_{16} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}. \end{aligned}$$

Note that

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i.$$

Therefore,

$$\begin{aligned} u(0) &\geq -c_{12} + c_{16} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\ &= -c_{12} + c_{16} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} - \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \right) \\ &= -c_{12} + c_{16} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} r_{i-1}^{-\frac{N-p}{p-1}} - \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \right) \\ &\geq -c_{12} + c_{16} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\ &\geq -c_{17} + c_{18} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}. \end{aligned}$$

Since U_1 is the maximal solution in K^c , u satisfies the same equation in $B \setminus K$ and $U_1 \geq u = 0$ on ∂B , it follows that U_1 dominates u in $B \setminus K$. Then $U_1(0) \geq u(0)$ and we derive (3.4).

Step 2. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, \dots, J$. By [13, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{q_1} dx = C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_j).$$

We have, for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k \right)^r \leq \sum_{k=0}^{\infty} \theta_{k,r} a_k^r$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,r} = \begin{cases} 1 & \text{if } r \in (0, 1] \\ \left(\frac{\theta+1}{\theta}\right)^{r-1} (\theta+1)^{kr} & \text{if } r > 1. \end{cases}$$

Thus,

$$\begin{aligned} \int_{B_1(0)} \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] (x) \right)^{q_1} dx &\leq \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_{B_1(0)} (\mathbf{W}_{1,p}^1[\mu_k](x))^{q_1} dx \\ &\leq \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_R (\mathbf{W}_{1,p}^1[\mu_k](x))^{q_1} dx \\ &\leq c_{19} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} \int_R (G_p[\mu_k](x))^{q_1} dx \\ &= c_{19} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} C_{p, \frac{q_1}{q_1-p+1}}(K \cap S_k) \\ &\leq c_{20} \sum_{k=3}^J \theta_{k, \frac{1}{p-1}}^{q_1} \theta_{k, q_1} 2^{-k} \left(N - \frac{pq_1}{q_1-p+1} \right) \\ &\leq c_{21} \quad \text{for } \theta \text{ small enough,} \end{aligned}$$

where the constant c_{21} does not depend on J . Hence,

$$\left\| \left(\mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right)^q \right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1(0))} \leq c_{22} \left\| \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^J \mu_k \right] \right\|_{L^{q_1}(B_1(0))} \leq c_{23} \quad (3.10)$$

where c_{23} is independent of J . Take $B = B_{\frac{1}{4}}$. Note that $\frac{q_1}{q} > \frac{N}{p}$. By [2], 3.10, Corollary 2.5 and Theorem 2.8, there exists a renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + u^q &= \sum_{j=3}^J \mu_j && \text{in } B \\ u &= 0 && \text{on } \partial B. \end{aligned} \quad (3.11)$$

It belongs to $C_{loc}^{1,\alpha}(B \setminus K)$ and

$$u(0) \geq -c_{24} + c_{25} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^J \mu_j \right] (0).$$

As above, we also get that

$$u(0) \geq -c_{26} + c_{27} \int_0^1 \left(\frac{C_{p, \frac{q_1}{41-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Since U_2 is the maximal solution in U_2 in $B \setminus K$, it dominates the solution u in $B \setminus K$, and thus $U_2(0) \geq u(0)$. Therefore, we get (3.5). \square

4 Proof of the main results

Proof of Theorem 1. Let u be the maximal solution of

$$-\Delta_p u + e^u - 1 = 0 \quad \text{in } \Omega \quad (4.1)$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B}_{1/4}(z_0)$. Let $U_1 \in C^1(K^c)$ be the maximal solution of (3.2). We have $u \geq U_1$ in Ω . By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\ &\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \end{aligned}$$

Therefore,

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

\square

Proof of Theorem 2. Let u be the maximal solution to

$$-\Delta_p u + u^q = 0 \quad \text{in } \Omega. \quad (4.2)$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B}_{1/4}(z_0)$. Let $U_2 \in C^1(K^c)$ be the maximal solution of (3.3). We have $u \geq U_2$

in Ω . By Theorem 3.1,

$$\begin{aligned}
U_1(z_0) &\geq -C_1 + C_2 \int_{\delta}^1 \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -C_1 + C_2 \int_{\delta}^1 \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0)) \\
&\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -C_1 + C_2' \int_{\delta}^{1/2} \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.
\end{aligned}$$

Therefore,

$$\inf_{B_{\delta} \cap \Omega} u \geq \inf_{B_{\delta} \cap \Omega} U \geq -C_1 + C_2' \int_{\delta}^{1/2} \left(\frac{C_p, \frac{q_1}{q_1-p+1}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \rightarrow \infty \quad \text{as } \delta \rightarrow 0.$$

□

5 Large solutions of quasilinear Hamilton-Jacobi equations

In this section we used our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned}
-\Delta_p u + |\nabla u|^q + bu^{p-1} &= 0 && \text{in } \Omega \\
\lim_{\rho(x) \rightarrow 0} u(x) &= \infty, && (5.1)
\end{aligned}$$

where b is a real number and $p-1 < q < p$.

Lemma 5.1 *The maximal solution of (4.1) is a large solution if and only if for any $a > 0$ and $b < b_a := \theta_1 a^{1-p}$ the maximal solution of*

$$-\Delta_p v + e^{av} + b = 0 \quad \text{in } \Omega, \quad (5.2)$$

is a large solution, where θ_1 is a positive constant depending on N, p and Ω .

Proof. Since monotonicity and Vazquez' condition (1.4) hold, it is sufficient to exhibit a large subsolution (i.e. tending to infinity on the boundary) in order to conclude on the existence of a large solution to (5.2).

Assume $u := u_{1,-1}$ is a large solution of (4.1), then for any $\Lambda \geq 1$

$$-\Delta_p u_{1,-1} + e^{u_{1,-1}} - \Lambda = 1 - \Lambda \leq 0 \quad \text{in } \Omega,$$

thus $u_{1,-1}$ is a subsolution of the corresponding solution and there exists a larger solution which is necessarily a large solution $u_{1,-\Lambda}$ of

$$-\Delta_p u + e^u - \Lambda = 0 \quad \text{in } \Omega. \quad (5.3)$$

Set $\min\{u_{1,1}(x) : x \in \Omega\} = \theta > 0$. then, for any $c \in (0, 1)$ and $d \geq 0$ there holds

$$e^{u_{1,-1}} - 1 \geq m_{\theta} e^{cu_{1,-1}} \geq m_{\theta} e^{cu_{1,-1}} - d \quad \text{on } [\theta, \infty)$$

with $m_\theta = e^{(1-c)\theta} - e^{-c\theta}$. This implies that $-\Delta_p u_{1,-1} + m_\theta e^{cu_{1,-1}} - d \leq 0$, therefore $v := u_{1,-1} + c^{-1} \ln m_\theta$ satisfies $-\Delta_p u_{1,-1} + e^{cu_{1,-1}} - d \leq 0$. Therefore there exists a large solution $u_{c,-d}$ to

$$-\Delta_p u + e^{cu} - d = 0 \quad \text{in } \Omega. \quad (5.4)$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, set $u_{c,-d} = \alpha w + \beta$, then $-\Delta_p w + \alpha^{1-p} e^{\beta c} e^{\alpha c w} - d \alpha^{1-p} = 0$. If we take $\beta = \frac{p-1}{c} \ln \alpha$, then

$$-\Delta_p w + e^{\alpha c w} - d \alpha^{1-p} = 0 \quad \text{in } \Omega. \quad (5.5)$$

Since $\alpha > 0$ and $d \geq 0$ are arbitrary, we see that for any $a > 0$ and $b \geq 0$, there exists a large solution $u = u_{a,-b}$ to

$$-\Delta_p u + e^{au} - b = 0 \quad \text{in } \Omega. \quad (5.6)$$

We can notice that since $u_{a,0} = a^{-1} u_{1,0} + (1-p)a^{-1} \ln a$, the minimum $\theta := \theta_a$ of $u_{a,0}$ satisfies $\theta_a = a^{-1} \theta_1 + (1-p)a^{-1} \ln a$. For $\epsilon > 0$, there holds

$$e^{a u_{a,0}} - \epsilon e^{a u_{a,0}} = (1-\epsilon) e^{a u_{a,0}} \geq (1-\epsilon) \theta_1 a^{1-p}.$$

Therefore $-\Delta_p u_{a,0} + \epsilon e^{a u_{a,0}} + (1-\epsilon) \theta_1 a^{1-p} \leq 0$. Thus $v = u_{a,0} - a^{-1} \ln \epsilon$ satisfies

$$-\Delta_p v + e^{av} + (1-\epsilon) \theta_1 a^{1-p} \leq 0$$

which implies that there exists a large solution to the corresponding equation. Since ϵ is arbitrary, it follows that for any $b < b_a := \theta_1 a^{1-p}$, there exists a large solution $u = u_{a,b}$ to (5.2). \square

Theorem 5.2 *Assume $p-1 < q < p$ and (1.12) holds. Then there exists $b^* = b^*(p, q, N, \Omega) > 0$ such that for any $b \in (-\infty, b^*)$, problem (5.1) admits a solution.*

Proof. If (1.12) holds, for any $a > 0$ and $b < b_a$, there exists a large solution u to (5.2). We set $u = \alpha \ln w$ with $\alpha > 0$, then

$$-\Delta_p w + (p-1) \frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha a + p - 1} + b \alpha^{1-p} w^{p-1} = 0 \quad \text{in } \Omega. \quad (5.7)$$

By Hölder's inequality

$$(p-1) \frac{|\nabla w|^p}{w} \geq |\nabla w|^q - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}},$$

therefore

$$-\Delta_p w + |\nabla w|^q + \alpha^{1-p} w^{\alpha a + p - 1} + b \alpha^{1-p} w^{p-1} - \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}} \leq 0.$$

Since $q > p-1$, $\frac{q}{p-q} > p-1$. We choose α and a such that

$$\alpha a + p - 1 = \frac{q}{p-q} \quad \text{and} \quad \alpha^{1-p} = \frac{p-q}{p} \left(\frac{q}{p(p-1)} \right)^{\frac{q}{p-q}}.$$

Therefore w satisfies

$$-\Delta_p w + |\nabla w|^q + b \alpha^{1-p} w^{p-1} \leq 0.$$

This implies that there exists a large solution to (5.1). \square

Theorem 5.3 *Let $q > 1$ and assume that there exists a solution to*

$$\begin{aligned} -\Delta u + |\nabla u|^2 + |u|^{q-1} u &= 0 && \text{in } \Omega \\ \lim_{\rho(x) \rightarrow 0} u(x) &= \infty. \end{aligned} \quad (5.8)$$

Then for any $s > 1$ there holds

$$\int_0^1 \frac{C_{2,s}(B_r(x) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = \infty \quad \forall x \in \partial\Omega. \quad (5.9)$$

Proof. For $\delta > 0$ set $\Omega_\delta := \{x \in \Omega : \rho(x) < \delta\}$. There exists $\delta_0 > 0$ such that $u(x) > 1$ in Ω_{δ_0} . For $\sigma > 0$ we set $u = v^\sigma$, therefore

$$-\Delta v - (\sigma - 1) \frac{|\nabla v|^2}{v} + \sigma v^{\sigma-1} |\nabla v|^2 + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0.$$

Since $v > 1$ in Ω_{δ_0} , it follows

$$-\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = \sigma \frac{|\nabla v|^2}{v} \left(1 - \frac{1}{\sigma} - v^\sigma\right) \leq 0 \quad \text{in } \Omega_{\delta_0}.$$

For $0 < \delta < \delta_0$ and $m > \inf\{u^{\frac{1}{\sigma}}(x) : x \in \partial\Omega_{\delta_0}\}$ we denote by $v_{m,\delta}$ the solution of

$$\begin{aligned} -\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} &= 0 && \text{in } \Omega_{\delta,\delta_0} := \{x \in \Omega : \delta < \rho(x) < \delta_0\} \\ v &= m && \text{in } \partial\Omega_{\delta_0} \\ \lim_{\text{dist}(x,\partial\Omega_\delta) \rightarrow 0} v(x) &= \infty. \end{aligned} \quad (5.10)$$

Then $v_\delta \geq u^{\frac{1}{\sigma}}$ and $v_{\delta'} < v_\delta$ if $0 < \delta' < \delta$. Thus $v_m = \lim_{\delta \rightarrow 0} v_{m,\delta}$ satisfies

$$\begin{aligned} -\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} &= 0 && \text{in } \Omega_{\delta_0} \\ v &= m && \text{in } \partial\Omega_{\delta_0} \\ \lim_{\rho(x) \rightarrow 0} v(x) &= \infty. \end{aligned} \quad (5.11)$$

Notice that, since $\partial\Omega_{\delta_0}$ is Lipschitz, the boundary data is preserved in the approximation process. Letting $m \rightarrow \infty$ and using the monotonicity of $\{v_m\}$, it implies that there exists a large solution to

$$-\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0 \quad \text{in } \Omega_{\delta_0}.$$

By Labutin's result it implies in particular (5.9) with $s = \frac{(q-1)\sigma+1}{(q-1)\sigma}$ and $s > 1$ is arbitrary. \square

Remark. If we set $v = e^u$ in (5.8), then v satisfies

$$-\Delta v + e^{(q+1)v} = |\nabla v|^2 (1 - e^v) \quad \text{in } \Omega. \quad (5.12)$$

From this, we can construct a large solution of

$$-\Delta v + e^{(q+1)v} = 0 \quad \text{in } \Omega_{\delta_0} \setminus \Omega_\delta. \quad (5.13)$$

It would be interesting to see what Wiener type criterion the existence of such a large solution implies. We conjecture that this condition is

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x) \cap \Omega^c) dr}{r^{N-2}} = \infty \quad \forall x \in \partial\Omega. \quad (5.14)$$

Theorem 5.4 *Assume that for some $p > 0$ there exists a function $u \in C(\overline{\Omega})$ satisfying*

$$\begin{aligned} -\Delta u + u^{-p} &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (5.15)$$

Then for any $s > 1$ (5.9) holds.

Proof. We set $v = e^{-v}$, then v is a large solution of

$$-\Delta v + |\nabla v|^2 + e^{(p+1)v} = 0 \quad \text{in } \Omega \quad (5.16)$$

and we conclude using the preceding theorem (and the remark hereafter). \square

References

- [1] D. R. Adams, L. I. Hedberg, *Function spaces and potential theory*. Grundlehren der Mathematischen Wissenschaften **314**, Springer-Verlag, Berlin, 1996. xii+366 pp.
- [2] M.F. Bidaut-Véron, H. Nguyen Quoc, L. Véron: *Quasilinear Lane-Emden equations with absorption and measure data*, submitted.
- [3] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa **28**, 741-808 (1999).
- [4] A. Friedman, L. Véron: *Singular Solutions of Some Quasilinear Elliptic Equations*, Arch. Rat. Mech. Anal. **96**, 259-287 (1986).
- [5] P. Honzik, B. Jaye: *On the good- λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180 (2012).
- [6] T. Kilpelainen, J. Malý: *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).
- [7] J. B. Keller: *On solutions of $\Delta u = f(u)$* , Comm. Pure Appl. Math. **10**, 503-510 (1957).
- [8] D. Labutin: *Wiener regularity for large solutions of nonlinear equations*, Ark. Mat. **41**, no. 2, 307-39 (2003).
- [9] M. Marcus, L. Véron: *Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets*, J. Math. Pures Appl. **91**, 256295 (2009).
- [10] V. Maz'ya: *On the continuity at a boundary point of solutions of quasilinear equations*, Vestnik Leningrad Univ. Math. **3**, 225-242 (1976).
- [11] J. Malý, W.P. Ziemer: *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, A.M.S (1997).
- [12] R. Osserman: *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math. **7**, 1641-1647 (1957)
- [13] B. O. Turesson: *Nonlinear Potential Theory and Sobolev Spaces*, Lecture Notes in Mathematics **1736**, Springer-Verlag Berlin, Heidelberg (2000).

- [14] J. L. Vazquez: *An a priori interior estimate for the solution of a nonlinear problem representing weak diffusion*, Nonlinear Anal. T. M. A. **5**, 95103 (1981).
- [15] J. L. Vazquez, L. Véron: *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. **33**, 129-144 (1980).
- [16] L. Véron: *On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data*, Math. Z. **273** 1-2, 1-17 (2013).
- [17] N. Wiener: *The Dirichlet problem*, J. Math. Phys. **3**, 127-146 (1924).
- [18] W. Ziemer: *Weakly Differentiable Functions*, Graduate Texts in Mathematics **120**, Springer-Verlag, Berlin (1989).