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► **To cite this version:**

Didier Pinchon, Pierre Siohan. A simple commutativity condition for block decimators and expanders. 2013. hal-00851242v1

HAL Id: hal-00851242

<https://hal.science/hal-00851242v1>

Preprint submitted on 13 Aug 2013 (v1), last revised 10 Sep 2013 (v2)

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A simple commutativity condition for block decimators and expanders

Didier Pinchon and Pierre Siohan

Abstract

Commutativity rules play an essential role when building multirate signal processing systems. In this letter, we focus on the interchangeability of block decimators and expanders. We, formally, prove that commutativity between these two operators is possible if and only if the data blocks are of an equal length corresponding to the greatest common divisor of the integer decimation and expansion factors.

Index Terms

Block sampling, Decimation, Commutativity, Expansion.

I. INTRODUCTION

The polyphase decomposition, introduced by Bellanger et al. in 1976 [1], is a key element in multirate signal processing to reduce the computational complexity of digital equipments. Its implementation involves decimation and expansion operators. For an input sequence $x[n]$, a conventional decimator, with integer decimation factor p_1 , only retains the samples at time multiple of p_1 . A conventional expander, with integer expansion factor p_2 , inserts $p_2 - 1$ samples between each pair of consecutive $x[n]$ samples. It is well-known that conventional decimators and expanders can commute if and only if p_1 and p_2 are coprime. The important question of commutativity of these basic operators also occurs when designing either multidimensional [2], [3] or block processing systems [4], [5]. Block samplers are of a particular interest to deal with the class of incompatible nonuniform filter banks [6]. Block decimators (resp. expanders) are defined by parameters that, in addition to the decimation (resp. expansion) factors, include the block length. In [5] the authors consider a block decimator $\downarrow(q_1, p_1)$ and an expander $\uparrow(q_2, p_2)$, with q_1, q_2 the block size and p_1, p_2 the decimation and expansion factor, respectively. Assuming q_1 and q_2 are not necessarily equal and p_1, p_2 non necessarily integers, they give three joint conditions that are necessary and sufficient conditions to insure the commutativity of $\downarrow(q_1, p_1)$ and $\uparrow(q_2, p_2)$. However no examples are given with unequal block lengths and/or rational non integer sampling ratios where the three conditions are satisfied. In [4], the commutativity of up and down sampling is studied when the sampling ratios are integers but with unequal block lengths. Again, the authors do not provide any example where the commutativity is obtained with unequal block lengths.

Our notations slightly differ from those in [5]. For integers q_1, p_1 such that $0 < q_1 < p_1$, let us denote by $D(q_1, p_1)$ the decimator with block length q_1 and sampling ratio p_1/q_1 . Such a decimator takes a sequence of p_1 consecutive input symbols of a signal, keeps the first q_1 ones and discards the last $p_1 - q_1$ symbols. For integers q_2, p_2 such that $0 < q_2 < p_2$, $E(q_2, p_2)$ denotes the expander with block length q_2 and sampling ratio p_2/q_2 : each block with length q_2 of the input signal is transmitted with the addition of $p_2 - q_2$ zero taps.

In this letter, the following theorem is proved.

Theorem. $D(q_1, p_1)$ and $E(q_2, p_2)$ commute if and only if $q_1 = q_2 = \gcd(p_1, p_2)$.

The result that when $q_1 = q_2 = \gcd(p_1, p_2)$ then $D(q_1, p_1)$ and $E(q_2, p_2)$ commute is already well known for $q_1 = q_2 = 1$ (see for example [7], pages 119 and 179). Its extension to arbitrary equal block lengths corresponds to

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the easiest part of our theorem. However, as we could not find any proof of it, in order to provide a self contained paper, we demonstrate it in our Lemmas 2 and 3.

II. PROOF OF THE THEOREM

The proof of the theorem follows as a consequence of several lemmas.

Let q_1, p_1, q_2, p_2 be four integers such that $0 < q_1 < p_1$ and $0 < q_2 < p_2$. Applied to a block of $p_1 q_2$ input entries $x = (x[n], 0 \leq n < p_1 q_2)$ the decimator $D(q_1, p_1)$ returns a sequence of $q_1 q_2$ symbols $t = (t[m], 0 \leq m < q_1 q_2)$ such that

$$(a, b) = \text{div}(n, p_1), 0 \leq a < q_2, 0 \leq b < q_1, \quad (1)$$

$$m = a q_1 + b \text{ if } b < q_1, \quad (2)$$

$$t[m] = x[n], \quad (3)$$

where a notation like $(a, b) = \text{div}(n, p_1)$ means that a and b are the quotient and the remainder, respectively, of the Euclidean division of n by p_1 such that $0 \leq a < q_2$ and $0 \leq b < p_1$. When $q_1 \leq b < p_1$ the symbol $x[n]$ is discarded. The expander $E(q_2, p_2)$ is then applied to the sequence $t[m], 0 \leq m < q_1 q_2$, to produce a sequence of $q_1 p_2$ symbols $y_1[k], 0 \leq k < q_1 p_2$ such that

$$(\alpha, \beta) = \text{div}(m, q_2), 0 \leq \alpha < q_1, 0 \leq \beta < q_2, \quad (4)$$

$$y_1[\alpha p_2 + \beta] = t[m]. \quad (5)$$

Unassigned symbols in the sequence $y_1[k], 0 \leq k < q_1 p_2$ are zero taps and the overall transformation is denoted by $E(q_2, p_2)D(q_1, p_1)$.

The action of $E(q_2, p_2)$ on input sequence x producing a sequence $z = (z[l], 0 \leq l < p_1 p_2)$ is defined by

$$(c, d) = \text{div}(n, q_2) 0 \leq c < p_1, 0 \leq d < q_2, \quad (6)$$

$$l = c p_2 + d, \quad (7)$$

$$z[l] = x[n]. \quad (8)$$

We then apply the decimator $D(q_1, p_1)$ to the input sequence t to produce the sequence $y_2[k], 0 \leq k < q_1 p_2$ using the relations

$$(\gamma, \delta) = \text{div}(l, p_1), 0 \leq \gamma < p_2, 0 \leq \delta < p_1, \quad (9)$$

$$y_2[\gamma q_1 + \delta] = z[l], \text{ if } \delta < q_1. \quad (10)$$

Again unassigned symbols in the sequence $y_2[k], 0 \leq k < q_1 p_2$ are zero taps. This transformation is denoted by $D(q_1, p_1)E(q_2, p_2)$.

Equations (1)-(5) allow us to define a function f_1 , depending on parameters q_1, p_1, q_2, p_2 , and defined on integers $0 \leq n \leq p_1 q_2 - 1$ that are expressed in an intuitive algorithmic language in (11) such that, when $f_1(n) \geq 0$, $y_1[f_1(n)] = x[n]$. When $f_1(n) = -1$, the symbol $x[n]$ is discarded and $y_1[m] = 0$ when m is not in the image of f_1 . In a similar manner, when $f_2(n) \geq 0$, the function f_2 defined in (11) is such that $y_2[f_2(n)] = x[n]$. When $f_2(n) = -1$, the symbol $x[n]$ is discarded and $y_2[m] = 0$ when m is not in the image of f_2 .

$$\left[\begin{array}{l} f_1 = \text{proc}(n) \\ \text{local } \alpha, \beta, a, b \\ (a, b) = \text{div}(n, p_1) \\ \text{if } b < q_1 \text{ then} \\ \quad (\alpha, \beta) = \text{div}(a q_1 + b, q_2) \\ \quad \text{return } \alpha p_2 + \beta \\ \text{else return } -1 \\ \text{end} \end{array} \right. \quad \left[\begin{array}{l} f_2 = \text{proc}(n) \\ \text{local } \gamma, \delta, c, d \\ (c, d) = \text{div}(n, q_2) \\ (\gamma, \delta) = \text{div}(c p_2 + d, p_1) \\ \text{if } \delta < q_1 \text{ then} \\ \quad \text{return } \gamma q_1 + \delta \\ \text{else return } -1 \\ \text{end} \end{array} \right. \quad (11)$$

Inverting equations (1)–(5) (resp. (6)–(10)) for given parameters q_1, p_1, q_2, p_2 , we may introduce the function $g_1(m)$ (resp. $g_2(m)$) defined for $0 \leq m \leq q_1 p_2$ such that $y_1[m] = 0$ when $g_1(m) = -1$ and $y_1[m] = x[g_1(m)]$ elsewhere (resp. $y_2[m] = 0$ when $g_2(m) = -1$ and $y_2[m] = x[g_2(m)]$ elsewhere).

$$\left[\begin{array}{l} g_1 = \text{proc}(m) \\ \text{local } \alpha, \beta, a, b \\ (\alpha, \beta) = \text{div}(m, p_2) \\ \text{if } \beta < q_2 \text{ then} \\ \quad (a, b) = \text{div}(\alpha q_2 + \beta, q_1) \\ \quad \text{return } a p_1 + b \\ \text{else return } -1 \\ \text{end} \end{array} \right. \quad \left[\begin{array}{l} g_2 = \text{proc}(m) \\ \text{local } \gamma, \delta, c, d \\ (\gamma, \delta) = \text{div}(m, q_1) \\ (c, d) = \text{div}(\gamma p_1 + \delta, p_2) \\ \text{if } d < q_2 \text{ then} \\ \quad \text{return } c q_2 + d \\ \text{else return } -1 \\ \text{end} \end{array} \right. \quad (12)$$

It is now obvious that $D(q_1, p_1)$ and $E(q_2, p_2)$ commute if and only if $y_1[m] = y_2[m]$, $0 \leq m < q_1 p_2$, or equivalently $f_1 = f_2$ or $g_1 = g_2$. This is clearly stated in Theorem 1 in [4] which amounts to say that $g_1 = g_2$ for integer sampling ratios if and only if the up and down sampling commute.

Remark. The action of $E(q_2, p_2)D(q_1, p_1)$ on the sequence $x = (x[n], 0 \leq n < p_1 q_2)$ to produce the sequence $y_1 = (y_1[k], 0 \leq k < q_1 p_2)$ may be represented (cf. [5]) as $y_1^T = \mathbf{A} x^T$ where \mathbf{A} is a matrix with $q_1 p_2$ rows, $p_1 q_2$ columns and 0-1 entries. f_1 gives a sparse representation of \mathbf{A} , defined in an algorithmic way, such that $\mathbf{A}_{i,j} = 1$ if and only if $f_1[i] = j$. The column of index j in \mathbf{A} is null if the symbol $x[j]$ is discarded, i.e. $f_1(j) = -1$. In the same way f_2 gives an algorithmic sparse representation of a matrix \mathbf{B} with the same dimensions as \mathbf{A} that corresponds to the action of $D(q_1, p_1)E(q_2, p_2)$. To emphasize the dependence of $f_1(n)$ on parameters q_1, p_1, q_2, p_2 we may instead use the notation $f_1(q_1, p_1, q_2, p_2, n)$ and the same for functions f_2, g_1 and g_2 .

The following symmetry property will be useful.

Lemma 1. *Let q_1, p_1, q_2, p_2 be integers with $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$. Then $D(q_1, p_1)$ and $E(q_2, p_2)$ commute if and only if $D(q_2, p_2)$ and $E(q_1, p_1)$ commute.*

Proof.– It is sufficient to observe that, for $0 \leq n < p_1 q_2$,

$$f_1(q_1, p_1, q_2, p_2, n) = g_1(q_2, p_2, q_1, p_1, n), \quad (13)$$

$$f_2(q_1, p_1, q_2, p_2, n) = g_2(q_2, p_2, q_1, p_1, n), \quad (14)$$

for any set of parameters q_1, p_1, q_2, p_2 and to use afterwards as a commutativity criterion the equality of functions f_1 and f_2 or the equality of g_1 and g_2 . \square

The decimator $D(1, p_1)$ is the conventional decimator of factor p_1 and the expander $E(1, p_2)$ is the conventional expander of factor p_2 . The following lemma is a classical result.

Lemma 2. *$D(1, p_1)$ and $E(1, p_2)$ commute if and only if p_1 and p_2 are relatively prime integers.*

Proof.– For $0 \leq n < p_1$, it is straightforward that $f_1(n) = 0$ if $n = 0$ and $f_1(n) = -1$ if $n > 0$. Now in the evaluation of $f_2(n)$, we get $c = n, d = 0$ and then $\delta = 0$ and $f_2(n) \neq -1$ if and only if $n p_2$ is a multiple of p_1 . This is not possible when $\text{gcd}(p_1, p_2) = 1$ and when $d = \text{gcd}(p_1, p_2) > 1$, with $p_1 = d p'_1$ and $p_2 = d p'_2$, choosing $n = p'_1$ gives $n p_2 = p'_1 p_2 = p'_2 p_1$ and thus $f_2(p'_1) = p'_2$. \square

The following lemma allows us to multiply the parameters q_1, p_1, q_2, p_2 by a same integer which is an already well known result.

Lemma 3. *Let q_1, p_1, q_2, p_2 be integers with $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$ and $d > 1$ an integer. Then $D(q_1, p_1)$ and $E(q_2, p_2)$ commute if and only if $D(d q_1, d p_1)$ and $E(d q_2, d p_2)$ commute.*

Proof.– Let us introduce an operator K_d that collects d consecutive symbols of the input sequence in a unique symbol of a new type

$$K_d(x[dn + r], 0 \leq r < d) = X[n] = \sum_{r=0}^{d-1} x[dn + r] T^r, \quad (15)$$

where T is a variable, and L_d the inverse operator

$$L_d\left(\sum_{r=0}^{d-1} x[dn+r]T^r\right) = (x[dn+r], 0 \leq r < d). \quad (16)$$

The following equations are then verified

$$\begin{aligned} D(dq_1, dp_1) &= L_d D(q_1, p_1) K_d, \\ E(dq_2, dp_2) &= L_d E(q_2, p_2) K_d, \end{aligned}$$

and thus

$$\begin{aligned} D(dq_1, dp_1)E(dq_2, dp_2) &= [L_d D(q_1, p_1)K_d][L_d E(q_2, p_2)K_d], \\ &= L_d D(q_1, p_1)E(q_2, p_2)K_d, \end{aligned}$$

while

$$\begin{aligned} E(dq_2, dp_2)D(dq_1, dp_1) &= [L_d E(q_2, p_2)K_d][L_d D(q_1, p_1)K_d], \\ &= L_d E(q_2, p_2)D(q_1, p_1)K_d, \end{aligned}$$

which proves the lemma. \square

Lemma 4. *Let q_1, p_1, q_2, p_2 be integers with $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$. If $q_1 > q_2$ then $f_1(q_1 - 1) > f_2(q_1 - 1)$.*

Proof.– With $n = q_1 - 1$ in function f_1 , we get $a = 0$ and $b = q_1 - 1$. As $b < q_1$, α and β are such that $\alpha q_2 + \beta = q_1 - 1$, $0 \leq \beta < q_2$ and $\alpha > 0$ because $q_1 > q_2$. then $f_1(q_1 - 1) = \alpha p_2 + \beta$.

On the other hand, in function f_2 for $n = q_1 - 1$, $c = \alpha$ and $d = \beta$. Then $\alpha p_2 + \beta = \gamma p_1 + \delta$ with $0 \leq \delta < p_1$. If $\gamma = 0$, then $\delta = \alpha p_2 + \beta > \alpha q_2 + \beta = q_1 - 1$ and thus $f_2(q_1 - 1) = -1$. When $\gamma > 0$, if $\delta \geq q_1$ then $f_2(q_1 - 1) = -1$. If $\gamma > 0$ and $\delta < q_1$ then $f_2(q_1 - 1) = \gamma q_1 + \delta < \gamma p_1 + \delta = \alpha p_2 + \beta = f_1(q_1 - 1)$. \square

Lemma 5. *Let q_1, p_1, q_2, p_2 be integers with $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$. If $q_1 < q_2$ then $g_1(q_2 - 1) > g_2(q_2 - 1)$.*

Proof.– The result immediately follows from Lemma 4 and Lemma 1. \square

Let us now consider the case of equal block lengths $q_1 = q_2 = q$.

Lemma 6. *Let q, p_1, q_2, p_2 be integers, $1 \leq q \leq p_2 < p_1$ such that $p_1 = kq + r, p_2 = lq + s$ with $0 \leq r < s < q$. Then $f_1(p_1 + q - s) = p_2 + q - s$ and $f_2(p_1 + q - s) = -1$.*

Proof.– In the computation of $f_1(p_1 + q - s)$, we get $a = 1$, $b = q - s$ and as $0 < b < q$. Now $aq + b = 2q - s$ and thus $\alpha = 1$ and $\beta = q - s$ and we obtain $f_1(p_1 + q - s) = \alpha p_2 + \beta = p_2 + q - s$.

From $f_2(p_1 + q - s) = f_2(kq + r + q - s)$, we get $c = k$ and $d = r + q - s$ because $0 \leq r \leq r + q - s < q$. Then

$$cp_2 + d = k(lq + s) + r + q - s = l(kq + r) + (k - 1)s - (l - 1)r + q,$$

Condition $p_1 > p_2$ implies that $k > l$ and thus $(k - 1)s - (l - 1)r + q > q$. As $s < q$, $(k - 1)s - (l - 1)r + q < kq < p_1$, we get $cp_2 + d = \gamma p_1 + \delta$ with $\gamma = l$ and $\delta = (k - 1)s - (l - 1)r + q > q$. So $f_2(p_1 + q - s) = -1$. \square

Lemma 7. *Let q, p_1, q_2, p_2 be integers, $1 \leq q \leq p_2 < p_1$ such that $p_1 = kq + r, p_2 = lq + s$ with $r > 0$ and $0 \leq s \leq r < q$. Then $f_1(p_1 + q - 1) > f_2(p_1 + q - 1) \geq 0$.*

Proof.– As before, we get $f_1(p_1 + q - 1) = p_2 + q - 1 = (l + 1)q + s - 1$. In the computation of $f_2(p_1 + q - 1)$, we have $p_1 + q - 1 = (k + 1)q + r - 1$ and since $r > 0$, $c = k + 1, d = r - 1$. Then γ and δ are determined by $\gamma p_1 + \delta = cp_2 + d = (k + 1)(lq + s) + r - 1$. If $q \leq \delta < p_1$, $f_2(p_1 + q - 1) = -1$ and the lemma is proved. If $0 \leq \delta < q$, we use

$$\begin{aligned} (k + 1)(lq + s) + r - 1 &= \\ (l + 1)(kq + r) - (k - l)q - (l + 1)r + (k + 1)s + r - 1. \end{aligned}$$

Then we remark that

$$\begin{aligned} (k-l)q + (l+1)r - (k+1)s &\geq (k-l)q \\ &> (k-l)q + (l+1)(r-s) > 0, \end{aligned}$$

because $r \geq s, k \geq l$ and $k = l, r = s$ is not possible as $p_1 > p_2$. It follows that

$$\gamma p_1 + \delta < (l+1)p_1 + r - 1, \quad 0 \leq \delta < q,$$

and thus, because the application ϕ defined on integers n such that $n = kp_1 + d$ with $0 \leq d < q$ by $\phi(n) = \phi(kp_1 + d) = kq + d$ is a strictly increasing function,

$$f_2(p_1 + q - 1) = \gamma q + \delta < (l+1)q + r - 1 = f_1(p_1 + q - 1).$$

□

Proof of the theorem.— When $q_1 \neq q_2$ Lemmas 2 and 3 prove that $D(q_1, p_1)$ and $E(q_2, p_2)$ cannot commute. Then, when $p_2 < p_1$, the only case not considered in lemmas 6 and 7 is the case where $r = s = 0$, i.e. $p_1 = kq$ and $p_2 = lq$ with $l < k$, and thus $D(q_1, p_1)$ and $E(q_2, p_2)$ cannot commute when p_1 or p_2 are not multiples of q . Using the symmetry property given by lemma 1, we obtain the same result for $p_2 > p_1$. When $q_1 = q_2 = q$ and $p_1 = p_2 = p$ with $q < p$, $D(q, p)$ do not commute with $E(q, p)$ because $D(q, p)E(q, p)$ is the identity while in $E(q, p)D(q, p)$ the symbol $x[q]$ is discarded, i.e. $f_1(q) = -1$, $f_2(q) = q$. Using Lemma 3 and then Lemma 2, we then deduce that if $D(q_1, p_1)$ and $E(q_2, p_2)$ commute then $q_1 = q_2 = \gcd(p_1, p_2) = 1$ which is the direct part of the theorem. The converse part also results immediately from Lemmas 2 and 3. □

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