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A predictive Stein's effect even in the low-dimensional case

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Abstract: In this work, we are concerned with the estimation of the predictive density of a Gaussian random vector where both the mean and the variance are unknown. In such a context, we prove the inadmissibility of the best equivariant predictive density under the Kullback-Leibler risk in a nonasymptotic framework. Our result stands whatever the dimension d of the vector is, even when $d \leq 2$, which can be somewhat surprising compared to the known variance setting. We also propose a class of priors leading to a Bayesian predictive density that dominates the best equivariant one. Throughout the article, we give several elements that we believe are useful for establishing the parallel between the prediction and the estimation problems, as it was done in the known variance framework.

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1. Context

The most natural way to estimate an unobserved quantity $\boldsymbol{\mu}$ is to use observed averages. However, [Stein \(1956\)](#) demonstrated the inadmissibility of such estimators for dimensions strictly greater than two, that is, he showed that there exist better estimators. The notion of better estimator means here that it has a lower quadratic risk

$$R_Q(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}) = \mathbb{E}_{\boldsymbol{\mu}}[\|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2],$$

where $\hat{\boldsymbol{\mu}}$ is an estimator of $\boldsymbol{\mu}$ and the expectation is taken with respect to observed random quantities used in the construction of $\hat{\boldsymbol{\mu}}$. This phenomenon is the now very well-known Stein effect, which was first brought to light in the context of parameter estimation. For such a problem, many classes of estimators dominating the average have been proposed.

In parallel, a similar phenomenon has been observed for the predictive density estimation problem. For instance, let us observe n independent vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ of size d , each supposedly normally distributed, $\boldsymbol{x}_i \sim \mathcal{N}_d(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_d)$, $1 \leq i \leq n$, where the common mean $\boldsymbol{\mu}$ is unknown, and the variance is first assumed to be known. The aim is to estimate the density of a future vector \boldsymbol{y} , which is also assumed to be normal with same mean and variance, $\boldsymbol{y} \sim \mathcal{N}_d(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_d)$. In this context, the mean $\bar{\boldsymbol{x}} = n^{-1} \sum_{i=1}^n \boldsymbol{x}_i$ is a sufficient statistic used for the

estimation of the predictive density by a function $\hat{p}(\mathbf{y}|\bar{\mathbf{x}})$. The quality of such an estimator is often measured by the Kullback-Leibler risk, given by

$$R_{KL}(\boldsymbol{\mu}, \hat{p}) = \int p(\bar{\mathbf{x}}|\boldsymbol{\mu}) \int p(\mathbf{y}|\boldsymbol{\mu}) \log \frac{p(\mathbf{y}|\boldsymbol{\mu})}{\hat{p}(\mathbf{y}|\bar{\mathbf{x}})} d\mathbf{y} d\bar{\mathbf{x}},$$

where $p(\cdot|\boldsymbol{\mu})$ is the Gaussian density. The most natural way to estimate $p(\mathbf{y}|\boldsymbol{\mu})$ here would be to just plug in an estimator of $\boldsymbol{\mu}$, $\hat{p}(\mathbf{y}|\bar{\mathbf{x}}) = p(\mathbf{y}|\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}(\bar{\mathbf{x}}))$. However, [Aitchison \(1975\)](#) proved that plug-in densities are uniformly dominated under the Kullback-Leibler risk by the *best equivariant predictive density*, given with respect to the uniform prior $\pi_U(\boldsymbol{\mu}) = 1$ by

$$\hat{p}_U(\mathbf{y}|\bar{\mathbf{x}}) = \frac{1}{(2\pi\sigma^2)^{d/2}} \left\{ \frac{n}{n+1} \right\}^{d/2} \exp\left(-\frac{n}{n+1} \frac{\|\mathbf{y} - \bar{\mathbf{x}}\|^2}{2\sigma^2}\right).$$

[George, Liang and Xu \(2006\)](#) have shown that predictive density estimation and parameter estimation are in fact related. The authors also argue that the best equivariant predictive density $\hat{p}_U(\mathbf{y}|\bar{\mathbf{x}})$ shares many properties with the maximum likelihood estimator: minimaxity, invariance, constant risk, and inadmissibility for dimension greater or equal to 3. Hence, it should be taken as the reference estimator to be improved upon. [Komaki \(2001\)](#) actually showed that, when $d \geq 3$, it is dominated by the Bayesian predictive density with respect to the *harmonic prior* (also known as the *shrinkage prior*)

$$\pi_H(\boldsymbol{\mu}) = \|\boldsymbol{\mu}\|^{-(d-2)}. \quad (1.1)$$

A more interesting and practical setting is the one where the variance is unknown. In such a case, the sufficient statistic is $(\bar{\mathbf{x}}, s)$, where $s = \sum_{i=1}^n \|\mathbf{x}_i - \bar{\mathbf{x}}\|^2$ estimates the variance (up to a factor). The best equivariant predictive density is now taken with respect to the right invariant prior $\pi_R(\boldsymbol{\mu}, \sigma) = 1/\sigma$ and is given by

$$\hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}, s) = c_R \left\{ \frac{1}{\pi s} \cdot \frac{n}{n+1} \right\}^{d/2} \left(1 + \frac{1}{s} \frac{n}{n+1} \|\mathbf{y} - \bar{\mathbf{x}}\|^2 \right)^{-nd/2}, \quad (1.2)$$

where $c_R = \Gamma(nd/2)/\Gamma(\{n-1\}d/2)$ and Γ denotes the Gamma function. Note that \hat{p}_R is actually a Student t -distribution (see for instance [Liang and Barron, 2004](#)). The Kullback-Leibler risk now becomes

$$R_{KL}(\{\boldsymbol{\mu}, \sigma^2\}, \hat{p}) = \iint p(\bar{\mathbf{x}}|\boldsymbol{\mu}, \eta) p(s|\eta) \int p(\mathbf{y}|\boldsymbol{\mu}, \eta) \log \frac{p(\mathbf{y}|\boldsymbol{\mu}, \eta)}{\hat{p}(\mathbf{y}|\bar{\mathbf{x}}, \eta)} d\mathbf{y} d\bar{\mathbf{x}} ds.$$

Sofar, there are very few results on admissibility of \hat{p}_R in the literature. Three works attracted our attention, though, and have been of great influence for the present paper. The first one is due to [Kato \(2009\)](#), who showed that, for $d \geq 3$ and under the Kullback-Leibler risk, the best equivariant density \hat{p}_R is

dominated by the Bayesian predictive density with respect to the prior

$$\begin{aligned}\pi_{LT}(\boldsymbol{\mu}, \eta) &= \int_0^1 \pi(\boldsymbol{\mu}|\eta, \lambda) \pi(\eta, \lambda) d\lambda \\ \boldsymbol{\mu}|\eta, \lambda &\sim \mathcal{N}_d(0, \eta^{-1} \lambda^{-1} \{1 - \lambda\} \mathbf{I}_d) \\ \pi(\eta, \lambda) &= \eta^{-2} \lambda^{-2} \mathbf{1}_{(0,1)}(\lambda),\end{aligned}\tag{1.3}$$

where $\eta = 1/\sigma^2$. This prior corresponds to an extension of the harmonic prior given in Equation (1.1) and of Komaki (2001)'s work to the unknown variance assumption since π_{LT} is proportional to $\|\boldsymbol{\mu}\|^{-(d-2)} \eta^{-1}$.

Komaki (2006, 2007) himself proposed another extension of his own work. His idea is the following one: noticing that the harmonic prior π_H , given in Equation (1.1), is actually proportional to the so-called Green function in the Euclidean d -dimensional space (see Komaki, 2006, for a definition), he proposes to consider as a prior the Green function of the manifold of the unknown variance model, which is a hyperbolic plane. Although the form of this prior can look complicated, its limits actually correspond to the right invariant prior on one side and, when $d = 1$, to the Cauchy density on the other side. But the most interesting part of this article is that the best equivariant predictive density is shown to be *asymptotically* inadmissible even in the low-dimensional case where $d = 1$ or 2 .

Finally, another interesting work is the one by Maruyama and Strawderman (2012), where the authors uses an extension of the prior given by Equation (1.3) with the following change

$$\pi(\eta, \lambda) = \eta^a \lambda^a (1 - \lambda)^b \mathbf{1}_{(0,1)}(\lambda).\tag{1.4}$$

This prior leads to an estimator of the density dominating the best equivariant density also in the low dimensional case, where $d = 1$ or 2 . Unlike Komaki (2007)'s result, the latter one was obtained in a nonasymptotic framework, and under the following risk

$$R_1(\{\boldsymbol{\mu}, \eta\}, \hat{p}) = \iint p(\bar{\boldsymbol{x}}|\boldsymbol{\mu}, \eta) p(s|\eta) \int \hat{p}(\mathbf{y}|\bar{\boldsymbol{x}}, s) \log \left(\frac{\hat{p}(\mathbf{y}|\bar{\boldsymbol{x}}, s)}{p(\mathbf{y}|\boldsymbol{\mu}, \eta)} \right) d\mathbf{y} d\bar{\boldsymbol{x}} ds,$$

which is denoted by R_1 because it corresponds to Csiszár's α -divergence with $\alpha = 1$.

In the present work, we consider the Kullback-Leibler risk, which is also a special case of Csiszár's α -divergence with $\alpha = -1$. We are interested in the context of unknown variance and we show that the inadmissibility of the best equivariant predictive density \hat{p}_R for any d , even $d = 1$ or 2 , is still true under the Kullback-Leibler risk, hence enlarging Kato (2009)'s result. We also partially solve Problem 2-2 stated by Maruyama and Strawderman (2012), namely, *under $d = 1, 2$ and the R_α risk with $-1 \leq \alpha < 1$, does the best invariant predictive density keep inadmissibility? If so, which Bayesian predictive densities improve it?* We consider a different extension of the shrinkage prior than that of

Komaki (2007), but the major difference lies in the fact that we have nonasymptotic results. Such results are actually derived under the Gaussian assumption, whereas Komaki (2007) considered a more general distributional assumption. In this work, we also establish a preliminary basis for comparing the estimation and the prediction problems for the unknown variance setting. Such a comparison is however merely qualitative, and proving it formally is still an open problem.

The rest of this paper is organized as follows. First, Section 2 gives the general expression of the Bayesian predictive density with respect to any prior π as a function of the best equivariant predictive density for the unknown variance case. In Section 3, we specify the class of priors we consider and state our main result on inadmissibility of the best equivariant predictive density. Then, Section 4 shows visualizations of this result through a simulation study. Finally, Section 5 presents a discussion on a possible parallel between the estimation and the prediction problems.

2. The Bayesian predictive density under unknown variance

Recall that we want to estimate the predictive density of

$$\mathbf{y} \sim \mathcal{N}_d(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_d), \quad (2.1)$$

based on the observations $\mathbf{x}_1, \dots, \mathbf{x}_n$, where $\mathbf{x}_i \sim \mathcal{N}_d(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_d)$.

In this section, we aim at extending the famous result by George, Liang and Xu (2006) expressing a Bayesian predictive density with respect to a prior π as a function of the best equivariant one, which we recall here

$$\hat{p}_\pi(\mathbf{y}|\bar{\mathbf{x}}_n) = \frac{m_\pi(\bar{\mathbf{x}}_{n+1}; \{n+1\}^{-1}\eta^{-1})}{m_\pi(\bar{\mathbf{x}}_n; n^{-1}\eta^{-1})} \hat{p}_U(\mathbf{y}|\bar{\mathbf{x}}_n),$$

where $\bar{\mathbf{x}}_n = \bar{\mathbf{x}}$, $\bar{\mathbf{x}}_{n+1} = (n\bar{\mathbf{x}}_n + \mathbf{y})/(n+1)$, and $m_\pi(\mathbf{z}; \sigma^2)$ is the marginal density defined as

$$m_\pi(\mathbf{z}; \sigma^2) = \int \frac{1}{(2\pi\sigma^2)^{d/2}} \exp\left(-\frac{\|\mathbf{z} - \boldsymbol{\mu}\|^2}{2\sigma^2}\right) \pi(\boldsymbol{\mu}) d\boldsymbol{\mu}.$$

Theorem 2.1 gives a similar expression for the unknown variance case. Before stating it, note that, in this case, the expression relies on both the sufficient statistic $(\bar{\mathbf{x}}, s)$ and on the statistic $(\bar{\mathbf{x}}_{n+1}, s_{n+1})$, with

$$\begin{aligned} \bar{\mathbf{x}}_{n+1} &\sim \mathcal{N}_d(\boldsymbol{\mu}, \{n+1\}^{-1}\eta^{-1}\mathbf{I}_d) \\ s_{n+1} &= s_n + n\|\mathbf{y} - \bar{\mathbf{x}}_n\|^2/(n+1) \sim \eta^{-1}\chi_{nd}^2, \end{aligned} \quad (2.2)$$

where $s_n = s$. Moreover, the marginal density $m_\pi(\mathbf{z}, v; l)$ is now defined for $\mathbf{z} \sim \mathcal{N}_d(\boldsymbol{\mu}, \{l\eta\}^{-1}\mathbf{I}_d)$ and $v \sim \chi_{(l-1)d}^2$ by

$$\begin{aligned} m_\pi(\mathbf{z}, v; l) &= \iint \frac{l^{d/2}\eta^{d/2}}{(2\pi)^{d/2}} \exp\left(-\frac{l\eta\|\mathbf{z} - \boldsymbol{\mu}\|^2}{2}\right) \\ &\quad \times \frac{\eta^{(l-1)d/2}v^{(l-1)d/2-1}}{\Gamma(\{l-1\}d/2)} \frac{1}{2^{(l-1)d/2}} \exp\left(-\frac{\eta v}{2}\right) \pi(\boldsymbol{\mu}, \eta) d\boldsymbol{\mu} d\eta. \end{aligned} \quad (2.3)$$

We thus have the following extension of Lemma 2 of [George, Liang and Xu \(2006\)](#) for the unknown variance case. Unfortunately, we have not been able to link such an expression directly to the estimation problem, like it has been done by [George, Liang and Xu \(2006\)](#), but we believe that the link exists and we give in Section 5 some leads to uncover it.

Theorem 2.1. *For any prior $\pi(\boldsymbol{\mu}, \eta)$, the Bayesian predictive density $\hat{p}_\pi(\mathbf{y}|\bar{\mathbf{x}}_n, s_n)$ can be expressed as*

$$\hat{p}_\pi(\mathbf{y}|\bar{\mathbf{x}}_n, s_n) = \frac{\rho_\pi(\bar{\mathbf{x}}_{n+1}, s_{n+1}; n+1)}{\rho_\pi(\bar{\mathbf{x}}_n, s_n; n)} \cdot \hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n, s_n),$$

where $\hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n, s_n)$ is the best equivariant predictive density and ρ_π is the function defined as

$$\rho_\pi(\mathbf{z}, v; l) = v \cdot m_\pi(\mathbf{z}, v; l). \quad (2.4)$$

Furthermore, the difference between the Kullback-Leibler risks of \hat{p}_π and \hat{p}_R is given by

$$R_{KL}(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R_{KL}(\{\boldsymbol{\mu}, \eta\}, \hat{p}_\pi) = \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \frac{\rho_\pi(\bar{\mathbf{x}}_{n+1}, s_{n+1}; n+1)}{\rho_\pi(\bar{\mathbf{x}}_n, s_n; n)} \right],$$

provided the expectations exist.

Proof. The Bayesian predictive density is computed by

$$\hat{p}_\pi(\mathbf{y}|\bar{\mathbf{x}}_n, s_n) = p_\pi(\mathbf{y}, \bar{\mathbf{x}}_n, s_n) / m_\pi(\bar{\mathbf{x}}_n, s_n; n), \quad (2.5)$$

where $p_\pi(\mathbf{y}, \bar{\mathbf{x}}_n, s_n)$ is the joint density. Let us compute this joint density

$$\begin{aligned} p_\pi(\mathbf{y}, \bar{\mathbf{x}}_n, s_n) &= \iint p(\mathbf{y}|\boldsymbol{\mu}, \eta) p(\bar{\mathbf{x}}_n|\boldsymbol{\mu}, \eta) p(s_n|\eta) \pi(\boldsymbol{\mu}, \eta) d\boldsymbol{\mu} d\eta \\ &= \iint \frac{n^{d/2} \eta^{(n+1)d/2} s_n^{(n-1)d/2-1}}{\Gamma(\{n-1\}d/2) 2^{(n-1)d/2} (2\pi)^d} \\ &\quad \times \exp\left(-\frac{\eta}{2} \{\|\mathbf{y} - \boldsymbol{\mu}\|^2 + n\|\bar{\mathbf{x}}_n - \boldsymbol{\mu}\|^2 + s_n\}\right) \pi(\boldsymbol{\mu}, \eta) d\boldsymbol{\mu} d\eta. \end{aligned}$$

Further, noticing that

$$\|\mathbf{y} - \boldsymbol{\mu}\|^2 + n\|\bar{\mathbf{x}}_n - \boldsymbol{\mu}\|^2 = (n+1)\|\bar{\mathbf{x}}_{n+1} - \boldsymbol{\mu}\|^2 + \frac{n}{n+1}\|\mathbf{y} - \bar{\mathbf{x}}_n\|^2,$$

we can reexpress the joint density as

$$\begin{aligned} p_\pi(\mathbf{y}, \bar{\mathbf{x}}_n, s_n) &= \iiint \left\{ \frac{n}{n+1} \right\}^{d/2} \frac{s_n^{(n-1)d/2-1}}{\Gamma(\{n-1\}d/2) \pi^{d/2}} \\ &\quad \times \frac{(n+1)^{d/2} \eta^{d/2}}{(2\pi)^{d/2}} \exp\left(-\frac{\{n+1\}\eta}{2} \|\bar{\mathbf{x}}_{n+1} - \boldsymbol{\mu}\|^2\right) \\ &\quad \times \frac{\Gamma(nd/2) \eta^{nd/2}}{\Gamma(nd/2) 2^{nd/2}} \left\{ \frac{\frac{n}{n+1} \|\mathbf{y} - \bar{\mathbf{x}}_n\|^2 + s_n}{\frac{n}{n+1} \|\mathbf{y} - \bar{\mathbf{x}}_n\|^2 + s_n} \right\}^{nd/2-1} \\ &\quad \times \exp\left(-\frac{\eta}{2} \left\{ \frac{n}{n+1} \|\mathbf{y} - \bar{\mathbf{x}}_n\|^2 + s_n \right\}\right) \pi(\boldsymbol{\mu}, \eta) d\boldsymbol{\mu} d\eta. \end{aligned}$$

Now, we easily recognize the densities of $\bar{\mathbf{x}}_{n+1}$ and s_{n+1} from Equation (2.2) and the best equivariant density $\hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n)$, yielding

$$p_\pi(\mathbf{y}, \bar{\mathbf{x}}_n, s_n) = \hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n, s_n) \iint \frac{s_{n+1}}{s_n} p(\bar{\mathbf{x}}_{n+1}|\boldsymbol{\mu}, \eta) p(s_{n+1}|\eta) \pi(\boldsymbol{\mu}, \eta) d\boldsymbol{\mu} d\eta.$$

Replacing the joint density by this latter expression in Equation (2.5), we obtain the desired result. The result on the difference in risks follows immediately from the fact that

$$R_{KL}(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R_{KL}(\{\boldsymbol{\mu}, \eta\}, \hat{p}_\pi) = \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \frac{\hat{p}_\pi(\mathbf{y}|\bar{\mathbf{x}}_n, s_n)}{\hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n, s_n)} \right]. \quad (2.6)$$

□

In the following section, we specify the class of priors we consider for this work and prove the inadmissibility of the best equivariant predictive density.

3. Main result

In this section, we propose to study the Bayesian predictive density with respect to the following improper prior

$$\begin{aligned} \pi_{GM}(\boldsymbol{\mu}, \eta) &= \int_0^1 \pi(\boldsymbol{\mu}|\eta, \lambda) \pi(\eta, \lambda) d\lambda \\ \boldsymbol{\mu}|\eta, \lambda &\sim \mathcal{N}_d(0, \eta^{-1} \lambda^{-1} \{1 - \lambda\} \mathbf{I}_d) \\ \pi(\eta, \lambda) &= \eta^a \lambda^a \tilde{\pi}(\lambda) \mathbf{1}_{(0,1)}(\lambda), \end{aligned} \quad (3.1)$$

where we recall that $\eta = 1/\sigma^2$ and the subscript *GM* stands for Gaussian mixture. Note that the priors studied by Kato (2009) and Maruyama and Strawderman (2012) are special cases of this new prior with $a = -2$ and $\tilde{\pi}(\lambda) = 1$ for the former one and $\tilde{\pi}(\lambda) = (1 - \lambda)^b$ for the latter one.

We first give the expression of the Bayesian predictive density associated with π_{GM} in the following lemma, and then state our main results of domination and inadmissibility in the subsequent theorems. In the sequel, we denote the Kullback-Leibler risk by R instead of R_{KL} for ease of notation.

3.1. Expression of the new Bayesian predictive density

The prior π_{GM} defined in Equation (3.1) leads to the following Bayesian predictive density.

Lemma 3.1. *Under the assumptions of Model (2.1), the Bayesian predictive density with respect to the prior π_{GM} in Equation (3.1) is given by*

$$\hat{p}_{GM}(\mathbf{y}|\bar{\mathbf{x}}_n, s_n) = c_{GM} \left\{ \frac{s_{n+1}}{s_n} \right\}^{-\nu} \frac{J_{n+1}}{J_n} \hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n, s_n),$$

where $c_{GM} = \{1 + 1/n\}^\nu B(\nu, \{n-1\}d/2) / B(\nu, nd/2)$ is a constant and the term J_l , $l \in \{n; n+1\}$, is defined as

$$J_l = \int_0^1 \frac{\lambda^{\nu-1} \{1 + (l-1)\lambda\}^{d/2-\nu-1}}{\{1 + w_l \lambda\}^{(l-1)d/2+\nu}} \tilde{\pi}\left(\frac{l\lambda}{1 + \{l-1\}\lambda}\right) d\lambda, \quad (3.2)$$

where $\nu = d/2 + a + 1$, $B(\cdot, \cdot)$ denotes the Beta function and $w_l = l\|\bar{\mathbf{x}}_l\|^2/s_l$.

Proof. From Theorem 2.1, we only have to compute $\rho_\pi(\bar{\mathbf{x}}_{n+1}, s_{n+1}; n+1)$ and $\rho_\pi(\bar{\mathbf{x}}_n, s_n; n)$. Let us start by computing the expression of $\rho_\pi(\mathbf{z}, v; l)$ with respect to the prior defined in Equation (3.1):

$$\begin{aligned} \rho_\pi(\mathbf{z}, v; l) &= \iiint \frac{1}{\Gamma(\{l-1\}d/2)} \frac{l^{d/2} v^{(l-1)d/2}}{(2\pi)^d 2^{(l-1)d/2}} \eta^{a+(l+1)d/2} \lambda^{a+d/2} \tilde{\pi}(\lambda) \\ &\quad \times (1-\lambda)^{-d/2} \exp\left(-\frac{\eta}{2} \left\{ l\|\mathbf{z} - \boldsymbol{\mu}\|^2 + v + \frac{\lambda}{1-\lambda} \|\boldsymbol{\mu}\|^2 \right\}\right) d\boldsymbol{\mu} d\eta d\lambda. \end{aligned}$$

We first integrate with respect to $\boldsymbol{\mu}$. In order to do so, we develop the squares involving $\boldsymbol{\mu}$ in the exponential term

$$l\|\mathbf{z} - \boldsymbol{\mu}\|^2 + \frac{\lambda}{1-\lambda} \|\boldsymbol{\mu}\|^2 = \frac{l-(l-1)\lambda}{1-\lambda} \left\| \boldsymbol{\mu} - \frac{(1-\lambda)l\mathbf{z}}{l-(l-1)\lambda} \right\|^2 + \frac{l\lambda}{l-(l-1)\lambda} \|\mathbf{z}\|^2.$$

Noticing that the integration with respect to the location parameter $\boldsymbol{\mu}$ corresponds to the Gaussian density with scale parameter $(1-\lambda)/\{l-(l-1)\lambda\}\eta$, we can easily compute the following integral

$$\int \exp\left(-\frac{\eta}{2} \frac{l-(l-1)\lambda}{1-\lambda} \left\| \boldsymbol{\mu} - \frac{(1-\lambda)l\mathbf{z}}{l-(l-1)\lambda} \right\|^2\right) d\boldsymbol{\mu} = \left\{ \frac{2\pi}{\eta} \frac{1-\lambda}{l-(l-1)\lambda} \right\}^{d/2}.$$

Hence, the function ρ_π becomes

$$\begin{aligned} \rho_\pi(\mathbf{z}, v; l) &= \iint \frac{1}{\Gamma(\{l-1\}d/2)} \frac{l^{d/2} v^{(l-1)d/2}}{(2\pi)^{d/2} 2^{(l-1)d/2}} \eta^{a+ld/2} \lambda^{a+d/2} \tilde{\pi}(\lambda) \\ &\quad \times \{l-(l-1)\lambda\}^{-d/2} \exp\left(-\frac{\eta}{2} \left\{ \frac{l\lambda}{l-(l-1)\lambda} \|\mathbf{z}\|^2 + v \right\}\right) d\eta d\lambda. \end{aligned}$$

We now turn to the integration with respect to the precision parameter η . It can be easily seen that such an integral is related to Gamma distributions, giving the following equality:

$$\begin{aligned} &\int_0^\infty \eta^{(l-1)d/2+\nu-1} \exp\left(-\frac{\eta}{2} \left\{ \frac{l\lambda}{l-(l-1)\lambda} \|\mathbf{z}\|^2 + v \right\}\right) d\eta \\ &= \Gamma\left(\frac{\{l-1\}d}{2} + \nu\right) \left\{ \frac{1}{2} \left(\frac{l\lambda}{l-(l-1)\lambda} \|\mathbf{z}\|^2 + v \right) \right\}^{-(l-1)d/2-\nu}, \end{aligned}$$

where we recall that $\nu = d/2 + a + 1$. Replacing the integral in η by this latter expression in the function ρ_π thus yields

$$\begin{aligned} \rho_\pi(\mathbf{z}, v; l) &= \int \frac{\Gamma(\{l-1\}d/2 + \nu)}{\Gamma(\{l-1\}d/2)} \frac{l^{d/2} 2^\nu}{(2\pi)^{d/2} v^\nu} \lambda^{\nu-1} \tilde{\pi}(\lambda) \\ &\quad \times \{l - (l-1)\lambda\}^{-d/2} \left\{ 1 + \frac{\lambda}{l - (l-1)\lambda} \frac{l\|\mathbf{z}\|^2}{v} \right\}^{-(l-1)d/2-\nu} d\lambda. \end{aligned}$$

Finally, we apply the change in variable $t = \lambda/\{l - (l-1)\lambda\}$, with $d\lambda = l/\{1 + (l-1)t\}^2 dt$

$$\begin{aligned} \rho_\pi(\mathbf{z}, v; l) &= \int \frac{\Gamma(\{l-1\}d/2 + \nu)}{\Gamma(\{l-1\}d/2)} \frac{l^\nu 2^\nu}{(2\pi)^{d/2} v^\nu} t^{\nu-1} \tilde{\pi}\left(\frac{lt}{1 + \{l-1\}t}\right) \\ &\quad \times \{1 + (l-1)t\}^{d/2-\nu-1} \{1 + wt\}^{-(l-1)d/2-\nu} d\lambda \\ &= \frac{\Gamma(\{l-1\}d/2 + \nu)}{\Gamma(\{l-1\}d/2)} \frac{l^\nu 2^\nu}{(2\pi)^{d/2} v^\nu} J_l, \end{aligned}$$

where $w = l\|\mathbf{z}\|^2/v$ and J_l is given in Equation (3.2). Replacing (\mathbf{z}, v) by $(\bar{\mathbf{x}}_{n+1}, s_{n+1})$ and $(\bar{\mathbf{x}}_n, s_n)$ successively yields

$$\hat{p}_{GM}(\mathbf{y}|\bar{\mathbf{x}}_n, s_n) = \frac{B(\nu, \{n-1\}d/2)}{B(\nu, nd/2)} \left\{ \frac{n s_{n+1}}{(n+1) s_n} \right\}^{-\nu} \frac{J_{n+1}}{J_n} \hat{p}_R(\mathbf{y}|\bar{\mathbf{x}}_n, s_n),$$

where the ratio of Gamma functions has been reformulated thanks to the relationship between Gamma and Beta functions. \square

In the sequel, we separate two cases, the low-dimensional case where $1 \leq d \leq 4$, and the higher-dimensional case where $d \geq 3$. In both cases, we obtain a domination result.

3.2. Domination for the low dimensional case ($1 \leq d \leq 4$)

In this section, we assume $n = 2$, $0 < \nu < 1$, and we consider the following special case for the prior in Equation (3.1)

$$\tilde{\pi}(\lambda) = \{1 - \lambda\}^{(n-1)d/2-1} \left\{ 1 - \frac{n-1}{n} \lambda \right\}^{-(n-2)d/2-\nu}. \quad (3.3)$$

Theorem 3.1. *Let $n = 2$, $1 \leq d \leq 4$, $0 < \nu < 1$ and $\tilde{\pi}$ be specified by Equation (3.3). Then, there exists a constant ν_* depending only on the dimension d such that the inequality*

$$R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_{GM}) \geq 0$$

holds for any $\{\boldsymbol{\mu}, \eta\} \in \mathbb{R}^d \times \mathbb{R}_+^*$ and $0 < \nu \leq \nu_*$.

In view of Theorem 3.1, we can state our main result on the inadmissibility of the best equivariant predictive density \hat{p}_R . Note that this result solves Problem 2.2 stated by Maruyama and Strawderman (2012) for the Kullback-Leibler risk. Also, a similar result has been obtained by Komaki (2007), but only asymptotically with the number of observations ($n \rightarrow \infty$), whereas here we give it in a nonasymptotic framework (fixed n and d).

Theorem 3.2. *If $d \leq 4$ and $n = 2$, the best equivariant density \hat{p}_R is inadmissible under the Kullback-Leibler risk and the unknown variance setting.*

In order to prove Theorem 3.1, we first need the three following lemmas.

Lemma 3.2. *Assume $n \geq 2$, $0 < \nu < 1$, and $\tilde{\pi}$ be specified by Equation (3.3). Then*

$$\mathbb{E}_{(\mu, \eta)} \left[\log \left(\frac{B(\nu, \{n-1\}d/2)}{B(\nu, nd/2)} \frac{J_{n+1}}{J_n} \frac{\{1-u_n\}^\nu}{\{1-u_{n+1}\}^\nu} \right) \right] \geq \nu g(n, d, \nu) \mathbb{E}[1-u_{n+1}],$$

where $u_l = w_l/(1+w_l)$, $l \in \{n; n+1\}$, the expectation $\mathbb{E}[\cdot]$ is taken with respect to a noncentral Beta random variable, and the function g is defined by

$$g(n, d, \nu) = \begin{cases} \frac{1}{\nu} \log \frac{B(\nu, nd/2 - \{d-\nu\}/n)}{B(\nu, nd/2)} & d \leq n, \\ \frac{1}{n\nu} \frac{d-\nu}{nd/2-1} \log(1+\nu) & d \geq n+1. \end{cases} \quad (3.4)$$

Proof of Lemma 3.2. Note first that the choice done for $\tilde{\pi}$ leads, after some calculation, to the following equation

$$\begin{aligned} & \{1 + (l-1)\lambda\}^{d/2-\nu-1} \tilde{\pi} \left(\frac{l\lambda}{1 + \{l-1\}\lambda} \right) \\ &= \{1 - \lambda\}^{(n-1)d/2-1} \left\{ 1 + \left(\frac{l}{n} - 1 \right) \lambda \right\}^{-(n-2)d/2-\nu}. \end{aligned} \quad (3.5)$$

When $l = n$, the last term reduces to 1 and J_n thus becomes

$$J_n = \int_0^1 \lambda^{\nu-1} \{1 - \lambda\}^{(n-1)d/2-1} \{1 + w_n \lambda\}^{-(n-1)d/2-\nu} d\lambda.$$

Thus, applying Part 2 of Lemma A.1, we obtain

$$J_n = \{1 - u_n\}^\nu B(\nu, \{n-1\}d/2). \quad (3.6)$$

When $l = n+1$, the last term in Equation (3.5) is equal to a power of $(1 + \lambda/n)$ and can be bounded by

$$\{1 + \lambda/n\}^{-(n-2)d/2-\nu} \geq \{1 - \lambda\}^{(n-2)d/(2n)+\nu/n},$$

by Part 1 of Lemma A.2. Hence, we obtain for J_{n+1} the lower bound

$$J_{n+1} \geq \int_0^1 \lambda^{\nu-1} \{1 - \lambda\}^{nd/2-(d-\nu)/n-1} \{1 + w_{n+1} \lambda\}^{-nd/2-\nu} d\lambda.$$

Then, applying Part 1 of Lemma A.1, the right-hand side of the last inequality becomes

$$J_{n+1} \geq \{1 - u_{n+1}\}^\nu \int_0^1 t^{\nu-1} \{1 - t\}^{nd/2 - (d-\nu)/n - 1} \{1 - u_{n+1}t\}^{(d-\nu)/n} dt. \quad (3.7)$$

The term in $(1 - u_{n+1}t)$ still makes it complicated to compare to J_n and we next treat it in a different fashion for the cases $d \leq n$ and $d \geq n + 1$. In both cases, we assume that $0 < \nu < 1$.

Case $d \leq n$ In this case, since $0 < (d - \nu)/n < 1$, we can apply Part 2 of Lemma A.2 with $\alpha = (d - \nu)/n$, $\beta = u_{n+1}$ and $\gamma = t$, which gives

$$\{1 - u_{n+1}\}^{(d-\nu)/n} \geq \{1 - t\}^{(d-\nu)/n} + (1 - u_{n+1}) \left\{1 - (1 - t)^{(d-\nu)/n}\right\}.$$

This inequality results in the following lower bound for J_{n+1}

$$\begin{aligned} J_{n+1} &\geq \{1 - u_{n+1}\}^\nu \int_0^1 t^{\nu-1} \{1 - t\}^{nd/2 - (d-\nu)/n - 1} \\ &\quad \times \left(\{1 - t\}^{(d-\nu)/n} + (1 - u_{n+1}) \left\{1 - (1 - t)^{(d-\nu)/n}\right\} \right) dt \\ &= B(\nu, nd/2) \{1 - u_{n+1}\}^\nu \\ &\quad \times \left\{ 1 + (1 - u_{n+1}) \left(\frac{B(\nu, nd/2 - \{d - \nu\}/n)}{B(\nu, nd/2)} - 1 \right) \right\}. \end{aligned}$$

This new lower bound is more similar to the expression of J_n , but we can still simplify it further. Indeed, by Part 1 of Lemma A.2, we have

$$J_{n+1} \geq B(\nu, nd/2) \{1 - u_{n+1}\}^\nu \left\{ \frac{B(\nu, nd/2 - \{d - \nu\})}{B(\nu, nd/2)} \right\}^{1 - u_{n+1}}. \quad (3.8)$$

Case $d \geq n + 1$ In this case, since $(d - \nu)/n \geq 1$, we cannot apply the same technique. Instead, we will use Jensen's inequality.

First, notice that

$$\frac{1 - u_{n+1}t}{1 - t} = 1 + (1 - u_{n+1}) \frac{t}{1 - t}.$$

Hence, Inequality (3.7) becomes

$$J_{n+1} \geq \{1 - u_{n+1}\}^\nu \int_0^1 t^{\nu-1} \{1 - t\}^{nd/2 - 1} \left\{ 1 + \frac{(1 - u_{n+1})t}{1 - t} \right\}^{(d-\nu)/n} dt.$$

Applying the transformation $\lambda = t/(1 - t)$ to the integral of the right-hand side in the latter bound yields

$$\begin{aligned} &\int_0^1 t^{\nu-1} \{1 - t\}^{nd/2 - 1} \left\{ 1 + (1 - u_{n+1})t/(1 - t) \right\}^{(d-\nu)/n} dt \\ &= \int_0^\infty \lambda^{\nu-1} \{1 + \lambda\}^{-nd/2 - \nu} \{1 + (1 - u_{n+1})\lambda\}^{(d-\nu)/n} d\lambda \\ &= B(\nu, nd/2) \mathbb{E}_\lambda \left[\{1 + (1 - u_{n+1})\lambda\}^{(d-\nu)/n} \right], \end{aligned}$$

where the expectation \mathbb{E}_λ is taken under the Beta prime distribution with parameters ν and $nd/2$. Then, noticing that, since $(d - \nu)/n \geq 1$, the function $x \mapsto \{1 + x\}^{(d-\nu)/n}$ is convex, we can use Jensen's inequality

$$\begin{aligned} \mathbb{E}_\lambda \left[\{1 + (1 - u_{n+1})\lambda\}^{(d-\nu)/n} \right] &\geq \{1 + (1 - u_{n+1}) \mathbb{E}_\lambda[\lambda]\}^{(d-\nu)/n} \\ &= \left\{ 1 + (1 - u_{n+1}) \frac{\nu}{nd/2 - 1} \right\}^{(d-\nu)/n}, \end{aligned}$$

where the last bound directly follows from the mean of Beta prime random variables. Plugging this lower bound back into the one on J_{n+1} yields

$$J_{n+1} \geq \{1 - u_{n+1}\}^\nu B(\nu, nd/2) \left\{ 1 + \frac{\nu}{nd/2 - 1} (1 - u_{n+1}) \right\}^{(d-\nu)/n}.$$

Finally, by Part 1 of Lemma A.2, we obtain

$$\begin{aligned} J_{n+1} &\geq \{1 - u_{n+1}\}^\nu B(\nu, nd/2) \\ &\quad \times \exp \left(\nu \{1 - u_{n+1}\} \log \left(\{1 + \nu\}^{1/\nu} \frac{(d - \nu)/n}{nd/2 - 1} \right) \right). \end{aligned} \quad (3.9)$$

Combining (3.6), (3.8) and (3.9) together yields

$$\mathbb{E}_{(\mu, \eta)} \left[\log \left(\frac{B(\nu, \{n-1\}d/2)}{B(\nu, nd/2)} \frac{J_{n+1}}{J_n} \frac{\{1 - u_n\}^\nu}{\{1 - u_{n+1}\}^\nu} \right) \right] \geq \nu g(n, d, \nu) \mathbb{E}[1 - u_{n+1}],$$

where $g(n, d, \nu)$ is given by Equation (3.4), thus completing the proof. \square

The following result relies on the digamma function, defined by

$$\psi(\alpha) = \frac{d}{d\alpha} \log \Gamma(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \{\log x\} x^{\alpha-1} e^{-x} dx, \quad (3.10)$$

and on basic formulas about digamma functions given in Lemma A.3 in the Appendix.

Lemma 3.3. *Assume $n \geq 2$ and let $\tilde{\pi}$ be specified by Equation (3.3). Then,*

$$\mathbb{E}_{(\mu, \eta)} \left[\log \left(\frac{n+1}{n} \cdot \frac{s_n}{s_{n+1}} \cdot \frac{1 - u_{n+1}}{1 - u_n} \right) \right] \geq -h(n, d) \mathbb{E}[1 - u_{n+1}],$$

where $u_l = w_l/(1 + w_l)$, $l \in \{n; n+1\}$, the expectation $\mathbb{E}[\cdot]$ is taken with respect to a noncentral Beta random variable, and the function h is defined by

$$h(n, d) = \frac{1 + (n+1)d/2}{nd/2} \left\{ \psi \left(1 + \frac{\{n+1\}d}{2} \right) - \psi \left(1 + \frac{nd}{2} \right) \right\}. \quad (3.11)$$

Proof of Lemma 3.3. Let us first treat the term with the random variables. We have

$$\frac{s_n}{s_{n+1}} \cdot \frac{1 - u_{n+1}}{1 - u_n} = \frac{n \|\bar{\mathbf{x}}_n\|^2 + s_n}{(n+1) \|\bar{\mathbf{x}}_{n+1}\|^2 + s_{n+1}} = \frac{\chi_{nd}^2(n\eta \|\boldsymbol{\mu}\|^2)}{\chi_{(n+1)d}^2(\{n+1\}\eta \|\boldsymbol{\mu}\|^2)},$$

where $\chi_{df}^2(ncp)$ denotes a noncentral Chi-square random variable with degrees of freedom df and noncentral parameter ncp . Taking the expected logarithm yields

$$\begin{aligned} & \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{n+1}{n} \cdot \frac{s_n}{s_{n+1}} \cdot \frac{1-u_{n+1}}{1-u_n} \right) \right] \\ &= \mathbb{E} \left[\log \left(\frac{n+1}{n} \frac{\chi_{nd}^2(2\theta_n)}{\chi_{nd}^2(2\theta_{n+1})} \right) \right] + \mathbb{E} \left[\log \frac{\chi_{nd}^2(2\theta_{n+1})}{\chi_{(n+1)d}^2(2\theta_{n+1})} \right], \end{aligned} \quad (3.12)$$

where $\theta_l = l\eta\|\boldsymbol{\mu}\|^2/2$. By Lemma A.4, the first term of the right-hand side of Equation (3.12) is bounded from below by

$$\mathbb{E} \left[\log \left(\frac{n+1}{n} \frac{\chi_{nd}^2(2\theta_n)}{\chi_{nd}^2(2\theta_{n+1})} \right) \right] \geq \log \left(\frac{n+1}{n} \right) e^{-\theta_{n+1}}.$$

Then, Part 7 of Lemma A.3 enables us to rewrite the second term of the right-hand side of (3.12) as

$$\begin{aligned} \mathbb{E} \left[\log \frac{\chi_{nd}^2(2\theta_{n+1})}{\chi_{(n+1)d}^2(2\theta_{n+1})} \right] &= - \sum_{j=0}^{\infty} e^{-\theta_{n+1}} \frac{\theta_{n+1}^j}{j!} \mathbb{E} \left[\log \frac{\chi_{(n+1)d+2j}^2}{\chi_{nd+2j}^2} \right] \\ &= - \sum_{j=0}^{\infty} e^{-\theta_{n+1}} \frac{\theta_{n+1}^j}{j!} \Psi_d(j + \{n+1\}d/2), \end{aligned}$$

where $\Psi_d(x) = \psi(x) - \psi(x - d/2)$. We thus obtain

$$\begin{aligned} & \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{n+1}{n} \cdot \frac{s_n}{s_{n+1}} \cdot \frac{1-u_{n+1}}{1-u_n} \right) \right] \\ & \geq \log \frac{n+1}{n} e^{-\theta_{n+1}} - \sum_{j=0}^{\infty} e^{-\theta_{n+1}} \frac{\theta_{n+1}^j}{j!} \Psi_d(j + \{n+1\}d/2) \\ & = e^{-\theta_{n+1}} \left\{ \log \frac{n+1}{n} - \Psi_d(\{n+1\}d/2) \right\} \\ & \quad - \sum_{j=1}^{\infty} e^{-\theta_{n+1}} \frac{\theta_{n+1}^j}{j!} \Psi_d(j + \{n+1\}d/2). \end{aligned}$$

Further, Part 5 of Lemma A.3 states that, for $j \geq 1$,

$$\Psi_d(j + \{n+1\}d/2) \leq \frac{1 + (n+1)d/2}{j + (n+1)d/2} \Psi_d(1 + \{n+1\}d/2),$$

yielding

$$\begin{aligned}
& \mathbb{E}_{(\mu, \eta)} \left[\log \left(\frac{n+1}{n} \cdot \frac{s_n}{s_{n+1}} \cdot \frac{1-u_{n+1}}{1-u_n} \right) \right] \\
& \geq e^{-\theta_{n+1}} \left\{ \log \frac{n+1}{n} - \Psi_d \left(\frac{\{n+1\}d}{2} \right) \right\} \\
& \quad - \{1 + (n+1)d/2\} \Psi_d \left(1 + \frac{\{n+1\}d}{2} \right) \sum_{j=1}^{\infty} \frac{\theta_{n+1}^j}{j!} \frac{e^{-\theta_{n+1}}}{j + (n+1)d/2} \\
& = e^{-\theta_{n+1}} \left\{ \log \frac{n+1}{n} - \Psi_d \left(\frac{\{n+1\}d}{2} \right) + \frac{(n+1)d+2}{(n+1)d} \Psi_d \left(1 + \frac{\{n+1\}d}{2} \right) \right\} \\
& \quad - \{1 + (n+1)d/2\} \Psi_d \left(1 + \frac{\{n+1\}d}{2} \right) \sum_{j=0}^{\infty} \frac{\theta_{n+1}^j}{j!} \frac{e^{-\theta_{n+1}}}{j + (n+1)d/2},
\end{aligned}$$

where the last inequality is obtained by adding and subtracting the term corresponding to $j = 0$ in the series. Applying successively Parts 6 and 4 of Lemma A.3, we obtain

$$\begin{aligned}
& \mathbb{E}_{(\mu, \eta)} \left[\log \left(\frac{n+1}{n} \cdot \frac{s_n}{s_{n+1}} \cdot \frac{1-u_{n+1}}{1-u_n} \right) \right] \\
& \geq \frac{e^{-\theta_{n+1}}}{(n+1)d/2} \Psi_d \left(1 + \frac{\{n+1\}d}{2} \right) h(n, d) \sum_{j=0}^{\infty} \frac{\theta_{n+1}^j}{j!} \frac{(nd/2) e^{-\theta_{n+1}}}{j + (n+1)d/2} \\
& \geq -h(n, d) \sum_{j=0}^{\infty} \frac{\theta_{n+1}^j}{j!} \frac{(nd/2) e^{-\theta_{n+1}}}{j + (n+1)d/2},
\end{aligned}$$

where $h(n, d)$ is given in Equation (3.11). Finally, noticing that u_{n+1} is a non-central Beta random variable $B(d, nd, 2\theta_{n+1})$, we can see that the series in the right-hand side of the last bound actually corresponds to $\mathbb{E}[1 - u_{n+1}]$. Indeed, from the formula of moments of noncentral Beta random variables (see *e.g.* Marchand, 1997), we have that

$$\mathbb{E}[1 - u_{n+1}] = 1 - \sum_{j=0}^{\infty} \frac{\theta_{n+1}^j}{j!} \frac{(d/2 + j) e^{-\theta_{n+1}}}{(n+1)d/2 + j} = \sum_{j=0}^{\infty} \frac{\theta_{n+1}^j}{j!} \frac{(nd/2) e^{-\theta_{n+1}}}{j + (n+1)d/2}.$$

We thus obtain the desired result. \square

The last lemma gives the behavior of the difference $\{g(n, d, \nu) - h(n, d)\}$.

Lemma 3.4. *Let $n = 2$ and $1 \leq d \leq 4$. Then, there exists a constant ν_* depending only on the dimension d such that,*

$$g(n, d, \nu) - h(n, d) \geq 0$$

for any $0 < \nu \leq \nu_*$.

Proof of Lemma 3.4. By definition of Beta functions, we have that

$$\begin{aligned} & \log B(\nu, nd/2 - \{d - \nu\}/n) - \log B(\nu, nd/2) \\ &= \log \Gamma(nd/2 + \nu) - \log \Gamma(nd/2) \\ & \quad - \{\log \Gamma(nd/2 - \{d - \nu\}/n + \nu) - \log \Gamma(nd/2 - \{d - \nu\}/n)\}. \end{aligned}$$

Now, by definition of digamma functions, given in Equation (3.10), we deduce the limit

$$\nu^{-1} \log \frac{B(\nu, nd/2 - \{d - \nu\}/n)}{B(\nu, nd/2)} \xrightarrow{\nu \rightarrow 0} \psi(nd/2) - \psi(nd/2 - d/n). \quad (3.13)$$

Notice also that

$$\lim_{\nu \rightarrow 0} \log \left(\{1 + \nu\}^{1/\nu} \right) = 1. \quad (3.14)$$

Thus, gathering Equations (3.13) and (3.14) yields

$$\lim_{\nu \rightarrow 0} g(n, d, \nu) = \begin{cases} \psi(nd/2) - \psi(nd/2 - d/n) & d \leq n, \\ d/\{n(nd/2 - 1)\} & d \geq n + 1, \end{cases}$$

Computing the exact value of the function $\zeta(n, d)$, defined as

$$\zeta(n, d) = \lim_{\nu \rightarrow 0} g(n, d, \nu) - h(n, d), \quad (3.15)$$

for different values of n and d , as displayed by Figure 1, this limit turns out to be positive only when $n = 2$ and $1 \leq d \leq 4$. Under such assumptions, and from the continuity of $\{g(n, d, \nu) - h(n, d)\}$, there exists ν_* such that

$$g(n, d, \nu) - h(n, d) \geq 0$$

for $0 < \nu \leq \nu_*$. This completes the proof. □

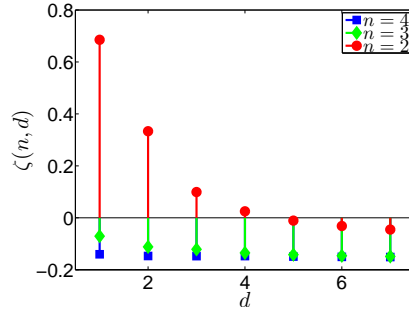


FIG 1. Computation of $\zeta(n, d)$ in Equation (3.15) as a function of d for $n \in \{2; 3; 4\}$.

Numerical computations give approximate values of the constant ν_* for $1 \leq d \leq 4$, displayed in Table 1.

We are now able to prove Theorem 3.1.

TABLE 1
Numerical approximation of ν_* for several values of the dimension d .

d	ν_*
1	0.253
2	0.325
3	0.175
4	0.052

Proof of Theorem 3.1. From Equation (2.6), the risk difference between the best equivariant predictive density and the Bayesian predictive density with respect to the prior π_{GM} , where the function $\tilde{\pi}$ is specified by Equation (3.3), is given by

$$\begin{aligned}
& R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_{GM}) \\
&= \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{B(\nu, \{n-1\}d/2)}{B(\nu, nd/2)} \left\{ \frac{n+1}{n} \right\}^\nu \left\{ \frac{s_{n+1}}{s_n} \right\}^{-\nu} \frac{J_{n+1}}{J_n} \right) \right] \\
&= \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{B(\nu, \{n-1\}d/2)}{B(\nu, nd/2)} \frac{J_{n+1}}{J_n} \left\{ \frac{1-u_n}{1-u_{n+1}} \right\}^\nu \right) \right] \\
&\quad + \nu \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{n+1}{n} \left\{ \frac{s_{n+1}}{s_n} \right\}^{-1} \frac{1-u_{n+1}}{1-u_n} \right) \right].
\end{aligned}$$

By Lemmas 3.2 and 3.3, the risk difference is bounded by

$$R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_{GM}) \geq \nu \mathbb{E}[1 - u_{n+1}] \{g(n, d, \nu) - h(n, d)\}.$$

Since $\nu > 0$ and $1 - u_{n+1} \geq 0$, then the sign of the lower bound is determined by the sign of $\{g(n, d, \nu) - h(n, d)\}$. Applying Lemma 3.4 completes the proof. \square

3.3. Domination in higher dimension ($d \geq 3$)

The domination of our Bayesian predictive density \hat{p}_{GM} over the best equivariant one \hat{p}_R does not only occur in the case where $d \leq 4$. Indeed, the following result shows that the domination is also true in higher dimension d , and larger number n of observations, when this time the function $\tilde{\pi}$ is subject to the following condition, for $0 < \nu \leq d/2 - 1$,

$$\left\{ 1 - \frac{n}{n+1} \lambda \right\}^{d/2-\nu-1} \leq \tilde{\pi}(\lambda) \leq \left\{ 1 - \frac{n-1}{n} \lambda \right\}^{d/2-\nu-1}. \quad (3.16)$$

Note that Condition (3.16) includes the extension of the harmonic prior when $\tilde{\pi} \equiv 1$ and $a = -2$, or $\nu = d/2 - 1$ equivalently, thus enlarging Kato (2009)'s work. Our result actually relies on the proof of Theorem 3 by Kato (2009).

Theorem 3.3. *Let $n \geq 2$, $d \geq 3$, $0 < \nu \leq d/2 - 1$ and $\tilde{\pi}$ be specified by Condition (3.16). Then, the inequality*

$$R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_{GM}) > 0$$

holds for any $\{\boldsymbol{\mu}, \eta\} \in \mathbb{R}^d \times \mathbb{R}_+^*$.

Remark 3.1. In this setting, the inadmissibility of the best equivariant predictive density was already proved by Kato (2009). However, only one improving Bayesian predictive density had been provided, whereas we give a class of such dominating densities.

Proof. From Equation (2.6), the difference in Kullback-Leibler risks between the best equivariant predictive density and the Bayesian predictive density with respect to the prior π_{GM} , where the function $\tilde{\pi}$ is specified by Equation (3.16), is given by

$$\begin{aligned} & R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_{GM}) \\ &= \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{B(\nu, \{n-1\}d/2)}{B(\nu, nd/2)} \left\{ \frac{n+1}{n} \right\}^\nu \left\{ \frac{s_{n+1}}{s_n} \right\}^{-\nu} \frac{J_{n+1}}{J_n} \right) \right]. \end{aligned}$$

By the change of variable $t = \lambda / \{l - (l-1)\lambda\}$ and taking first $l = n+1$, Condition (3.16) gives the upper bound

$$\tilde{\pi} \left(\frac{\{n+1\}\lambda}{1+n\lambda} \right) \geq \left\{ 1 - \frac{n}{n+1}\lambda \right\}^{-d/2+\nu+1},$$

so that

$$J_{n+1} \geq \int_0^1 t^{\nu-1} \{1 + w_{n+1}t\}^{-nd/2-\nu} dt.$$

Next, applying the change of variables $z = w_{n+1}t / (1 + w_{n+1}t)$, with $dt = w_{n+1}^{-1}(1-z)^{-2}dz$ yields

$$J_{n+1} \geq w_{n+1}^{-\nu} \int_0^{u_{n+1}} z^{\nu-1} \{1-z\}^{nd/2-1} dz, \quad (3.17)$$

where we recall that $w_{n+1} = (n+1)\|\bar{\boldsymbol{x}}_{n+1}\|^2/s_{n+1}$ and $u_{n+1} = w_{n+1}/(1+w_{n+1})$. Similarly for $l = n$, the upper bound of $\tilde{\pi}(\lambda)$ given in (3.16) is equivalent to

$$\tilde{\pi} \left(\frac{n\lambda}{1 + \{n-1\}\lambda} \right) \leq \{1 + (n-1)\lambda\}^{-d/2+\nu+1},$$

which in turn yields

$$J_n \leq w_n^\nu \int_0^{u_n} \lambda^{\nu-1} \{1-\lambda\}^{(n-1)d/2-1} d\lambda, \quad (3.18)$$

where $w_n = n\|\bar{\boldsymbol{x}}_n\|^2/s_n$ and $u_n = w_n/(1+w_n)$.

Gathering (3.17) and (3.18), the risk difference can thus be bounded as follows

$$\begin{aligned}
 & R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_R) - R(\{\boldsymbol{\mu}, \eta\}, \hat{p}_{GM}) \\
 & \geq -\nu \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \left(\frac{n}{n+1} \cdot \frac{s_{n+1}}{s_n} \cdot \frac{w_{n+1}}{w_n} \right) \right] \\
 & \quad + \mathbb{E} \left[\log \left(\frac{1}{B(\nu, nd/2)} \int_0^{u_{n+1}} \lambda^{\nu-1} \{1-\lambda\}^{nd/2-1} d\lambda \right) \right] \\
 & \quad - \mathbb{E} \left[\log \left(\frac{1}{B(\nu, \{n-1\}d/2)} \int_0^{u_n} \lambda^{\nu-1} \{1-\lambda\}^{(n-1)d/2-1} d\lambda \right) \right] \\
 & = \nu \mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \|\bar{\boldsymbol{x}}_n\|^2 - \log \|\bar{\boldsymbol{x}}_{n+1}\|^2 \right] + r_{n+1} - r_n,
 \end{aligned} \tag{3.19}$$

where $r_l = \mathbb{E}[\log F_l(u_l)]$, $l \in \{n; n+1\}$, with F_l being the distribution defined as

$$F_l(u) = \frac{1}{B(\nu, \{l-1\}d/2)} \int_0^u t^{\nu-1} \{1-t\}^{(l-1)d/2-1} dt,$$

and the expectation being taken with respect to F_l . Inequality (3.19) actually becomes an equality for the prior π_{LT} considered by Kato (2009). Noticing that $l\|\bar{\boldsymbol{x}}_l\|^2 \sim \chi_d^2(l\eta\|\boldsymbol{\mu}\|^2)$, Lemma A.4 gives

$$\mathbb{E}_{(\boldsymbol{\mu}, \eta)} \left[\log \|\bar{\boldsymbol{x}}_n\|^2 - \log \|\bar{\boldsymbol{x}}_{n+1}\|^2 \right] \geq 0.$$

Hence, it only remains to show that the difference ($r_{n+1} - r_n$) is positive. Now, the random variable u_l is actually a noncentral Beta random variable, that is,

$$u_l = \frac{\chi_d^2(l\eta\|\boldsymbol{\mu}\|^2)}{\chi_d^2(l\eta\|\boldsymbol{\mu}\|^2) + \chi_{(l-1)d}^2}.$$

From the properties of noncentral Chi-square random variables, we have that the ratio

$$\frac{\chi_d^2(ncp)}{\chi_d^2(ncp) + \chi_{(l-1)d}^2}$$

is stochastically increasing in the noncentral parameter ncp (see for instance Lehmann and Romano, 2005). Since $F_l(u)$ is a nondecreasing function of u (it can actually also be seen as the cumulative distribution of a Beta variable), we deduce that

$$\mathbb{E} \left[\log F_{n+1} \left(\frac{\chi_d^2(\{n+1\}\xi)}{\chi_d^2(\{n+1\}\xi) + \chi_{nd}^2} \right) \right] \geq \mathbb{E} \left[\log F_{n+1} \left(\frac{\chi_d^2(n\xi)}{\chi_d^2(n\xi) + \chi_{nd}^2} \right) \right],$$

where $\xi = \eta\|\boldsymbol{\mu}\|^2$. Hence, we obtain

$$r_{n+1} - r_n \geq \mathbb{E} \left[\log F_{n+1} \left(\frac{\chi_d^2(n\xi)}{\chi_d^2(n\xi) + \chi_{nd}^2} \right) \right] - \mathbb{E} \left[\log F_n \left(\frac{\chi_d^2(n\xi)}{\chi_d^2(n\xi) + \chi_{(n-1)d}^2} \right) \right].$$

Finally, Lemma A.5 yields

$$\mathbb{E} \left[\log F_{n+1} \left(\frac{\chi_d^2(n\xi)}{\chi_d^2(n\xi) + \chi_{nd}^2} \right) \right] \geq \mathbb{E} \left[\log F_n \left(\frac{\chi_d^2(n\xi)}{\chi_d^2(n\xi) + \chi_{(n-1)d}^2} \right) \right],$$

thus completing the proof. \square

Remark 3.2. It is interesting to notice that, for $n = 2$ and $d = 2$, the function $\tilde{\pi}$ in the prior in Equation (3.3) is equal to

$$\tilde{\pi}(\lambda) = \{1 - \lambda/2\}^{-\nu}. \quad (3.20)$$

On the other hand, in the case where $d \geq 3$, if we take $\tilde{\pi}$ to be equal to the upper bound in Condition (3.16), we have that

$$\tilde{\pi}(\lambda) = \{1 - (n-1)\lambda/n\}^{d/2-1-\nu}. \quad (3.21)$$

Taking $d = 2$ and $n = 2$ in this latter expression, we can see that Equations (3.20) and (3.21) are actually equal. Hence, even though Theorem 3.3 relies on the assumption that $d \geq 3$, the case where $d = n = 2$ seems to be making the link between the two settings.

In the following section, we verify numerically the domination of our new Bayesian predictive density over the best equivariant one.

4. Numerical experiment

In practice, the Bayesian predictive density \hat{p}_{GM} might be a little challenging to compute, because of the presence of integrals in J_{n+1} and J_n . We discuss how we can accurately approximate both terms in the first part of this section. Then, in the second part, we display the results of the numerical study.

4.1. Computational aspects

In the simulation study, we only considered the case where $1 \leq d \leq 4$, since Kato (2009) already drove a simulation study for the case $d \geq 3$.

First, we would like to point out that, in such a case, with the choice of $\tilde{\pi}$ defined by Equation (3.3), Equation (3.6) shows that J_n is exactly equal to

$$J_n = \{1 - u_n\}^\nu B(\nu, \{n-1\}d/2).$$

We thus were able to compare the different solutions discussed in the sequel to this true value of J_n .

We tested two options for computing J_l , both with Matlab: expected values of functions of Beta random variables, and functions computing numerical approximation of integrals. For the first option, notice that, after applying the

change of variables $t = w_l\lambda/(1 + w_l\lambda)$, both J_{n+1} and J_n can be expressed thanks to the function

$$\begin{aligned} f(m, p, x, w) &= \frac{1}{B(\nu, b + 1)} \int_0^1 \lambda^{\nu-1} \{1 - \lambda\}^b \{1 - x\lambda\}^p \{1 - u\lambda\}^{-md/2-\nu} d\lambda \\ &= \mathbb{E}_\lambda \left[\{1 - x\lambda\}^p \{1 - u\lambda\}^{-md/2-\nu} \right], \end{aligned}$$

where $b = (n - 1)d/2 - 1$, λ is a Beta random variable, that is, $\lambda \sim \text{Beta}(\nu, b + 1)$, and \mathbb{E}_λ is the corresponding expectation. Note that this expression can be generalized to any function $\tilde{\pi}$ by taking $b = 0$ and replacing $(1 - x\lambda)^p$ by the expression of $\tilde{\pi}$. We thus have

$$\begin{aligned} J_{n+1} &= B(\nu, b + 1) n^{-c} \{1 - u_{n+1}\}^{nd/2+\nu} f(n, c, 1 - n, u_{n+1}) \\ J_n &= B(\nu, b + 1) \{1 - u_n\}^{(n-1)d/2+\nu} f(n - 1, 0, 1 - 1/n, u_n), \end{aligned}$$

where $c = -(n - 2)d/2 - \nu$. In the simulation study, the approximation of f by averaging the bracket terms was very accurate when we drew $r_1 = 1\,000\,000$ replicates of λ and averaged the results over 10 trials. We tested it for several values of n , d and ν , and found an accuracy of the order of 10^{-4} in average, whatever ν we chose.

The second option we considered consists in directly estimating the integral with functions such as `quadl` or `quadgk` in Matlab. Those functions evaluate the integral thanks to the adaptive Lobatto quadrature and the Gauss-Kronrod quadrature, respectively. Both functions are much faster than the solution with Beta random variables, `quadgk` being the fastest of the two. However, when $\nu \leq 0.3$, their accuracy becomes very bad (an error of 0.2 has been found for $\nu = 0.1$ and of around 3 when $\nu = 0.05$, while Beta random variables' accuracy is still of the order of 10^{-3}).

Hence, we recommend the following strategy: if ν is taken in the range $(0; 0.3)$, then the solution with Beta random variables should be preferred ; whereas if $\nu \geq 0.3$, then `quadgk` is the most accurate (around 10^{-7} error accuracy) and the fastest solution (hundreds of times faster than Beta random variables). This is the strategy we adopted for the simulation study exposed in the sequel.

4.2. Simulation study

In order to verify in practice the results of Theorem 3.1, we considered the example given by Kato (2009) and we generalized it so as to include our working assumptions.

The experiment is driven for values of the noncentral parameter $\xi = \eta \|\boldsymbol{\mu}\|^2$ taken in $[0; 1\,000]$, and values of the shape parameter ν taken in $\{0.05; 0.25; 0.50; \dots; 1.75; 2\}$. We estimated the difference in risks over $r_2 = 5\,000$ replicates of the different random variables involved, namely (\bar{x}_l, s_l) , $l \in \{n; n + 1\}$, and averaged the results over 10 trials of such an experiment.

Figure 2 shows the behaviour of the thus estimated difference in risks as a function of the noncentral parameter ξ and of the shape parameter ν for each

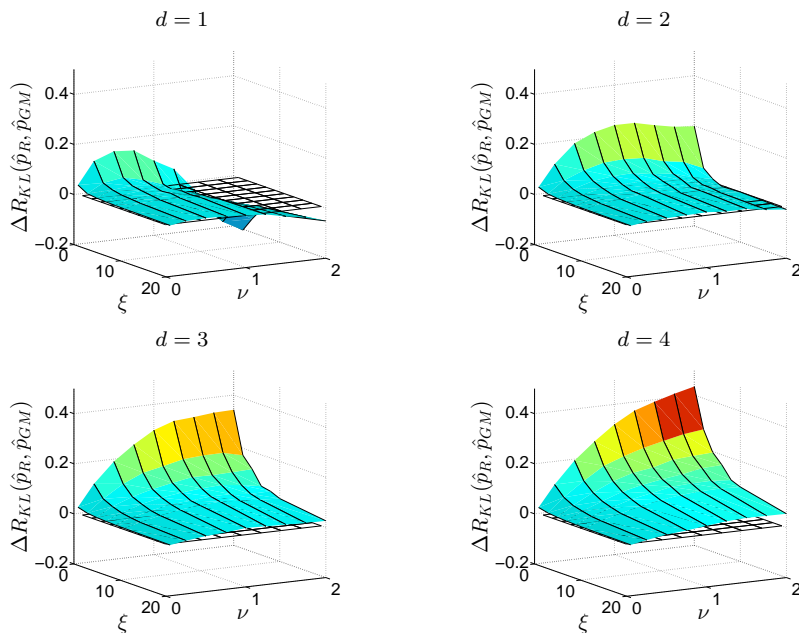


FIG 2. Difference in Kullback-Leibler risks $\Delta R_{KL}(\hat{p}_R, \hat{p}_{GM}) = R_{KL}(\{\hat{\mu}, \eta\}, \hat{p}_R) - R_{KL}(\{\hat{\mu}, \eta\}, \hat{p}_{GM})$ as a function of the noncentral parameter $\xi = \eta \|\mu\|^2$ and of the shape parameter ν , for the low dimensional case ($1 \leq d \leq 4$), with $n = 2$. The black grid shows the plane $z = 0$.

$d \in \{1; 2; 3; 4\}$ when $n = 2$. From this figure, we can see that the difference in risks is indeed positive for the values of ν defined by Theorem 3.1, whatever the noncentral parameter ξ is. This fact thus confirms our theoretical results. What is perhaps more surprising is that the difference in risks is still positive for larger values of ν . This is especially noticeable in the case where $d = 4$, where the largest improvement seems to be obtained for ν in the range $[1; 2]$, whereas in theory we could only prove the domination when $\nu \leq 0.052$. For the cases $d = 1$ and 2 , the difference in risks becomes negative for $\nu \geq 1$.

Note that, although the difference in risks is decreasing with respect to ξ for each fixed ν , it is still positive for the ranges of ν previously mentioned when $\xi = 1000$, as shown by Figure 3.

5. Discussion

Concluding remarks In the present work, we have shown that, for the prediction of a Gaussian random vector with unknown mean and variance, the best equivariant predictive density is inadmissible under the Kullback-Leibler risk even in the low-dimensional setting, that is, when the dimension d of the vector is 1 or 2. This result is more general than the one of Kato (2009) since the latter one proved the inadmissibility only for $d \geq 3$.

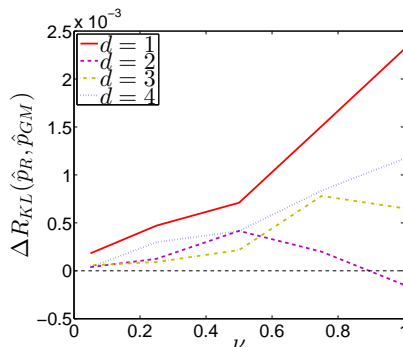


FIG 3. Difference in Kullback-Leibler risks $\Delta R_{KL}(\hat{p}_R, \hat{p}_{GM}) = R_{KL}(\{\hat{\boldsymbol{\mu}}, \eta\}, \hat{p}_R) - R_{KL}(\{\hat{\boldsymbol{\mu}}, \eta\}, \hat{p}_{GM})$ as a function of the shape parameter ν for $\xi = \eta \|\boldsymbol{\mu}\|^2 = 1000$, in the low dimensional case ($1 \leq d \leq 4$), with $n = 2$.

We exhibit a class of priors leading to a Bayesian predictive density that improves upon the best equivariant one. This class of priors generalizes the well-known shrinkage prior (or harmonic prior), but in a different fashion than what is done by Komaki (2007). The results of the simulation study seems to go even further than the theory since it shows positivity of the difference in risks between the best equivariant and the Bayesian predictive densities even when departing from the assumptions of Theorem 3.1.

A parallel between the estimation and prediction problem under the unknown variance setting We would like to point out that, although we have not been able to prove it sofar, it is our conviction that there exists a link between the prediction problem and the estimation problem in the unknown variance case, just as it has been shown in the known variance case (see George, Liang and Xu, 2006). In order to make our point of view clear, recall that Brown (1979) argues that some decision-theoretic properties do not seem to depend on the choice of the loss function. This claim concerned the estimation problem, but the work of Maruyama and Strawderman (2012), of Komaki (2007) and the present work, showing inadmissibility of the best equivariant predictive density for any dimension $d \geq 1$ under two different losses, seem to verify it also for the prediction problem. Such a relationship is particularly appealing since it was proved by Maruyama and Strawderman (2012) that there exists a direct link between the prediction problem under R_1 -risk and the simultaneous estimation of the mean $\boldsymbol{\mu}$ and the variance σ^2 . Indeed, the authors showed that, for π given in Equation (1.4),

$$R_1(\{\boldsymbol{\mu}, \sigma^2\}, \hat{p}_\pi) = c_{IQ} R_{IQ}(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_\pi) + c_S R_S(\sigma^2, \hat{\sigma}_\pi^2),$$

where R_{IQ} and R_S are respectively the invariant quadratic risk of the posterior mean $\hat{\boldsymbol{\mu}}_\pi$ and Stein's risk of the posterior variance $\hat{\sigma}_\pi^2$ defined by

$$R_{IQ}(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}) = \mathbb{E}_{(\boldsymbol{\mu}, \sigma^2)} \left[\frac{1}{\sigma^2} \|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2 \right]$$

$$R_S(\sigma^2, \hat{\sigma}^2) = \mathbb{E}_{(\boldsymbol{\mu}, \sigma^2)} \left[\frac{\hat{\sigma}^2}{\sigma^2} - \log \frac{\hat{\sigma}^2}{\sigma^2} - 1 \right],$$

and c_{IQ} and c_S are weights taking the dimension d into account. However, this parallel has been possible because the Bayesian predictive density under the risk R_1 actually reduces to a plug-in density, which is not the case here. Hence, it is not clear whether the link would still be possible under the Kullback-Leibler risk.

It is also worth noting that several arguments given by [George, Liang and Xu \(2006\)](#) are still true under the unknown variance setting. First, it can easily be shown that the unbiased estimator with minimum variance (commonly referred to as the UMVU estimator) of $(\boldsymbol{\mu}, \sigma^2)$ are equal to the posterior mean and variance under the right invariant prior, which led to the best equivariant predictive density. Also, both the UMVU estimators and the best equivariant distribution are minimax and inadmissible whatever the dimension is (this is true for the estimation problem since it is true for the UMVU estimator of σ^2 , see for instance [Brown, 1968](#)). Finally, Theorem 2.1 expressing any Bayesian predictive density as a function of the best equivariant one is very similar to Lemma 2 of [George, Liang and Xu \(2006\)](#). The main difference lies in the ratio of variance estimators in the factor, which does not seem unreasonable for the unknown variance case. Theorem 2.1 also gives an expression of the difference in risks, which can be rewritten as

$$R_{KL}(\{\boldsymbol{\mu}, \sigma^2\}, \hat{p}_R) - R_{KL}(\{\boldsymbol{\mu}, \sigma^2\}, \hat{p}_\pi) = \int_n^{n+1} \frac{\partial}{\partial l} Q(l, \boldsymbol{\mu}, \sigma^2) dl,$$

where $Q(l, \boldsymbol{\mu}, \sigma^2) = \mathbb{E}_{(\boldsymbol{\mu}, \sigma^2)} [\log \rho_\pi(\boldsymbol{z}, v; l)]$ and $\rho_\pi(\boldsymbol{z}, v; l)$ is given in Equation (2.4). Hence, a sufficient condition for the difference in risks to be positive is that, for all $(\boldsymbol{\mu}, \sigma^2)$ and all $l \in [n, n + 1]$,

$$\frac{\partial}{\partial l} Q(l, \boldsymbol{\mu}, \sigma^2) \geq 0,$$

with strict inequality at least for one value $(\boldsymbol{\mu}_0, \sigma_0^2)$. This type of expression was the key element that helped [George, Liang and Xu \(2006\)](#) explicitly express the link with the estimation problem, but is much more difficult to derive under the unknown variance setting. It thus remains an open problem to find a similar result when the variance is unknown.

Appendix A: Useful lemmas

The first lemma expresses hypergeometric-type functions in special cases.

Lemma A.1 (Maruyama and Strawderman (2005)).

1. For $\alpha > 0$ and $\beta > 0$,

$$\begin{aligned} & \int_0^1 \lambda^{\alpha-1} (1-\lambda)^{\beta-1} (1+w\lambda)^{-\gamma} d\lambda \\ &= (1-u)^\alpha \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \{1-ut\}^{-\alpha-\beta+\gamma} dt, \end{aligned}$$

for $u = w/(1+w)$.

2. For $\alpha > 0$ and $\beta > 0$,

$$\int_0^1 \lambda^{\alpha-1} (1-\lambda)^{\beta-1} (1+w\lambda)^{-\alpha-\beta} d\lambda = (1-u)^\alpha B(\alpha, \beta),$$

for $u = w/(1+w)$.

Proof. See [Maruyama and Strawderman \(2005\)](#). □

The following lemma gives bounds on power functions.

Lemma A.2.

1. Assume $\alpha > 0$ and $0 < \beta < 1$.

$$(1+\alpha)^\beta \leq 1+\alpha\beta \leq (1-\beta)^{-\alpha}.$$

2. For $0 < \alpha, \beta, \gamma < 1$,

$$(1-\beta\gamma)^\alpha \geq (1-\gamma)^\alpha + (1-\beta) \{1 - (1-\gamma)^\alpha\}.$$

Proof. [Part 1] The right-hand side inequality follows directly from the Taylor expansion of $(1-\beta)^{-\alpha}$, since we have

$$(1-\beta)^{-\alpha} = 1 + \alpha\beta + \sum_{i=2}^{\infty} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)} \frac{\beta^i}{i!} \geq 1 + \alpha\beta.$$

For the left-hand side of the inequality, notice first that

$$\log(1+\beta\alpha) = \log(1+\alpha) + \frac{(1-\beta)\alpha}{1+\alpha} \frac{1+\alpha}{(1-\beta)\alpha} \log\left(1 - \frac{\{1-\beta\}\alpha}{1+\alpha}\right),$$

and that the function $x \mapsto x^{-1} \log(1-x)$ is decreasing on $x \in (0, 1)$. Thus we obtain

$$\log(1+\beta\alpha) \geq \log(1+\alpha) + \frac{(1-\beta)\alpha}{1+\alpha} \frac{1+\alpha}{\alpha} \log\left(1 - \frac{\alpha}{1+\alpha}\right) = \beta \log(1+\alpha).$$

The desired inequality is derived by taking the exponential of the latter one.

[Part 2] Using again the Taylor expansion and the fact that $\beta \in (0, 1)$, we have on the one side that

$$(1 - \beta\gamma)^\alpha = 1 - \alpha \sum_{i=1}^{\infty} \frac{\Gamma(1 - \alpha + i)}{\Gamma(1 - \alpha)} \frac{\{\beta\gamma\}^i}{i!} \geq 1 - \alpha\beta \sum_{i=1}^{\infty} \frac{\Gamma(1 - \alpha + i)}{\Gamma(1 - \alpha)} \frac{\gamma^i}{i!},$$

and on the other side that

$$(1 - \gamma)^\alpha = 1 - \alpha \sum_{i=1}^{\infty} \frac{\Gamma(1 - \alpha + i)}{\Gamma(1 - \alpha)} \frac{\gamma^i}{i!}.$$

Gathering those two expressions yields

$$(1 - \beta\gamma)^\alpha \geq 1 - \beta \{1 - (1 - \gamma)^\alpha\} = (1 - \gamma)^\alpha + (1 - \beta) \{1 - (1 - \gamma)^\alpha\},$$

thus completing the proof. \square

The following lemma gives results on the digamma function $\psi(z)$ and related functions.

Lemma A.3. *Let $\chi_{2\alpha}^2$ be a chi-squared random variable with 2α degrees of freedom and let the functions $\psi(z)$ and $\Psi_d(x)$ be defined by*

$$\begin{aligned} \psi(\alpha) &= \frac{d}{d\alpha} \log \Gamma(\alpha) \\ \Psi_d(x) &= \psi(x) - \psi(x - d/2). \end{aligned}$$

Then,

1. $\psi(z)$ is increasing and concave;
2. for any positive z ,

$$\psi(z + 1) = \psi(z) + 1/z;$$

3. for any positive z ,

$$\frac{1}{2z + 1} \leq \psi(z + 1) - \psi(z + 1/2) \leq \frac{1}{2z};$$

4. for any $x > d/2$, $\Psi_d(x)$ is strictly positive ;
5. for any $x > d/2$,

$$(x + 1)\Psi_d(x + 1) - x\Psi_d(x) < 0;$$

6. for $nd \geq 2$,

$$\log \frac{n+1}{n} - \Psi_d\left(\frac{\{n+1\}d}{2}\right) + \Psi_d\left(1 + \frac{\{n+1\}d}{2}\right) \geq 0;$$

7. for any α such that $\alpha > 0$,

$$\mathbb{E} [\log \chi_{2\alpha}^2] = \psi(\alpha) + \log 2.$$

Proof. [Part 1 and 2] See [Abramowitz and Stegun \(1964\)](#).

[Part 3] By Part 1, we have

$$\psi(z+1) - \psi(z+1/2) \leq \psi(z+1/2) - \psi(z),$$

which in turn yields

$$\psi(z+1/2) \geq \{\psi(z+1) + \psi(z)\}/2.$$

Reinjecting this inequality in the former one gives

$$\psi(z+1) - \psi(z+1/2) \leq \{\psi(z+1) - \psi(z)\}/2.$$

Now, by Part 2, we obtain the right-hand side inequality. Similarly we have

$$\psi(z+1) - \psi(z+1/2) \geq \{\psi(z+3/2) - \psi(z+1/2)\}/2 = 1/(2z+1).$$

[Part 4] The positivity of Ψ_d directly follows from Part 1.

[Part 5] By the definition of Ψ_d , we can easily see that

$$\begin{aligned} & (x+1)\Psi_d(x+1) - x\Psi_d(x) \\ &= x \{ \psi(x+1) - \psi(x+1-d/2) - \psi(x) + \psi(x-d/2) \} \\ & \quad + \psi(x+1) - \psi(x+1-d/2). \end{aligned}$$

Now, Part 2 yields

$$\begin{aligned} & (x+1)\Psi_d(x+1) - x\Psi_d(x) \\ &= x \left(\frac{1}{x} - \frac{1}{x-d/2} \right) + \psi(x+1) - \psi(x+1-d/2) \\ &= -\frac{d/2}{x-d/2} + \psi(x+1) - \psi(x+1-d/2). \end{aligned}$$

For even $d = 2k$, $k \in \mathbb{N}$, Part 2 gives

$$\psi(x+1) - \psi(x+1-k) = \frac{1}{x} + \dots + \frac{1}{x+1-k} \leq \frac{k}{x+1-k},$$

which implies

$$(x+1)\Psi_d(x+1) - x\Psi_d(x) \leq -\frac{d}{2} \left(\frac{1}{x-d/2} - \frac{1}{x+1-d/2} \right) < 0.$$

For odd $d = 2k+1$, note that Part 3 gives

$$\psi(x+1) - \psi(x+1-d/2) \leq 1/2x + \psi(x+1/2) - \psi(x+1-d/2).$$

Similarly as for even d , we have

$$\psi(x+1/2) - \psi(x+1/2-k) \leq k/(x+1/2-k).$$

Hence, we obtain

$$(x+1)\Psi_d(x+1) - x\Psi_d(x) \leq -\frac{1}{2} \frac{d/2}{x(x-d/2)} - \frac{(d-1)/2}{(x-d/2)(x+1-d/2)},$$

which is also negative.

[Part 6] Notice first that

$$\log \frac{n+1}{n} = -\log \left(1 - \frac{1}{n+1}\right) = \sum_{i=1}^{\infty} \frac{(n+1)^{-i}}{i} > \frac{1}{n+1}.$$

Then, by definition of $\psi(x)$ and $\Psi_d(x)$, and by Part 2, we have that

$$\Psi_d \left(1 + \frac{\{n+1\}d}{2}\right) - \Psi_d \left(\frac{\{n+1\}d}{2}\right) = \frac{1}{(n+1)d/2} - \frac{1}{nd/2}.$$

Therefore

$$\log \frac{n+1}{n} + \Psi_d \left(1 + \frac{\{n+1\}d}{2}\right) - \Psi_d \left(\frac{\{n+1\}d}{2}\right) > \frac{1}{n+1} \left(1 - \frac{2}{nd}\right) \geq 0.$$

[Part 7] Making the change of variable $z = 2x$ in the definition of the digamma function given in Equation (3.10), $\psi(\alpha)$ is written as

$$\psi(\alpha) = \int_0^{\infty} \log z \frac{z^{\alpha-1} e^{-z/2}}{2^{\alpha} \Gamma(\alpha)} dz - \log 2 = \mathbb{E} [\log \chi_{2\alpha}^2] - \log 2,$$

which completes the proof. \square

The following result is essentially due to Komaki (2001).

Lemma A.4. For $m \geq 2$ and $l_1 \leq l_2$, the expected logarithm of the ratio of two noncentral Chi-square random variables is bounded by

$$\mathbb{E} \left[\log \left(\frac{l_2 \chi_m^2(l_1 \xi)}{l_1 \chi_m^2(l_2 \xi)} \right) \right] \geq \log \frac{l_2}{l_1} e^{-l_2 \xi/2}.$$

Proof. Let us first recall the following inequality due to Komaki (2001)

$$\frac{d}{dz} \mathbb{E} \left[\log \frac{\chi_m^2(2z)}{z} \right] < -z^{-1} e^{-z}.$$

Next, writing z as $2(z/2)$ yields

$$\mathbb{E} \left[\log \frac{\chi_m^2(2\{l_1 \xi/2\})}{l_1 \xi/2} \right] - \mathbb{E} \left[\log \frac{\chi_m^2(2\{l_2 \xi/2\})}{l_2 \xi/2} \right] \geq \int_{l_1 \xi/2}^{l_2 \xi/2} z^{-1} e^{-z} dz.$$

Finally, since the function $x \mapsto e^{-x}$ is decreasing, we get the following lower bound

$$\mathbb{E} \left[\log \left(\frac{l_2 \chi_m^2(l_1 \xi)}{l_1 \chi_m^2(l_2 \xi)} \right) \right] \geq e^{-l_2 \xi/2} \int_{l_1 \xi/2}^{l_2 \xi/2} z^{-1} dz = \log \frac{l_2}{l_1} e^{-l_2 \xi/2},$$

thus completing the proof. \square

The last result is essentially due to Kato (2009).

Lemma A.5. For $l > m$ and $d \geq \alpha$,

$$\mathbb{E} \left[\log F \left(\left\{ 1 + \chi_l^2 / \chi_d^2(\theta) \right\}^{-1}; \alpha, l \right) \right] \geq \mathbb{E} \left[\log F \left(\left\{ 1 + \chi_m^2 / \chi_d^2(\theta) \right\}^{-1}; \alpha, m \right) \right]$$

where

$$F(u; \alpha, \beta) = \frac{1}{B(\alpha/2, \beta/2)} \int_0^u t^{\alpha/2-1} (1-t)^{\beta/2-1} dt.$$

Proof. The proof mainly relies on the expression of noncentral Chi-square random variables as mixtures of Chi-square random variables with respect to Poisson mixture density, and on the fact that the distribution F is star-ordered with respect to the parameter β , that is, the ratio

$$\frac{F^{-1}(u; \alpha, \beta_1)}{F^{-1}(u; \alpha, \beta_2)}$$

is nondecreasing in $u \in (0, 1)$ for $1 < \beta_1 < \beta_2$.

See Kato (2009) for more details. □

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