A note on the stable equivalence problem

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PIERRE-MARIE POLONI

Abstract. We provide counterexamples to the stable equivalence problem in every dimension $d \geq 2$. That means that we construct hypersurfaces $H_1, H_2 \subset \mathbb{C}^{d+1}$ whose cylinders $H_1 \times \mathbb{C}$ and $H_2 \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{d+2}$, although H_1 and H_2 themselves are not equivalent by an automorphism of $\mathbb{C}^{d+1}$. We also give, for every $d \geq 2$, examples of two non-isomorphic algebraic varieties of dimension d which are biholomorphic.

1. Introduction

The well known generalized cancellation problem asks the following question.

Generalized cancellation problem. Given two complex affine varieties V_1 and V_2 with the property that $V_1 \times \mathbb{C}^m$ and $V_2 \times \mathbb{C}^m$ are isomorphic for some $m \in \mathbb{N}$. Does this imply that V_1 and V_2 are isomorphic?

An affirmative answer was given by Abhyankar, Eakin and Heinzer [1] for the case of affine curves. The cancellation property holds also in the case where V_1 (or V_2) has nonnegative logarithmic Kodaira dimension. This was shown by Iitaka and Fujita in [10]. However, the answer to the generalized cancellation problem turns out to be negative in general. The first counterexamples are surfaces due to Danielewski [2] (see also [6]). Later on, Danielewski's construction was generalized by Dubouloz [4] to produce counterexamples of every dimension $d \geq 2$ (see also [7] and [5] for factorial and contractible 3-dimensional examples).

In 2004, Makar-Limanov, van Rossum, Shpilrain and Yu [15] considered the following analogous problem.

Stable equivalence problem. If two hypersurfaces in $\mathbb{C}^n$ are stably equivalent, are they equivalent?

Recall that two algebraic varieties V_1, V_2 in $\mathbb{C}^n$ are said to be *equivalent* if there exists a polynomial automorphism of $\mathbb{C}^n$ which maps V_1 onto V_2, and that they are said to be *stably equivalent* if there is an integer $m \in \mathbb{N}$ such that the cylinders $V_1 \times \mathbb{C}^m$ and $V_2 \times \mathbb{C}^m$ are equivalent varieties in $\mathbb{C}^{n+m}$. The stable equivalent problem has a positive answer for affine plane curves, as already shown by Makar-Limanov, van Rossum, Shpilrain and Yu in [15]. In the same vein of the result of Iitaka-Fujita, Drylo proved in [3] that two stably equivalent hypersurfaces in $\mathbb{C}^n$ are equivalent, if one of them is not $\mathbb{C}$-uniruled. The first counterexamples in $\mathbb{C}^3$, consisting in families of Danielewski hypersurfaces, were provided by Moser-Jauslin and the author [17]. Also, contractible 3-dimensional counterexamples appeared in [5].

In this note, we complete the analogy between the results on the generalized cancellation and stable equivalence problems. Indeed, we produce counterexamples to the stable equivalence problem for every $n \geq 3$. These new examples are easy generalizations of those of [17], inspired by the construction in [4].

We will actually give two kinds of counterexamples. On one hand, polynomials $P, Q \in \mathbb{C}[X_1, \ldots, X_n]$ whose zero-sets $V(P)$ and $V(Q)$ are non-isomorphic varieties, but such that the cylinders $V(P) \times \mathbb{C}$ and $V(Q) \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{n+1}$. On the other hand, polynomials $P, Q \in \mathbb{C}[X_1, \ldots, X_n]$ with the properties that $V(P) \times \mathbb{C}$ and $V(Q) \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{n+1}$ and that $V(P)$ and $V(Q)$ are non-equivalent hypersurfaces in $\mathbb{C}^n$, although the fibers $V(P-c)$ and $V(Q-c)$ of P and Q are pairwise isomorphic for all $c \in \mathbb{C}$. More precisely, we will prove the following result.

Theorem. *The following assertions hold for every natural number $n \geq 1$.*

1. *The hypersurfaces $H_1, H_2 \subset \mathbb{C}^{n+2}$ defined by the equation $x_1^2 \cdots x_n^2 y + z^2 + x_1 \cdots x_n(z^2-1) = 1$ and $x_1^2 \cdots x_n^2 y + z^2 + x_1 \cdots x_n(z^2-2) = 1$, respectively, are non-isomorphic algebraic varieties such that $H_1 \times \mathbb{C}$ and $H_2 \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{n+3}$.*
2. *The polynomials $Q_k = x_1^2 \cdots x_n^2 y + z^2 + x_1 \cdots x_n(z^2-1)^k \in \mathbb{C}[x_1, \ldots, x_n, y, z]$ are stably equivalent for all $k \geq 1$, whereas the hypersurfaces $V(Q_k) \subset \mathbb{C}^{n+2}$ are pairwise non-equivalent. However, the varieties $V(Q_k - c)$ and $V(Q_{k'} - c)$ are isomorphic for all $k, k' \geq 1$ and every $c \in \mathbb{C}$.*

It is worth mentioning that the special case of affine spaces is still open, for both cancellation and stable equivalence problems. Recall that the question to know whether an isomorphism $V \times \mathbb{C}^m \simeq \mathbb{C}^{n+m}$ implies $V \simeq \mathbb{C}^n$ is usually referred to as the "Zariski cancellation problem". It has a positive solution for $n = 1$ and for $n = 2$ by the results of Fujita and Miyanishi-Sugie ([9], [16]), whereas it is still an unsolved problem for $n \geq 3$.

Similarly, it was asked in [15] if every hypersurface in $\mathbb{C}^{n+1}$, which is stably equivalent to a (linear) hyperplane, is already equivalent to this hyperplane. Note that it is true for $n = 1$ and also, using the cancellation property of the affine plane and a result of Kaliman [12], for $n = 2$. Moreover, as noticed in [15], a positive answer to this question for an integer $n \geq 3$ would imply that the n-dimensional affine space has the cancellation property.

2. Four hypersurfaces in $\mathbb{C}^{n+2}$

Let us fix some notations.

Notation 2.1. Given a ring R and an integer $m \in \mathbb{N}$, we denote by $R^{[m]}$ the polynomial ring in m variables over R. Throughout this paper, we fix a positive integer n and we denote by $\mathbb{C}[\underline{x}]$ the polynomial ring $\mathbb{C}[x_1, \ldots, x_n] \simeq \mathbb{C}^{[n]}$ in the variables $x_1, \ldots, x_n$.

For every integer $k \in \mathbb{N}$, we denote by $\underline{x}^{[k]}$ the element $\underline{x}^{[k]} = x_1^k \cdots x_n^k \in \mathbb{C}[\underline{x}]$ and, for every polynomial $q \in \mathbb{C}^{[1]}$, by P_q the polynomial of $\mathbb{C}[x_1, \ldots, x_n, y, z] = \mathbb{C}[\underline{x}][y, z]$ defined by

$$P_q = \underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} q(z^2).$$

The counterexamples to the stable equivalent problem mentioned in the introduction are realized as hypersurfaces in $\mathbb{C}^{n+2}$ given by the fibers $V(P_q - c)$ of some polynomials P_q. We will determine the isomorphism classes of these varieties. This will be done by using techniques mainly developed by Makar-Limanov in [14]. The idea is to exploit the fact that this kind of hypersurfaces admit additive group actions, but not too many. For instance, their Makar-Limanov invariants are non trivial.

It is in general very difficult and technical to compute such invariants. But we are in a good situation, since the method of Kaliman and Makar-Limanov ([13]) applies to the varieties that we are considering. Moreover, we can even use directly the results of Dubouloz [4], who already did the computation for the case where the polynomial q is constant. Remark that, thanks to the next lemma, it suffices to consider only this special case.

Lemma 2.2. *Let $R = \mathbb{C}[\underline{x}] \simeq \mathbb{C}^{[n]}$. Given $q \in \mathbb{C}^{[1]}$ and $c \in \mathbb{C}$, we let $g_c \in \mathbb{C}^{[1]}$ be the polynomial such that the equality $q(z^2) - q(c) = g_c(z^2)(z^2 - c)$ holds in $\mathbb{C}[z]$. Then, the endomorphism $\varphi_c \in \mathrm{End}_R R[y,z]$ of $R[y,z]$ fixing R and defined by*

$$\varphi_c(y) = \left(1 + \underline{x}^{[1]} g_c(z^2)\right) y + q(c) g_c(z^2) \quad \text{and} \quad \varphi_c(z) = z$$

induces an isomorphism between the rings $\mathbb{C}[\underline{x}, y, z]/(P_q - c)$ and $\mathbb{C}[\underline{x}, y, z]/(P_{q(c)} - c)$.

Proof. First, one checks that $\varphi_c\left(P_q - c\right) = \left(1 + \underline{x}^{[1]} g_c(z^2)\right)\left(P_{q(c)} - c\right)$. Thus, φ_c induces a morphism between $\mathbb{C}[\underline{x}, y, z]/(P_q - c)$ and $\mathbb{C}[\underline{x}, y, z]/(P_{q(c)} - c)$. The latter is invertible. To see this, one checks that the inverse morphism is induced by the endomorphism $\psi_c \in \mathrm{End}_R R[y,z]$ defined by $\psi_c(y) = \left(1 - \underline{x}^{[1]} g_c(z^2)\right) y - q(z^2) g_c(z^2)$ and $\varphi_c(z) = z$. □

We will now compute, for all $q \in \mathbb{C}^{[1]}$ and all $c \in \mathbb{C}$, the set $\mathrm{LND}(B_{q,c})$ of locally nilpotent derivations on the coordinate ring $B_{q,c}$ of the varieties $V(P_q - c)$. Recall that a derivation δ of a $\mathbb{C}$-algebra B is called *locally nilpotent* if there exists, for every element $b \in B$, an integer $m = m(b) \geq 1$ such that $\delta^m(b) = 0$. Let Δ be the derivation of $\mathbb{C}[\underline{x}, y, z]$ defined by

$$\Delta = \underline{x}^{[2]} \frac{\partial}{\partial z} - 2z(1 + \underline{x}^{[1]} q'(z^2)) \frac{\partial}{\partial y},$$

where q' denotes the derivative of q. Note that Δ is locally nilpotent (it is a triangular derivation) and that it annihilates the polynomial $P_q - c$. Therefore, it induces a locally nilpotent derivation on $B_{q,c}$, which we still denote by Δ. It turns out that all other locally nilpotent derivations on $B_{q,c}$ are multiple of Δ by elements of $\mathbb{C}[\underline{x}]$.

Proposition 2.3. *Let $q \in \mathbb{C}^{[1]}$, $c \in \mathbb{C}$ and $B_{q,c} = \mathbb{C}[\underline{x}, y, z]/(P_q - c)$, where $P_q = \underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} q(z^2) \in \mathbb{C}[\underline{x}, y, z]$. Then, the following hold for every nonzero locally nilpotent derivation δ of $B_{q,c}$.*

1. $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$ *and* $\mathrm{Ker}(\delta^2) = \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$.
2. *There exists $h(\underline{x}) \in \mathbb{C}[\underline{x}]$ such that $\delta = h(\underline{x})\Delta$, where Δ is the locally nilpotent derivation on $B_{q,c}$ defined above.*

Proof. (1) First of all, remark that we can suppose that q is a constant polynomial. Indeed, take the isomorphism $\phi : B_{q,c} \to B_{q(c),c}$ given by Lemma 2.2 and let $\delta \in \mathrm{LND}(B_{q,c}) \setminus \{0\}$ be a nonzero locally nilpotent derivation. Then, $\widetilde{\delta} = \phi \circ \delta \circ \phi^{-1} \in \mathrm{LND}(B_{q(c),c}) \setminus \{0\}$ and we have $\mathrm{Ker}(\delta) = \phi^{-1}(\mathrm{Ker}(\widetilde{\delta}))$ and $\mathrm{Ker}(\delta^2) = \phi^{-1}(\mathrm{Ker}(\widetilde{\delta}^2))$. Since ϕ^{-1} maps $\mathbb{C}[\underline{x}]$ onto $\mathbb{C}[\underline{x}]$ and $\mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$ onto $\mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$, it suffices to prove $\mathrm{Ker}(\widetilde{\delta}) = \mathbb{C}[\underline{x}]$ and $\mathrm{Ker}(\widetilde{\delta}^2) = \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$.

So, let $q, c \in \mathbb{C}$ be two constants and let δ be a nonzero locally nilpotent derivation on $B_{q,c}$. We are now in the case considered by Dubouloz in [4], where he proved (see paragraph 2.7 in [4]) that $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$ and $\mathrm{Ker}(\delta^2) \subset \mathbb{C}[\underline{x}, z]$ hold. This implies easily $\mathrm{Ker}(\delta^2) = \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$.

Indeed, let $a \in \mathrm{Ker}(\delta^2) \setminus \mathrm{Ker}(\delta)$ and write $a = \sum_{i=0}^{d} \alpha_i(\underline{x}) z^i$ with $d \geq 1$ and $\alpha_i(\underline{x}) \in \mathbb{C}[\underline{x}]$. Then $\delta(a) = \delta(z) \sum_{i=1}^{d} i\alpha_i(\underline{x}) z^{i-1}$ is a nonzero element of $\mathrm{Ker}(\delta)$. Since the kernel of a locally nilpotent derivation is factorially closed, it follows that $\delta(z)$ lies in $\mathrm{Ker}(\delta)$. Thus, $z \in \mathrm{Ker}(\delta^2)$ and so $\mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}] \subset \mathrm{Ker}(\delta^2)$. On the other hand, $\delta(a) \in \mathrm{Ker}(\delta)$ implies $d = 1$, since $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$. Therefore, $a \in \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$ and (1) is proved.

(2) Let $\delta \in \mathrm{LND}(B_{q,c}) \setminus \{0\}$. By (1), $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$ and there exists a polynomial $a(\underline{x}) \in \mathbb{C}[\underline{x}] \setminus \{0\}$ such that $\delta(z) = a(\underline{x})$. To prove (2), it suffices to find an element $h(\underline{x}) \in \mathbb{C}[\underline{x}]$ such that $a(\underline{x}) = \underline{x}^{[2]} h(\underline{x})$, since

$$0 = \delta(P_q - c) = \underline{x}^{[2]} \delta(y) + a(\underline{x}) 2z(1 + \underline{x}^{[1]} q'(z^2)).$$

The equality above means that there exist polynomials $F, R \in \mathbb{C}^{[n+2]}$ such that

$$\underline{X}^{[2]} F(\underline{X}, Y, Z) + a(\underline{X}) 2Z(1 + \underline{X}^{[1]} q'(Z^2)) = R(\underline{X}, Y, Z)(\underline{X}^{[2]} Y + Z^2 + \underline{X}^{[1]} q(Z^2) - c).$$

From this, it follows that $a(\underline{X})$ and $R(\underline{X}, Y, Z)$ are both divisible by $\underline{X}^{[1]}$. Setting $a(\underline{X}) = \underline{X}^{[1]} \widetilde{a}(\underline{X})$ and $R(\underline{X}, Y, Z) = \underline{X}^{[1]} \widetilde{R}(\underline{X}, Y, Z)$, we obtain the equality

$$\underline{X}^{[1]} F(\underline{X}, Y, Z) + \widetilde{a}(\underline{X}) 2Z(1 + \underline{X}^{[1]} q'(Z^2)) = \widetilde{R}(\underline{X}, Y, Z)(\underline{X}^{[2]} Y + Z^2 + \underline{X}^{[1]} q(Z^2) - c).$$

The latter implies that $\widetilde{a}(\underline{X})$ is divisible by $\underline{X}^{[1]}$. Thus, $a(\underline{x}) = \underline{x}^{[2]} h(\underline{x})$ for an element $h(\underline{x}) \in \mathbb{C}[\underline{x}]$. This completes the proof. □

We are now in position to classify all hypersurfaces in $\mathbb{C}^{n+2}$ given by an equation of the form $P_q = c$. They have exactly four isomorphism classes. Each of them is given by one of the following varieties.

Notation 2.4. We denote by $V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1}$ the hypersurfaces in $\mathbb{C}^{n+2}$ defined by the equation $\underline{x}^{[2]} y + z^2 = 0$, $\underline{x}^{[2]} y + z^2 - 1 = 0$, $\underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} = 0$ and $\underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} - 1 = 0$, respectively.

These varieties are pairwise non-isomorphic and we have the following result, which was already proved in [17] for the case $n = 1$.

Proposition 2.5. *Let $q \in \mathbb{C}^{[1]}$, $c \in \mathbb{C}$. and let $P_q = \underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} q(z^2) \in \mathbb{C}[\underline{x}, y, z]$ as in Notation 2.1. Then, the variety $V(P_q - c)$ is isomorphic to:*

1. *$V_{0,0}$ if an only if $c = 0$ and $q(c) = 0$;*
2. *$V_{1,0}$ if an only if $c = 0$ and $q(c) \neq 0$;*
3. *$V_{0,1}$ if an only if $c \neq 0$ and $q(c) = 0$;*
4. *$V_{1,1}$ if an only if $c \neq 0$ and $q(c) \neq 0$.*

Proof. By Lemma 2.2, the variety $V(P_q - c)$ is isomorphic to the hypersurface of equation

$$\underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} q(c) - c = 0.$$

The "if parts" of the proposition follow then easily.

In order to prove that $V_{0,0}, V_{0,1}, V_{1,0}$ and $V_{1,1}$ are non-isomorphic, we consider two polynomials $q_1, q_2 \in \mathbb{C}^{[1]}$ and two constants $c_1, c_2 \in \mathbb{C}$. For $j = 1, 2$, let B_j denotes the ring $B_j = \mathbb{C}[\underline{x}, y, z]/(P_{q_j} - c_j)$ and let x_{i_j}, y_j, z_j denote the images of x_i, y, z in B_j. We also denote by $\mathbb{C}\left[\underline{x}_j\right]$ the ring $\mathbb{C}[x_{1_j}, \ldots, x_{n_j}]$. Suppose now that $\varphi : B_1 \to B_2$ is an isomorphism.

Let $\delta \in \mathrm{LND}(B_1) \setminus \{0\}$ be a nonzero locally derivation on B_1. Then, $\widetilde{\delta} = \varphi \circ \delta \circ \varphi^{-1}$ is a nonzero locally derivation on B_2 and we have $\mathrm{Ker}(\widetilde{\delta}) = \varphi(\mathrm{Ker}(\delta))$ and $\mathrm{Ker}((\widetilde{\delta})^2) = \varphi(\mathrm{Ker}(\delta^2))$. By Proposition 2.3, we have $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}_1]$ and $\mathrm{Ker}(\widetilde{\delta}) = \mathbb{C}[\underline{x}_2]$. Thus, φ restricts to an isomorphism between $\mathbb{C}[\underline{x}_1]$ and $\mathbb{C}[\underline{x}_2]$. Moreover $\varphi(z_1) \in \mathrm{Ker}((\widetilde{\delta})^2) = \mathbb{C}[\underline{x}_2]z_2 + \mathbb{C}[\underline{x}_2]$. Therefore, $\varphi(z_1) = \alpha(\underline{x}_2)z_2 + \beta(\underline{x}_2)$ for some polynomials α and β. Repeating the same argument with φ^{-1}, we obtain that $\varphi^{-1}(z_2) = a(\underline{x}_1)z_1 + b(\underline{x}_1)$ for some polynomials a and b. From this, we get that the elements $\alpha(\underline{x}_2) \in \mathbb{C}[\underline{x}_2]$ and $a(\underline{x}_1) \in \mathbb{C}[\underline{x}_1]$ are in fact invertible, thus nonzero constants.

If we take the derivation $\delta = \Delta$ (see Proposition 2.3), one checks that $\widetilde{\delta}(z_2) = \varphi(\Delta(az_1 + b(\underline{x}_1))) = a\varphi(\underline{x}_1^{[2]})$. Consequently, there exists, again by Proposition 2.3, a polynomial h such that $a\varphi(\underline{x}_1^{[2]}) = h(\underline{x}_2)\underline{x}_2^{[2]}$. Since $\varphi : \mathbb{C}[\underline{x}_1] \to \mathbb{C}[\underline{x}_2]$ is an isomorphism, this implies that there exist a bijection σ of the set $\{1, \ldots, n\}$ and nonzero constants $\lambda_i \in \mathbb{C}^*$ such that $\varphi(x_i) = \lambda_i x_{\sigma(i)}$ for all $1 \le i \le n$.

Let $\lambda = \prod_{i=1}^n \lambda_i$ and suppose from now on that q_1 and q_2 are constant. Since $\lambda^2 \underline{x}_2^{[2]} \varphi(y_1) + (\alpha z_2 + \beta(\underline{x}_2))^2 + \lambda \underline{x}_2^{[1]} q_1 - c_1 = \varphi(\underline{x}_1^{[2]} y_1 + z_1^2 + \underline{x}_1^{[1]} q_1 - c_1) = 0$ in B_2, there exist polynomials $F, A \in \mathbb{C}^{[n+2]}$ such that

$$\lambda^2 \underline{x}^{[2]} F(\underline{x}, y, z) + (\alpha z + \beta(\underline{x}))^2 + \lambda q_1 \underline{x}^{[1]} - c_1 = A(\underline{x}, y, z)(\underline{x}^{[2]} y + z^2 + q_2 \underline{x}^{[1]} - c_2).$$

Looking at this equality modulo $(\underline{x}^{[2]})$, it follows that $\beta(\underline{x})$ lies in the ideal of $\mathbb{C}[\underline{x}]$ generated by $\underline{x}^{[2]}$, and that $c_1 = A(\underline{0}, 0, 0)c_2$ and $\lambda q_1 = A(\underline{0}, 0, 0)q_2$. This shows that $V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1}$ are pairwise non-isomorphic and proves the proposition. □

Remark 2.6. Even if they are non-isomorphic, the varieties $V_{0,1}$ and $V_{1,1}$ are biholomorphic. Indeed, the analytic automorphism Ψ of $\mathbb{C}[\underline{x}, y, z]$ defined by $\Psi(x_i) = x_i$ for all $1 \le i \le n$,

$$\Psi(y) = \exp(-\underline{x}^{[1]})y - \frac{\exp(-\underline{x}^{[1]}) - 1 + \underline{x}^{[1]}}{\underline{x}^{[2]}} \quad \text{and} \quad \Psi(z) = \exp(-\frac{1}{2}\underline{x}^{[1]})z,$$

satisfies $\Psi(\underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} - 1) = \exp(-\underline{x}^{[1]})(\underline{x}^{[2]} y + z^2 - 1)$. The case $n = 1$ is due to Freudenburg and Moser-Jauslin [8] and it was, to our knowledge, the first explicit example in the literature of two algebraically non-isomorphic varieties that are holomorphically isomorphic. Note that Jelonek [11] has recently constructed other examples, in every dimension $d \ge 2$, of rational varieties with these properties.

3. Stable equivalence

In this paper, we will consider two notions of equivalence.

Definition 3.1.

(1) Two hypersurfaces $H_1, H_2 \subset \mathbb{C}^n$ are said to be *equivalent* if there exists a polynomial automorphism Φ of $\mathbb{C}^n$ such that $\Phi(H_1) = H_2$.

(2) Two polynomials $P, Q \in \mathbb{C}^{[n]}$ are said to be *equivalent* if there exists a polynomial automorphism Φ of $\mathbb{C}^n$ such that $\Phi^*(P) = Q$.

These two notions are of course closely related, the zero-sets $V(P)$ and $V(Q)$ of irreducible polynomials $P, Q \in \mathbb{C}^{[n]}$ being equivalent hypersurfaces in $\mathbb{C}^n$ if and only if there exists a nonzero constant $\mu \in \mathbb{C}^*$ such that P and μQ are equivalent polynomials in $\mathbb{C}^{[n]}$.

The next proposition gives the classification, up to equivalence, of all polynomials P_q (see Notation 2.1) and of their fibers $V(P_q - c)$. It is an easy generalization of results of [17] to the case $n \geq 2$.

Proposition 3.2. *Let $q_1, q_2 \in \mathbb{C}^{[1]}$ be two polynomials and $c_1, c_2 \in \mathbb{C}$ be two constants. For $i = 1, 2$, let $P_{q_i} = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}q_i(z^2) \in \mathbb{C}[\underline{x}, y, z]$ as in Notation 2.1. Then, the following hold.*

1. *The polynomials $P_{q_1} - c_1$ and $P_{q_2} - c_2$ of $\mathbb{C}^{[n+2]}$ are equivalent if and only if $c_1 = c_2$ and there exists a nonzero constant $\lambda \in \mathbb{C}^*$ such that $q_2 = \lambda q_1$.*
2. *The hypersurfaces $H_1 = V(P_{q_1} - c_1), H_2 = V(P_{q_2} - c_2) \subset \mathbb{C}^{n+2}$ are equivalent if and only if there exist two nonzero constants $\lambda, \mu \in \mathbb{C}^*$ such that $c_2 = \mu^{-1}c_1$ and such that the equality $q_2(t) = \lambda q_1(\mu t)$ holds in $\mathbb{C}[t]$.*

Proof. (1) Suppose that $P_{q_1} - c_1$ and $P_{q_2} - c_2$ are equivalent polynomials of $\mathbb{C}[\underline{x}, y, z]$ and let Φ be an automorphism of $\mathbb{C}[\underline{x}, y, z]$ such that $\Phi(P_{q_1} - c_1) = P_{q_2} - c_2$. The key of the proof is to show that $\Phi(\underline{x}^{[1]}) = \lambda\underline{x}^{[1]}$ for some constant $\lambda \in \mathbb{C}^*$. Afterwards, we can conclude exactly as in [17].

Remark that Φ induces, for every $c \in \mathbb{C}$, an isomorphism Φ_c between the rings $B_1 = \mathbb{C}[\underline{x}, y, z]/(P_{q_1} - c_1 - c)$ and $B_2 = \mathbb{C}[\underline{x}, y, z]/(P_{q_2} - c_2 - c)$. Therefore, as we have seen in the proof of Proposition 2.5, the element $\Phi_c(\underline{x}^{[1]})$ lies in the ideal $\underline{x}^{[1]}B_2$. Thus,

$$\Phi(\underline{x}^{[1]}) \in \bigcap_{c \in \mathbb{C}} \left(\underline{x}^{[1]}, P_{q_2} - c_2 - c\right) = \bigcap_{c \in \mathbb{C}} \left(\underline{x}^{[1]}, z^2 - c_2 - c\right) = \left(\underline{x}^{[1]}\right).$$

Since Φ is an automorphism, this implies that there exists a nonzero constant $\lambda \in \mathbb{C}^*$ such that $\Phi(\underline{x}^{[1]}) = \lambda\underline{x}^{[1]}$, as desired.

Now, since $\Phi(P_{q_1} - c_1 + \alpha\underline{x}^{[1]} - c) = P_{q_2} - c_2 + \alpha\lambda\underline{x}^{[1]} - c$, the varieties $V(P_{q_1+\alpha} - c_1 - c)$ and $V(P_{q_2+\alpha\lambda} - c_2 - c)$ are isomorphic for all $\alpha, c \in \mathbb{C}$. By Proposition 2.5, this implies that $c_1 = c_2$ and then that the zeros of the polynomials $q_1 + \alpha$ and $q_2 + \alpha\lambda$ are the same for all $\alpha \in \mathbb{C}$. Thus, $q_2 = \lambda q_1$.

Conversely, if $q_2 = \lambda q_1$ for some $\lambda \in \mathbb{C}^*$, it suffices to check that $\Phi(P_{q_1}) = P_{q_2}$, where Φ is the automorphism of $\mathbb{C}[\underline{x}, y, z]$ defined by $\Phi(x_1) = \lambda x_1$, $\Phi(x_i) = x_i$ for all $2 \leq i \leq n$, $\Phi(y) = \lambda^{-2}y$ and $\Phi(z) = z$. This proves the assertion (1).

(2) The hypersurfaces $H_1 = V(P_{q_1} - c_1)$ and $H_2 = V(P_{q_2} - c_2)$ are equivalent if and only if there exists a nonzero constant $\mu \in \mathbb{C}^*$ such that the polynomials $P_{q_1} - c_1$ and $\mu(P_{q_2} - c_2)$ are equivalent. Then, Assertion (2) follows from Assertion (1), noting that $\mu(P_{q_2} - c_2)$ is equivalent to the polynomial $P_{\widetilde{q_2}} - \mu c_2$, where $\widetilde{q_2}$ denotes the element of $\mathbb{C}[t]$ defined by $\widetilde{q_2}(t) = q_2(\mu^{-1}t)$. Indeed, one checks that this equivalence is realized by the automorphism of $\mathbb{C}^{n+2}$ defined by $(x_1, x_2, \ldots, x_n, y, z) \mapsto (\mu^{-1}x_1, x_2, \ldots, x_n, \mu y, \epsilon z)$, where ϵ is any complex number such that $\epsilon^2 = \mu^{-1}$. □

Before we state the next result, let us recall the notion of *stable equivalence*.

Definition 3.3.

1. Two hypersurfaces $H_1, H_2 \subset \mathbb{C}^n$ are said to be *stably equivalent* if there exists a $m \in \mathbb{N}$ such that $H_1 \times \mathbb{C}^m$ and $H_2 \times \mathbb{C}^m$ are equivalent hypersurfaces in $\mathbb{C}^{n+m}$.

(2) Two polynomials $P, Q \in \mathbb{C}^{[n]}$ are said to be *stably equivalent* if there exists a $m \in \mathbb{N}$ such that P and Q are equivalent polynomials of $\mathbb{C}^{[n+m]}$.

In this context, we have the following obvious generalization of Theorem 2.5' of [17].

Lemma 3.4. *For every $q \in \mathbb{C}^{[1]}$, the polynomials P_q and $P_{q(0)}$ are stably equivalent.*

Proof. The case $n = 1$ was proved in [17], where an explicit automorphism Φ of $\mathbb{C}[x, y, z, w]$, fixing x and satisfying $\Phi(x^2y+z^2+xq(z^2)) = x^2y+z^2+xq(0)$, is constructed. Since this automorphism fixes x, it suffices to replace formally x by $\underline{x}^{[1]}$ to get an automorphism of $\mathbb{C}[\underline{x}, y, z, w]$ which maps P_q onto $P_{q(0)}$. For the sake of completeness, let us give the formula.

Let $r \in \mathbb{C}[t]$ be the polynomial such that the equality $q(t) - q(0) = 2tr(t)$ holds. We let $\Phi(x_i) = x_i$ for all $1 \leq i \leq n$, $\Phi(z) = (1 - \underline{x}^{[1]}r(P_{q(0)}))z + \underline{x}^{[2]}w$ and $\Phi(w) = (1+\underline{x}^{[1]}r(P_{q(0)}))w - (r(P_{q(0)}))^2z$. Note that $\Phi(z^2+\underline{x}^{[1]}q(z^2)) \equiv z^2+\underline{x}^{[1]}q(0) \mod (\underline{x}^{[2]})$. Therefore, we can choose $\Phi(y) \in \mathbb{C}[\underline{x}, y, z, w]$ such that $\Phi(P_q) = P_{q(0)}$. Doing so, we get an endomorphism (we will show that it is in fact an automorphism) Φ of $\mathbb{C}[\underline{x}, y, z, w]$ which maps P_q onto $P_{q(0)}$.

Similarly, we define an endomorphism Ψ of $\mathbb{C}[\underline{x}, y, z, w]$ such that $\Psi(P_{q(0)}) = P_q$ by posing $\Psi(x_i) = x_i$ for all $1 \leq i \leq n$, $\Psi(z) = (1 + \underline{x}^{[1]}r(P_q))z - \underline{x}^{[2]}w$ and $\Psi(w) = (1 - \underline{x}^{[1]}r(P_q))w + (r(P_q))^2z$.

Now, one checks that $\Phi \circ \Psi(z) = z$ and that $\Phi \circ \Psi(w) = w$. Moreover, since $\Phi \circ \Psi(P_{q(0)}) = P_{q(0)}$, we have $\underline{x}^{[2]}\Phi \circ \Psi(y) + z^2 + \underline{x}^{[1]}q(0) = \Phi \circ \Psi(P_{q(0)}) = P_{q(0)} = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}q(0)$. This implies that $\Phi \circ \Psi(y) = y$. Therefore, Ψ is the inverse morphism of Φ. This proves the lemma. □

Together with Propositions 2.5 and 3.2, Lemma 3.4 leads to many counterexamples to the "stable equivalence problem" of every dimension $d \geq 2$. Finally, let us emphasize two particular examples.

Example 3.5.

(1) The polynomials $P = \underline{x}^{[2]}y+z^2+\underline{x}^{[1]}(z^2-1)-1$ and $Q = \underline{x}^{[2]}y+z^2+\underline{x}^{[1]}(z^2-1)-1$ of $\mathbb{C}[\underline{x}, y, z]$ are stably equivalent, but the hypersurfaces $V(P)$ and $V(Q)$ in $\mathbb{C}^{n+2}$ are not equivalent. Indeed, they are even non-isomorphic varieties.

(2) The polynomials $Q_k = \underline{x}^{[2]}y+z^2+\underline{x}^{[1]}(z^2-1)^k \in \mathbb{C}[\underline{x}, y, z]$ are stably equivalent for all $k \geq 1$, whereas the hypersurfaces $V(Q_k) \subset \mathbb{C}^{n+2}$ are pairwise non-equivalent. However, the varieties $V(Q_k - c)$ and $V(Q_{k'} - c)$ are isomorphic for all $k, k' \geq 1$ and every $c \in \mathbb{C}$.

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Universität Basel, Mathematisches Institut, Rheinsprung 21, CH-4051 Basel, Switzerland.

E-mail address: pierre-marie.poloni@unibas.ch