Data dispersion near the boundaries: can it partially explain the problems of decision and utility theories?

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An existence theorem for non-zero restrictions on the mean of a function on a finite interval in the presence of a non-zero dispersion of the function is proved. Here, "restriction" means the existence of a fixed distance from a border of the interval. Within this distance, it is impossible for the mean to be located. Equivalently, this is the width of a "forbidden zone" for the mean. The aims are to use this theorem in experiments interpretation, probability theory, statistics, economics and management. The ultimate aims are to help answer the Aczél-Luce question of whether $W(1)=1$ and to explain, at least partially, the well-known problems of decision and utility theory, such as risk aversion, the underweighting of high and the overweighting of low probabilities, the Allais paradox, the four-fold pattern paradox, etc. All these ultimate aims should be attained by purely mathematical methods.

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Introduction

This article presents the first part of the whole research. The research was motivated by the paradoxes of decision and utility theory. The analysis of such paradoxes was started in 1738 by Bernoulli in [1]. Examples of these paradoxes include the Allais paradox [2], the Ellsberg paradox [3], the "four-fold pattern" paradox (see, e.g., [4]), etc. In 2002 Kahneman received the Prize in Economic Sciences in Memory of Alfred Nobel for research in this field. In 2006, Kahneman and Thaler [5] pointed out that preferences inconsistencies in the paradoxes have still not been adequately overcome.

One possible way to solve these paradoxes was widely discussed, e.g., in [6]-[8]. The essence of this way consists in a proper attention to noise, imprecision, and other reasons that might cause dispersion, scattering, or spread of the data.

Aczél and Luce [9] emphasized a fundamental question: whether $W(1)=1$ (whether Prelec's weighting function is equal to 1 at $p=1$). This question opens one more way which consists in paying proper attention to the boundaries.

The research partially presented in this article combines these two ways. That is, it considers **a dispersion of the data near the boundaries** (or influence of a dispersion of the data near the boundaries).

Simple considerations of the research have an applied mathematical character but a significant practical importance. The ultimate aims are to provide pure mathematical support both for works that are based on the dispersion of data and for works that are concerned the Aczél-Luce question (or Luce question).

This article, as the first part of the research, deals with the general case of the restrictions on the mean of a function on finite intervals in the presence of a non-zero dispersion of the function.

The second part of the research will deal with the estimation of restriction values.

The third part of the research will deal with restrictions on the probability estimation and for the probability.

The fourth part of the research will deal with possible explanations of the above mentioned paradoxes of utility and decision theory and with the Aczél-Luce question.

The results may be used to estimate preferences, choices, decisions, (ir)rational behavior at data uncertainty, noises, experimental errors in management, investment, insurance, etc.

An illustrative example of restrictions

Two points

Let us suppose given an interval $[A, B]$ (see Figure 1). Let us suppose that two points are determined on this interval: a left point x_{Left} and a right point x_{Right} : $x_{Left}<x_{Right}$. The coordinates of the middle, mean point may be calculated as $M=(x_{Left}+x_{Right})/2$.

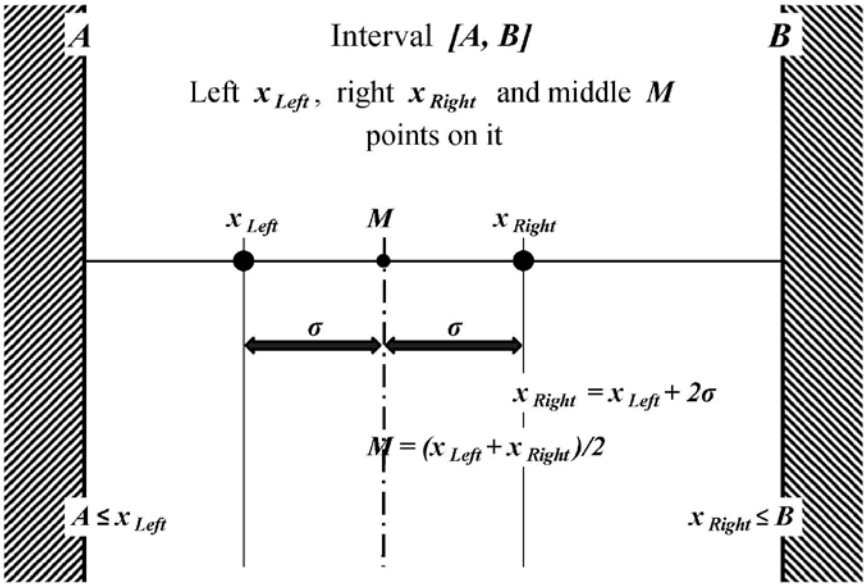

Figure 1. An interval $[A, B]$. Left x_{Left}, right x_{Right} and middle, mean M points on it

Let us suppose that $x_{Right}-x_{Left}\geq 2\sigma=2Const_{\sigma}>0$. So, of course, $x_{Right}\geq x_{Left}+2\sigma$ and $x_{Left}\leq x_{Right}-2\sigma$. For the sake of simplicity, Figures 1-3 represent the case of the equality $x_{Right}-x_{Left}=2\sigma$ and also, of course, $x_{Right}=x_{Left}+2\sigma$, $x_{Left}=x_{Right}-2\sigma$ and $M-x_{Left}=x_{Right}-M=\sigma=Const_{\sigma}>0$.

So, $M=x_{Left}+\sigma>x_{Left}$ and $M=x_{Right}-\sigma<x_{Right}$.

Suppose further that $x_{Left}\geq A$ and $x_{Right}\leq B$.

One can easily see that two types of zones for M can exist in the interval:

1) The mean point M can be located only in the zone which will be referred to as "allowed" (see Figure 2).

2) The mean point M cannot be located in the zones which will be referred to as "forbidden" (see Figure 3).

Allowed zone

Due to the conditions of the example, the left point x_{Left} may not be located further left than the left border of the interval $x_{Left} \geq A$ and the right point x_{Right} may not be located further right than the right border of the interval $x_{Right} \leq B$.

For M, we have $M = x_{Left} + \sigma \geq A + \sigma > A$ and $M = x_{Right} - \sigma \leq B - \sigma < B$ (see Figure 2).

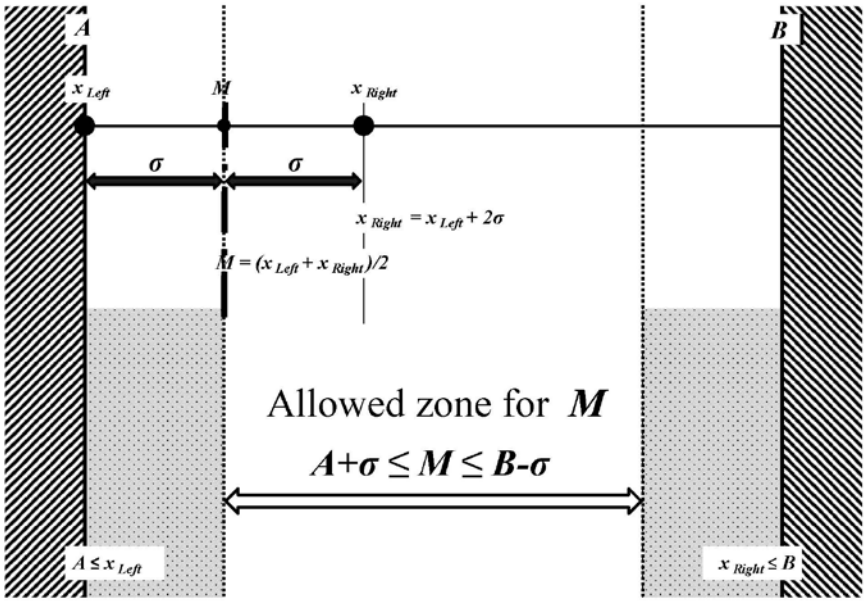

Figure 2. Allowed zone for M

The width of the allowed zone for M is equal to

$$B - \sigma - (A + \sigma) = (B - A) - 2\sigma .$$

It is less than the width $(B\text{-}A)$ of the total interval $[A, B]$ by 2σ. Also, the allowed zone is a proper subset of the total interval.

If the distance 2σ between the left x_{Left} and right x_{Right} points is non-zero, then the difference between the width of the allowed zone and the width of the interval is non-zero also. If the distance is greater than 2σ, then the difference is greater than 2σ also.

So, the mean point M can be located only in the allowed zone of the interval.

Forbidden zones, restrictions

Let us define the term "restriction" for the purposes of this article:

Definition. A **restriction** (more exactly, a **restriction on the mean**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "restriction."

If $A \le x_{Left}$, $x_{Right} \le B$ and $x_{Right}-x_{Left}=2\sigma$, then restrictions, forbidden zones with the width of one sigma σ exist between the mean point and the borders of the interval (see Figure 3). So there are two forbidden zones, located near the borders of the interval. The mean point M can not be located in these forbidden zones.

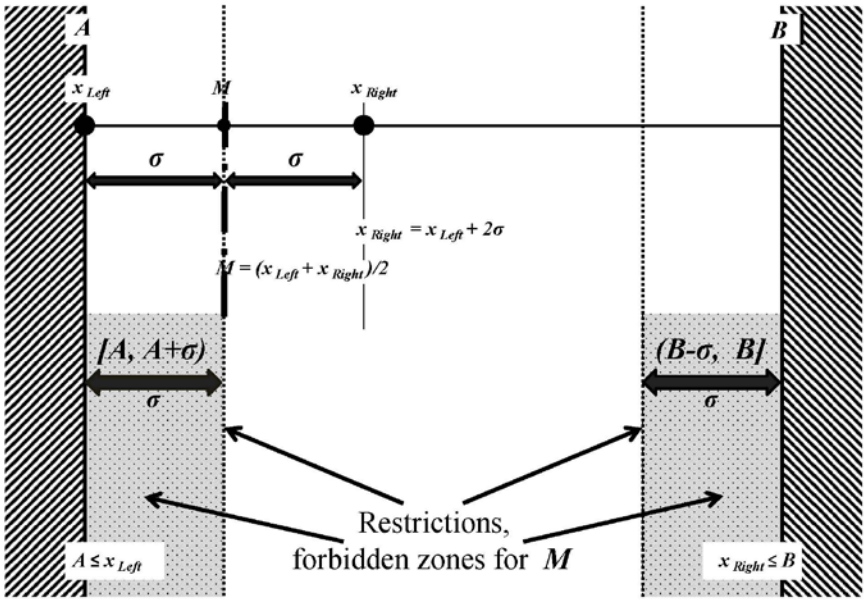

Figure 3. Forbidden zones, restrictions on M

The restrictions, the forbidden zones are shown by two dotted lines and by painting in the bottom part of Figure 3.

As we can easily see, restrictions on the mean or forbidden zones exist between the allowed zone of the mean M and the borders A and B of the interval $[A; B]$. The value of the restriction, or, equivalently, the width of the forbidden zone, is equal to σ.

So, the restrictions of the value σ on the mean point M exist near the borders of the interval.

1. Preliminary notes

1.1. Interval and function

Let us suppose given a finite interval, $X=[A, B] : 0<Const_{AB}\leq(B-A)<\infty$, a set of points $x_k : k=1, 2, \ldots K : 2\leq K\leq\infty$, and a finite non-negative function $f_K(x_k)$ such that for $x_k<A$ and $x_k>B$ the statement $f_K(x_k)\equiv 0$ is true; for $A\leq x_k\leq B$ the statement $0\leq f_K(x_k)<\infty$ is true, and

$$\sum_{k=1}^{K} f_K(x_k) = W_K ,$$

where W_K (the total weight of $f_K(x_k)$) is a constant such that

$$0 < W_K < \infty .$$

Without loss of generality, the function $f_K(x_k)$ may be normalized so that $W_K=1$.

1.2. Analog of moments

Definition 1.1. Let us define an analog of the moment of n-th order of the function $f_K(x_k)$ relative to a point x_0:

$$E(X - X_0)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - x_0)^n f_K(x_k) = \sum_{k=1}^{K}(x_k - x_0)^n f_K(x_k) .$$

From now on, for brevity, we refer to this analog of the moment of n-th order as simply the moment of n-th order.

Let us suppose the mean $M\equiv E(X)$ of the function $f_K(x_k)$ exists

$$E(X) \equiv \frac{1}{W_K}\sum_{k=1}^{K} x_k f_K(x_k) = \sum_{k=1}^{K} x_k f_K(x_k) \equiv M .$$

Let us suppose at least one central moment $E(X-M)^n : 2\leq n<\infty$, of the function $f_K(x_k)$ exists

$$E(X - M)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - M)^n f_K(x_k) = \sum_{k=1}^{K}(x_k - M)^n f_K(x_k) .$$

2. Maximality

Let us search for a function which attains the maximal possible central moment. The intuitively evident maximal possible absolute value of a central moment is obtained for the function which is concentrated at the borders of the interval.

2.1. A couple of elements

Let us consider the mean M of the function $f_K(x_k)$, a couple of points x_A and x_B, such as

$$A \le x_A \le M \le x_B \le B \ ,$$

and a couple of elements $f_K(x_A)$ and $f_K(x_B)$ (of the function $f_K(x_k)$) such that they are tied together by the conditions of having a constant total weight f and a constant mean point M

$$f_K(x_A) + f_K(x_B) = f \ ,$$
$$(M - x_A) f_K(x_A) = (x_B - M) f_K(x_B) \ .$$

A central moment $E_{Couple}(X\text{-}M)^n$ of this couple of elements may be written as

$$E_{Couple}(X - M)^n \equiv (x_A - M)^n f_K(x_A) + (x_B - M)^n f_K(x_B) .$$

Its absolute value does not exceed the sum of the absolute values of its parts

$$| E_{Couple}(X - M)^n | \le | (x_A - M)^n | f_K(x_A) + | (x_B - M)^n | f_K(x_B) =$$
$$= (M - x_A)^n f_K(x_A) + (x_B - M)^n f_K(x_B) \ .$$

2.2. Modification of the basic expression

After replacing $f_K(x_B)$ by

$$f_K(x_B) = \frac{M - x_A}{x_B - M} f_K(x_A) = f - f_K(x_A)$$

and replacing $f_K(x_A)$ and $f_K(x_B)$ by functions of x_A and x_B

$$f_K(x_A) + f_K(x_B) = \frac{x_B - M + M - x_A}{x_B - M} f_K(x_A) = \frac{x_B - x_A}{x_B - M} f_K(x_A) = f \ ,$$

we obtain the function $f_K(x_k)$:

$$f_K(x_A) = \frac{x_B - M}{x_B - x_A} f \quad \text{and} \quad f_K(x_B) = \frac{M - x_A}{x_B - x_A} f \ .$$

Hence, the expression for the central moment $E_{Couple}(X\text{-}M)^n$ of the couple may be rewritten as an expression which depends only on x_A and x_B

$$| E_{Couple}(X - M)^n | \le (M - x_A)^n \frac{x_B - M}{x_B - x_A} f + (x_B - M)^n \frac{M - x_A}{x_B - x_A} f \ .$$

2.3. Derivatives

Let us use the analysis of derivatives to find a maximum of the absolute value of the central moments $|E_{Couple}(X\text{-}M)^n|$ of this couple of elements.

Let us differentiate the expression for the absolute value of a central moment with respect to x_A

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_A} =$$

$$= \frac{-n(M-x_A)^{n-1}(x_B-x_A)+(M-x_A)^n}{(x_B-x_A)^2}(x_B-M) f +$$

$$+ (x_B-M)^n \frac{-(x_B-x_A)+(M-x_A)}{(x_B-x_A)^2} f =$$

$$= \{[-n(x_B-x_A)+(M-x_A)](M-x_A)^{n-1} +$$

$$+ [-(x_B-x_A)+(M-x_A)](x_B-M)^{n-1}\} \frac{(x_B-M)}{(x_B-x_A)^2} f$$

Since $n \geq 2$:

If $(x_B\text{-}x_A)=(M\text{-}x_A)$, that is, if $x_B=M$, then from

$$(M-x_A) f_K(x_A) = (x_B-M) f_K(x_B) .$$

we obtain

$$(M-x_A) = (M-M) \frac{f_K(x_B)}{f_K(x_A)} = 0$$

or $x_A=M$, and hence all the central moments are zero.

So, at $(x_B\text{-}x_A)=(M\text{-}x_A)$, that is, at $x_B=M=x_A$,

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_A} = 0 .$$

This is a stable equilibrium state. If $x_B=M=x_A$ then all the central moments are independent of x_A and are equal to zero.

If $(x_B\text{-}x_A)>(M\text{-}x_A)$, that is, if $x_B>M$, then

$$\{[-n(x_B-x_A)+(M-x_A)](M-x_A)^{n-1} +$$

$$+ [-(x_B-x_A)+(M-x_A)](x_B-M)^{n-1}\} < 0$$

and

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_A} < 0 .$$

So, for non-zero $E_{Couple}(X\text{-}M)^n$, the first derivative with respect to x_A is strictly less than zero for any $A \leq x_A < M$, independently of x_B, for any $M < x_B \leq B$. The closer is x_A to A, the greater is $|E_{Couple}(X\text{-}M)^n|$.

Hence, for any $x_B : M < x_B \leq B$, the maxima of the absolute value of a central moment $E_{Couple}(X\text{-}M)^n$ are attained at the minimal x_A, that is, at $x_A=A$.

Let us differentiate the expression for the absolute value of a central moment $E_{Couple}(X\text{-}M)^n$ of the couple f with respect to x_B

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_B} =$$

$$= (M-x_A)^n \frac{(x_B-x_A)-(x_B-M)}{(x_B-x_A)^2} f +$$

$$+ \frac{n(x_B-M)^{n-1}(x_B-x_A)-(x_B-M)^n}{(x_B-x_A)^2}(M-x_A) f = .$$

$$= \{[(x_B-x_A)-(x_B-M)](M-x_A)^{n-1} +$$

$$+ [n(x_B-x_A)-(x_B-M)](x_B-M)^{n-1}\} \frac{(M-x_A)}{(x_B-x_A)^2} f$$

Since $n \geq 2$:

If $(x_B\text{-}x_A)=(x_B\text{-}M)$, that is, if $x_A=M$, then $x_B=M$ also, and hence all the central moments are zero.

So, at $(x_B\text{-}x_A)=(M\text{-}x_A)$, that is, at $x_B=M=x_A$,

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_A} = 0 .$$

This is a stable equilibrium state. If $x_A=M=x_B$ then all the central moments are independent of x_B and are equal to zero.

If $(x_B\text{-}x_A)>(x_B\text{-}M)$, that is, if $x_A<M$, then

$$\{[(x_B-x_A)-(x_B-M)](M-x_A)^{n-1} +$$

$$+ [n(x_B-x_A)-(x_B-M)](x_B-M)^{n-1}\} > 0$$

and

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_B} > 0 .$$

So, for non-zero $E_{Couple}(X\text{-}M)^n$, the first derivative with respect to x_B is strictly greater than zero for any $M<x_B \leq B$, independently of x_A for any $A \leq x_A < M$. The closer is x_B to B the greater is $|E_{Couple}(X\text{-}M)^n|$.

Hence, for any $x_A : A \leq x_A < M$, the maxima of the absolute value of a central moment $E_{Couple}(X\text{-}M)^n$ are attained at the maximal x_B, that is, at $x_B=B$.

So, for non-zero central moments $E_{Couple}(X\text{-}M)^n$ of a couple f of elements $f_K(x_A)$ and $f_K(x_B)$, the maxima of the absolute value of $E_{Couple}(X\text{-}M)^n$ are attained at $x_A=A$ and $x_B=B$. That is, they are attained for the functions that are concentrated at the borders of the interval $[A, B]$ and

$$Max(|E_{Couple}(X-M)^n|) \leq (M-A)^n \frac{B-M}{B-A} f + (B-M)^n \frac{M-A}{B-A} f .$$

2.4. Dividing by couples

Let us analyze whether any function of Chapter 1 and its central moments may be completely divided and represented by such couples of elements.

Let us divide the points x_k into three groups: $x_{k(A)}<M$, $x_{k(M)}=M$ (zero central moments) and $x_{k(B)}>M$. Hence, $k(A)\le K_A$, $k(M)\le K_M$, $k(B)\le K_B$ and

$$K_A + K_M + K_B = K .$$

For $E(X-M)^n>0$ we have $K_A\ge 1$ and $K_B\ge 1$ and hence $k(A)=1, \ldots, K_A$ and $k(B)=1, \ldots, K_B$.

The definition of the mean

$$E(X) \equiv \sum_{k=1}^{K} x_k f_K(x_k) \equiv M$$

may be transformed to the first central moment

$$\sum_{k\le K} (x_k - M) f_K(x_k) = 0$$

and to

$$\sum_{k\le K} (x_k - M) f_K(x_k) = \sum_{k(A)\le K_A}^{K_A} (x_{k(A)} - M) f_K(x_{k(A)}) +$$

$$+ \sum_{k(M)\le K_M} (x_{k(M)} - M) f_K(x_{k(M)}) + \sum_{k(B)\le K_B} (x_{k(B)} - M) f_K(x_{k(B)}) = 0 .$$

Due to

$$x_{k(M)} - M \equiv 0 ,$$

we have

$$\sum_{k\le K} (x_k - M) f_K(x_k) =$$

$$= \sum_{k(A)\le K_A} (x_{k(A)} - M) f_K(x_{k(A)}) + \sum_{k(B)\le K_B} (x_{k(B)} - M) f_K(x_{k(B)}) = 0$$

and it may be transformed to a balance

$$\sum_{x_{k(A)}\le K_A} (M - x_{k(A)}) f_K(x_{k(A)}) = \sum_{x_{k(B)}\le K_B} (x_{k(B)} - M) f_K(x_{k(B)}) .$$

Let us consider sequentially cases with various numbers of elements $K_{AB}=K_A+K_B$ from $K_{AB}=0$ to a general case.

Case 0. If $K_{AB}=0$, then $E(X-M)^n=0$.

Case 1. Evidently, due to the definition of the mean, the case $K_A=0$ and $K_B\geq 1$ and the case $K_A\geq 1$ and $K_B=0$ cannot occur.

Case 2. If $K_{AB}=2$, $K_A=1$ and $K_B=1$, then

$$(M-x_{1(A)})\,f_K(x_{1(A)})=(x_{1(B)}-M)\,f_K(x_{1(B)})$$

and the pair $f_K(x_{1(A)})$ and $f_K(x_{1(B)})$ is the required couple of the previous subchapters.

Let us enumerate the points $x_{k(A)}$ and $x_{k(B)}$, for example, from those furthest from M and with maximal weights, to those closest to M and with minimal weights.

Case 3. If, for example, $K_A=2$ and $K_B=1$, then we divide the element $f_K(x_{1(B)})$ into two parts $f_{K.1}(x_{1(B)})$ and $f_{K.2}(x_{1(B)})$ such that

$$f_K(x_{1(B)})=f_{K.1}(x_{1(B)})+f_{K.2}(x_{1(B)})$$

and

$$(M-x_{1(A)})\,f_K(x_{1(A)})=(x_{1(B)}-M)\,f_{K.1}(x_{1(B)})\ .$$

The pair $f_K(x_{1(A)})$ and $f_{K.1}(x_{B1})$ is the required couple of the previous subchapters. The balance remains

$$(M-x_{1(A)})\,f_K(x_{1(A)})+(M-x_{2(A)})\,f_K(x_{2(A)})=$$
$$=(x_{1(B)}-M)\,f_{K.1}(x_{1(B)})+(x_{1(B)}-M)\,f_{K.2}(x_{1(B)})\ ,$$

and we come to Case 2

$$(M-x_{2(A)})\,f_K(x_{2(A)})=(x_{1(B)}-M)\,f_{K.2}(x_{1(B)})\ .$$

The pair $f_K(x_{2(A)})$ and $f_{K.2}(x_{B1})$ is the required couple of the previous subchapters also.

The general case L. Suppose we are in the case $K_{AB}=L\geq 4$, $K_A\geq 1$ and $K_B\geq 1$.

If

$$(M-x_{1(A)})\,f_K(x_{1(A)})=(x_{1(B)}-M)\,f_K(x_{1(B)})\ ,$$

then the pair $f_K(x_{1(A)})$ and $f_K(x_{1(B)})$ is the required couple of the previous subchapters. The number of uncoupled elements is diminished by two and we come to Case L-2.

If

$$(M-x_{1(A)})\,f_K(x_{1(A)})\neq(x_{1(B)}-M)\,f_K(x_{1(B)})\ ,$$

then let us divide the appropriate element as in Case 3 and we diminish the number of uncoupled elements by one and come to Case L-1.

So, we may consecutively diminish the number of uncoupled elements from any L to 2 and, so, we may come to fully coupled elements. Hence, any function of Chapter 1 may be completely divided by couples of elements, except $x_{k(M)}$.

So, any function of Chapter 1 and its central moments may be completely divided and represented by couples of elements except for points $x_{k(M)}$ which do not contribute to the central moment.

So, the function $f_{Max.K}(x_k)$, which possesses maximal central moments (in modulus), should be concentrated at the borders $x_1=A$ and $x_2=B$ of the interval.

Under the condition of norm one ($W_K=1$) of Chapter 1, the function $f_{Max.K}(x_k)\equiv f_{Borders.K}(x_k)$ should have the form

$$f_{Borders.K}(A)=\frac{B-M}{B-A}\qquad\text{and}\qquad f_{Borders.K}(B)=\frac{M-A}{B-A}\ .$$

The central moments $E_{Borders}(X\text{-}M)^n$ of the function $f_{Borders.K}(x_k)$ are

$$E_{Borders}(X-M)^n=(A-M)^n\frac{B-M}{B-A}+(B-M)^n\frac{M-A}{B-A}.$$

The modulus of a central moment $E_{Borders}(X\text{-}M)^n$ of the function $f_{Borders.K}(x_k)$ is not greater than

$$|\,E_{Borders}(X-M)^n\,|\leq(M-A)^n\frac{B-M}{B-A}+(B-M)^n\frac{M-A}{B-A}\,.$$

For the even orders $2n$ of the central moments $E_{Borders}(X\text{-}M)^n$ the inequality is reduced to an equality without the modulus

$$E_{Borders}(X-M)^{2n}=(M-A)^{2n}\frac{B-M}{B-A}+(B-M)^{2n}\frac{M-A}{B-A}\,.$$

So, the moduli of the central moments of any function $f_K(x_k)$ of Chapter 1 are not greater than

$$Max(|\,E(X-M)^n\,|)\leq(M-A)^n\frac{B-M}{B-A}+(B-M)^n\frac{M-A}{B-A}\qquad(2.1).$$

2.5. Two notes

Let us analyze the maximal central moments $E_{Borders}(X\text{-}M)^n$ for

$$M = \frac{B-A}{2}$$

and for

$$M \approx A \qquad \text{or} \qquad M \approx B.$$

The mean is in the center of the interval

Let us analyze the maximal central moments for

$$M = \frac{B-A}{2}.$$

Let us differentiate the expression for the absolute value of a central moment $E_{Borders}(X\text{-}M)^n$ with respect to M

$$\frac{1}{B-A}[(M-A)^n(B-M)+(B-M)^n(M-A)]'_M =$$
$$= \frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$
$$-n(B-M)^{n-1}(M-A)+(B-M)^n]$$

and, at $M=(B\text{-}A)/2$,

$$\frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$
$$-n(B-M)^{n-1}(M-A)+(B-M)^n] =$$
$$= \frac{1}{B-A}[n(\frac{B-A}{2})^{n-1}(\frac{B-A}{2})-(\frac{B-A}{2})^n - .$$
$$-n(\frac{B-A}{2})^{n-1}(\frac{B-A}{2})+(\frac{B-A}{2})^n] =$$
$$= \frac{1}{B-A}(\frac{B-A}{2})^n[n-1-n+1] = 0$$

So, at $M=(B\text{-}A)/2$, for any $n \geq 2$ there is an extremum or a point of inflection.

Let us differentiate $E_{Borders}(X\text{-}M)^n$ once more

$$\frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n-$$
$$-n(B-M)^{n-1}(M-A)+(B-M)^n]'_M=$$
$$=\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-n(M-A)^{n-1}-n(M-A)^{n-1}+$$
$$+n(n-1)(B-M)^{n-2}(M-A)-n(B-M)^{n-1}-n(B-M)^{n-1}]=$$
$$=\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1}+$$
$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}]$$

and, at $M=(B\text{-}A)/2$,

$$\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1}+$$
$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}]=$$
$$=\frac{1}{B-A}(\frac{B-A}{2})^{n-1}[n(n-1)-2n+n(n-1)-2n]= \quad .$$
$$=\frac{1}{B-A}(\frac{B-A}{2})^{n-1}2[n(n-1)-2n]=$$
$$=(\frac{B-A}{2})^{n-2}[n(n-3)]$$

That is, at $M=(B\text{-}A)/2$:

For $n=2$ there is a well-known maximum, the moment of inertia of two material points whose weights are equal to each other

$$E_{Borders}(X-\frac{B-A}{2})^2=(M-A)^2\frac{B-M}{B-A}+(B-M)^2\frac{M-A}{B-A}=$$
$$=(\frac{B-A}{2})^2\frac{1}{2}+(\frac{B-A}{2})^2\frac{1}{2}=(\frac{B-A}{2})^2 \quad .$$

For $n=3$ there is a point of inflection.

For $n>3$ there are minima.

The mean is near a border of the interval

Let us search maximums which are close to the borders of the interval.

Let us differentiate the absolute value of the central moment $E_{Borders}(X\text{-}M)^n$ with respect to M for $M \approx A$ and $n >> 1$

$$\frac{1}{B-A}[(M-A)^n(B-M)+(B-M)^n(M-A)]'_M =$$
$$= \frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$
$$-n(B-M)^{n-1}(M-A)+(B-M)^n] \approx$$
$$\approx \frac{1}{B-A}[(B-M)^n - n(B-M)^{n-1}(M-A)]$$

and

$$\frac{1}{B-A}[(B-M)^n - n(B-M)^{n-1}(M-A)] =$$
$$= \frac{(B-M)^{n-1}}{B-A}[(B-M)-n(M-A)] =$$
$$= \frac{(B-M)^{n-1}}{B-A}[(B-A-(M-A))-n(M-A)] =$$
$$= \frac{(B-M)^{n-1}}{B-A}[(B-A)-(n+1)(M-A)] = 0$$

and

$$\boldsymbol{M - A \approx \frac{B-A}{n+1}}.$$

The second derivative is

$$\frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$
$$-n(B-M)^{n-1}(M-A)+(B-M)^n]'_M =$$
$$= \frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-n(M-A)^{n-1}-n(M-A)^{n-1}+$$
$$+n(n-1)(B-M)^{n-2}(M-A)-n(B-M)^{n-1}-n(B-M)^{n-1}] =$$
$$= \frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1}+$$
$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}]$$

For $M \approx A$ and $n>>1$

$$\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1}+$$
$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}] \approx$$
$$\approx n\frac{(B-M)^{n-2}}{B-A}[(n-1)(M-A)-2(B-M)]=$$
$$=n\frac{(B-M)^{n-2}}{B-A}[(n-1)(M-A)-2((B-A)-(M-A))]=$$
$$=n\frac{(B-M)^{n-2}}{B-A}[(n+1)(M-A)-2(B-A)]$$

and, for $M \approx A+(B-A)/(n+1)$ and $n>>1$,

$$n\frac{(B-M)^{n-2}}{B-A}[(n+1)(M-A)-2(B-A)] \approx$$
$$\approx n\frac{(B-M)^{n-2}}{B-A}[(n+1)\frac{B-A}{n+1}-2(B-A)]=.$$
$$=n(B-M)^{n-2}[1-2]<0$$

So, the second derivative is negative and there are maxima at the points $M \approx A+(B-A)/(n+1)$.

The analog of the central moments $E_{Borders}(X-M)^n$ of the function $f_{Borders.K}(x_k)$ for $M-A \approx (B-A)/(n+1)$ and $n>>1$ may be taken as

$$|E_{Borders}(X-M)^n| \le (M-A)^n\frac{B-M}{B-A}+(B-M)^n\frac{M-A}{B-A} \approx$$
$$\approx (B-M)^n\frac{M-A}{B-A} \approx (B-A-\frac{B-A}{n+1})^n\frac{1}{n+1}= \quad .$$
$$=(1-\frac{1}{n+1})^n\frac{(B-A)^n}{n+1}$$

For $n>>1$

$$(1-\frac{1}{n+1})^n=(1-\frac{1}{n+1})^{n+1}(1-\frac{1}{n+1})^{-1} \approx \frac{1}{e}.$$

So, for $M \approx A+(B-A)/(n+1)$ and $n>>1$, the maxima (those are attained for even n) of $E_{Borders}(X-M)^n$ are curiously

$$Max(|E_{Borders}(X-(A+\frac{B-A}{n+1}))^n|) \approx \frac{1}{e}\frac{(B-A)^n}{n+1}.$$

Evidently, for $M \approx B-(B-A)/(n+1)$, at $n>>1$, the maxima (those are attained for even n) of $E_{Borders}(X-M)^n$ are analogously

$$Max(|E_{Borders}(X-(B-\frac{B-A}{n+1}))^n|) \approx \frac{1}{e}\frac{(B-A)^n}{n+1}.$$

3. Theorem

3.1. Lemma about the tendency to zero for central moments

Lemma 3.1. If, for the function $f_K(x_k)$ defined in Section 1, $M \equiv E(X)$ tends to A or to B,

then, for $n : 2 \le n < \infty$, $E(X-M)^n$ tends to zero.

Proof. For $M \to A$, the estimate (2.1) gives

$$| E(X-M)^n | \le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} =$$

$$= [(M-A)^{n-1} + (B-M)^{n-1}] \frac{(M-A)(B-M)}{B-A} <$$

$$< [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \le$$

$$\le 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0$$

This rough estimate is already sufficient for the purpose of this article. But a more precise estimate may be obtained:

Let us transform

$$[(M-A)^{n-1} + (B-M)^{n-1}] \frac{(M-A)(B-M)}{B-A} =$$

$$= [(\frac{M-A}{B-A})^{n-1} + (\frac{B-M}{B-A})^{n-1}](B-A)^{n-1} \frac{(M-A)(B-M)}{B-A}$$

Let us consider the terms $(M-A)/(B-A)$ and $(B-M)/(B-A)$. Keeping in mind that $A \le M \le B$ we obtain $0 \le (M-A)/(B-A) \le 1$ and $0 \le (B-M)/(B-A) \le 1$. For $n \ge 2$ we have

$$(\frac{M-A}{B-A})^{n-1} + (\frac{B-M}{B-A})^{n-1} \le$$

$$\le \frac{M-A}{B-A} + \frac{B-M}{B-A} = \frac{B-A}{B-A} \equiv 1$$

So,

$$[(\frac{M-A}{B-A})^{n-1} + (\frac{B-M}{B-A})^{n-1}](B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \le$$

$$\le (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \le (B-A)^{n-1}(M-A)$$

So,

$$| E(X-M)^n | \le (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \qquad (3.1).$$

For $M \to B$, the proof is similar and gives

$$| E(X-M)^n | \le (B-A)^{n-1}(B-M) \xrightarrow[M \to B]{} 0 \qquad (3.2).$$

So, if $(B-A)$ and n are finite and $M \to A$ or $M \to B$, then $E(X-M)^n \to 0$.

The lemma has been proved.

3.2. Existence theorem for restrictions on the mean

Let us define two terms for the purposes of this article:

Definition 3.1. A **restriction on the mean** r_{Mean} (or, simply, a **restriction**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "restriction."

Definition 3.2. Let us define "**restriction on dispersion** of the n-th order" $r^n_{Dispersion.n} \equiv \boldsymbol{r^n_{Disp.n}} : r_{Disp.n}>0$ (where dispersion is taken in the broad sense, as scattering, spread, variation, etc.) to be the minimal absolute value of the analog of the n-th order central moment $E(X-M)^n$ such that $|E(X-M)^n| \geq r^n_{Disp.n}>0$.

For $n=2$ the restriction on the dispersion of second order is the minimal possible dispersion (in the particular sense) $r^2_{Disp.2}=\sigma^2_{Min}$.

Note $r_{Disp.n}<(B-A)$. This follows from

$$E(X-M)^n = \sum_{k=1}^{K}(x_k - M)^n f_K(x_k) < (B-A)^n \sum_{k=1}^{K} f_K(x_k) = (B-A)^n .$$

Theorem 3.2. If, for the finite non-negative discrete function $f_K(x_k)$ defined in Section 1, with the mean $M \equiv E(X)$ and the analog of an n-th ($2 \leq n < \infty$) order central moment $E(X-M)^n$ of the function, a non-zero restriction on dispersion of the n-th order $r^n_{Disp.n}=Const_{Disp.n}>0 : |E(X-M)^n| \geq r^n_{Disp.n}$, exists,

then the non-zero restriction $r_{Mean}>0$ on the mean $E(X)$ exists and $A<(A+r_{Mean}) \leq M \equiv E(X) \leq (B-r_{Mean})<B$.

Proof. From the conditions of the theorem and from (3.1) for $M \to A$, we have

$$0 < r^n_{Disp.n} \leq |E(X-M)^n| \leq (B-A)^{n-1}(M-A)$$

and

$$0 < \frac{r^n_{Disp.n}}{(B-A)^{n-1}} \leq (M-A) .$$

So,

$$(M-A) \geq r_{Mean} \equiv \frac{r^n_{Disp.n}}{(B-A)^{n-1}} > 0 \qquad (3.3).$$

For $M \to B$, the proof is similar and gives

$$(B-M) \geq r_{Mean} \equiv \frac{r^{n}_{Disp.n}}{(B-A)^{n-1}} > 0 \qquad (3.4).$$

So, as long as $(B\text{-}A)$ and n are finite and $r^{n}_{Disp.n}=Const_{Disp.n}>0$, then $r_{Mean}=Const_{M}>0$ and $A<(A+r_{Mean})\leq M\leq(B-r_{Mean})<B$.
The theorem has been proved.

Note

This estimate is an ultra-reliable one. It is, in a sense, as ultra-reliable as the Chebyshev inequality. Preliminary calculations [10] which were performed for real cases, such as the normal, uniform and exponential distributions with the minimal values σ^{2}_{Min} of the analog of the dispersion (in the particular sense), gave the restrictions r_{Mean} on the mean of the function, which are not worse than

$$r_{Mean} \geq \frac{\sigma_{Min}}{3} \ .$$

3.3. Comments on the theorem

We may reformulate the essence of the theorem in some variants:

Suppose the analog of a finite $(n<\infty)$ central moment $E(X\text{-}M)^{n}$ of a finite non-negative function, which is defined for a finite interval, cannot approach zero closer than a non-zero constant $r^{n}_{Disp.n}: |E(X\text{-}M)^{n}|\geq r^{n}_{Disp.n}>0$. Then the mean of the function also cannot approach any border of this interval closer than the other non-zero constant $r_{Mean}>0$.

More particularly: Suppose the analog of the dispersion (in the particular sense) $E(X\text{-}M)^{2}$ of a finite non-negative function, which is defined for a finite interval, cannot approach zero closer than a non-zero constant σ^{2}_{Min} : $E(X\text{-}M)^{2}\geq\sigma^{2}_{Min}>0$. Then the mean of the function also cannot approach any border of this interval closer than the other non-zero constant $r_{Mean}>0$.

In other words: Suppose for a finite non-negative function, which is defined on a finite interval, a non-zero constant restriction $r^n_{Disp.n}>0$ on dispersion exists. That is: between the zone of possible values of the analog of a finite $(n<\infty)$ central moment $E(X-M)^n$ of the function and zero there is a forbidden zone for the dispersion values, i.e., $|E(X-M)^n|\geq r^n_{Disp.n}>0$. Then other non-zero constant restrictions $r_{Mean}>0$ on the mean exist also. That is: between the zone of possible values of the mean of this function and any border of the interval, there is another forbidden zone for the mean, i.e., $A<(A+r_{Mean})\leq M\leq(B-r_{Mean})<B$.

More particularly: Suppose for a finite non-negative function, which is defined on a finite interval, a non-zero constant restriction $\sigma^2_{Min}>0$ on dispersion exists. That is: between the zone of possible values of the analog of the dispersion (in the particular sense) $E(X-M)^2$ of the function and zero, there is a forbidden zone for the dispersion values, i.e., $E(X-M)^2\geq\sigma^2_{Min}>0$. Then other non-zero constant restrictions $r_{Mean}>0$ on the mean exist also. That is: between the zone of possible values of the mean of this function and any border of the interval there is another forbidden zone for the mean, i.e., $A<(A+r_{Mean})\leq M\leq(B-r_{Mean})<B$.

In other words: If there is a zero restriction $r^n_{Disp.n}=0$ on the dispersion (in the broad sense) $E(X-M)^n$ of a function, then there are zero restrictions $r_{Mean}=0$ on the mean of the function. The greater restriction $r^n_{Disp.n}>0$ there is on the dispersion, the greater restrictions $r_{Mean}>0$ there are on the mean.

So, a restriction $r^n_{Disp.n}>0$ on the dispersion biases the boundaries of the zone of possible values of the mean away from the borders of the interval, towards the middle of the interval.

So, a restriction $r^n_{Disp.n}>0$ on the dispersion biases the mean away from the borders, towards the middle of the interval.

To put the matter more simply: A non-zero dispersion of a finite non-negative function leads to the existence of non-zero restrictions on the mean of this function.

To put the matter even more simply:

A non-zero dispersion leads to a non-zero bias of the mean.

In its simplest form, this can be stated:

Dispersion biases the mean.

4. Applications of the theorem to economics and management

Sketches of the proof of the theorem have also been provided (see, e.g., [11]) for estimating probabilities. In the presence of data dispersion, scattering, spread, variation, etc., restrictions can exist on probability estimation and on the probability near the borders of the probability scale.

The estimation of a probability possesses the properties assumed for the function of Chapter 1 (in fact, these properties have been chosen to be those satisfied by probability estimations). So, the theorem is true for the estimation of a probability.

The probability is a limit of the estimation of probability as the number of attempts tends to infinity. If the probability estimation is greater (less) than or equal to a quantity, then a limit of the probability estimation is greater (less) than or equal to the quantity also.

The hypothesis of the restrictions and the sketches of the proof of the theorem have been used in economics. The first international paper [12] of the item was published in 2005. The hypothesis and the sketches of the proof have explained the well-known problems and paradoxes of decision theory and utility theory, such as risk aversion, the underweighting of high and the overweighting of low probabilities, the four-fold pattern paradox, etc. (see, e.g., [13]-[16]). They may be used also in management for the same and analogous aims.

In the presence of data dispersion, the restrictions, that can exist on the probability near the borders of the probability scale, can bias the results of experiments in comparison with those without data dispersion. The preliminary researches, including considerations of the restrictions as a hypothesis, showed that this bias can explain (at least partially) the well-known problems and paradoxes of decision and utility theories. It should be noted that this explanation is true not only for a particular combination of parameters but both for high and low probabilities and both for gains and losses (see, e.g., [13]-[16]).

The new fields of application of the theorem might be concerned with the Aczél-Luce question [9] whether $W(1)=1$ (whether Prelec's weighting function is equal to 1 at $p=1$).

Conclusions

The possibility of the existence of non-zero restrictions in the presence of non-zero dispersion (both in the particular sense, as the analog of the second central moment, and in the broad sense, as scattering or spread) has been analyzed in this article.

The existence of non-zero restrictions on the mean of a discrete finite non-negative function on an interval $X=[A, B]$, when the analog of a central moment of the function takes a non-zero value, has been proved. Suppose there is a non-zero restriction $r_{Dispersion.n} \equiv r_{Disp.n} > 0$ on the analog of the n-th central moment $|E(X-M)^n|$ $(\infty > n \geq 2)$ of a discrete finite non-negative function, that is, $|E(X-M)^n| \geq r^n_{Disp.n} > 0$. Then other non-zero restrictions $r_{Mean} > 0$ on the mean M of this function exist at the borders A and B of the interval $[A, B]$. The value of the restriction r_{Mean} at A is (see (3.3))

$$(M - A) \geq r_{Mean} \equiv \frac{r^{\,n}_{Disp.n}}{(B-A)^{n-1}} > 0 \ .$$

The value of the restriction r_{Mean} at B is also (see (3.4))

$$(B - M) \geq r_{Mean} \equiv \frac{r^{\,n}_{Disp.n}}{(B-A)^{n-1}} > 0 \ .$$

For $n=2$ the analog of the central moment is the analog of the dispersion (in the particular sense), and r_{Mean} at A may be rewritten for the minimum σ_{Min} of the analog of the standard deviation σ, i.e., $\sigma \geq \sigma_{Min} \equiv r_{Disp.2} > 0$, as

$$(M - A) \geq r_{Mean} = \frac{r^{\,2}_{Disp.2}}{(B-A)} \equiv \frac{\sigma^{\,2}_{Min}}{(B-A)} > 0 \ .$$

The value of the restriction r_{Mean} at B may be also rewritten for the minimum σ_{Min} of the analog of the standard deviation σ as

$$(B - M) \geq r_{Mean} = \frac{r^{\,2}_{Disp.2}}{(B-A)} \equiv \frac{\sigma^{\,2}_{Min}}{(B-A)} > 0 \ .$$

The above estimates for the restrictions r_{Mean} on the mean are, in a sense, as ultra-reliable as the Chebyshov inequality. For real cases such as the normal distribution, for the minimal values σ^2_{Min} of the analog of the dispersion (in the particular sense), the preliminary calculations of [10] give the restrictions r_{Mean} on the mean, which are no worse than

$$r_{Mean} \geq \frac{\sigma_{Min}}{3} \ .$$

The theorem may have a significant practical value (it is considered and proved here mainly due to this value):

The considered non-zero restrictions on the mean are equivalent to non-zero biases of the mean.

Such biases are observed in a wealth of experiments and in practice.

So, the theorem may explain, at least partially, a number of the well-known problems of decision theory and utility theory.

Sketches of the proof of the theorem have been provided (see, e.g., [11]) for estimating probabilities. In the presence of data dispersion, scattering, spread, variation, etc, restrictions can exist on probability estimation and on the probability (as the limit of the probability estimation) near the borders of the probability scale.

The hypothesis of the restrictions and the sketches of the proof of the theorem have been used in economics and have qualitatively explained the well-known problems and paradoxes of decision theory and utility theory, such as risk aversion, the underweighting of high and the overweighting of low probabilities, the four-fold pattern paradox, etc. (see, e.g., [13]-[16]).

New applications of the theorem might be concerned with research into the Aczél-Luce question (or Luce question) [9] whether Prelec's weighting function is equal to 1 at $p=1$.

The ultimate aims are to use the theorem and its applications in estimating preferences, choices, decisions, (ir)rational behavior at data uncertainty, noises, experimental errors in management, investment, insurance, etc.

So, the theorem of this article states:

Data dispersion near the boundaries may not be excluded to be able to partially explain the problems of decision and utility theories.

The ultimate aim of the whole research is to conclude:

Data dispersion near the boundaries can (at least partially) explain the problems of decision and utility theories.

References

[1] Bernoulli D. (1738), "Specimen Theoriae Novae de Mensura Sortis", Commentarii Academiae Scientiarum Imperialis Petropolitanae, 5 (1738): 175-192. English translation: "Exposition of a New Theory on the Measurement of Risk", *Econometrica*, 22(1): 23-36.

[2] Allais, M. (1953), "Le comportement de l'homme rationnel devant le risque: Critique le postulats et axioms de L'École Américaine", *Econometrica*, 21: 503-546.

[3] Ellsberg L. (1961), "Risk, Ambiguity, and the Savage Axioms", *The Quarterly Journal of Economics*, 75 (4): 643-669.

[4] Tversky, A. and Wakker, P. (1995) "Risk attitudes and decision weights", *Econometrica*, 63: 1255-1280.

[5] Kahneman, D. and Thaler, R. (2006) "Anomalies: Utility Maximization and Experienced Utility", *Journal of Economic Perspectives*, 20, #1: 221-234.

[6] Schoemaker, P., and Hershey J. (1992), "Utility measurement: Signal, noise, and bias", *Organizational Behavior and Human Decision Processes*, 1992, vol. 52 (3): 397-424.

[7] Hey, J., and Orme C. (1994), "Investigating Generalizations of Expected Utility Theory Using Experimental Data", *Econometrica*, 62 (6): 1291-1326.

[8] Butler, D. J., and Loomes G. C. (2007), "Imprecision as an Account of the Preference Reversal Phenomenon", *American Economic Review*, 97 (1): 277-297.

[9] Aczél, J. and Luce R. D. (2007), "A behavioral condition for Prelec's weighting function on the positive line without assuming W(1)=1", *Journal of Mathematical Psychology*, Volume 51, Issue 2, April 2007, Pages 126–129.

[10] Harin A. (2009a), "About existence of ruptures in the probability scale: Calculation of ruptures' values", *Proceedings of the Ninth International Scientific School "Modelling and Analysis of Safety and Risk in complex systems"*, 9, 458-464, 2009.

[11] Harin A. (2010), "Theorem of existence of ruptures in the probability scale", *Proceedings of the 9th International conference "Financial and Actuarial Mathematics and Eventoconverging Technologies"*, 9, 312-315, (2010).

[12] Harin A. (2005), "A new approach to solve old problems", *Game Theory and Information from Economics Working Paper Archive at WUSTL*, 2005.

[13] Harin A. (2009b), "Taking into account boundary effects of noises as a new way to solution of problems of the utility theory", *First Russian Economic Congress (REC-2009)*, 2009.

[14] Harin A. (2012a), "Theorem of existence of ruptures in probability scale and utility theory. "Four-fold-pattern" paradox", *19th International Conference on "Mathematics. Computer. Education"*, 2012.

[15] Harin A. (2012b), "Data dispersion in economics (I) – Possibility of restrictions", *Review of Economics & Finance*, 2(3): 59-70. http://www.bapress.ca/ref/ref-2012-3/Data%20Dispersion%20in%20Economics%20(I)%20---%20Possibility%20of%20Restrictions.pdf

[16] Harin A. (2012c), "Data dispersion in economics (II) – Inevitability and Consequences of Restrictions", *Review of Economics & Finance*, 2(4): 24-36. http://www.bapress.ca/ref/ref-2012-4/Data%20Dispersion%20in%20Economics%20(II)%20--%20Inevitability.pdf