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On structural computations until fracture based on an anisotropic and unilateral damage theory

Martin Genet^{a,1,*}, Lionel Marcin^{a,2}, Pierre Ladevèze^{a,b}

^a*LMT-Cachan, ENS-Cachan/CNRS UMR8535/Paris VI University/UniverSud Paris PRES, 61 avenue du Président Wilson, 94235 Cachan Cedex, France*

^b*EADS Foundation Chair “Advanced Computational Structural Mechanics”*

Abstract

This paper describes the formulation and numerical implementation of a family of anisotropic and unilateral damage models for the prediction of damage and final rupture in engineering structures. The damage can be load-oriented, microstructure-oriented or (for the first time within this modeling framework) softening. The local equations are solved using a combination of fixed-point and Newton-Raphson algorithms, whose efficiencies are drastically improved through Aitken’s relaxation and BFGS approximation. A delay-effect method is used to control the localization of damage, which leads to an objective calculation of the final rupture of structures.

Keywords:

Continuum damage mechanics; Anisotropic and unilateral damage; Structural computation; Rupture; Delay-effect localization control

*Corresponding author. E-mail address: genet@lmt.ens-cachan.fr

¹Current affiliation: Marie-Curie Fellow, Cardiac Biomechanics Laboratory, Department of Surgery, University of California at San Francisco, USA

²Current affiliation: Snecma Villaroche, France

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1. Introduction

Since continuum damage mechanics (CDM) first appeared [Kachanov, 1958, 1999], it has been studied and improved by many research groups worldwide [Leckie, 1978; Ladevèze, 1983; Murakami, 1983; Lemaitre, 1985; Simo and Ju, 1987; Chaboche, 1988], taught in many universities [Kachanov, 1986; Lemaitre, 1992; Krajcinovic, 1996], and has become one of the classical tools of the structural mechanics community [Lemaitre and Desmorat, 2005].

In practice, CDM consists in developing a model of the macroscopic behavior of a material in which microscopic damage is represented as a stiffness reduction. The form of the stiffness reduction is defined by the *damage kinematics* using *damage variables*, which are internal variables in thermodynamical terminology. The stiffness reduction, *i.e.* the magnitude of the damage variable, is defined as a function of the applied stress or strain using the *damage kinetics*. Damage models, and in particular their damage kinetics, usually involve several parameters (in addition to the classical Young and shear moduli and Poisson ratios) which must be identified experimentally through classical coupon tests. Then these models can be used to predict damage in engineering structures subjected to possibly complex loading and, for example, become part of a structural optimization process. Let us point out that homogenization techniques were recently used to derive damage kinematics and kinetics formally from an analysis on the damage scale, leading to generic models (*i.e.* models which are valid for a whole family of materials) for laminated composites [Ladevèze and Lubineau, 2002].

While early works concerned “only” unidirectional damage models, with applications to the creep rupture of metals [Kachanov, 1958, 1999; Leckie, 1978], CDM was rapidly extended to multidirectional models

1 involving damage anisotropy [Cordebois and Sidoroff, 1982; Ladevèze,
2 1983; Murakami, 1983; Chaboche, 1984; Chow and Wang, 1987b], with
3 applications to the ductile [Lemaître, 1985; Chow and Wang, 1987a] and
4 fatigue [Chow and Wei, 1991] rupture of metals. More recently, CDM was
5 applied to laminated [Talreja, 1985, 1986; Voyiadjis and Kattan, 1993;
6 Ladevèze and Lubineau, 2002], woven [Lesne and Saanouni, 1993; Aubard,
7 1995; Ladevèze, 1995] and braided [Gorbatikh et al., 2007] composites,
8 to concrete [Peerlings et al., 1998; Pensée et al., 2002; Desmorat et al.,
9 2007; Badel et al., 2007], *etc.* However, only a few published models are
10 capable of reproducing all aspects of cracks, including anisotropy and
11 unilaterality, especially when the orientation of damage is determined
12 by the loading, *i.e.* when the damage kinematics is not known *a priori*.
13 Indeed, in that case, anisotropy requires a tensor damage variable and the
14 construction of a continuously differentiable potential involving tensors,
15 and tension/compression partitioning is not straightforward [Ladevèze,
16 1983; Chaboche, 1992; Carol and Willam, 1996; Desmorat, 2000; Ladevèze
17 and Letombe, 2000; Ladevèze, 2002]. This theoretical problem was solved
18 by Ladevèze within the anisotropic and unilateral damage theory, first
19 for second-order damage tensors [Ladevèze, 1983], then for more general
20 representations of damage [Ladevèze and Letombe, 2000; Ladevèze, 2002].
21 Basically, the approach relies on a specific tension/compression partitioning
22 of stress or strain which takes into account the damage state in order
23 to ensure the continuity of the state law. Let us mention that another
24 solution to this problem for fourth-order damage tensors was proposed in
25 [Chaboche, 1995].

26

27 In this paper, we discuss several aspects of the formulation and nu-

1 merical implementation of the family of models which belong to the sec-
2 ond version of the anisotropic and unilateral damage theory. This discus-
3 sion follows the classical Newton-Raphson/Finite Element framework. Al-
4 though other methods can be used to solve nonlinear partial differential
5 equations over a domain [Ladevèze, 1999; Passieux et al., 2010], this is the
6 most common method for dealing with material nonlinearities in science and
7 engineering. All the developments presented here were carried out within
8 Abaqus/Standard as C++ UMats based on the software development plat-
9 form of [Leclerc, 2010].

10 In the first part of the paper, we review the theory's basis and present
11 several damage evolution laws which differ in the nature of the damage
12 mechanisms they represent. Regarding the modeling, the first fundamental
13 question concerning a crack network is whether its orientation is governed
14 by the loading (*e.g.* the inter-yarn cracking of woven ceramic matrix com-
15 posites (CMCs) [Ladevèze et al., 1994; Ladevèze, 1995; Lamon, 2001], the
16 cracking of concrete [Desmorat et al., 2007], *etc.*) or by the microstruc-
17 ture (*e.g.* the intra-yarn cracking of woven CMCs [Ladevèze et al., 1994;
18 Ladevèze, 1995; Lamon, 2001], the cracking of laminated composites [Tal-
19 reja, 1985, 1986; Lafarie-Frenot et al., 2001; Ladevèze and Lubineau, 2002],
20 *etc.*). The second question is whether the crack network becomes saturated
21 or reaches a critical value beyond which it localizes to form a macroscopic
22 crack [Needleman, 1988; Pijaudier-Cabot and Benallal, 1993; Peerlings et al.,
23 1998; Ladevèze et al., 2000]. Examples are presented for each situation. It
24 is worth mentioning that the softening case is addressed in this modeling
25 framework for the first time.

26 In the second part, we discuss the resolution of the local behavior at
27 the integration point level. We use a fixed-point algorithm to solve the

1 local equations; we also present and evaluate several relaxation schemes,
 2 including Aitken's. We also focus on the inversion of the state law. As will
 3 be seen later on, this law is nonlinear even if all the internal variables are
 4 fixed, and it requires a specific solver. Therefore, we use a Newton-Raphson
 5 algorithm; we also present and evaluate several optimization techniques,
 6 including BFGS operator updating and Aitken's relaxation.

7 In the third part, we discuss the control of damage localization be-
 8 yond the critical point of a softening model. We use a delay-effect method
 9 [Ladevèze et al., 2000] to overcome the loss of ellipticity [Pijaudier-Cabot and
 10 Benallal, 1993; Peerlings et al., 1998] and control the localization of damage
 11 in the form of a macroscopic crack, which eliminates any pathological mesh
 12 dependency. Again, let us mention that the simulation of localization in the
 13 context of this damage framework is presented here for the first time.

14 We end up with a relatively complete, efficient and robust computa-
 15 tional environment for anisotropic and unilateral damage within the popular
 16 Abaqus/Standard finite element code.

17 **2. The anisotropic and unilateral damage theory**

18 *2.1. State potential and state law*

19 *State potential.* The damage framework introduced in [Ladevèze and
 20 Letombe, 2000; Ladevèze, 2002] is based on the following general form of
 21 the potential of elastic energy:

$$2\rho\phi\left(\underline{\underline{\underline{\sigma}}}, \underline{\underline{\underline{S}}}, \underline{\underline{\underline{Z}}}\right) = \langle \underline{\underline{\underline{\sigma}}} \rangle_+^S : \underline{\underline{\underline{S}}} : \langle \underline{\underline{\underline{\sigma}}} \rangle_+^S + \langle \underline{\underline{\underline{\sigma}}} \rangle_-^{S_0} : \underline{\underline{\underline{S}}_0} : \langle \underline{\underline{\underline{\sigma}}} \rangle_-^{S_0} + \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{Z}}} : \underline{\underline{\underline{\sigma}}} \quad (1)$$

22 where $\underline{\underline{\underline{S}}_0}$ is the compliance tensor of the undamaged material, which is
 23 always active in compression; $\underline{\underline{\underline{S}}}$ is a damaged compliance, which is active

1 only in tension (initially, $\underline{\underline{S}}(t = 0) = \underline{\underline{S}}_0$); and $\underline{\underline{Z}}$ is an additional compliance,
2 which is active both in tension and in compression (initially, $\underline{\underline{Z}}(t = 0) = 0$).

3 Let us first observe that this potential gives maximum freedom in terms
4 of damage modeling, which enables one to deal with load-oriented damage,
5 *i.e.* damage whose direction is not known *a priori*: the damage kinematics
6 is not set *a priori*, but is defined completely by the damage evolution laws.
7 Actually, there are no damage variables associated with specific damage
8 mechanisms; the damage variables of the model are the whole compliance
9 tensors $\underline{\underline{S}}$ and $\underline{\underline{Z}}$. Thus, any compliance can be reached from the initial
10 compliance, which makes the model equivalent to an eighth-order damage
11 tensor model [Lemaître et al., 2009].

12 Let us mention that we also considered strain-based formulations, but we
13 had to abandon that idea because it could lead to cases in which the damaged
14 stiffness tensor ceases to be positive definite before the actual stiffness in the
15 loading direction gets to zero, which would make it impossible to model final
16 rupture.

17 In order to deal with the crack closure effect, the model distinguishes
18 clearly the tension state from the compression state: the stress tensor is
19 divided into a positive part and a negative part, each associated with a dif-
20 ferent compliance operator. To ensure the continuous differentiability of the
21 state potential, *i.e.* the continuity of the state law, this partitioning is carried
22 out in a specific way which takes into account the middle operator. It is well-
23 known that the coupling between tensorial damage and tension/compression
24 partitioning is not straightforward [Ladevèze, 1983; Chaboche, 1992]. There-

1 fore, we use the following definitions of positive and negative stresses:

$$\left\{ \begin{array}{l} \langle \underline{\underline{\underline{\sigma}}} \rangle_+^S = \underline{\underline{\underline{S}}}^{-1/2} : \langle \underline{\underline{\underline{S}}}^{1/2} : \underline{\underline{\underline{\sigma}}} \rangle_+ \\ \langle \underline{\underline{\underline{\sigma}}} \rangle_-^{S_0} = \underline{\underline{\underline{S_0}}}^{-1/2} : \langle \underline{\underline{\underline{S_0}}}^{1/2} : \underline{\underline{\underline{\sigma}}} \rangle_- \end{array} \right. \quad (2)$$

2 where $\langle \rangle_{+/-}$ denotes the positive/negative decomposition of a second-
3 order symmetric tensor (obtained by taking the positive/negative eigen-
4 values alone). It is important to note that unless both the positive part
5 and the negative part are defined that way the state law is discontinuous.
6 With these definitions, the continuous differentiability of the potential can
7 be clearly shown by introducing Equation (2) into Equation (1):

$$2\rho\phi\left(\underline{\underline{\underline{\sigma}}}, \underline{\underline{\underline{S}}}, \underline{\underline{\underline{Z}}}\right) = \langle \underline{\underline{\underline{S}}}^{1/2} : \underline{\underline{\underline{\sigma}}} \rangle_+ + \langle \underline{\underline{\underline{S}}}^{1/2} : \underline{\underline{\underline{\sigma}}} \rangle_+ + \langle \underline{\underline{\underline{S_0}}}^{1/2} : \underline{\underline{\underline{\sigma}}} \rangle_- + \langle \underline{\underline{\underline{S_0}}}^{1/2} : \underline{\underline{\underline{\sigma}}} \rangle_- + \underline{\underline{\underline{\sigma}}} : \underline{\underline{\underline{Z}}} : \underline{\underline{\underline{\sigma}}} \quad (3)$$

8 whose continuous differentiable property is trivial [Ladevèze and Letombe,
9 2000; Desmorat, 2000]. As a direct consequence of this property, the state
10 law (which is the first derivative of the potential) will always be contin-
11 uous, and the compliance operator (the second derivative) will always be
12 symmetric.

13 Finally, our framework also enables us to distinguish between dam-
14 age which is highly dependent on the tension/compression state (*i.e.* ten-
15 sion damage, added to $\underline{\underline{\underline{S}}}$) and damage which is independent of the ten-
16 sion/compression state (*i.e.* shear damage, added to $\underline{\underline{\underline{Z}}}$).

17 *State law.* The state law, derived from the elastic energy potential, is simply:

18

$$\underline{\underline{\underline{\epsilon}}} = \frac{\partial \rho\phi}{\partial \underline{\underline{\underline{\sigma}}}} = \underline{\underline{\underline{S}}} : \langle \underline{\underline{\underline{\sigma}}} \rangle_+^S + \underline{\underline{\underline{S_0}}} : \langle \underline{\underline{\underline{\sigma}}} \rangle_-^{S_0} + \underline{\underline{\underline{Z}}} : \underline{\underline{\underline{\sigma}}} \quad (4)$$

1 *2.2. Thermodynamic forces*

2 First of all, let us define the following two thermodynamic forces, which
 3 are linked directly to the energy release rates associated with the evolutions
 4 of the damage variables:

$$\begin{cases} \underline{\underline{Y}}_S = 2 \frac{\partial \rho \phi}{\partial \underline{\underline{S}}} = \langle \underline{\underline{\sigma}} \rangle_+^S \otimes \langle \underline{\underline{\sigma}} \rangle_+^S \\ \underline{\underline{Y}}_Z = 2 \frac{\partial \rho \phi}{\partial \underline{\underline{Z}}} = \underline{\underline{\sigma}} \otimes \underline{\underline{\sigma}} \end{cases} \quad (5)$$

5 These thermodynamic forces or the corresponding damage variables cannot
 6 be associated with specific damage mechanisms and, therefore, cannot be
 7 used as such to drive all damage mechanisms; several additional thermody-
 8 namic forces must be defined in order to drive any type of damage.

For example, we will see that while tension damage can be driven cor-
 rectly by $\underline{\underline{Y}}_S$ (which is highly load-oriented, see Equation (5)) or its projec-
 tions onto specific directions, this is not the case for shear damage. Because
 of the positive part of $\underline{\underline{\sigma}}$, only the tension part (*i.e.* the positive eigenvalue)
 drives the evolution of damage. Besides, since $\underline{\underline{Y}}_Z$ is active also in compres-
 sion, we will not use it to drive any type of damage. Therefore, we must
 define another thermodynamic force to drive shear damage (see Sections
 2.3.1 and 2.3.2). This additional force is simply a rotation of $\underline{\underline{Y}}_S$ defined as:

$$\underline{\underline{Y}}_{S'} = \left(\underline{\underline{R}}_{\pi/2} \langle \underline{\underline{\sigma}} \rangle_+^S \right)_{\text{sym}} \otimes \left(\underline{\underline{R}}_{\pi/2} \langle \underline{\underline{\sigma}} \rangle_+^S \right)_{\text{sym}} \quad (6)$$

with $\underline{\underline{R}}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9 Since all the previous thermodynamic forces are proportional to stresses,
 10 they could not be used to model the localization of damage (*i.e.* the soften-

ing phenomenon). Therefore, we will need to define an additional thermo-
dynamic force which is proportional to strains (see Section 2.3.3):

$$\underline{\underline{X}}_S = \underline{\underline{S}} \underline{\underline{Y}}_S \underline{\underline{S}} \quad (7)$$

2.3. Damage evolution laws

Several damage evolution laws can be defined depending on the very
nature of the damage they represent: is this damage load-oriented or
microstructure-oriented? Does it reach saturation or does it localize into
a macroscopic crack? We present examples for each situation, including
(for the first time) localizing damage. The proposed formulation is closely
modeled after the associated classical framework with normality rule and
isotropic hardening [Lemaître et al., 2009].

2.3.1. The case of load-oriented damage

For the sake of simplicity, from here on, we will use essentially the clas-
sical engineering notations $\widehat{\cdot}$ for second- and fourth-order tensors [Lemaître
et al., 2009].

In the case of load-oriented damage, one can use the load-oriented ther-
modynamic forces directly to drive the evolution of damage. Thus, we define
the following effective thermodynamic force and its maximum over time:

$$\begin{cases} z = \left(a \operatorname{Tr} \left(\widehat{\underline{\underline{Y}}_S} \right)^{n+1} + (1-a) \operatorname{Tr} \left(\widehat{\underline{\underline{Y}}_S}^{n+1} \right) \right)^{1/n+1} \\ \bar{z}(t) = \sup_{\tau \leq t} z(\tau) \end{cases} \quad (8)$$

Then the corresponding evolution of the damage variables is:

$$\begin{cases} \dot{\underline{\underline{S}}} = \dot{\alpha} \frac{\partial z}{\partial \widehat{\underline{\underline{Y}}_S}} = \dot{\alpha} \frac{a \operatorname{Tr} \left(\widehat{\underline{\underline{Y}}_S} \right)^n \underline{\underline{1}} + (1-a) \widehat{\underline{\underline{Y}}_S}^n}{\bar{z}^n} \\ \dot{\underline{\underline{Z}}} = \dot{\alpha} \frac{b \widehat{\underline{\underline{Y}}_S}^n}{\bar{z}^n} \end{cases} \quad (9)$$

1 where α is the “hardening variable”, a function of \bar{z} which must be cali-
 2 brated experimentally. Usually, one assumes a shape depending on several
 3 parameters which must be identified. For example, the following shape was
 4 used in [Ladevèze and Letombe, 2000; Ladevèze, 2002; Genet et al., 2012]:

$$\alpha = k \left\langle \frac{\sqrt{\bar{z}} - \sigma_0}{\sigma_0} \right\rangle_+^2 \quad (10)$$

5 where σ_0 characterizes the onset of cracking and k its intensity. (In the case
 6 of a unidirectional load, $\sqrt{\bar{z}}$ is simply the magnitude of the stress.) The
 7 corresponding responses of this and other laws are compared in Figure 3 of
 8 Section 2.3.3.

9 The dissipation associated with this evolution law is:

$$\begin{cases} \omega_S = \frac{\partial \rho \phi}{\partial \underline{\underline{\hat{S}}}} : \dot{\underline{\underline{\hat{S}}}} = \frac{\dot{\alpha}}{2\bar{z}^n} \left(a \operatorname{Tr} \left(\underline{\underline{\hat{Y}}}_S \right)^{n+1} + (1-a) \operatorname{Tr} \left(\underline{\underline{\hat{Y}}}_S^{n+1} \right) \right) \\ \omega_Z = \frac{\partial \rho \phi}{\partial \underline{\underline{\hat{Z}}}} : \dot{\underline{\underline{\hat{Z}}}} = \frac{b\dot{\alpha}}{2\bar{z}^n} \operatorname{Tr} \left(\underline{\underline{\sigma}} \left(\underline{\underline{R}}_{\pi/2} \left\langle \underline{\underline{\sigma}} \right\rangle_+^S \right)_{\text{sym}} \right)^2 \operatorname{Tr} \left(\underline{\underline{\hat{Y}}}_{S'}^{n-2} \right) \end{cases} \quad (11)$$

10 which is trivially positive.

As can be seen in Equation (9), the evolution of damage can be tuned using parameters a , b and n and can be made highly load-oriented. For example, if $a = 1$, the tension damage is fully isotropic; but if $a = 0$, the larger n , the more anisotropic the tension damage. Regarding shear damage, its magnitude is directly proportional to b . To illustrate the influence of a

and b , let us define the relative angular tension and compression moduli:

$$\begin{cases} \tilde{E}_t(\theta) = \frac{\underline{N}(\theta) : \underline{S}_0 : \underline{N}(\theta)}{\underline{N}(\theta) : \left(\underline{S} + \underline{Z} \right) : \underline{N}(\theta)} \\ \tilde{E}_c(\theta) = \frac{\underline{N}(\theta) : \underline{S}_0 : \underline{N}(\theta)}{\underline{N}(\theta) : \left(\underline{S}_0 + \underline{Z} \right) : \underline{N}(\theta)} \end{cases} \quad (12)$$

with $\underline{N}(\theta) = \underline{n}(\theta) {}^t \underline{n}(\theta)$

$$\text{and } \underline{n}(\theta) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

1 These relative angular tension and compression moduli corresponding to
 2 a given amount of damage in different directions for an initially isotropic
 3 material are presented in Figure 1 for several values of a and b . (In all
 4 cases, $n = 2$; due to symmetry, only one quadrant was required for each set
 5 of parameters.) One can see that a rotation of the load leads simply to a
 6 rotation of the damage morphology.

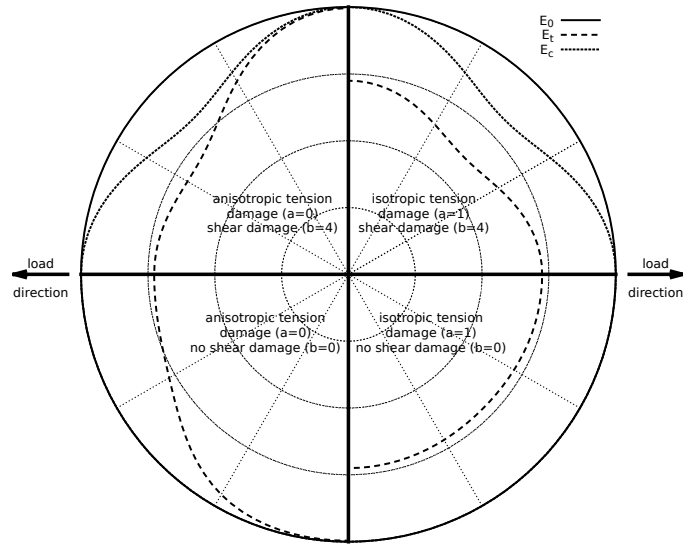
7 2.3.2. The case of microstructure-oriented damage

8 In this case, the load-oriented thermodynamic forces cannot be used
 9 directly, but must be projected onto the *a priori* known damage directions.
 10 For example, for a crack network which is orthogonal to \underline{n} and parallel to \underline{t} ,
 11 we define the projectors:

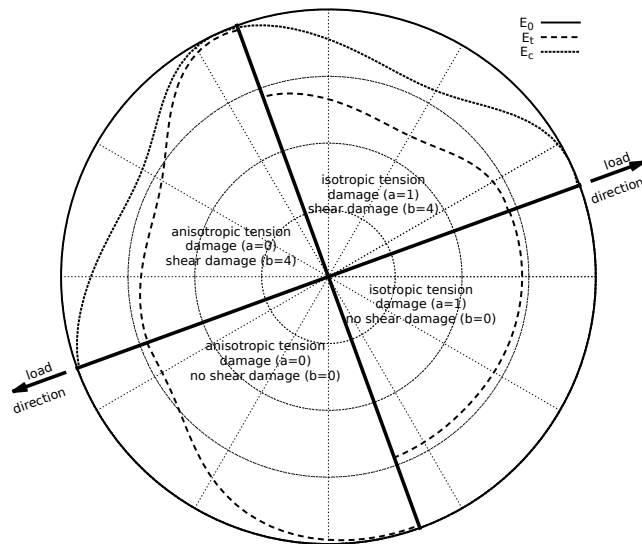
$$\begin{cases} \underline{P}_{nn} = \underline{n} {}^t \underline{n}, \underline{P}_{tt} = \underline{t} {}^t \underline{t}, \underline{P}_{nt} = (\underline{n} {}^t \underline{t})_{\text{sym}} \\ \widehat{\underline{P}}_{nnnn} = \widehat{\underline{P}}_{nn} {}^t \widehat{\underline{P}}_{nn}, \widehat{\underline{P}}_{tttt} = \widehat{\underline{P}}_{tt} {}^t \widehat{\underline{P}}_{tt}, \widehat{\underline{P}}_{ntnt} = \widehat{\underline{P}}_{nt} {}^t \widehat{\underline{P}}_{nt} \end{cases} \quad (13)$$

12 and the effective thermodynamic force:

$$\begin{cases} z = \left(a \text{Tr} \left(\widehat{\underline{Y}}_{\underline{S}} \right)^{n+1} + (1-a) \text{Tr} \left(\widehat{\underline{P}}_{nnnn} \widehat{\underline{Y}}_{\underline{S}}^{n+1} \widehat{\underline{P}}_{nnnn} \right) \right)^{1/n+1} \\ \bar{z}(t) = \sup_{\tau \leq t} z(\tau) \end{cases} \quad (14)$$



(a) First load direction



(b) Second load direction

Figure 1: Damaged angular tension and compression moduli using a load-oriented damage evolution law and two load directions for several values of a and b (top: shear damage activated, $b = 4$; bottom: no shear damage, $b = 0$; left: anisotropic tension damage, $a = 0$; right: isotropic tension damage, $a = 1$)

1 The corresponding evolution of the damage variables is:

$$\begin{cases} \dot{\underline{\underline{S}}} = \dot{\alpha} \frac{\partial z}{\partial \underline{\underline{Y}}_S} = \dot{\alpha} \frac{a \operatorname{Tr}(\widehat{\underline{\underline{Y}}_S})^n \underline{\underline{1}} + (1-a) \widehat{\underline{\underline{P}}_{nnnn}} \widehat{\underline{\underline{Y}}_S}^n \widehat{\underline{\underline{P}}_{nnnn}}}{\bar{z}^n} \\ \dot{\underline{\underline{Z}}} = \dot{\alpha} \frac{b \operatorname{Tr}(\widehat{\underline{\underline{P}}_{nnnn}} \widehat{\underline{\underline{Y}}_S}^n \widehat{\underline{\underline{P}}_{nnnn}}) \widehat{\underline{\underline{P}}_{ntnt}}}{2^n \bar{z}^n} \end{cases} \quad (15)$$

2 where α is again the “hardening variable” to be calibrated experimentally
3 (see Section 2.3.1).

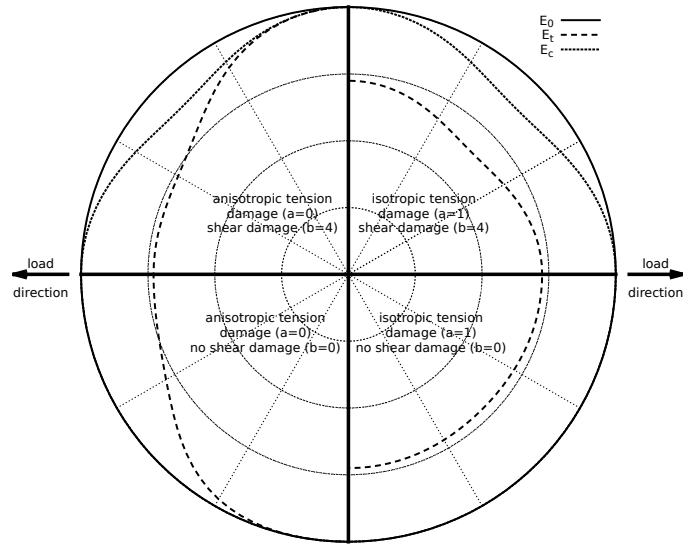
4 Let us note that in the case of a unidirectional tension load normal to the
5 crack this evolution law is almost the same as the load-oriented law defined
6 in Section 2.3.1. The only small difference is due to the positive stress,
7 which is slightly different from the complete stress even for a unidirectional
8 tension load and, therefore, slightly modifies the effect of projections. This
9 difference is not significant, as can be seen in Figure 3 of Section 2.3.3 where
10 the evolution laws are compared.

11 In this case, the dissipation is:

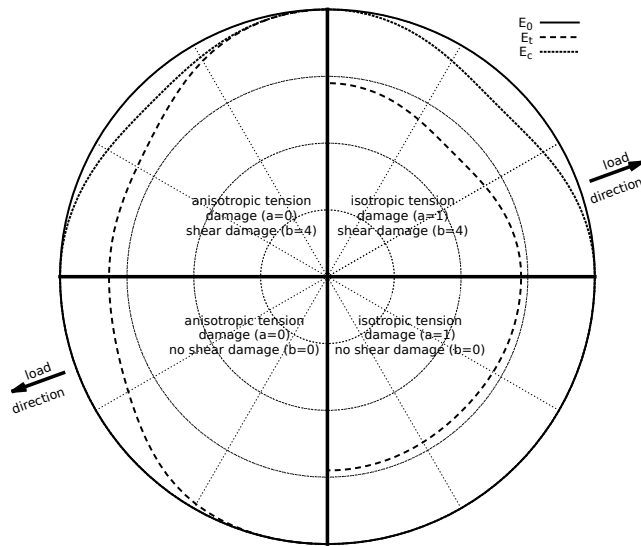
$$\begin{cases} \omega_S = \frac{\partial \rho \phi}{\partial \underline{\underline{S}}} : \dot{\underline{\underline{S}}} = \frac{\dot{\alpha}}{2\bar{z}^n} \left(a \operatorname{Tr}(\widehat{\underline{\underline{Y}}_S})^{n+1} + (1-a) \operatorname{Tr}(\widehat{\underline{\underline{P}}_{nnnn}} \widehat{\underline{\underline{Y}}_S}^n \widehat{\underline{\underline{P}}_{nnnn}} \widehat{\underline{\underline{Y}}_S}) \right) \\ \omega_Z = \frac{\partial \rho \phi}{\partial \underline{\underline{Z}}} : \dot{\underline{\underline{Z}}} = \frac{b\dot{\alpha}}{2^{n+1}\bar{z}^n} \operatorname{Tr}(\widehat{\underline{\underline{P}}_{nnnn}} \widehat{\underline{\underline{Y}}_S}^n \widehat{\underline{\underline{P}}_{nnnn}}) \operatorname{Tr}(\widehat{\underline{\underline{P}}_{ntnt}} \widehat{\underline{\underline{Y}}_Z}) \end{cases} \quad (16)$$

12 which is also positive.

13 Equation (15) clearly shows that in this case the evolution of damage is
14 oriented by the microstructure. Relative angular tension and compression
15 moduli corresponding to a given amount of damage in different directions
16 are presented in Figure 2 for several values of a and b (in all cases, $n = 2$).
17 One can see that in this case a change in the load direction modifies the
18 amount of damage, but not its morphology.



(a) Load direction perpendicular to the crack



(b) Load direction at 20° of the normal to the crack

Figure 2: Damaged angular tension and compression moduli using a microstructure-oriented damage evolution law and two load directions for several values of a and b (top: shear damage activated, $b = 4$; bottom: no shear damage, $b = 0$; left: anisotropic tension damage, $a = 0$; right: isotropic tension damage, $a = 1$)

1 *2.3.3. The case of softening damage*

2 This is the first presentation of a softening damage evolution law in the
3 proposed anisotropic and unilateral damage theory. In this case, formula-
4 tions similar to those presented in Section 2.3.1 for load-oriented damage
5 and in Section 2.3.2 for microstructure-oriented damage can be applied, but
6 thermodynamic forces proportional to strains rather than stresses must be
7 used (see Section 2.2). Indeed, beyond the critical point, the stress decreases
8 and, thus, a thermodynamic force dependent on the stress alone could not
9 cause the damage to increase. Therefore, one must consider a force based
10 either on the effective stress or on the strain. We chose the latter. Thus, for
11 a load-oriented mechanism, we use the effective thermodynamic force:

$$\begin{cases} z = \left(a \operatorname{Tr} \left(\underline{\underline{\widehat{X}_S}} \right)^{n+1} + (1-a) \operatorname{Tr} \left(\underline{\underline{\widehat{X}_S}}^{n+1} \right) \right)^{1/n+1} \\ \bar{z}(t) = \sup_{\tau \leq t} z(\tau) \end{cases} \quad (17)$$

12 with the evolution of the damage variable:

$$\underline{\underline{\dot{S}}} = \dot{\alpha} \frac{\partial z}{\partial \underline{\underline{\widehat{X}_S}}} = \dot{\alpha} \frac{a \operatorname{Tr} \left(\underline{\underline{\widehat{X}_S}} \right)^n \underline{\underline{1}} + (1-a) \underline{\underline{\widehat{X}_S}}^n}{\bar{z}^n} \quad (18)$$

13 For a microstructure-oriented mechanism, we use the effective thermody-
14 namic force:

$$\begin{cases} z = \left(a \operatorname{Tr} \left(\underline{\underline{\widehat{X}_S}} \right)^{n+1} + (1-a) \operatorname{Tr} \left(\underline{\underline{\widehat{P}_{nnnn}}} \underline{\underline{\widehat{X}_S}}^{n+1} \underline{\underline{\widehat{P}_{nnnn}}} \right) \right)^{1/n+1} \\ \bar{z}(t) = \sup_{\tau \leq t} z(\tau) \end{cases} \quad (19)$$

15 with the evolution of the damage variable:

$$\underline{\underline{\dot{S}}} = \dot{\alpha} \frac{\partial z}{\partial \underline{\underline{\widehat{X}_S}}} = \dot{\alpha} \frac{a \operatorname{Tr} \left(\underline{\underline{\widehat{X}_S}} \right)^n \underline{\underline{1}} + (1-a) \underline{\underline{\widehat{P}_{nnnn}}} \underline{\underline{\widehat{X}_S}}^n \underline{\underline{\widehat{P}_{nnnn}}}}{\bar{z}^n} \quad (20)$$

1 α is still the “hardening variable” to be calibrated experimentally (see Sec-
 2 tion 2.3.1), but it must be redefined because the effective thermodynamic
 3 forces are now proportional to the strains. For example, one can use the
 4 following shape:

$$\alpha = \begin{cases} 0 & \text{if } \sqrt{\bar{z}} \leq \epsilon_0 \\ k \left(\frac{\sqrt{\bar{z}} - \epsilon_0}{\epsilon_1 - \sqrt{\bar{z}}} \right)^2 & \text{if } \epsilon_0 \leq \sqrt{\bar{z}} \leq \epsilon_1 \\ +\infty & \text{if } \sqrt{\bar{z}} \geq \epsilon_1 \end{cases} \quad (21)$$

5 where ϵ_0 characterizes the onset of cracking, ϵ_1 the final rupture, and k the
 6 amount of damage. (In the case of a unidirectional load, $\sqrt{\bar{z}}$ is simply the
 7 longitudinal strain.) The responses of this and previous laws are compared
 8 in Figure 3.

9 In this case, the dissipation is:

$$\omega = \frac{\partial \rho \phi}{\partial \underline{\underline{\hat{S}}}} : \dot{\underline{\underline{\hat{S}}}} = \frac{\dot{\alpha}}{2\bar{z}^n} \left(a \operatorname{Tr} \left(\underline{\underline{\hat{X}}}_S \right)^n \operatorname{Tr} \left(\underline{\underline{\hat{Y}}}_S \right) + (1 - a) \operatorname{Tr} \left(\underline{\underline{\hat{P}}}_{nnnn} \underline{\underline{\hat{X}}}_S^n \underline{\underline{\hat{P}}}_{nnnn} \underline{\underline{\hat{Y}}}_S \right) \right) \quad (22)$$

10 which is also positive.

11 Let us mention that only tension damage was considered here because
 12 the introduction of shear damage would lead to the same problem as strain-
 13 based formulations (see the remark in Section 2.1): sometimes the damaged
 14 stiffness tensor could cease to be positive definite before the actual stiffness
 15 in the loading direction gets to zero.

16 Another element must be added to this law in order to control the end
 17 of the damaging process. Since tension damage alone is being considered,
 18 the only internal variable is the tensor $\underline{\underline{\hat{S}}}$ itself. (The case of shear damage
 19 with a softening law, which involves the second internal variable $\underline{\underline{\hat{Z}}}$, has not
 20 yet been addressed.) Thus, the full damage criterion can be based directly

1 on the eigenvalues of $\hat{\underline{S}}$:

$$\exists i \quad / \quad \frac{1}{M E_0} \leq \lambda_{\hat{\underline{S}}}^i \leq \frac{M}{E_0} \implies \text{rupture} \quad (23)$$

2 where $\lambda_{\hat{\underline{S}}}^i$ are the eigenvalues of $\hat{\underline{S}}$ and M is a large number (in practice, we
 3 use $M = 10^3$). When this limit is reached, the corresponding integration
 4 point is considered to be broken and the damage ceases to evolve. It is worth
 5 mentioning that this criterion can be applied at no substantial additional
 6 cost because $\hat{\underline{S}}$ must be diagonalized anyway in order to calculate $\sqrt{\hat{\underline{S}}}$.

7 The responses of the model under unidirectional tension-compression
 8 loading using the damage evolution laws presented in Sections 2.3.1, 2.3.2
 9 and 2.3.3 are compared in Figure 3. (In the case of the microstructure-
 10 oriented law, the load was orthogonal to the cracks.) One can see that
 11 every model recovers its stiffness in compression, that the load-oriented and
 12 microstructure-oriented non-softening models are nonseparable, and that
 13 the law presented in this section does have a softening character.

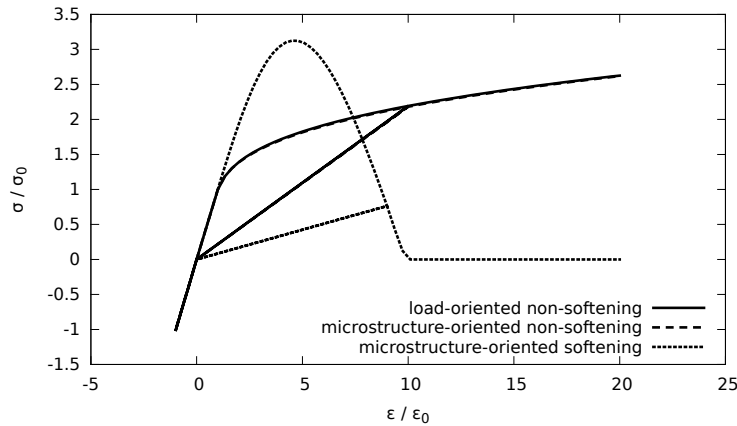


Figure 3: The responses of the model under unidirectional tension-compression loading ($k = 10^{-5}$, $\epsilon_1/\epsilon_0 = 10$) using different damage evolution laws

1 **3. Calculation of the local behavior**

2 Let us now present the key aspects of the numerical implementation of
3 the family of models presented in Section 2 based on the anisotropic and
4 unidirectional damage theory. In this section, only non-softening damage
5 evolution laws are considered. The softening case will be discussed in Section
6 4.

7 *3.1. The local loop*

8 The local loop is run for each integration point and at each global itera-
9 tion of each load increment. The input is $\underline{\underline{\epsilon}}^{l,i}$, the total strain tensor at load
10 increment l and global iteration i , and the output consists of the damage
11 tensors $\underline{\underline{S}}^{l,i}$ and $\underline{\underline{Z}}^{l,i}$, the corresponding damage “hardening” variable $\alpha^{l,i}$,
12 and the stress tensor $\underline{\underline{\sigma}}^{l,i}$. Starting here, in order to simplify the notation,
13 the subscripts l, i will be omitted.

14 *3.1.1. The fixed-point solver*

15 This set of nonlinear equations could be solved using a Newton-Raphson
16 method, but the very different units and magnitudes of the unknowns would
17 lead to very unbalanced problems. Besides, most of the derivatives in the
18 equations are difficult to calculate or even to approximate. For these reasons,

1 we chose to use the following fixed-point algorithm instead:

$$\left\{ \begin{array}{l}
 \text{initialization: } j = 0 ; \alpha^j = \alpha^{l-1} ; \underline{\underline{S}}^j = \underline{\underline{S}}^{l-1} ; \underline{\underline{Z}}^j = \underline{\underline{Z}}^{l-1} \\
 \text{loop:} \\
 \quad \text{stress: } \underline{\underline{\sigma}}^j / \underline{\underline{\epsilon}} = \underline{\underline{S}}^j : \langle \underline{\underline{\sigma}}^j \rangle_+^{S^j} + \underline{\underline{S}}_0 : \langle \underline{\underline{\sigma}}^j \rangle_-^{S_0} + \underline{\underline{Z}}^j : \underline{\underline{\sigma}}^j \\
 \quad \text{residual: } R^j = \alpha(\underline{\underline{\sigma}}^j) - \alpha^j \\
 \quad \text{exit test: } \frac{|R^j|}{|\alpha^j - \alpha^{l-1}|} < \text{tolerance} \implies \text{exit} \\
 \quad \text{damage: } \begin{cases} \alpha^{j+1} = \alpha^j + R^j \\ \underline{\underline{S}}^{j+1} = \underline{\underline{S}} \left(\underline{\underline{S}}^{l-1}, \alpha^{j+1} - \alpha^{l-1} \right) \\ \underline{\underline{Z}}^{j+1} = \underline{\underline{Z}} \left(\underline{\underline{Z}}^{l-1}, \alpha^{j+1} - \alpha^{l-1} \right) \end{cases} \\
 \text{end loop: } j = j + 1
 \end{array} \right. \quad (24)$$

2 where α is the function defined in Equations (10) or (21), and $\underline{\underline{S}}$ and $\underline{\underline{Z}}$ are
3 the functions defined in Equations (9) or (15). Concerning the tolerance,
4 in the absence of a reference value for the magnitude of the residual of the
5 proposed algorithm, we used a stagnation criterion with a tolerance of 10^{-3} .

6 3.1.2. Acceleration of the solver

7 The fixed-point method can be viewed as a Newton-Raphson method
8 with a unit search direction and, therefore, can oscillate greatly. This is
9 particularly true in our case because of the presence of unilateral conditions
10 (damage can only increase, see Equations (8) and (14)) and because in
11 practice Algorithm (24) has very poor convergence or no convergence at all.
12 Therefore, we propose two relaxation methods which improve the algorithm
13 convergence drastically. The only modification to Algorithm (24) concerns

1 the damage increase line, which becomes:

$$\alpha^{j+1} = \alpha^j + s^j R^j \quad (25)$$

2 where s^j is defined as follows:

3 *Basic relaxation.* One can consider that relaxation is required only if con-
4 vergence fails to occur after a given number of iterations. This leads to the
5 following very simple relaxation scheme:

$$s^j = \frac{1}{1 + E(j/N)} \quad (26)$$

6 where E is the classical integer part operator and N is the predefined number
7 of iterations.

8 *Aitken's relaxation.* Another acceleration scheme considered here is Aitken's
9 relaxation, whose excellent performance was demonstrated in recent works
10 [Kassiotis et al., 2010]. This can be viewed as a search direction optimization
11 based on previous iterations. Then, s^j is defined as:

$$s^j = \begin{cases} 1 & \text{if } j = 0 \\ -s^{j-1} \frac{R^{j-1}}{R^j - R^{j-1}} & \text{if } j > 0 \end{cases} \quad (27)$$

12 3.1.3. Performance

13 The performance of each of the options proposed for the resolution of
14 the model's equations was evaluated using a very simple problem involving
15 no structural effect: a single linear quadrangular element with four integra-
16 tion points was subjected to pure unidirectional tension through symmetry
17 conditions and prescribed displacements (see Figure 4). The damage law
18 considered was that of Section 2.3.1, which leads to the most difficult local
19 loop because damage is stress-driven (see Figure 3). Figure 5 shows how the

1 methods compare in terms of the number of iterations and time. Clearly,
 2 Aitken’s relaxation was found to be much more efficient than fixed-value
 3 relaxation, and it will be used from now on. For example, Figure 5(b) shows
 4 a 75% gain compared to the case where relaxation occurs after 10 uncon-
 5 verged iterations. Figure 5(c) also shows that this acceleration reduces the
 6 cost of the local loop to about 10% of that of the global loop, which is a
 7 very satisfactory ratio.

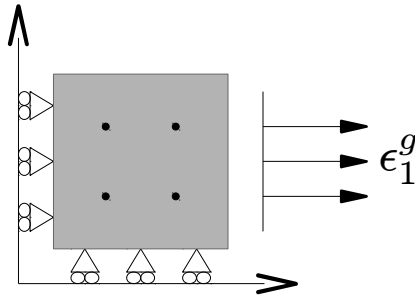
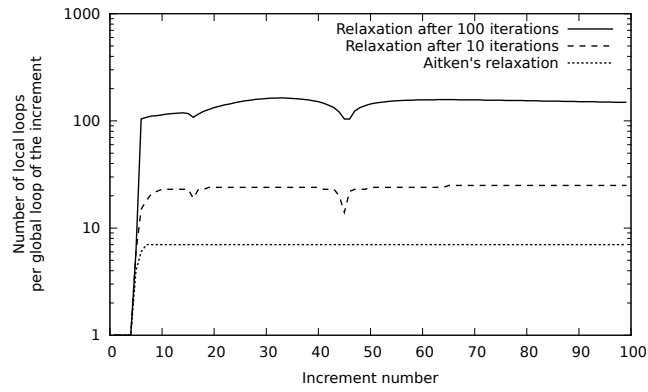


Figure 4: The problem used for the evaluation of the performance of the solver (ϵ_1^g denotes the applied strain)

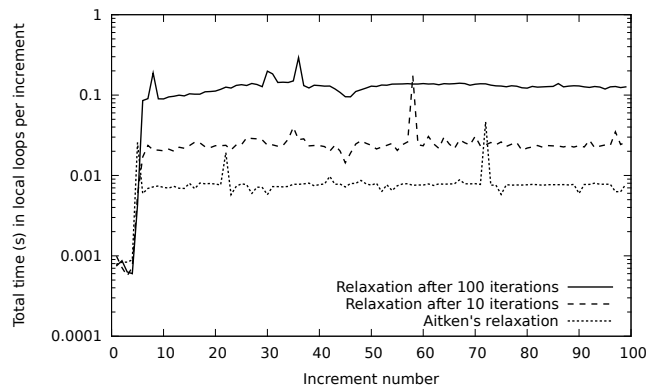
8 3.2. The behavior loop

9 In the previous discussion of the local loop (see Section 3.1), we never
 10 explained how to calculate $\underline{\underline{\sigma}}$ when all the internal variables are fixed (see
 11 the second line of Algorithm (24)). While this step is straightforward for
 12 most existing models, it is not for ours. Because of the partitioning of $\underline{\underline{\sigma}}$
 13 into positive and negative parts, Equation (4) is nonlinear even with fixed
 14 operators, so the problem can be formulated as follows:

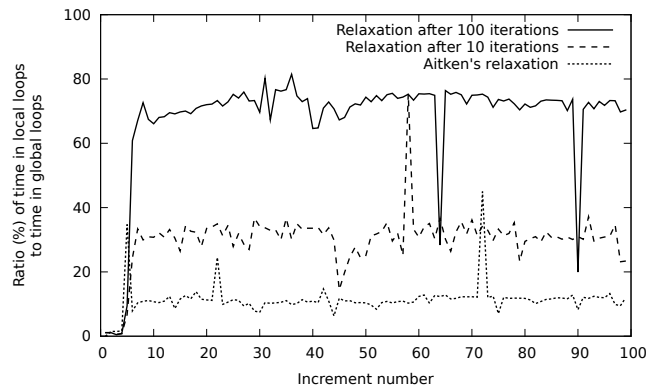
$$\left\{ \begin{array}{l} \text{with } \underline{\underline{\epsilon}}, \underline{\underline{S}}, \underline{\underline{S}}_0 \text{ and } \underline{\underline{Z}} \text{ known, find } \underline{\underline{\sigma}} \text{ such that} \\ \underline{\underline{\epsilon}} = \underline{\underline{S}} : \langle \underline{\underline{\sigma}} \rangle_+^S + \underline{\underline{S}}_0 : \langle \underline{\underline{\sigma}} \rangle_-^{S_0} + \underline{\underline{Z}} : \underline{\underline{\sigma}} \end{array} \right. \quad (28)$$



(a) iterations



(b) time



(c) time ratio

Figure 5: Performances of the local loop acceleration methods

1 3.2.1. The Newton-Raphson solver

2 This nonlinear problem is solved using a Newton-Raphson algorithm:

$$\left\{ \begin{array}{l}
 \text{initialization: } k = 0 ; \underline{\underline{\sigma}}^k = \underline{\underline{\sigma}}^{l-1} \\
 \text{loop:} \\
 \quad \left| \begin{array}{l}
 \text{residual: } \underline{\underline{R}}^k = \underline{\underline{\epsilon}} - \underline{\underline{S}} : \langle \underline{\underline{\sigma}}^k \rangle_+^S - \underline{\underline{S}}_0 : \langle \underline{\underline{\sigma}}^k \rangle_-^{S_0} - \underline{\underline{Z}} : \underline{\underline{\sigma}}^k \\
 \text{exit test: } \|\underline{\underline{R}}^k\| < \text{tolerance} \implies \text{exit} \\
 \text{stress: } \underline{\underline{\sigma}}^{k+1} = \underline{\underline{\sigma}}^k + \underline{\underline{D}}^k : \underline{\underline{R}}^k
 \end{array} \right. \\
 \text{end loop: } k = k + 1
 \end{array} \right. \quad (29)$$

3 for which several search directions $\underline{\underline{D}}^k$ can be used. The actual tangent
4 direction is not an option because one cannot derive the state law (4) with
5 respect to $\underline{\underline{\sigma}}$ in the general case where $\langle \underline{\underline{\sigma}} \rangle_+^S \neq 0$ and $\langle \underline{\underline{\sigma}} \rangle_-^{S_0} \neq 0$. One can
6 choose, for example, the initial operator $\underline{\underline{H}}_0 = \underline{\underline{S}}_0^{-1}$, which is not a good
7 direction, especially when the damage is significant, but which is fast because
8 the calculation of the operator is very inexpensive. Another option is to use
9 the quasi-secant operator, defined as:

$$\underline{\underline{D}}^k = \begin{cases} \left(\underline{\underline{S}}^{l-1} + \underline{\underline{Z}}^{l-1} \right)^{-1} & \text{if } \text{Tr}(\underline{\underline{\sigma}}^k) > 0 \\ \left(\underline{\underline{S}}_0 + \underline{\underline{Z}}^{l-1} \right)^{-1} & \text{if } \text{Tr}(\underline{\underline{\sigma}}^k) < 0 \end{cases} \quad (30)$$

10 which is a much better search direction, but also a more expensive one
11 because these operators are usually not saved and must be recalculated at
12 each iteration. Other means of drastically improving the performance of the
13 solver will be presented in Section 3.2.2.

14 Regarding tolerance, in practice, since the residual of the proposed al-
15 gorithm is a strain, we use 10^{-9} .

1 3.2.2. Acceleration of the solver

2 In practice, the initial operator converges very slowly, or even does not
 3 converge at all if the damage is significant. The secant operator defined
 4 in Equation (30) generally converges very poorly, too. Therefore, we pro-
 5 pose two acceleration methods in order to improve the convergence of the
 6 algorithm drastically.

7 *BFGS search direction.* The first acceleration scheme one can consider is
 8 the BFGS method [Matthies and Strang, 1979], whose ability to provide a
 9 very good compromise between cost and quality for the search direction is
 10 well-known: a quasi-tangent direction is generated at the cost of $4k$ addi-
 11 tional scalar products compared to the initial direction [Matthies and Strang,
 12 1979]. The algorithm for the calculation of $\underline{\underline{D}}^k$ is well-known and will not
 13 be recalled here; the only specificity is that the secant operator defined in
 14 Equation (30) is used for the central operator because the tangent operator
 15 cannot be derived formally.

Aitken's relaxation. Another acceleration considered here is Aitken's relax-
 ation, which was already used in Section 3.1.2 and which can also be viewed
 as a search direction optimization based on previous iterations. The only
 modification to Algorithm (29) is the last line, which becomes:

$$\underline{\underline{\sigma}}^{k+1} = \underline{\underline{\sigma}}^k + s^k \underline{\underline{D}}^k : \underline{\underline{R}}^k \quad (31)$$

$$\text{with } s^k = \begin{cases} 1 & \text{if } k = 0 \\ -s^{k-1} \frac{\underline{\underline{R}}^{k-1} : (\underline{\underline{R}}^k - \underline{\underline{R}}^{k-1})}{(\underline{\underline{R}}^k - \underline{\underline{R}}^{k-1}) : (\underline{\underline{R}}^k - \underline{\underline{R}}^{k-1})} & \text{if } k > 0 \end{cases}$$

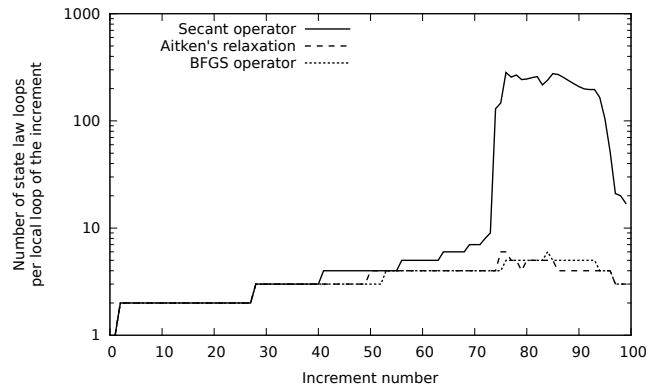
16 For the same reason as for the BFGS acceleration, we use the secant operator
 17 defined in Equation (30) for $\underline{\underline{D}}^k$.

1 3.2.3. Performance

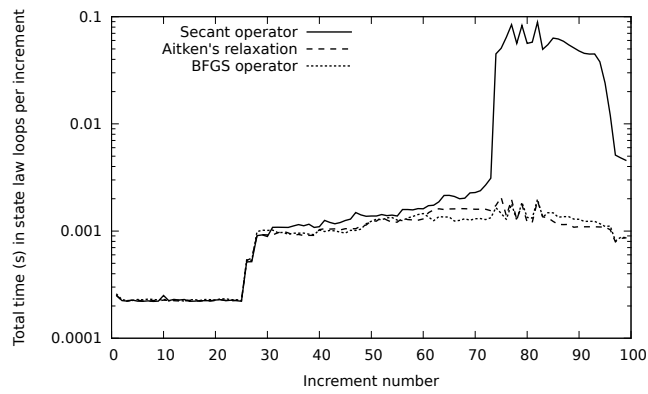
2 Let us now evaluate the performances of the different methods proposed
3 for the inversion of the state law of the model. We used the same test case
4 as in Section 3.1.3 (see Figure 4), but the damage law considered was that
5 of section 2.3.3, which is the most difficult behavior loop because damage
6 increases until it reaches the criterion defined in (23) (see Figure 3). The
7 methods are compared in terms of the number of iterations and time in
8 Figure 6. Clearly, the secant operator performs very poorly and should not
9 be used. The Aitken and BFGS operators have comparable performance in
10 terms of both the number of iterations and time. Figure 6(b) shows that
11 they lead to a gain by a factor 2 to 200 compared to the secant operator and
12 reduce the cost of the inversion of the state law to approximately 50% of that
13 of the local loop (see Figure 6(c)) even when damage approaches saturation.
14 The BFGS operator, which is slightly more efficient than Aitken's, will be
15 used from now on. In order to appreciate the importance of this gain, one
16 should bear in mind that this loop is run at each local iteration of each
17 global iteration of each time increment, which makes it the key factor in the
18 performance of the implementation.

19 4. Control of the damage localization

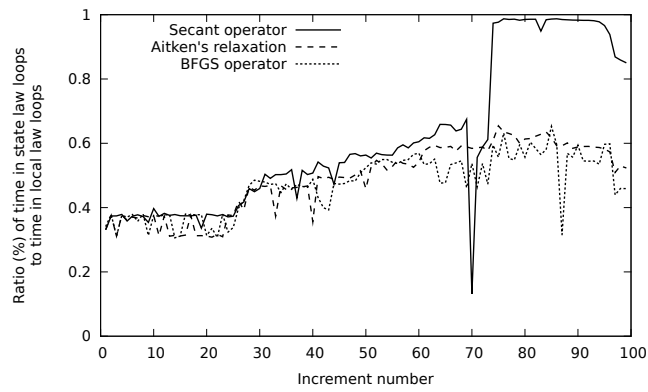
20 The calculation of softening models such as those defined in Section 2.3.3
21 over a domain goes through a critical point where a loss of ellipticity, *i.e.*
22 a loss of uniqueness of the solution [Needleman, 1988; Pijaudier-Cabot and
23 Benallal, 1993; Peerlings et al., 1998], occurs and leads to pathological mesh
24 dependencies [Ladevèze et al., 2000; Desmorat et al., 2010]. Several remedies
25 exist for damage models, including nonlocal formulations [Pijaudier-Cabot



(a) iterations



(b) time



(c) time ratio

Figure 6: Performance of the state law inversion methods

1 and Benallal, 1993; Peerlings et al., 1998; Voyiadjis et al., 2004; Desmorat
 2 et al., 2010] and delay-effect approaches [Ladevèze et al., 2000; Kerfriden
 3 et al., 2009; Desmorat et al., 2010]. The latter was chosen for our work
 4 because it is a local method, which is easy to implement in FE codes such
 5 as Abaqus/Standard.

6 4.1. The delay-effect method

7 First, in order to have a damage variable with an order of magnitude of
 8 1, let us rewrite Equation (21) as follows:

$$\begin{cases} d = \begin{cases} 0 & \text{if } \sqrt{\bar{z}} \leq \epsilon_0 \\ \left(\frac{\sqrt{\bar{z}} - \epsilon_0}{\epsilon_1 - \epsilon_0}\right)^2 & \text{if } \epsilon_0 \leq \sqrt{\bar{z}} \leq \epsilon_1 \\ 1 & \text{if } \sqrt{\bar{z}} \geq \epsilon_1 \end{cases} \\ \alpha = k \frac{d}{1-d} \end{cases} \quad (32)$$

9 Now, the delay-effect method for controlling the localization of damage con-
 10 sists in replacing the evolution of this damage variable by an evolution with
 11 a bounded rate:

$$\dot{d} = \frac{1}{\tau_c} \left(1 - e^{-\langle d^{\text{static}} - d \rangle_+}\right) \quad (33)$$

12 where d^{static} is the static function defined in Equation (32) and τ_c a ficti-
 13 tious time parameter which must be chosen in order to avoid uncontrolled
 14 localization [Ladevèze et al., 2000; Kerfriden et al., 2009]. The numerical
 15 resolution of such a simple nonlinear, but scalar, equation is straightforward
 16 and will not be discussed here.

17 The behavior law is modified with the introduction of this viscous-like
 18 parameter. Figure 7 shows the stress-strain curves obtained with different
 19 values of τ_c .

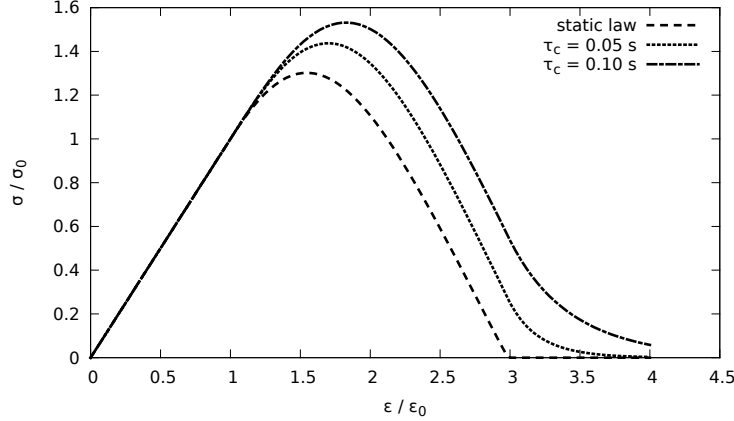


Figure 7: The response of the model under unidirectional tension loading ($k = 10^{-5}$, $\epsilon_1/\epsilon_0 = 3$, $\dot{\epsilon}_1^q = 4\epsilon_0/s$) using different characteristic times for the delay-effect law

1 4.2. Illustrative examples

2 *CT specimen.* The effectiveness of the delay-effect method in controlling the
3 localization of damage was evaluated using a simple CT-like test, *i.e.* a pre-
4 cracked specimen subjected to tension (see Figure 8). The model consisted of
5 an initially isotropic material (with $E_0 = 250$ MPa and $\nu_0 = 0.3$) whose state
6 evolution was modeled using a microstructure-oriented softening damage
7 law (see Equations (19) and (20) of Section 2.3.3, with $a = 0$, $n = 2$ and
8 \underline{n} orthogonal to the initial crack) and the limited-rate kinetics of the delay-
9 effect method (see Equations (32) and (33) of Section 4.1, with $\epsilon_0 = 1 \cdot 10^{-3}$,
10 $\epsilon_1 = 3 \cdot 10^{-3}$, $k = 10^{-5}$ and $\tau_c = 3 \cdot 10^{-2}$ s). The response and the damage
11 fields for different meshes (generated with GMSH [Geuzaine and Remacle,
12 2009]) are shown in Figure 9. One can clearly observe that the solution is
13 objective, *i.e.* mesh-independent, in terms of both the mechanical response
14 and the damage fields. Indeed, while the response of the coarse mesh is
15 slightly different, the responses of the medium and fine meshes are very

1 similar.

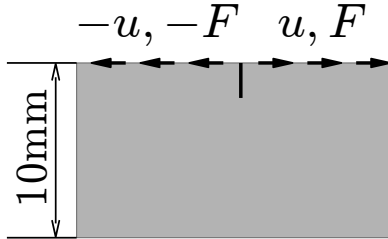
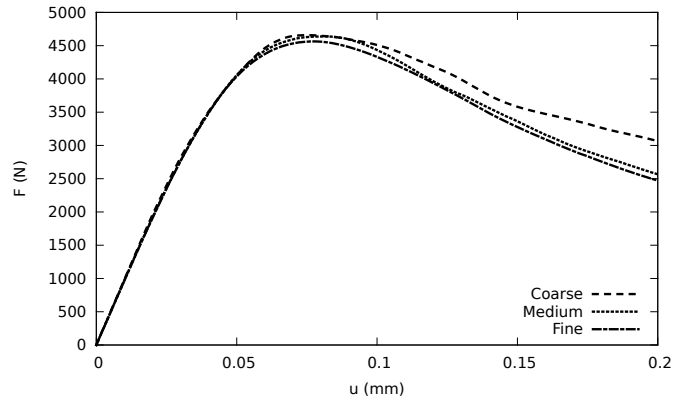


Figure 8: The problem used for the evaluation of localization control based on delay-effect damage (u is the applied displacement, F is the calculated force, and $\dot{u} = 0.2$ mm/s)

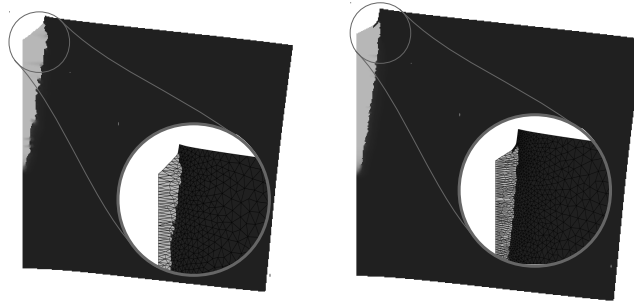
2 *Plate with an open hole.* Now let us consider a first structural example which
3 consists in a plate with an open hole (see Figure 10). The model was the
4 same as for the CT specimen, except for $\tau_c = 1 \cdot 10^{-2}$ s. Figure 11 shows the
5 response of the model using different meshes (again generated with GMSH
6 [Geuzaine and Remacle, 2009]). Once again, the response of the coarse mesh
7 is slightly different, but the responses of the medium and fine meshes are
8 very similar, which attests to the fact that the method can be used to make
9 an objective prediction of the structure's final rupture.

10 5. Conclusion

11 This paper examined several key aspects of the formulation and numer-
12 ical implementation of a family of models belonging to the anisotropic and
13 unilateral damage theory. This modeling framework is based on a general
14 expression of the damaged elastic energy potential which enables one to
15 distinguish between tension behavior and compression behavior throughout

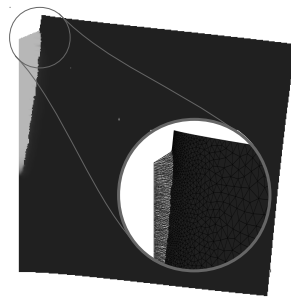


(a) Response



(b) Coarse mesh (2250 DOFs)

(c) Medium mesh (3500 DOFs)



(d) Fine mesh (4550 DOFs)

Figure 9: The response and the damage field for the CT specimen at $t = 0.75$ s (dark grey: no damage; light gray: full damage criterion reached, see Equation (23) of Section 2.3.3): objectivity of the mesh and prediction of the crack's propagation

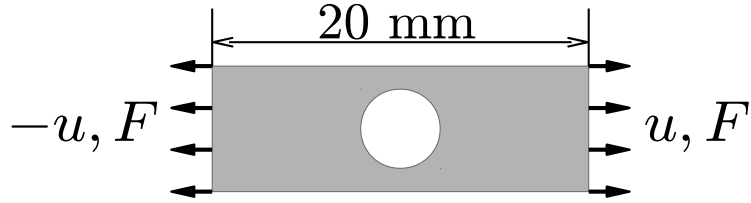
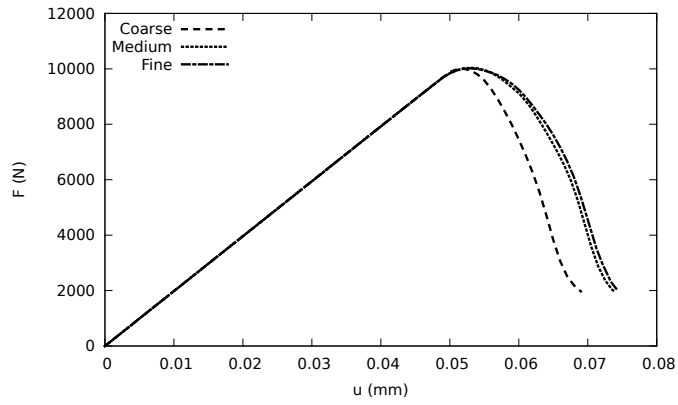


Figure 10: Structural case: plate with an open hole (u is the applied displacement, F is the calculated force, and $\dot{u} = 0.1$ mm/s)

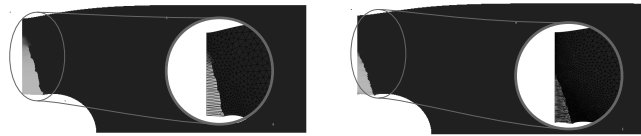
1 the evolution of the damage, even when the orientation of the damage is not
 2 known *a priori* (see Section 2.1).

3 Several damage evolution laws were presented in order to deal with all
 4 types of damage: load-oriented or microstructure-oriented; reaching saturation
 5 or localizing into a macroscopic crack (see Section 2.3). The formulation
 6 of the microstructure-oriented damage law was modified slightly compared
 7 to the previous formulation [Ladevèze, 2002] in order to make it equivalent
 8 to the load-oriented laws when the load is normal to the crack (see Sec-
 9 tion 2.3.2). In addition, softening laws were presented in this framework for
 10 the first time (see Section 2.3.3). Let us note that even though this paper
 11 discusses only models with a single damage evolution law, it is possible to
 12 define models with several laws, each associated with a different damage
 13 mechanism. For example, in the model proposed for CMCs in [Ladevèze,
 14 2002; Genet et al., 2012], three damage evolution laws were defined: one
 15 load-oriented law for inter-yarn cracking and two microstructure-oriented
 16 laws for intra-yarn cracking of the longitudinal and transversal yarns.

17 Then we presented our current implementation of the whole family of
 18 models in Abaqus/Standard. The local equations are solved using a fixed-
 19 point method. Among several relaxation schemes which were considered,



(a) Response



(b) Coarse mesh (1330 DOFs)

(c) Medium mesh (5850 DOFs)



(d) Fine mesh (7100 DOFs)

Figure 11: The response and the damage field for the open hole specimen at $t = 1$ s (dark grey: undamaged; light grey: full damage criterion reached, see Equation (23) of Section 2.3.3): objectivity of the mesh and prediction of the final rupture

1 Aitken's appears to be the most efficient and reduces the cost of the local
2 resolutions to approximately 10% of that of the global iterations (see Section
3 3.1). Since the model's state law is nonlinear even when all the internal
4 variables are fixed, its inversion is carried out using a Newton-Raphson
5 method. Once again, several acceleration schemes were considered, among
6 which the BFGS quasi-tangent method appears to be the most efficient.
7 This reduces the cost of inverting the state law to approximately 50% of
8 that of the local iterations, even with significant damage (see Section 3.2).

9 Finally, a delay-effect method was used to control damage localization
10 in the case of softening evolution laws. This enabled us to simulate the
11 propagation of a macroscopic crack and the final rupture of a structure
12 with no mesh dependency (see Section 4).

13

14 Regarding perspectives, since this framework has already been applied to
15 the case of CMCs [Ladevèze, 2002; Genet et al., 2012], the new capabilities
16 of simulating the creation and propagation of macroscopic cracks will replace
17 the simple criterion used until now (see [Cluzel et al., 2009; Genet et al.,
18 2012]). In addition, microanalyses of cracked media will be carried out in
19 order to derive in a generic way some of the model parameters, such as those
20 which define the damage anisotropy (see Sections 2.3.1 and 2.3.2) or the
21 projectors (see Section 2.3.2). This has already been done for the kinematics
22 of a macroscopic damage model for concrete [Delaplace and Desmorat, 2007]
23 and for both the kinematics and the kinetics of a mesoscopic damage model
24 for laminated composites [Ladevèze and Lubineau, 2002].

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