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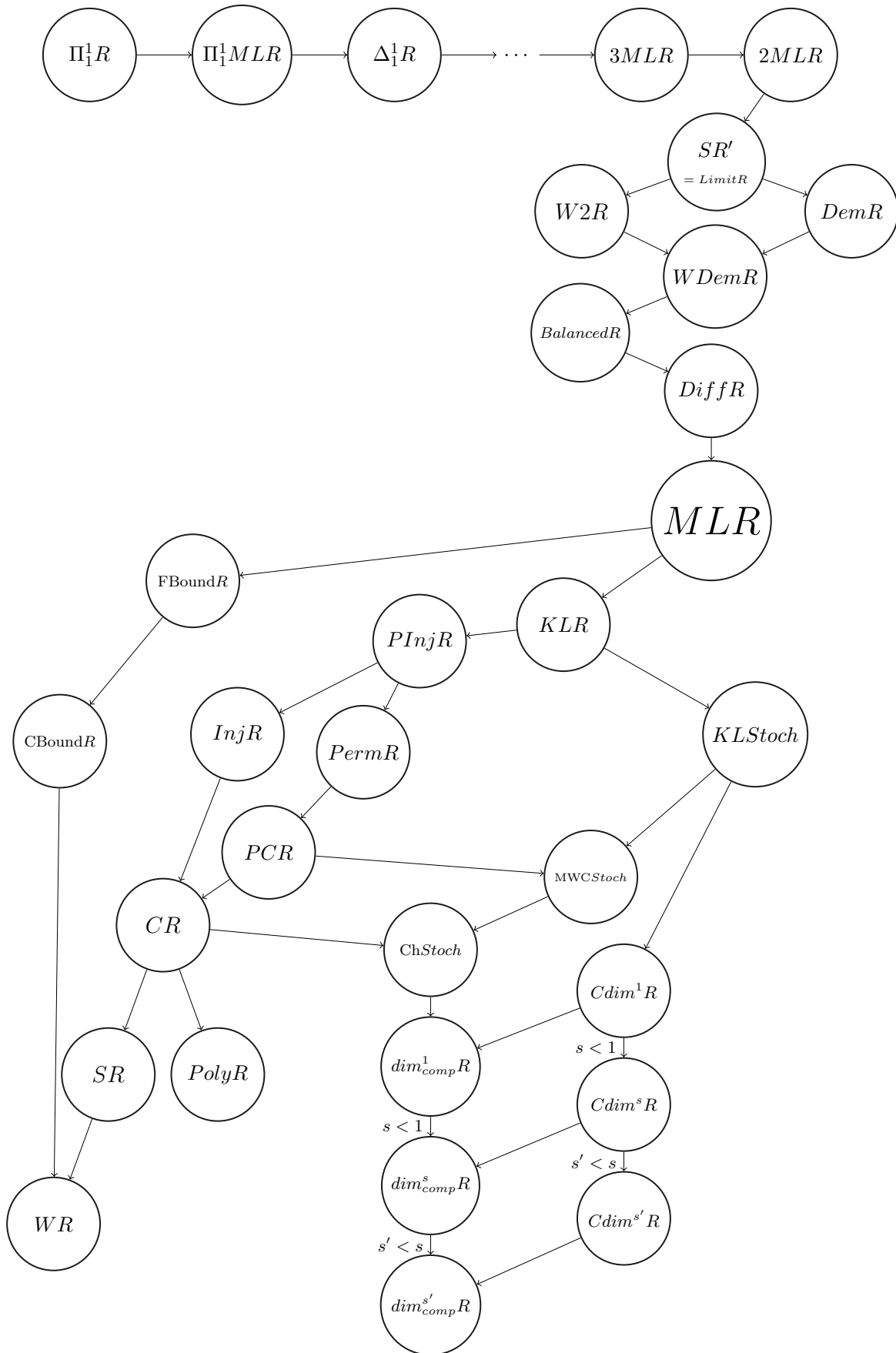
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# Randomness Zoo

Antoine Taveneaux



**Definition 1 (MLR: Martin-Löf Randomness (Definition 3.2.1 in [ML66])).**

1. A Martin-Löf test, a ML-test for short, is a uniformly c.e. sequence  $(G_m)_{m \in \mathbb{N}}$  of open sets such that  $\forall n \mu(G_m) \leq 2^{-m}$ .
2. A set  $R$  fails the test if  $R \in \bigcap_{m \in \mathbb{N}} G_m$  otherwise  $R$  passes the test.
3.  $R$  is Martin-Löf Random if  $R$  passes each ML-test.

## 1 Weaker than MLR

**Definition 2 (Scan rule function).** For a partial function  $f : 2^{<\omega} \rightarrow \{\text{scan, select}\} \times \mathbb{N}$  we denote  $n : 2^{<\omega} \rightarrow \mathbb{N}$  and  $\delta : 2^{<\omega} \rightarrow \{\text{scan, select}\}$ .

And we say that  $f$  is a scan rule if for all  $\sigma, \rho \in 2^{<\omega}$  such that  $\sigma \prec \rho$  we have  $n(\sigma) \neq n(\rho)$ .

The sequence of string observed by  $f$  on  $Z \in cs$  is defined by (if  $f$  is well defined on this points):

$$\begin{aligned} V_f^A(0) &= A(n(\varepsilon)) \\ V_f^A(k+1) &= V_f^A(k).A(n(V_f^A(k))) \end{aligned}$$

and bits selected by  $f$  are:

$$\begin{aligned} T_f^A(0) &= \varepsilon \\ T_f^A(n+1) &= \begin{cases} T_f^A(n) & \text{if } \delta(V_f^A(n)) = \text{scan} \\ T_f^A(n).V_f^A(n+1) & \text{if } \delta(V_f^A(n)) = \text{select} \end{cases} \end{aligned}$$

and we say that  $f$  is well defined on  $Z$  if  $V_f^A(k)$  is well define for all  $k$  and  $(T_f^A(n))$  converge to an infinite string and we denote this infinite string by  $T_f^A$ .

**Definition 3 (Martingale).** A martingale is a function  $\mathcal{M} : 2^{<\omega} \rightarrow \mathbb{R}^+ \cup 0$  such that for all  $\sigma \in 2^{<\omega}$

$$\mathcal{M}(\sigma) = \frac{\mathcal{M}(\sigma 0) + \mathcal{M}(\sigma 1)}{2}$$

We say that  $\mathcal{M}$  succeed on  $Z \in 2^\omega$  if

$$\limsup_{n \rightarrow \infty} \mathcal{M}(Z \upharpoonright n) = \infty$$

**Definition 4 (KLR: KolmogorovLoveland randomness (Definition in [Kol63] and [Lov66])).**  $R \in 2^\omega$  is KL-random if for any partial computable scan rule function  $f$  and any partial computable martingale  $\mathcal{M}$  such that  $f$  is well defined on  $R$  the martingale  $\mathcal{M}$  does not succeed on  $T_f^A$ .

**Definition 5 (KLStoch:** KolmogorovLoveland stochasticity (Definition in [Lov66])). Let  $\#0 : 2^{<\omega} \rightarrow \mathbb{N}$  the function giving the number of “0” in a string.

A sequence  $R$  is KolmogorovLoveland stochastic if for all partial computable scan rule  $f$  such that  $f$  is well defined on  $Z$  we have:

$$\lim_{n \rightarrow \infty} \frac{\#0(T_f^A \upharpoonright n)}{n} = \frac{1}{2}$$

**Definition 6 (MWCStoch:** Mises-Wald-Church stochasticity (Definition in [vM19] and [Chu40])). A sequence  $R$  is Mises-Wald-Church stochastic if for all partial computable monotonic scan rule function  $f$  is well defined on  $Z$  we have:

$$\lim_{n \rightarrow \infty} \frac{\#0(T_f^A \upharpoonright n)}{n} = \frac{1}{2}$$

**Definition 7 (ChStoch:** Church stochasticity (Definition in [vM19] and [Chu40])). A sequence  $R$  is Church stochastic if for all total computable monotonic scan rule function  $f$  we have:

$$\lim_{n \rightarrow \infty} \frac{\#0(R_{f(0)}R_{f(1)} \dots R_{f(n)})}{n} = \frac{1}{2}$$

**Definition 8 (PIInjR:** partial injective randomness (Definition in [MN06])).

A sequence  $R$  is partial injective random if for any total computable injective function  $g : \mathbb{N} \rightarrow \mathbb{N}$  and any partial computable martingale  $\mathcal{M}$  this martingale is defined and does not succeed on the sequence  $R_{f(1)}R_{f(2)} \dots R_{f(n)}R_{f(n+1)} \dots$ .

**Definition 9 (InjR:** injective randomness (Definition in [?])).

A sequence  $R$  is injective random if for any total computable injective function  $g : \mathbb{N} \rightarrow \mathbb{N}$  and any total computable martingale  $\mathcal{M}$  this martingale does not succeed on the sequence  $R_{f(1)}R_{f(2)} \dots R_{f(n)}R_{f(n+1)} \dots$ .

**Definition 10 (PermR:** partial permutation randomness (Definition in [BHKM09])).

A sequence  $R$  is partial permutation random if for any total computable bijective function  $g : \mathbb{N} \rightarrow \mathbb{N}$  and any partial computable martingale  $\mathcal{M}$  this martingale is defined and does not succeed on the sequence  $R_{f(1)}R_{f(2)} \dots R_{f(n)}R_{f(n+1)} \dots$ .

**Definition 11 (PCR:** partial computable randomness (Definition in [AS97])).

A sequence  $R$  is partial computable random if for all partial computable martingale  $\mathcal{M}$  if  $\mathcal{M}(R \upharpoonright n)$  is define for all  $n$  and  $\mathcal{M}$  does not succeed on  $R$ .

**Definition 12 (CR: computable randomness (Definition in [Sch71])).** A sequence  $R$  is computable random if for all total computable martingale  $\mathcal{M}$  this martingale succeed on  $R$ .

**Definition 13 (SR: Schnorr randomness(Definition in [Sch71])).** A Schnorr test is a uniformly c.e. sequence  $(G_m)_{m \in \mathbb{N}}$  of open sets such that  $\forall n \mu(G_m) = 2^{-m}$ .

$R$  is Schnorr random if for any Schnorr test  $(G_m)_{m \in \mathbb{N}}$   $R \notin \bigcap_{m \in \mathbb{N}} G_m$

**Definition 14 (FBoundR: finitely bounded randomness (Definition in [BDN12])).**  $R$  is finitely bounded random if  $R$  passes any Martin-Löf test  $(U_n)$  such that for every  $n$ ,  $\#U_n < \infty$  (with  $\#U_n$  the number of sting enumerated in  $U_n$ ).

**Definition 15 (CBoundR: computably bounded randomness (Definition in [BDN12])).** A Martin-Löf test  $(U_n)$  is computably bounded if there is some total computable function  $f$  such that  $\#U_n \leq f(n)$  for every  $n$ .

$R$  is computably bounded random if  $R$  passes every computably bounded Martin-Löf test.

**Definition 16 (WR: weakly randomness (Definition in [Kur81])).**  $R$  is weakly random if  $R \in U$  for every  $\Sigma_1^0$  set  $U \subseteq 2^\omega$  of measure 1.

**Definition 17 (PolyR: polynomial randomness (Definition in [Wan96])).** A sequence  $R$  is polynomially random if any martingale computable in polynomial time  $\mathcal{M}$  do not succeed on  $R$ .

**Definition 18 ( $\text{dim}_{\text{comp}}^s \mathbf{R}$ : computable  $s$ -randomness (Definition in [Lut03] and [May02])).** A computable  $s$ -test is a uniformly computable sequence  $(G_m)_{m \in \mathbb{N}}$  of computable open sets such that for all  $n$

$$\sum_{x \in G_m} 2^{-s|x|} \leq 2^{-n}.$$

$R$  is computably  $s$ -random if for all  $s' < s$  and computable  $s$ -tests  $(G_m)$  we have:

$$R \notin \bigcap_{m \in \mathbb{N}} G_m$$

**Definition 19 ( $\mathbf{Cdim}^s \mathbf{R}$ : constructive  $s$ -randomness (Definition in [Lut03] and [May02])).** A constructive  $s$ -test is a uniformly computable sequence  $(G_m)_{m \in \mathbb{N}}$  of computable enumerable open sets such that for all  $n$

$$\sum_{x \in G_m} 2^{-s|x|} \leq 2^{-n}.$$

$R$  is computably  $s$ -random if for all  $s' < s$  and computable  $s$ -tests  $(G_m)$  we have:

$$R \notin \bigcap_{m \in \mathbb{N}} G_m$$

## 2 Stronger than MLR

**Definition 20 (DiffR: difference randomness (Definition in [FN])).** A difference test is given by a sequence  $(V_m)_{m \in \mathbb{N}}$  of uniformly c.e. sets and a  $\Pi_1^0$  set  $P$  such that  $\mu(P \cap V_m) \leq 2^{-m}$  for every  $m$ .

A sequence  $R$  is difference random if for any difference test  $(V_m)_{m \in \mathbb{N}}, P$  we have

$$R \notin P \bigcap \left( \bigcap_{m \in \mathbb{N}} V_m \right).$$

**Definition 21 (BalancedR: balanced randomness (Definition in [FHM<sup>+</sup>10])).**

A balanced test is a sequence  $(V_m)_{m \in \mathbb{N}}$  of c.e. sets such that  $V_i = W_{f(i)}$  for some  $2^n$ -c.e. function  $f$  and  $\mu(V_m) \leq 2^{-m}$  for every  $m$ .

A sequence  $R$  is balanced random if  $R$  passes any balanced test.

**Definition 22 (WDemR: weak Demuth randomness (Definition in [Dem82])).**

Demuth test is a sequence  $(V_m)_{m \in \mathbb{N}}$  of c.e. sets such that  $V_i = W_{f(i)}$  for some  $\omega$ -c.e. function  $f$  and  $\mu(V_m) \leq 2^{-m}$  for every  $m$ .

A sequence  $R$  is weak Demuth random if for any Demuth test  $(V_m)$  we have  $R \notin \bigcap_{m \in \mathbb{N}} V_m$ .

**Definition 23 (DemR: Demuth randomness (Definition in [Dem82])).**

A Demuth test is a sequence  $(V_m)_{m \in \mathbb{N}}$  of c.e. sets such that  $V_i = W_{f(i)}$  for some  $\omega$ -c.e. function  $f$  and  $\mu(V_m) \leq 2^{-m}$  for every  $i$ .

A sequence  $R$  is Demuth random if for any Demuth test  $(V_m)$  we have  $R \notin V_m$  for almost all  $m$ .

**Definition 24 (W2R: weak 2-randomness (Definition in [Kau91])).**

A sequence  $R$  is weak 2-random if  $R \notin U$  for every  $\Pi_2^0$  set  $U \subset 2^\omega$  of measure 0.

**Definition 25 (LimitR: limit randomness (Definition in [KN11])).**

A limit test is a sequence  $(V_m)_{m \in \mathbb{N}}$  of c.e. sets such that  $V_i = W_{f(i)}$  for some  $\Delta_2^0$ -computable function  $f$  and  $\mu(V_m) \leq 2^{-m}$  for every  $m$ .

A sequence  $R$  is limit random if for any limit test  $(V_m)$  we have  $R \notin V_m$  for almost all  $m$ .

**Definition 26 ( $\Delta_1^1$ R:  $\Delta_1^1$  randomness (Definition in [ML70])).**  $R$  is  $\Delta_1^1$ -random if  $R$  avoids each null  $\Delta_1^1$ -class.

**Definition 27** ( $\Pi_1^1$ MLR:  $\Pi_1^1$ -Martin-Löf Randomness (Definition in [HN07])).  
 A  $\Pi_1^1$ -Martin-Löf test is a sequence  $(G_m)_{m \in \mathbb{N}}$  of open sets such that  $\forall n \mu(G_m) \leq 2^{-m}$  and the relation  $\{\langle m, \sigma \rangle \mid [\sigma] \subseteq G_m\}$  is  $\Pi_1^1$ .  
 $R$  is  $\Pi_1^1$ -Martin-Löf Random if  $R$  passes each  $\Pi_1^1$ -ML-test.

**Definition 28** ( $\Pi_1^1$ R:  $\Pi_1^1$ -Randomness (Definition in [HN07])).  $R$  is  $\Pi_1^1$ -random if  $R$  avoids each null  $\Pi_1^1$ -class.

## References

- [AS97] K. Ambos-Spies. Algorithmic randomness revisited. In *Language, Logic and Formalization of Knowledge*, pages 33–52, 1997.
- [BDN12] P. Brodhead, R. Downey, and K. M. Ng. Bounded randomness. In *Computation, Physics and Beyond*, pages 59–70, 2012.
- [BHKM09] L. Bienvenu, R. Hölzl, T. Kräling, and W. Merkle. Separations of non-monotonic randomness notions. In *6th Int’l Conf. on Computability and Complexity in Analysis*, 2009.
- [Chu40] A. Church. On the concept of a random sequence. *Bull. Amer. Math. Soc.*, 46:130–135, 1940.
- [Dem82] O. Demuth. Some classes of arithmetical real numbers. *Comment. Math. Univ. Carolin.*, 23(3):453–465, 1982.
- [FHM<sup>+</sup>10] S. Figueira, D. Hirschfeldt, J. Miller, Selwyn Ng, and A Nies. Counting the changes of random  $\Delta_2^0$  sets. In *CiE 2010*, pages 1–10, 2010. Journal version to appear in *J.Logic. Computation*.
- [FN] J. Franklin and K. M. Ng. Difference randomness. *Proceedings of the American Mathematical Society*. To appear.
- [HN07] G Hjorth and A Nies. Randomness in effective descriptive set theory. *J. London. Math. Soc.*, 75(2):495–508, 2007.
- [Kau91] S. Kautz. *Degrees of random sets*. Ph.D. Dissertation, Cornell University, 1991.
- [KN11] A. Kučera and A Nies. Demuth randomness and computational complexity. *Ann. Pure Appl. Logic*, 162:504–513, 2011.
- [Kol63] A. N. Kolmogorov. On tables of random numbers. *Sankhyā Ser. A*, 25:369–376, 1963. Reprinted in *Theoret. Comput. Sci.* 207(2) (1998) 387–395.
- [Kur81] S. Kurtz. *Randomness and genericity in the degrees of unsolvability*. Ph.D. Dissertation, University of Illinois, Urbana, 1981.
- [Lov66] D. Loveland. A new interpretation of the von Mises’ concept of random sequence. *Z. Math. Logik Grundlagen Math.*, 12:279–294, 1966.

- [Lut03] J. H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187, 2003.
- [May02] Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002.
- [ML66] P. Martin-Löf. The definition of random sequences. *Inform. and Control*, 9:602–619, 1966.
- [ML70] P. Martin-Löf. On the notion of randomness. In *Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968)*, pages 73–78. North-Holland, Amsterdam, 1970.
- [MN06] J. Miller and A. Nies. Randomness and computability: Open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.
- [Sch71] C.P. Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 218.
- [vM19] R. von Mises. Grundlagen der Wahrscheinlichkeitsrechnung. *Math. Zeitschrift*, 5:52–99, 1919.
- [Wan96] Yongge Wang. *Randomness and Complexity*. PhD dissertation, University of Heidelberg, 1996.