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# Classification of positive solutions of heat equation with supercritical absorption

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## Abstract

Let  $q \geq 1 + \frac{2}{N}$ . We prove that any positive solution of (E)  $\partial_t u - \Delta u + u^q = 0$  in  $\mathbb{R}^N \times (0, \infty)$  admits an initial trace which is a nonnegative Borel measure, outer regular with respect to the fine topology associated to the Bessel capacity  $C_{\frac{2}{q}, q'}$  in  $\mathbb{R}^N$  ( $q' = q/q - 1$ ) and absolutely continuous with respect to this capacity. If  $\nu$  is a nonnegative Borel measure in  $\mathbb{R}^N$  with the above properties we construct a positive solution  $u$  of (E) with initial trace  $\nu$  and we prove that this solution is the unique  $\sigma$ -moderate solution of (E) with such an initial trace. Finally we prove that every positive solution of (E) is  $\sigma$ -moderate.

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**Key words:** Nonlinear parabolic equation; Initial trace; Representation formula; Bessel capacities; Borel measure; fine topology.

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# 1 Introduction

Let  $q > 1$ ,  $Q_T = \mathbb{R}^N \times (0, T)$  with  $T > 0$  and  $Q = \mathbb{R}^N \times (0, \infty)$ . It is proved by Marcus and Véron [19] that for any positive function  $u \in C^{2,1}(Q_T)$  solution of

$$\partial_t u - \Delta u + u^q = 0 \quad (1.1)$$

there exists a unique couple  $(\mathcal{S}, \mu)$  where  $\mathcal{S}$  is a closed subset of  $\mathbb{R}^N$  and  $\mu$  a positive Radon measure on  $\mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$  such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{O}} u(x, t) dx = \infty \quad (1.2)$$

for all open set  $\mathcal{O}$  of  $\mathbb{R}^N$  such that  $\mathcal{S} \cap \mathcal{O} \neq \emptyset$ , and

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) \quad \forall \zeta \in C_0^\infty(\mathcal{R}). \quad (1.3)$$

To this couple  $(\mathcal{S}, \mu)$  it is associated a unique outer Borel measure  $\nu$  called *the initial trace* of  $u$  and denoted by  $tr(u)$ . The set  $\mathcal{S}$  is *the singular set* of  $\nu$  and the measure  $\mu$  is *the regular set* of  $\nu$ . Conversely, to any outer Borel measure  $\nu$  we can associate its singular part  $\mathcal{S}(\nu)$  which is a closed subset of  $\mathbb{R}^N$  and its regular part  $\mu_\nu$  which is a positive Radon measure on  $\mathcal{R}(\nu)$ . We denote  $\nu \approx (\mathcal{S}, \mu)$ . When  $1 < q < q_c := \frac{N+2}{N}$  Marcus and Véron [19] proved that the trace operator  $tr$  defines a one to one correspondence between the set  $\mathcal{U}_+(Q_T)$  of positive solutions of (1.1) in  $Q_T$  and the set  $\mathfrak{B}^{reg}(\mathbb{R}^N)$  of positive outer Borel measures in  $\mathbb{R}^N$ . This no longer the case if  $q \geq q_c$  since not any closed subset of  $\mathbb{R}^N$  (resp. any positive Radon measure) is eligible for being the singular set (resp. the regular part) of the the initial trace of some positive solution of (1.1). It is proved in [4] that the initial value problem

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1} u &= 0 && \text{in } Q \\ u(\cdot, 0) &= \mu && \text{in } \mathbb{R}^N \end{aligned} \quad (1.4)$$

where  $\mu$  is a positive bounded Radon measure admits a solution if and only if  $\mu$  satisfies

$$C_{\frac{2}{q}, q'}(E) = 0 \implies \mu(E) = 0 \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}, \quad (1.5)$$

where  $C_{\frac{2}{q}, q'}$  stands for the Bessel capacity in  $\mathbb{R}^N$  ( $q' = q/(q-1)$ ). It is shown in [19] that this result holds even if  $\mu$  is unbounded; this solution is unique and denoted  $u_\mu$ . If  $G$  is a Borel subset of  $\mathbb{R}^N$  we denote by  $\mathfrak{M}_q(G)$  the set of Borel measures  $\mu$  in  $G$  with the property that

$$C_{\frac{2}{q}, q'}(E) = 0 \implies \mu(E) = 0 \quad \forall E \subset G, E \text{ Borel}, \quad (1.6)$$

In the same article it is proved that a necessary and sufficient condition in order  $\nu \approx (\mathcal{S}, \mu)$  to be the initial trace of a positive solution of (1.1) is

$$\mu \in \mathfrak{M}_q(\mathcal{R}) \quad (1.7)$$

and

$$\mathcal{S} = \partial_\mu \mathcal{S} \cup \mathcal{S}^* \quad (1.8)$$

where

$$\partial_\mu \mathcal{S} = \{z \in \mathcal{S} : \mu(B_r(z) \cap \mathcal{S}) = \infty, \forall r > 0\} \quad (1.9)$$

and

$$\mathcal{S}^* = \{z \in \mathcal{S} : C_{\frac{2}{q}, q'}((B_r(z) \cap \mathcal{S})) > 0, \forall r > 0\}. \quad (1.10)$$

The meaning of (1.8) is that the singular set is created either by the local unboundedness of the Radon measure or because the singular set is locally non-removable. Furthermore the solution which is constructed is the maximal solution with initial trace  $(\mathcal{S}, \mu)$ .

A striking result due to Le Gall [15] shows that if  $q = 2$  and  $N \geq 2$ , a positive solution of (1.1) is not uniquely determined by its initial trace  $\nu \approx (\mathcal{S}, \mu)$  if  $\mathcal{S} \neq \emptyset$ . The result is actually extended to any  $q \geq q_c$  in [19]. The main point in this counter-example relies on the construction of a positive solution  $u$  of (1.1) with a singular set  $\mathcal{S} = \mathbb{R}^N$ , with a blow-up set at  $t = 0$  which is the union of a countable of closed balls  $\overline{B}_{\epsilon_n}(a_n)$  where  $\{a_n\}$  is a dense set in  $\mathbb{R}^N$  and the  $\epsilon_n$  are chosen small enough so that  $u(0, 1) \leq \alpha$  for some  $\alpha > 0$  fixed. If  $U_{\overline{B}_{\epsilon_n}(a_n)}$  denotes the solution with initial trace  $(\overline{B}_{\epsilon_n}(a_n), 0)$ , then  $U_{\overline{B}_{\epsilon_n}(a_n)}(0, 1) \leq C(\epsilon_n)$  with  $\lim_{\epsilon \rightarrow 0} C(\epsilon) = \infty$ . This is a consequence of the supercriticality assumption and the estimates in [22]. The solution  $u$  is constructed between a sub-solution and a super-solution

$$\sup_n \{U_{\overline{B}_{\epsilon_n}(a_n)}\} \leq u \leq \sum_{n=0}^{\infty} U_{\overline{B}_{\epsilon_n}(a_n)}, \quad (1.11)$$

the right-hand side being chosen so that  $\sum_{n=0}^{\infty} C(\epsilon_n) \leq \alpha$ . Denoting  $E = \cup_n \overline{B}_{\epsilon_n}(a_n)$ , then  $|E| < \infty$  and  $u$  satisfies

$$\lim_{t \rightarrow 0} u(x, t) = 0 \quad \forall x \in \mathbb{R}^N \setminus E \text{ where } |E| < \infty, \quad (1.12)$$

and

$$\lim_{t \rightarrow 0} t^{\frac{1}{q-1}} u(x, t) = c_q = (q-1)^{\frac{1}{1-q}} \quad \text{uniformly for } x \in K \subset \bigcup_n B_{\epsilon_n}(a_n), \text{ } K \text{ compact.} \quad (1.13)$$

Thus (1.2) holds for any nonempty open set  $\mathcal{O} \subset \mathbb{R}^N$ . This counter-example points out that the trace process associated to averaging a positive solution  $u$  of (1.1) on open sets and letting  $t \rightarrow 0$  is not sharp enough to distinguish among solutions; this process is now called the *rough trace*. This is why the introduction of a finer averaging appears to be needed. This finer averaging method is constructed by using the *fine topology* associated to the capacity  $C_{\frac{2}{q}, q'}$ . It will lead us to the notion of precise trace.

A similar approach has been carried out if one considers the boundary trace problem for the positive solutions of the elliptic equation

$$-\Delta u + |u|^{q-1} u = 0 \quad \text{in } \Omega \quad (1.14)$$

where  $\Omega$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $q > 1$ . The boundary trace is defined in a somewhat similar way as the initial trace, by considering the limit in the weak sense of measures, of the restriction of  $u$  to the set  $\Sigma_\epsilon := \{x \in \Omega : \text{dist}(x, \Omega^c) = \epsilon\}$ , when  $\epsilon \rightarrow 0$ . The boundary trace  $tr_{\partial\Omega}(u)$  is a uniquely determined outer regular Borel measure on  $\partial\Omega$ , with singular part  $\mathcal{S}$ , a closed subset of  $\partial\Omega$  and regular part  $\mu$ , a positive Radon measure on  $\mathcal{R} = \partial\Omega \setminus \mathcal{S}$ . This equation possesses a critical exponent  $q_e = (N + 1)/(N - 1)$ . The main contributions which lead to a complete picture of the boundary trace problem over a period of twenty years are due to Gmira and Véron [11], Le Gall [13], [14], Dynkin and Kuznetsov [5],[6], [7] [8], [9],[12], Marcus and Véron [17],[18],[20],[21],[23], [22], [16], and Mselati [24]. These contributions can be summarized as follows:

(i) If  $1 < q < q_e$  the boundary trace operator establishes a one to one correspondence between the set  $\mathcal{U}_+(\Omega)$  of positive solutions of (1.14) and the set of positive outer regular Borel measures on  $\partial\Omega$ .

(ii) If  $q \geq q_e$  the boundary value problem

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega \end{aligned} \quad (1.15)$$

where  $\mu$  is a positive Radon measure on  $\partial\Omega$  admits a solution (always unique) if and only if

$$C_{\frac{2}{q}, q'}(E) = 0 \implies \mu(E) = 0 \quad \forall E \subset \partial\Omega, E \text{ Borel}, \quad (1.16)$$

where  $C_{\frac{2}{q}, q'}$  is the Bessel capacity in  $\mathbb{R}^{N-1}$ .

(iii) If  $q \geq q_e$ , a outer regular Borel measure  $\nu \approx (\mathcal{S}, \mu)$  on  $\partial\Omega$  is the boundary trace of a positive solution of (1.14) if and only if

$$C_{\frac{2}{q}, q'}(E) = 0 \implies \mu(E) = 0 \quad \forall E \subset \mathcal{S}, E \text{ Borel},$$

and (1.8) holds with (1.9) and (1.10) where the capacity is relative to dimension  $N-1$ .

(iv) If  $q \geq q_e$  a solution is not uniquely determined by its boundary trace whenever  $\mathcal{S} \neq \emptyset$ .

However in [23] Marcus and Véron have defined a notion of *precise trace* for the case  $q \geq q_e$  with the following properties,

(v) If we denote by  $\mathfrak{T}_q$  the fine topology of  $\partial\Omega$  associated with the  $C_{\frac{2}{q}, q'}$ -capacity, there exists a  $\mathfrak{T}_q$ -closed subset  $\mathcal{S}_q$  of  $\partial\Omega$  such that for every  $z \in \mathcal{S}_q$

$$\lim_{\epsilon \rightarrow 0} \int_{\Xi} u(\epsilon, \sigma) dS = \infty \quad (1.17)$$

for every  $\mathfrak{T}_q$ -open neighborhood  $\Xi$  of  $z$  where  $(r, \sigma) \in [0, \epsilon_0] \times \partial\Omega$  are the flow coordinates near  $\partial\Omega$ , and for every  $z \in \mathcal{R}_q := \partial\Omega \setminus \mathcal{S}_q$ , there exists a  $\mathfrak{T}_q$ -open neighborhood  $\Xi$  of  $z$  such that

$$\limsup_{\epsilon \rightarrow 0} \int_{\Xi} u(\epsilon, \sigma) dS < \infty. \quad (1.18)$$

(vi) There exists a nonnegative Borel measure  $\mu$  on  $\mathcal{R}_q$ , outer regular for the  $\mathfrak{T}_q$ -topology, such that

$$\lim_{\epsilon \rightarrow 0} u_\epsilon^\Xi = u_{\mu\chi_\Xi} \quad \text{locally uniformly in } \Omega, \quad (1.19)$$

where  $u_\epsilon^\Xi$  is the solution of

$$\begin{aligned} -\Delta v + |v|^{q-1}v &= 0 & \text{in } \Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\} \\ v &= u(\epsilon, \cdot)\chi_\Xi & \text{in } \Sigma_\epsilon = \partial\Omega_\epsilon. \end{aligned} \quad (1.20)$$

The couple  $(\mathcal{S}_q, \mu)$  is uniquely determined and it is called the precise boundary trace of  $u$ . It can also be represented by a Borel measure with the  $\mathfrak{T}_q$ -outer regularity. It is denoted by  $tr_{\partial\Omega}^q(u)$ .

Concerning uniqueness Dynkin and Kuznetsov introduced in [9] the notion of  $\sigma$ -moderate solutions, which are elements  $u$  of  $\mathcal{U}_+(\Omega)$  with the property that there exists an increasing sequence  $\{\mu_n\}$  of nonnegative Radon measures on  $\partial\Omega$  such that  $u_{\mu_n} \rightarrow u$  when  $n \rightarrow \infty$ . In [23] Marcus and Véron proved that a  $\sigma$ -moderate positive solution of (1.14) is uniquely determined by its precise boundary trace. This precise trace is essentially the same, up to a set of zero  $C_{\frac{2}{\epsilon}, q'}$ -capacity, as the *fine trace* that Dynkin and Kuznetsov introduced in [9] using probabilistic tools such as the Brownian motion; however their construction is only valid in the range  $(1, q]$  of values of  $q$ . Finally, in [16], Marcus proved that any positive solution is  $\sigma$ -moderate. Notice that this result was already obtained by Mselati [24] in the case  $q = 2$  and then by Dynkin [6] for  $q_e \leq q \leq 2$  by using a combination of analytic and probabilistic techniques.

In this article we define a notion of *precise initial trace* for positive solutions of (1.1) associated to the  $\mathfrak{T}_q$ -topology, which denotes the  $C_{\frac{2}{q}, q'}$  fine topology of  $\mathbb{R}^N$ . We denote by  $\mathbb{H}[\cdot]$  the heat potential in  $Q$  expressed by

$$\mathbb{H}[\xi](x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \xi(y) dy, \quad (1.21)$$

for all  $\xi \in L^1(\mathbb{R}^N)$ . We define the *singular set* of  $u \in \mathcal{U}_+(Q_T)$  as the set of  $z \in \mathbb{R}^N$  such that for any  $\mathfrak{T}_q$ -open neighborhood  $\mathcal{O} \subset \mathbb{R}^N$  of  $z$ , there holds

$$\iint_{Q_T} \mathbb{H}[\chi_{\mathcal{O}}] u^q dx dt = \infty. \quad (1.22)$$

The singular set, denoted by  $\mathcal{S}_q = \mathcal{S}_q(u)$ , is  $\mathfrak{T}_q$ -closed. The regular set is  $\mathcal{R}_q := \mathbb{R}^N \setminus \mathcal{S}_q$ ; it is  $\mathfrak{T}_q$ -open. If  $z \in \mathcal{S}_q$  and  $\mathcal{O} \subset \mathbb{R}^N$  is a  $\mathfrak{T}_q$ -open neighborhood of  $z$  such that

$$\iint_{Q_T} \mathbb{H}[\chi_{\mathcal{O}}] u^q dx dt < \infty, \quad (1.23)$$

then for any  $\eta \in L^\infty \cap W^{\frac{2}{q}, q'}(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support contained in  $\mathcal{O}$  there exists

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) (\eta(x))^{2q'} dx := \ell_{\mathcal{O}}(\eta). \quad (1.24)$$

As a consequence there exists a positive Borel measure  $\mu$  on  $\mathcal{R}_q$ , outer regular for the  $\mathfrak{T}_q$ -topology, such that for  $\mathfrak{T}_q$ -open subset  $\Xi \subset \mathcal{R}_q$  there holds

$$\lim_{\epsilon \rightarrow 0} u_{\epsilon, \chi_\Xi}(\cdot, t) = u_{\chi_\Xi \mu} \quad (1.25)$$

where  $u_{\epsilon, \chi_{\Xi}}$  is the solution of

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1} v &= 0 & \text{in } Q^\epsilon := \mathbb{R}^N \times (\epsilon, \infty) \\ v(\cdot, \epsilon) &= \chi_{\Xi} & \text{in } \mathbb{R}^N. \end{aligned} \quad (1.26)$$

The set  $(\mathcal{S}_q, \mu)$  is called the *precise initial trace* of  $u$  and denoted by  $\text{tr}^c(u)$ . To this set we can associate a Borel measure  $\nu$  on  $\mathbb{R}^N$ . It is absolutely continuous with respect to the  $C_{\frac{2}{q}, q'}$ -capacity in the following sense

$$\forall Q \subset \mathbb{R}^N, \mathfrak{T}_q\text{-open}, \forall A \subset \mathbb{R}^N, A \text{ Borel}, C_{\frac{2}{q}, q'}(A) = 0 \implies \mu(Q \setminus A) = \mu(Q). \quad (1.27)$$

It is also outer regular with respect to the  $\mathfrak{T}_q$ -topology in the sense that for every Borel set  $E \subset \mathbb{R}^N$

$$\mu(E) = \inf\{\mu(Q) : Q \supset E, Q \mathfrak{T}_q\text{-open}\} = \sup\{\mu(K) : K \subset E, K \text{ compact}\}. \quad (1.28)$$

A measure with the above properties is called  $\mathfrak{T}_q$ -*perfect*. Similarly to Dynkin, we say that a positive solution  $u$  of (1.1) is  $\sigma$ -moderate if there exists an increasing sequence  $\{\mu_n\}$  of nonnegative Radon measures in  $\mathbb{R}^N$  such that  $u_{\mu_n} \rightarrow u$  when  $n \rightarrow \infty$ . It is proved in [22] that if  $F \subset \mathbb{R}^N$  is a closed subset, the maximal solution  $U_F$  with initial trace  $(F, 0)$  coincides with the maximal  $\sigma$ -moderate solution  $V_F$  with the same trace and which is defined by

$$V_F = \sup\{u_\mu : \mu \in \mathfrak{M}_q(\mathbb{R}^N), \mu(F^c) = 0\}. \quad (1.29)$$

It is indeed  $\sigma$ -moderate. Following Dynkin we define an addition among the elements of  $\mathcal{U}_+(Q_T)$  by

$$\forall (u, v) \in \mathcal{U}_+(Q_T) \times \mathcal{U}_+(Q_T), u \oplus v \text{ is the largest element of } \mathcal{U}_+(Q_T) \text{ dominated by } u + v. \quad (1.30)$$

The main results of this article are the following

**Theorem A.** *If  $\nu$  is a  $\mathfrak{T}_q$ -perfect measure with singular part  $\mathcal{S}_q$  and regular part  $\mu$  on  $\mathcal{R}_q$  then  $u_\mu \oplus U^{\mathcal{S}_q}$  is the only  $\sigma$ -moderate element of  $\mathcal{U}_+(Q)$  with precise trace  $\nu$ .*

In order to extend Marcus's result we need a parabolic counterpart of Ancona's characterization of positive solutions of Schrödinger equation with singular potential [1]. We prove a representation theorem valid for any positive solution of

$$\partial_t u - \Delta u + V(x, t)u = 0 \quad \text{in } Q, \quad (1.31)$$

where  $V$  is a Borel function which satisfies, for some  $c \geq 0$ ,

$$0 \leq V(x, t) \leq \frac{c}{t} \quad \text{for almost all } (x, t) \in Q. \quad (1.32)$$

Let  $T$  be fixed and let  $\psi$  be defined by

$$\psi(x, t) = \int_t^T \int_{\mathbb{R}^N} \frac{1}{(4\pi(s-t))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}} V(y, s) dy ds \quad \text{in } Q_T.$$



**Theorem B.** *There exists a kernel  $\Gamma$  defined in  $Q_T \times Q_T$  satisfying*

$$c_1 \frac{e^{-a_1 \frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{N}{2}}} \leq \Gamma(x, t, y, s) \leq c_2 \frac{e^{-a_2 \frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{N}{2}}} \quad \forall (x, t), (y, s) \in Q_T \times Q_T \text{ with } s \leq t. \quad (1.33)$$

where the  $a_j$  and  $c_j$  are positive constants depending on  $T$  and  $V$ , such that for any positive solution  $u$  of (1.31), there exists a positive Radon measure  $\mu$  in  $\mathbb{R}^N$  such that

$$u(x, t) = e^{\psi(x, t)} \int_{\mathbb{R}^N} \Gamma(x, t, y, 0) d\mu(y) \quad \text{for almost all } (x, t) \in Q_T. \quad (1.34)$$

The next result, combined with Theorem A, shows that in the case  $q \geq q_c$  the precise trace operator realizes a one to one correspondence between the set of positive solutions of (1.1) and the set of  $\mathfrak{T}_q$ -perfect Borel measures in  $\mathbb{R}^N$ .

**Theorem C** *Any positive solution of (1.1) is  $\sigma$ -moderate.*

Several proofs in this work are transposition to the parabolic framework of the constructions performed in [23] and [16]. However, for the sake of completeness and due to the technicalities involved, we kept many of them, sometimes under an abridged form.

## 2 The $\mathfrak{T}_q$ -fine topology

We assume that  $q \geq 1 + \frac{2}{N}$  and set  $q' = \frac{q}{q-1}$ . We recall that a set  $E \subset \mathbb{R}^N$  is  $(\frac{2}{q}, q')$ -thin at a point  $a$  if

$$\int_0^1 \left( \frac{C_{\frac{2}{q}, q'}(E \cap B_s(a))}{s^{N - \frac{2}{q-1}}} \right)^{q-1} \frac{ds}{s} < \infty. \quad (2.35)$$

If the value of the above integral is infinite, the set  $E$  is called  $(\frac{2}{q}, q')$ -thick at  $a$ . A set  $U$  is a  $(\frac{2}{q}, q')$ -fine neighborhood of one of its point  $a$  if  $U^c$  is thin at  $a$ . It is  $(\frac{2}{q}, q')$ -finely open, if  $U^c$  is thin at any point  $a \in U$ . It is  $(\frac{2}{q}, q')$ -finely closed if its complement is  $(\frac{2}{q}, q')$ -finely open. For simplicity we will denote by  $\mathfrak{T}_q$  the  $(\frac{2}{q}, q')$ -fine topology associated to these notions (see [2, Chap 6] for a thorough discussion of these notions). We say that a set  $E \subset \mathbb{R}^N$  is  $\mathfrak{T}_q$ -open (resp  $\mathfrak{T}_q$ -closed) if it is open (resp. closed) in the  $\mathfrak{T}_q$ -topology.

**Notation 2.1** *Let  $A, B \subset \mathbb{R}^N$ .*

a)  *$A$  is  $\mathfrak{T}_q$ -essentially contained in  $B$ , denoted  $A \subset^q B$ , if*

$$C_{\frac{2}{q}, q'}(A \setminus B) = 0.$$

b) *The sets  $A, B$  are  $\mathfrak{T}_q$ -equivalent, denoted  $A \sim^q B$ , if*

$$C_{\frac{2}{q}, q'}(A \Delta B) = 0.$$

c) *The  $\mathfrak{T}_q$ -closure of a set  $A$  is denoted by  $\tilde{A}$ . The  $\mathfrak{T}_q$ -interior of  $A$  is denoted by  $A^\diamond$ .*

d) *Given  $\varepsilon > 0$ ,  $A^\varepsilon$  denotes the  $\varepsilon$ -neighbourhood of  $A$  for the standard Euclidean distance in*

$\mathbb{R}^N$

e) The set of  $\mathfrak{T}_q$ -thick points of  $A$  is denoted by  $b_q(A)$ . The set of  $\mathfrak{T}_q$ -thin points of  $A$  is denoted by  $e_q(A)$ .

$$A \text{ is } \mathfrak{T}_q\text{-open} \Leftrightarrow A \subset e_q(A^c), \quad B \text{ is } \mathfrak{T}_q\text{-closed} \Leftrightarrow b_q(B) \subset B.$$

Consequently,

$$\tilde{A} = A \bigcup b_q(A), \quad A^\diamond = A \cap e_q(A^c).$$

The capacity  $C_{\frac{2}{q}, q'}$  possesses the Kellogg property (see [2, Cor. 6.3.17]), namely,

$$C_{\frac{2}{q}, q'}(A \setminus b_q(A)) = 0. \quad (2.36)$$

Therefore

$$A \subset^q b_q(A) \sim^q \tilde{A},$$

but, in general,  $b_q(A)$  does not contain  $A$ .

**Proposition 2.2** (i) If  $Q$  is a  $\mathfrak{T}_q$ -open, then  $e_q(Q^c)$  is the largest  $\mathfrak{T}_q$ -open set that is  $\mathfrak{T}_q$ -equivalent to  $Q$ .

(ii) If  $F$  is a  $\mathfrak{T}_q$ -closed then  $b_q(F)$  is the smallest  $\mathfrak{T}_q$ -closed set that is  $\mathfrak{T}_q$ -equivalent to  $F$ .

The proof is [23, Prop. 2.1]. We collect below several facts concerning the  $\mathfrak{T}_q$ -topology that are used throughout the paper.

**Proposition 2.3** Let  $q \geq 1 + \frac{2}{N}$ .

i) Every  $\mathfrak{T}_q$ -closed set is  $\mathfrak{T}_q$ -quasi closed ([2, Prop 6.4.13]).

ii) If  $E$  is  $\mathfrak{T}_q$ -quasi closed then  $E \sim^q \tilde{E}$  ([2, Prop 6.4.12]).

iii) A set  $E$  is  $\mathfrak{T}_q$ -quasi closed if and only if there exists a sequence  $\{E_m\}$  of closed subsets of  $E$  such that  $C_{\frac{2}{q}, q'}(E \setminus E_m) \rightarrow 0$  ([2, Prop. 6.4.9]).

iv) There exists a positive constant  $c$  such that, for every set  $E$ ,

$$C_{\frac{2}{q}, q'}(\tilde{E}) \leq c C_{\frac{2}{q}, q'}(E),$$

([2, Prop 6.4.11]).

v) If  $E$  is  $\mathfrak{T}_q$ -quasi closed and  $F \sim^q E$  then  $F$  is  $\mathfrak{T}_q$ -quasi closed.

vi) If  $\{E_i\}$  is an increasing sequence of arbitrary Borel sets then

$$C_{\frac{2}{q}, q'}\left(\bigcup E_i\right) = \lim_{i \rightarrow \infty} C_{\frac{2}{q}, q'}(E_i).$$

vii) If  $\{K_i\}$  is a decreasing sequence of compact sets then

$$C_{\frac{2}{q}, q'}\left(\bigcap K_i\right) = \lim_{i \rightarrow \infty} C_{\frac{2}{q}, q'}(K_i).$$

viii) Every Suslin set and, in particular, every Borel set  $E$  satisfies

$$\begin{aligned} C_{\frac{2}{q}, q'}(E) &= \inf\{C_{\frac{2}{q}, q'}(G) : E \subset G, G \text{ open}\} \\ &= \sup\{C_{\frac{2}{q}, q'}(K) : K \subset E, K \text{ compact}\}. \end{aligned}$$

For the last three statements see [2, Sec. 2.3]. Statement (v) is an easy consequence of [2, Prop. 6.4.9]. However note that this assertion is no longer valid if "Fr<sub>q</sub>-quasi closed" is replaced by "Fr<sub>q</sub>-closed." Only the following weaker statements holds:

If  $E$  is Fr<sub>q</sub>-closed and  $A$  is a set such that  $C_{\frac{2}{q}, q'}(A) = 0$  then  $E \cup A$  is Fr<sub>q</sub>-closed.

The next corollary is an easy consequence of (iii).

**Corollary 2.4** *A set  $E$  is Fr<sub>q</sub>-quasi closed if and only if there exists a sequence  $\{E_m\}$  of Fr<sub>q</sub>-quasi closed subsets of  $E$  such that  $C_{\frac{2}{q}, q'}(E \setminus E_m) \rightarrow 0$ .*

**Definition 2.5** *Let  $E$  be a Fr<sub>q</sub>-quasi closed set. An increasing sequence  $\{E_m\}$  of closed subsets of  $E$  such that  $C_{\frac{2}{q}, q'}(E \setminus E_m) \rightarrow 0$  is called a Fr<sub>q</sub>-stratification of  $E$ .*

(i) *We say that  $E_m$  is a proper Fr<sub>q</sub>-stratification of  $E$  if*

$$C_{\frac{2}{q}, q'}(E_{m+1} \setminus E_m) \leq \frac{1}{2^{m+1}}.$$

(ii) *If  $V$  is a Fr<sub>q</sub>-open set such that  $C_{\frac{2}{q}, q'}(E \setminus V) = 0$  we say that  $V$  is a Fr<sub>q</sub>-quasi neighborhood of  $E$ .*

The following separation statement is valid in any locally compact metric space.

**Lemma 2.6** *Let  $K$  be a closed subset of an open set  $A$ . Then there exists an open set  $G$  such that*

$$K \subset G \subset \overline{G} \subset A.$$

*Proof.* Let  $x \in K$ . We set  $B_n = B_n(x)$ ;  $n \in \mathbb{N}$  and  $K_n = \overline{B_n} \cap K$ . Since  $K_n$  is compact, we can easily show that there exists a decreasing sequence  $\{\varepsilon_n\}$  converging to 0 such that  $K_n^{\varepsilon_n} \subset \overline{K_n^{\varepsilon_n}} \subset A$ . Now we have

$$\bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_n}{2}} \subset \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}} \subset \bigcup_{n=1}^{\infty} K_n^{\varepsilon_n} \subset A.$$

If we prove that the set

$$\bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$$

is closed then the proof follows with  $G = \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$ . We will prove it by contradiction. We assume that there exists a sequence  $x_n \in \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$  such that  $x_n \rightarrow x$  and  $x \notin \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$ . We have  $x_1 = x_{n_1}$  such that  $\text{dist}(x_{n_1}, K) = \inf\{|x_{n_1} - y| : y \in K\} \leq \frac{\varepsilon_1}{2}$ . Also we assert that there exists  $x_{n_2}$  such that  $\text{dist}(x_{n_2}, K) \leq \frac{\varepsilon_2}{2}$ . Indeed, If this is not valid then  $\forall n \in \mathbb{N}$  we have  $\frac{\varepsilon_2}{2} < \text{dist}(x_n, K) \leq \frac{\varepsilon_1}{2}$ , which implies  $x \in K_1$ . Thus we have clearly a contradiction. Inductively, we can construct a subsequence  $\{x_{n_k}\}$  such that  $\text{dist}(x_{n_k}, K) \leq \frac{\varepsilon_k}{2}$ ,  $\forall k \in \mathbb{N}$ . If we send  $k$  to

infinite, we reach to a contradiction, since we would have  $\text{dist}(x, K) = 0$  and using the fact that  $K$  is closed, we would obtain that  $x \in K$ .  $\square$

In the framework of the  $\mathfrak{T}_q$ -topology, the preceding result admits the following counterpart.

**Lemma 2.7** *Let  $E$  be a  $\mathfrak{T}_q$ -closed set. Then:*

(i) *Let  $D$  be an open set such that  $C_{\frac{2}{q}, q'}(E \setminus D^c) = 0$ . Then there exists an open set  $O$  such that*

$$E \subset^q O \subset \tilde{O} \subset^q D. \quad (2.37)$$

(ii) *Let  $D$  be a  $\mathfrak{T}_q$ -open set such that  $E \subset^q D$ . Then there exists a  $\mathfrak{T}_q$ -open set  $O$  such that (2.37) holds.*

*Proof.* (i) Since  $E \cap D \sim^q E$ ,  $E \cap D$  is  $\mathfrak{T}_q$ -quasi closed, (see the discussion of the quasi topology in [2, sec. 6.4]). Thus there exists a proper  $\mathfrak{T}_q$ -stratification of  $E \cap D$ , say  $\{E_m\}$  and  $E \sim^q E' = \bigcup_{i=1}^{\infty} E_i$ . If  $E'$  is closed the result follows by Lemma 2.6. We assume that  $E'$  is not closed. Thus, we can assume without loss of generality that

$$E_{m+1} \setminus E_m \neq \emptyset \quad \forall m \in \mathbb{N}.$$

We set  $E'_m = G$ , where  $G$  is the open set of Lemma 2.6 with  $K = E_m$  and  $A = D$ . Now since  $C_{\frac{2}{q}, q'}(E_m \setminus E_{m-1}) < \frac{1}{2^{m+1}}$ , there exists an open set  $D_m \supset E_m \setminus E_{m-1}$ ;  $m \geq 2$ , such that  $C_{\frac{2}{q}, q'}(D_m) < \frac{1}{2^m}$ . Also we set  $D_1 = E'_1$ . Also we have by Lemma (2.6),

$$D_m \cap E_m \subset \widetilde{D_m \cap E_m} \subset \widetilde{E_m} \subset D \quad \forall m \in \mathbb{N}.$$

Also, since  $E' = E_1 \cup \bigcup_{m=2}^{\infty} (E_m \setminus E_{m-1})$  we have that

$$E' \subset \bigcup_{m=1}^{\infty} D_m \cap E'_m \subset \bigcup_{m=1}^{\infty} \widetilde{D_m \cap E'_m} \subset D.$$

Thus, it is enough to prove that the set  $\bigcup_{m=1}^{\infty} \widetilde{D_m \cap E'_m}$  is  $\mathfrak{T}_q$ -quasi closed. Indeed, for each  $n > 1$ , we have

$$\begin{aligned} C_{\frac{2}{q}, q'} \left( \bigcup_{m=1}^{\infty} \widetilde{D_m \cap E'_m} \setminus \bigcup_{m=1}^n \widetilde{D_m \cap E'_m} \right) &\leq C_{\frac{2}{q}, q'} \left( \bigcup_{m=n+1}^{\infty} \widetilde{D_m \cap E'_m} \right) \leq \sum_{m=n+1}^{\infty} C_{\frac{2}{q}, q'}(\widetilde{D_m}) \\ &\leq c \sum_{m=n+1}^{\infty} C_{\frac{2}{q}, q'}(D_m) \leq c \sum_{m=n+1}^{\infty} 2^{-m}. \end{aligned}$$

And the result follows by Corollary 2.4, since  $\bigcup_{m=1}^n \widetilde{D_m \cap E'_m}$  is  $\mathfrak{T}_q$ -quasi closed. The proof of (ii) is same as in [23, Lemma 2.4 (ii)].  $\square$

**Lemma 2.8** (I) *Let  $E$  be a  $\mathfrak{T}_q$ -closed set and  $\{E_m\}$  a proper  $\mathfrak{T}_q$ -stratification for  $E$ . Then there exists a decreasing sequence of open sets  $\{Q_j\}$  such that  $\bigcup E_m := E' \subset Q_j$  for every  $j \in \mathbb{N}$  and*

$$(i) \cap_j Q_j = E', \quad \tilde{Q}_{j+1} \subset^q Q_j,$$

$$(ii) \lim_{j \rightarrow \infty} C_{\frac{2}{q}, q'}(Q_j) = C_{\frac{2}{q}, q'}(E).$$

(II) If  $A$  is a  $\tilde{\Sigma}_q$ -open set, there exists a decreasing sequence of open sets  $\{A_m\}$  such that

$$A \subset \bigcap_m A_m =: A', \quad C_{\frac{2}{q}, q'}(A_m \setminus A') \rightarrow 0 \text{ as } m \rightarrow \infty, \quad A \sim^q A'.$$

Furthermore there exists an increasing sequence of closed sets  $\{F_j\}$  such that  $F_j \subset A'$  and

$$(i) \cup F_j = A', \quad F_j \subset^q F_{j+1}^\circ$$

$$(ii) C_{\frac{2}{q}, q'}(F_j) \rightarrow C_{\frac{2}{q}, q'}(A') \text{ as } j \rightarrow \infty.$$

*Proof.* Let  $\{D_j\}$  be a decreasing sequence of open sets such that  $D_j \supset E, \forall j \in \mathbb{N}$  and

$$\lim_{j \rightarrow \infty} C_{\frac{2}{q}, q'}(D_j) = C_{\frac{2}{q}, q'}(E') = C_{\frac{2}{q}, q'}(E).$$

*Case 1:*  $E$  is closed (thus  $E_m = E$  for any  $m \in \mathbb{N}$ ).

By Lemma 2.6 there exists a decreasing sequence  $\{\varepsilon_{1,n}\}$  converging to 0, such that  $\varepsilon_{1,1} < 1$ , and

$$E \subset Q_1 = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{1,n}}{2}} \subset \overline{Q}_1 \subset D_1,$$

where  $K_n = B_n(x) \cap E, x \in E$ . Also we have proven in Lemma 2.6 that the set  $\bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_{1,n}}{2}}}$  is closed.

Again by Lemma 2.6 there exists a decreasing sequence  $\{\varepsilon_{2,n}\}$  converging to 0, such that  $\varepsilon_{2,n} \leq \varepsilon_{1,n}$  for all  $n$  and

$$E \subset Q_2 = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{2,n}}{4}} \subset \overline{Q}_2 \subset D_2.$$

We note here that

$$\overline{Q}_2 \subset \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_{2,n}}{4}}} \subset \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{1,n}}{2}},$$

and since  $\overline{K_n^{\frac{\varepsilon_{2,n}}{4}}}$  is closed we have

$$Q_2 \subset \overline{Q}_2 \subset Q_1.$$

By induction, we construct a decreasing sequence  $\{\varepsilon_{j,n}\}$  converging to 0 with respect to  $n$ , such that  $\forall n \in \mathbb{N} : \varepsilon_{j,n} \leq \varepsilon_{k,n}$  for all  $j \geq k$ ,

$$E \subset Q_j = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{j,n}}{2^{j+1}}} \subset \overline{Q}_j \subset D_j,$$

and

$$Q_j \subset \overline{Q}_j \subset Q_{j-1}.$$

Now note that

$$E \subset Q_j \subset E^{\frac{1}{2^j}},$$

thus  $E = \cap Q_j$ . Finally,

$$C_{\frac{2}{q}, q'}(E) \leq \lim C_{\frac{2}{q}, q'}(Q_j) \leq \lim C_{\frac{2}{q}, q'}(D_j) = C_{\frac{2}{q}, q'}(E),$$

and the result follows in this case.

*Case 2:*  $E$  is not closed.

There exists a proper  $\mathfrak{T}_q$ -stratification of  $E$ , say  $\{E_m\}$  and  $E \sim^q E' = \bigcup_{i=1}^{\infty} E_i$ . Also by the Case 1, we can assume without loss of generality that

$$E_{m+1} \setminus E_m \neq \emptyset \quad \forall m \in \mathbb{N}.$$

Let us denote by  $Q_j^m$  the sets denoted by  $Q_j$  in the previous case if we replace  $E$  by  $E_m$ . Since there holds  $C_{\frac{2}{q}, q'}(\widetilde{E_m \setminus E_{m-1}}) \leq cC_{\frac{2}{q}, q'}(E_m \setminus E_1)$ , we can choose an open set  $D_m^1$  such that  $C_{\frac{2}{q}, q'}(D_m^1) \leq \frac{c}{2^m}$ . In view of Lemma (2.7) the set

$$Q_1 = \bigcup_{m=1}^{\infty} D_m^1 \cap Q_1^m$$

is open and

$$E' \subset Q_1 \subset \widetilde{Q}_1 \subset D_1.$$

Furthermore the set

$$\bigcup_{m=1}^{\infty} \widetilde{D_m^1 \cap Q_1^m}$$

is  $\mathfrak{T}_q$ -quasi closed. By Lemma 2.7 there exists an open set  $D_m^2$  such that

$$D_m^2 \subset \widetilde{D}_m^2 \subset D_m^1.$$

By induction, we construct a sequence of open sets  $\{D_m^j\}$  such that

$$D_m^j \subset \widetilde{D}_m^j \subset D_m^{j-1} \quad C_{\frac{2}{q}, q'}(D_m^j) \leq \frac{c}{2^m}.$$

Thus in view of Lemma 2.7 the set

$$Q_j = \bigcup_{m=1}^{\infty} D_m^j \cap Q_j^m$$

is open and the set

$$\bigcup_{m=1}^{\infty} \widetilde{D_m^j \cap Q_j^m}$$

is  $\mathfrak{T}_q$ -quasi closed. For any  $m$  we have

$$D_m^j \cap Q_j^m \subset \widetilde{D_m^j \cap Q_j^m} \subset \widetilde{D}_m^j \cap \widetilde{Q}_j^m \subset D_m^{j-1} \cap Q_{j-1}^m.$$

Thus

$$Q_j \subset \widetilde{Q}_j \subset \bigcup_{m=1}^{\infty} \widetilde{D_m^j \cap Q_j^m} \subset \bigcup_{m=1}^{\infty} D_m^{j-1} \cap Q_{j-1}^m \subset D_j.$$

Since the set  $\bigcup_{m=1}^{\infty} \widetilde{D_m^j \cap Q_j^m}$  is  $\mathfrak{T}_q$ -quasi closed we have

$$Q_j \subset \widetilde{Q}_j \subset Q_{j-1}.$$

Finally

$$E' \subset Q_j \subset E'^{\frac{1}{2^j}},$$

thus  $E' = \bigcap Q_j$ . The result follows in this case since

$$C_{\frac{2}{q}, q'}(E) \leq \lim C_{\frac{2}{q}, q'}(Q_j) \leq \lim C_{\frac{2}{q}, q'}(D_j) = C_{\frac{2}{q}, q'}(E).$$

(II) The proof is same as in [23, Lemma 2.6 (b)] and we omit it.  $\square$

The next results are respectively proved in [23, Lemma 2.5] and [23, Lemma 2.7].

**Proposition 2.9** *Let  $E$  be a bounded  $\mathfrak{T}_q$ -open set and let  $\mathcal{D}$  be a cover of  $E$  consisting of  $\mathfrak{T}_q$ -open sets. Then, for every  $\varepsilon > 0$  there exists an open set  $O_\varepsilon$  such that  $C_{\frac{2}{q}, q'}(O_\varepsilon) < \varepsilon$  and  $E \setminus O_\varepsilon$  is covered by a finite subfamily of  $\mathcal{D}$ .*

**Proposition 2.10** *Let  $Q$  be a  $\mathfrak{T}_q$ -open set. Then, for every  $\xi \in Q$ , there exists a  $\mathfrak{T}_q$ -open set  $O_\xi$  such that*

$$\xi \in Q_\xi \subset \widetilde{Q}_\xi \subset Q. \quad (2.38)$$

### 3 Lattice structure of $\mathcal{U}_+(Q)$

Consider the equation

$$\partial_t u - \Delta u + |u|^{q-1}u = 0, \quad \text{in } Q_\infty = \mathbb{R}^N \times (0, T], \quad \text{where } q \geq 1 + \frac{2}{N}. \quad (3.1)$$

A function  $u \in L_{loc}^q(Q_T)$  is a subsolution (resp. supersolution) of the equation if  $\partial_t u - \Delta u + |u|^{q-1}u \leq 0$  (resp.  $\geq 0$ ) holds in the sense of distributions.

If  $u \in L_{loc}^q(Q_T)$  is a subsolution of the equation then by Kato's inequality  $(\partial_t - \Delta)|u| + |u|^q \leq 0$  in the sense of distributions. Thus  $|u|$  is a subsolution of the heat equation and consequently  $u \in L_{loc}^\infty(Q_T)$ . If  $u \in L_{loc}^q(Q_T)$  is a solution then  $u \in C^{2,1}(Q_T)$ .

**Proposition 3.1** *Let  $u$  be a non-negative function in  $L_{loc}^\infty(Q_T)$ .*

(i) *If  $u$  is a subsolution of (3.1), there exists a minimal solution  $v$  dominating  $u$ , i.e.  $u \leq v \leq U$  for any solution  $U \geq u$ .*

(ii) *If  $u$  is a supersolution of (3.1), there exists a maximal solution  $w$  dominated by  $u$ , i.e.  $V \leq w \leq u$  for any solution  $V \leq u$ .*

*All the above inequalities hold almost everywhere .*

*Proof.* (i) Let  $\{J_\varepsilon\}$  be a filter of mollifiers in  $\mathbb{R}^{N+1}$ . If  $u$  is extended by zero outside of  $Q_T$ , then the function  $u_\varepsilon = J_\varepsilon * u$  belong to  $C^\infty(\mathbb{R}^{N+1})$ ,  $\lim_{\varepsilon \rightarrow 0} u_\varepsilon = \tilde{u} = u$  a.e. in  $\mathbb{R}^{N+1}$  and  $u_\varepsilon \rightarrow u$  in  $L^q_{loc}(Q_T)$ . We note that we can choose  $\varepsilon > 0$  small enough such that the function  $u_\varepsilon$  is a subsolution in  $B_R(0) \times (s, \infty)$  where  $R > 0$  and  $0 < s$ . Let  $v_\varepsilon$  be the positive solution of

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v &= 0, & \text{in } B_R(0) \times (s, \infty), \\ v &= u_\varepsilon, & \text{on } \partial B_R(0) \times (s, \infty), \\ v(\cdot, s) &= u_\varepsilon(\cdot, s) & \text{in } B_R(0). \end{aligned} \quad (3.2)$$

In view of the proof of Lemma 2.4 and Remark 2.5 in [19] we can prove that  $v_\varepsilon \geq u_\varepsilon$ . Since  $v_\varepsilon$  is a subsolution of the heat equation, we have  $v_\varepsilon \leq \|u_\varepsilon\|_{L^\infty(B_R(0) \times (s, T])} \leq \|u\|_{L^\infty(B_R(0) \times (s, T])}$ . Thus there exists a decreasing sequence  $\varepsilon_j$  converging to 0 such that  $v_{\varepsilon_j} \rightarrow v$  in  $L^q(B_R(0) \times (s, T])$ ,  $u \leq v \leq \|u\|_{L^\infty(B_R(0) \times (s, T])}$ ;  $0 < s < T < \infty$  and  $v$  is a positive solution of

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v &= 0, & \text{in } B_R(0) \times (s, T], \\ v &= u, & \text{on } \partial B_R(0) \times (s, T], \\ v(\cdot, s) &= u(\cdot, s) & \text{in } B_R(0). \end{aligned} \quad (3.3)$$

Let  $\{R_j\}$  be an increasing sequence tending to infinity and  $s_j$  be a decreasing one converging to 0. Let  $v_j$  be the positive solution of the above problem with  $R = R_j$  and  $s = s_j$ . Since  $v_j \geq u$ , we have by the maximum principle that  $v_{j+1} \geq v_j$ . Thus, by Keller-Osserman inequality and standard parabolic regularity results, there exists a subsequence, say  $\{v_j\}$ , such that  $v_j \rightarrow v$  locally uniformly in  $Q_T$ . The results follows in this case by the construction of  $v$ .

(ii) Since  $u \in L^q(B_R(0) \times (s, T])$  there exists a solution  $w$  of the problem

$$\begin{aligned} \partial_t w - \Delta w + |u|^q &= 0, & \text{in } B_R(0) \times (s, T] \\ w &= 0, & \text{on } \partial B_R(0) \times (s, T] \\ w(\cdot, s) &= 0 & \text{in } B_R(0). \end{aligned} \quad (3.4)$$

Hence  $u+w$  is supersolution of the heat equation with boundary and initial data  $u$ . Consequently,  $u+w \geq z$  where  $z$  is the solution of the heat equation with boundary and initial data  $u$ . Also, the function  $z-w$  is a subsolution, thus there exists a solution  $v \leq u$  of the problem (3.3) with boundary and initial data  $u$ . As before, let  $\{R_j\}$  be an increasing sequence tending to infinity and  $s_j$  be a decreasing sequence tending to 0. Let  $v_j$  be the positive solution of the problem (3.3) with  $R = R_j$  and  $s = s_j$ . Since  $v_j \leq u$ , we have by maximum principle that  $v_{j+1} \leq v_j$ . Thus by standard parabolic arguments, there exists a subsequence, say  $\{v_j\}$ , such that  $v_j \rightarrow v$  locally uniformly in  $Q_\infty$ . Again, the construction of  $v$  implies the result.  $\square$

**Proposition 3.2** *Let  $u$  and  $v$  be nonnegative, locally bounded functions in  $Q_T$ .*

(i) *If  $u$  and  $v$  are subsolutions (resp. supersolutions) then  $\max(u, v)$  is a subsolution (resp.  $\min(u, v)$  is a supersolution).*

(ii) *If  $u$  and  $v$  are supersolutions then  $u+v$  is a supersolution.*

(iii) *If  $u$  is a subsolution and  $v$  is a supersolution then  $(u-v)_+$  is a subsolution.*

*Proof.* The first two statements are immediate consequence of the parabolic Kato's inequality. The third statement is verified in a similar way since

$$\left(\frac{d}{dt} - \Delta\right)(u-v)_+ \leq \text{sign}_+(u-v)\left(\frac{d}{dt} - \Delta\right)(u-v) \leq -\text{sign}_+(u-v)(u^q - v^q) \leq -(u-v)_+^q.$$



□

**Notation 3.3** Let  $u, v$  be nonnegative, locally bounded functions in  $Q_T$ .

- (a) If  $u$  is a subsolution,  $[u]_{\dagger}$  denotes the smallest solution dominating  $u$ .
- (b) If  $u$  is a supersolution,  $[u]^{\dagger}$  denotes the largest solution dominated by  $u$ .
- (c) If  $u, v$  are subsolutions then  $u \vee v := [\max(u, v)]_{\dagger}$ .
- (d) If  $u, v$  are supersolutions then  $u \wedge v := [\inf(u, v)]^{\dagger}$  and  $u \oplus v = [u + v]^{\dagger}$ .
- (e) If  $u$  is a subsolution and  $v$  is a supersolution then  $u \ominus v := [(u - v)_+]_{\dagger}$ .

**Proposition 3.4** (i) Let  $\{u_k\}$  be a sequence of positive, continuous subsolutions of (3.1). Then  $U := \sup u_k$  is a subsolution. The statement remains valid if subsolution is replaced by supersolution and  $\sup$  by  $\inf$ .

(ii) ([5]) Let  $\mathcal{T}$  be a family of positive solutions of (3.1). Suppose that, for every  $u_1$  and  $u_2$  belonging to  $\mathcal{T}$  there exists  $v \in \mathcal{T}$  such that

$$\max(u_1, u_2) \leq v, \quad \text{resp.} \quad \min(u_1, u_2) \geq v.$$

Then there exists a monotone sequence  $\{u_n\}$  in  $\mathcal{T}$  such that

$$u_n \uparrow \sup \mathcal{T}, \quad \text{resp.} \quad u_n \downarrow \inf \mathcal{T}.$$

Thus  $\sup \mathcal{T}$  (resp.  $\inf \mathcal{T}$ ) is a solution.

*Proof.* (i) Set  $v_j = \max(u_1, u_2, \dots, u_j) = \max(\max(u_1, u_2), \max(\max(u_1, u_2), u_3), \dots, \max(\max(\dots), u_j))$ . By proposition 3.2  $v_j$  is a subsolution and  $v_{j+1} \geq v_j$ . Thus the positive solution  $[v_j]_{\dagger}$  is increasing with respect to  $j$ . Also by Keller-Osserman inequality, we have that  $[v_j]_{\dagger} \rightarrow \tilde{v}$ , where  $\tilde{v}$  is a positive solution. Thus  $v_j \rightarrow v$  where  $v$  is a subsolution of (3.1). Now since  $u_i \leq v$  for each  $i \in \mathbb{N}$ , we have that  $U \leq v$ . But  $v_j \leq U$  for each  $j \in \mathbb{N}$ , which implies  $v \leq U$ . And thus  $v = U$ . The proof for "inf" is similar and we omit it.

(ii) The proof is similar as the one in [5]. Let  $A = (x_n, t_n)$  be a countable dense subset of  $Q_T$  and let  $u_{nm} \in \mathcal{T}$  satisfy the condition  $\sup_m u_m(x_n, t_n) = w(x_n, t_n)$ . Since  $\mathcal{T}$  is closed with respect to  $\vee$ , there exists an increasing sequence of  $v_n \in \mathcal{T}$  such that  $v = \lim_{n \rightarrow \infty} v_n$ , coincides with  $w$  on  $A$ . We claim that  $v = w$  everywhere. Indeed,  $v \leq u$ . Suppose  $u \in \mathcal{T}$ . Then  $u \leq w$  and therefore  $u \leq v$  on  $A$ . Since  $A$  is everywhere dense and  $u, v$  are continuous,  $u \leq v$  everywhere in  $Q_{\infty}$ , which implies  $u \geq w = \sup u$ . □

As a consequence we have the following result which extends to equation (1.1) what Dynkin proved for (1.14) [5, Theorem 5.1].

**Theorem 3.5** The set  $\mathcal{U}_+(Q_T)$  is a complete lattice stable for the laws  $\oplus$  and  $\ominus$ .

## 4 Partition of unity in Besov spaces

**Lemma 4.1** Let  $U \subset \mathbb{R}^N$  be a  $\mathfrak{I}_q$ -open set and  $z \in U$ . Then there exists a function  $f \in W^{\frac{2}{q}, q'}(\mathbb{R}^N)$  with compact support in  $U$  such that  $f(z) > 0$ . In particular, there exists a bounded  $\mathfrak{I}_q$ -open set  $V$  such that  $\bar{V} \subset U$ .

*Proof.* We suppose that  $z$  is not an interior point of  $U$  with respect to Euclidean topology, since otherwise the result is obvious. Since  $U$  is  $\mathfrak{T}_q$ -open we have that  $U^c$  is thin at  $z$ . Also by the assumption on  $z$ , we have that  $z \in \overline{U^c} \setminus U$ . By [2, p. 174], we can find an open set  $W \supset U^c$ ,  $z \in \overline{W} \setminus W$  and  $W$  is thin at  $z$ .

We recall that for a set  $E$  with positive  $C_{\frac{2}{q}, q'}$ -capacity,  $F^E := \mathcal{V}^{\mu_E} = G_{\frac{1}{q}} * (G_{\frac{1}{q}} * \mu_E)^{p-1}$  where  $\mu_E$  is the capacitary measure on  $E$ . Then, by [2, Proposition 6.3.14], there exists  $r > 0$  small enough such that

$$\mathcal{V}^\mu(z) < \frac{1}{2},$$

where  $\mu$  is the capacitary measure of  $B(z, r) \cap W$  and  $\mathcal{V}^\mu$  the corresponding Besov potential (see [2, Theorems 2.2.7, 2.5.6]). By [2, Theorem 6.3.9],  $\mathcal{V}^\mu \geq 1$  quasi everywhere (abr. q.a.e.) on  $B(z, r) \cap W$ , and by [2, Proposition 2.6.7]  $\mathcal{V}^\mu \geq 1$  everywhere on  $B(z, r) \cap W$ . Thus

$$\mathcal{V}^\mu(z) < \frac{1}{2} < 1 \leq \mathcal{V}^\mu(x), \quad \forall x \in B(z, r) \cap W.$$

Thus we can find  $r_0 > 0$  small enough such that

$$\mathcal{V}^\mu(z) < \frac{1}{2} < 1 \leq \inf\{\mathcal{V}^\mu(x) : x \in B(z, r_0) \setminus U\}.$$

Now let  $0 \leq H(t)$  be a smooth nondecreasing function such that  $H(t) = t$  for  $t \geq \frac{1}{4}$  and  $H(t) = 0$  for  $t \leq 0$ . Also let  $\eta \in C_0^\infty(\mathbb{R}^N)$  such that  $0 \leq \eta \leq 1$ ,  $\text{supp } \eta \subset B(z, r_0)$  and  $\eta(z) = 1$ . Then the function

$$f(z) = \eta H(1 - \mathcal{V}^\mu),$$

belongs to  $W^{\frac{2}{q}, q'}(\mathbb{R}^N)$ . Since by definition  $\mathcal{V}^\mu$  is lower semicontinuous, the set  $\{1 - u \geq 0\}$  is closed. Hence the support of  $f$  is compact and

$$\text{supp } f \subset \text{supp } \eta \cap \{1 - u \geq 0\} \subset U.$$

□

**Lemma 4.2** *Let  $U$  be a  $\mathfrak{T}_q$ -open set and  $z \in U$ . Then there exists a  $\mathfrak{T}_q$ -open set  $V$ , such that  $z \in V \subset U$ , and a function  $\psi \in W^{\frac{2}{q}, q'}(\mathbb{R}^N)$  such that  $\psi = 1$  q.a.e. on  $V$  and  $\psi = 0$  outside  $U$ .*

*Proof.* As before, we assume that  $z$  is not an interior point of  $U$ . Let  $\mathcal{V}^\mu$  be the Besov potential of the previous lemma, with

$$\mathcal{V}^\mu(z) < \frac{1}{4}, \quad \mathcal{V}^\mu = 1 \quad \text{on } B(z, r_0) \setminus U.$$

By [2, Proposition 6.3.10]  $\mathcal{V}^\mu$  is quasi continuous, that we can find a  $\mathfrak{T}_q$ -open set  $W$  which contains  $z$  such that

$$\mathcal{V}^\mu(x) \leq \frac{1}{4}, \quad \text{q.a.e. on } W.$$

Let  $\eta \in C_0^\infty(\mathbb{R}^N)$  such that  $0 \leq \eta \leq 1$ ,  $\text{supp } \eta \subset B(z, r_0)$  and  $\eta(x) = 1, \forall x \in B(z, \frac{r_0}{2})$ . Set

$$f = 2\eta H\left(1 - H\left(\frac{1}{2} - \mathcal{V}^\mu(x)\right) - \mathcal{V}^\mu(x)\right).$$

Then  $f \in W^{\frac{2}{q}, q'}(\mathbb{R}^N)$ ,  $0 \leq f \leq 1$  and  $f = 0$  on  $B(z, r_0) \setminus U$ . Also,  $f = 1$  on  $B(z, \frac{r_0}{2}) \cap W$  and  $f = 0$  outside of  $B(z, r_0) \cap U$ .  $\square$

**Lemma 4.3** *Let  $\frac{2}{q} \leq 1$ ,  $K$  be a compact set and  $U$  be a  $\mathfrak{T}_q$ -open set such that  $K \subset U$ . Also, let  $\{U_j\}$  be a sequence of  $\mathfrak{T}_q$ -open subsets of  $U$  covering  $U$  up to a set of zero  $C_{\frac{2}{q}, q'}$ -capacity  $Z$ .*

*We assume that there exists a nonnegative  $u \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -supp  $u \subset K \subset U$ . Then there exist  $m(k) \in \mathbb{N}$  and nonnegative functions  $u_{k,j} \in L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -supp  $u_{k,j} \subset U_j$ , such that*

$$\sum_{j=1}^{m(k)} u_{k,j} \leq u \quad (4.1)$$

and

$$\lim_{k \rightarrow \infty} \|u - \sum_{j=1}^{m(k)} u_{k,j}\|_{W^{\frac{2}{q}, q'}(\mathbb{R}^N)} = 0.$$

*Remark.* If  $u$  changes sign, the conclusion of Lemma remains valid without inequality (4.1).

*Proof.* Without loss of generality we can assume that  $U$  and the  $\cup_j U_j$  are bounded. For any  $j \geq 0$ , there exists open sets  $G_{k,j}$  such that  $C_{\frac{2}{q}, q'}(G_{k,j}) \leq 2^{-k-j}$ ,  $Z \subset G_{k,0}$  and for  $j \geq 1$ , the sets  $U_j \cup G_{k,j}$  are open. Also the sets

$$G_k = \bigcup_{j=0}^{\infty} G_{k,j}, \quad \bigcup_{j=1}^{\infty} G_k \cup U_j$$

are open and  $C_{\frac{2}{q}, q'}(G_k) \rightarrow 0$  when  $k \rightarrow \infty$ .

Since  $G_k$  is open, its Besov potential  $F^{G_k}$  is larger or equal to 1 everywhere on  $G_k$  [2, Theorems 2.5.6, 2.6.7]). Also we have

$$\|\mathcal{V}^{\mu_k}\|_{W^{\frac{2}{q}, q'}(\mathbb{R}^N)}^{q'} \leq AC_{\frac{2}{q}, q'}(G_k),$$

where  $A$  is a positive constant which depends only on  $n, q$ . Now consider a smooth nondecreasing function  $H$  such that  $H(t) = 1$  for  $t \geq 1$  and  $H(t) = t$  for  $t \leq \frac{1}{2}$ , then the function  $\phi_k = H(\mathcal{V}^{\mu_k})$  belongs to  $W^{\frac{2}{q}, q'}(\mathbb{R}^N)$ , satisfies  $0 \leq \phi_k \leq 1$ ,  $\phi_k = 1$  on  $G_k$  and there exists a constant  $A'(n, q) > 0$  such that

$$\|\phi_k\|_{W^{\frac{2}{q}, q'}(\mathbb{R}^N)}^{q'} \leq A'C_{\frac{2}{q}, q'}(G_k).$$

Set  $\psi_k = 1 - \phi_k$ . By Lebesgue's dominated theorem

$$\|u - \psi_k u\|_{W^{\frac{2}{q}, q'}(\mathbb{R}^N)}^{q'} \rightarrow 0. \quad (4.2)$$

Thus it is enough to prove that

$$u\psi_k = \sum_{j=1}^{m(k)} u_{k,j}. \quad (4.3)$$

Fix  $k \in \mathbb{N}$ . Then there exist open balls  $B_{k,j,i}$ , for  $i, j = 1, 2, \dots$ , such that

$$\overline{B}_{k,j,i} \subset U_j \cup G_k, \quad \text{and} \quad \bigcup_{j=1}^{\infty} G_k \cup U_j = \bigcup_{i,j=1}^{\infty} B_{k,j,i}.$$

Since  $K$  is compact, there exists  $m(k) \in \mathbb{N}$  such that

$$K \subset \bigcup_{i,j=1}^{m(k)} B_{k,j,i}.$$

Now consider  $w_{k,j,i} \in C_0^\infty(\mathbb{R}^N)$  such that

$$\{w_{k,j,i} > 0\} = B_{k,j,i}.$$

If we set

$$u_{k,j} = w_{k,j} \frac{\sum_{i=1}^{m(k)} w_{k,j,i}}{\sum_{i,j=1}^{m(k)} w_{k,j,i}},$$

then  $u_{k,j} \in L^\infty(\mathbb{R}^N)$ , satisfies 1 and

$$\mathfrak{T}_q\text{-supp} u_{k,j} \subset (K \setminus G_k) \cap B_{k,j,i} \subset U_j.$$

□

*Remark.* We conjecture that the result still holds if  $\frac{2}{q} > 1$ , but we have not been able to prove (4.2).

## 5 The regular set and its properties

Let  $q > 1$ ,  $T > 0$ . If  $Q_T = \mathbb{R}^N \times (0, T)$ , we recall that  $\mathcal{U}_+(Q_T)$  is the set of positive solutions  $u$  of

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T. \quad (5.1)$$

If a function  $\zeta$  is defined in  $\mathbb{R}^N$ . We denote by  $\mathfrak{T}_q\text{-supp}(\zeta)$  the  $\mathfrak{T}_q$ -closure of the set where  $|\zeta| > 0$ . Let  $U$  be a Borel subset of  $\mathbb{R}^N$  and  $\chi_U$  be the characteristic function of  $U$ . We set

$$\mathbb{H}(\chi_U)(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \chi_U dy.$$

For any  $\xi \in \mathbb{R}^N$  the following dichotomy occurs:

(i) either there exists a  $\mathfrak{T}_q$ -open bounded neighborhood  $U = U_\xi$  of  $\xi$  such that

$$\int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dx dt < \infty, \quad (5.2)$$

where  $q' = \frac{q}{q-1}$ ,

(ii) or for any  $\mathfrak{T}_q$ -open neighborhood  $U$  of  $\xi$

$$\int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dx dt = \infty. \quad (5.3)$$

**Definition 5.1** The set of  $\xi \in \mathbb{R}^N$  such that (i) occurs is  $\mathfrak{T}_q$ -open. It is denoted by  $\mathcal{R}_q(u)$  and called the regular set of  $u$ . Its complement  $\mathcal{S}_q(u) = \mathbb{R}^N \setminus \mathcal{R}_q(u)$  is  $\mathfrak{T}_q$ -closed and called the singular set of  $u$ .

**Proposition 5.2** Let  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in a  $\mathfrak{T}_q$ -open bounded set  $U$ . Also let  $u \in \mathcal{U}_+(Q_T)$  satisfy

$$M_U = \int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dx dt < \infty.$$

Then there exists

$$l(\eta) := \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \mathbb{H}[\eta]_+^{2q'} dx. \quad (5.4)$$

Furthermore

$$|l(\eta)| \leq C(M_U, q) \left( \|\eta\|_{W^{\frac{2}{q}, q'}}^{2q'} + \|\eta\|_{L^\infty(\mathbb{R}^N)}^{2q'} \right). \quad (5.5)$$

*Proof.* Put  $h = \mathbb{H}[\eta]$  and  $\phi(r) = r_+^{2q'}$ . Since  $|\eta| \leq \|\eta\|_{L^\infty} \chi_U$ , there holds

$$\left| \int_0^T \int_{\mathbb{R}^N} u^q \phi(h) dx dt \right| \leq \|\eta\|_{L^\infty}^{2q'} \int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dx dt := \|\eta\|_{L^\infty}^{2q'} M_U < \infty. \quad (5.6)$$

Moreover

$$\int_s^t \int_{\mathbb{R}^N} (-u(\partial_t \phi(h) + \Delta \phi(h))) + u^q \phi(h) dx d\tau = \int_{\mathbb{R}^N} u \phi(h)(\cdot, s) dx - \int_{\mathbb{R}^N} u \phi(h)(\cdot, t) dx. \quad (5.7)$$

But

$$\partial_t \phi(h) + \Delta \phi(h) = 2q' \phi(h) h_+^{-2} (2h_+ \partial_t h + (2q' - 1) |\nabla h|^2).$$

By Hölder

$$\begin{aligned} & \left| \int_s^t \int_{\mathbb{R}^N} u(\partial_t \phi(h) + \Delta \phi(h)) dx d\tau \right| \\ & \leq \left( \int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau \right)^{\frac{1}{q}} \left( \int_s^t \int_{\mathbb{R}^N} \phi(h)^{-\frac{q'}{q}} |(\partial_t \phi(h) + \Delta \phi(h))|^{q'} dx d\tau \right)^{\frac{1}{q'}} \\ & \leq 4q' \left( \int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau \right)^{\frac{1}{q}} \left( \int_s^t \int_{\mathbb{R}^N} (h_+ |\partial_t h| + |\nabla h|^2)^{q'} dx d\tau \right)^{\frac{1}{q'}}. \end{aligned}$$

By standard regularity properties of the heat kernel

$$\int_s^t \int_{\mathbb{R}^N} |\partial_t h|^{q'} dx d\tau \leq \int_0^T \int_{\mathbb{R}^N} |\partial_t h|^{q'} dx d\tau \leq \|\eta\|_{W^{\frac{2}{q}, q'}}^{q'},$$

and by Gagliardo-Nirenberg inequality and the maximum principle

$$\int_s^t \int_{\mathbb{R}^N} |\nabla h|^{2q'} dx d\tau \leq \int_0^T \int_{\mathbb{R}^N} |\nabla h|^{2q'} dx d\tau \leq C \|\eta\|_{L^\infty}^{q'} \|\Delta h\|_{L^{q'}}^{q'} = C \|\eta\|_{L^\infty}^{q'} \|\partial_t h\|_{L^{q'}}^{q'}.$$

Therefore,

$$\left| \int_s^t \int_{\mathbb{R}^N} u(\partial_t \phi(h) + \Delta \phi(h)) dx d\tau \right| \leq C \left( \int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau \right)^{\frac{1}{q}} \|\eta\|_{L^\infty} \|\eta\|_{W^{\frac{2}{q}, q'}}. \quad (5.8)$$

This implies that the left-hand side of (5.7) tends to 0 when  $s, t \rightarrow 0$ , thus there exists

$$l(\eta) := \lim_{s \rightarrow 0} \int_{\mathbb{R}^N} u \phi(h)(x, s) dx.$$

From (5.7) it follows

$$\int_0^T \int_{\mathbb{R}^N} (-u(\partial_t \phi(h) + \Delta \phi(h))) + u^q \phi(h) dx d\tau + \int_{\mathbb{R}^N} u \phi(h)(\cdot, T) dx = l(\eta). \quad (5.9)$$

Since  $|u \phi(h)(\cdot, T)| \leq C(T) \|\eta\|_{L^\infty}^{2q'}$ , we derive

$$|l(\eta)| \leq C_1 \|\eta\|_{L^\infty}^{2q'} + C \|\eta\|_{L^\infty}^{q'} \|\eta\|_{W^{\frac{2}{q}, q'}}^{q'} \leq C \left( \|\eta\|_{L^\infty} + \|\eta\|_{W^{\frac{2}{q}, q'}} \right)^{2q'}. \quad (5.10)$$

**Proposition 5.3** *Let the assumptions of Lemma 5.2 be satisfied. Then*

$$\lim_{t \rightarrow 0} \int_U u(x, t) \eta_+^{2q'}(x) dx = l(\eta). \quad (5.11)$$

*Proof.* Using (5.6) with  $h$  replaced by  $h_s(x, t) := \mathbb{H}[\eta](x, t - s)$ , we get

$$\int_s^T \int_{\mathbb{R}^N} (-u(\partial_t \phi(h_s) + \Delta \phi(h_s))) + u^q \phi(h_s) dx d\tau + \int_{\mathbb{R}^N} u \phi(h_s)(\cdot, T) dx = \int_{\mathbb{R}^N} u \phi(h_s)(\cdot, s) dx. \quad (5.12)$$

When  $s \rightarrow 0$

$$\int_{\mathbb{R}^N} u \phi(h_s)(\cdot, T) dx \rightarrow \int_{\mathbb{R}^N} u \phi(h)(\cdot, T) dx,$$

and

$$\int_s^T \int_{\mathbb{R}^N} u^q \phi(h_s) dx d\tau \rightarrow \int_0^T \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau,$$

by the dominated convergence theorem. Furthermore,

$$\begin{aligned} & \left| \int_0^{T-s} \int_{\mathbb{R}^N} (u(x, t+s) - u(x, t)) (\partial_t \phi(h) + \Delta \phi(h)) dx dt \right| \\ & \leq C \left( \int_0^{T-s} \int_{\mathbb{R}^N} |u(x, t+s) - u(x, t)|^q h_+^{2q'} dx dt \right)^{\frac{1}{q}} \|\eta\|_{L^\infty}^{q'} \|\eta\|_{W^{\frac{2}{q}, q'}}^{q'}, \end{aligned}$$

which tends to zero with  $s$ . Finally,

$$\lim_{s \rightarrow 0} \int_{T-s}^T \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau = 0.$$

Subtracting (5.7) to (5.12), we derive

$$\lim_{s \rightarrow 0} \int_{\mathbb{R}^N} u(\cdot, s)(\phi(h)(\cdot, s) - \phi(\eta)) dx = 0,$$

which implies the claim.  $\square$

The next statement obtained by contradiction with the use of Lemma 5.2 and Lemma 5.3 will be very useful in the sequel.

**Proposition 5.4** *Assume that  $U$  is a bounded  $\mathfrak{T}_q$ -open set and*

$$\lim_{t \rightarrow 0} \int_U u(x, t) \eta^{2q'}(x) dx = \infty, \quad (5.13)$$

for some  $0 \leq \eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in  $U$ , then

$$\int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\eta]^{2q'} dx dt = \infty. \quad (5.14)$$

**Proposition 5.5** *Let  $\xi \in \mathcal{S}_q(u)$ . Then for any  $\mathfrak{T}_q$ -open set  $G$  which contains  $\xi$ , there holds*

$$\lim_{t \rightarrow 0} \int_G u(x, t) dx = \infty. \quad (5.15)$$

*Proof.* If  $\xi \in \mathcal{S}_q(u)$  and if  $G$  is  $\mathfrak{T}_q$ -open and contains  $\xi$ , then by Lemma 4.2 there exist  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and a  $\mathfrak{T}_q$ -open set  $D \subset G$  such that  $\eta = 1$  on  $D$ ,  $\eta = 0$  outside of  $G$  and  $0 \leq \eta \leq 1$ . Thus

$$\infty = \int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_D]^{2q'} dx dt \leq \int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\eta]^{2q'} dx dt,$$

Therefore

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u \mathbb{H}[\eta]^{2q'} dx = \infty,$$

which implies

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u \eta^{2q'} dx = \infty,$$

and the result follows by the properties of  $\eta$ .  $\square$

## 5.1 Moderate solutions

We first recall some classical results concerning initial value problem with initial measure data. A solution  $u$  of (3.1) is called *moderate* if  $u \in L^q(K)$  for any compact  $K \subset \overline{Q_\infty}$ . Then there exists a unique Radon measure  $\mu$  such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu \quad \forall \zeta \in C_0^\infty(\mathbb{R}^N). \quad (5.16)$$

Equivalently

$$- \int \int_{Q_\infty} u(\phi_t + \Delta \phi) dx dt + \int \int_{Q_\infty} |u|^{q-1} u \phi dx dt = \int_{\mathbb{R}^N} \phi(x, 0) d\mu,$$

for all  $\phi \in C^{1,1;1}(\overline{Q_\infty})$ , with compact support.

The above measure has the property that it vanishes on Borel sets with  $C_{\frac{2}{q}, q}$ -capacity zero.

There exists an sequence  $\{\mu_n\} \subset W^{-\frac{2}{q}, q}(\mathbb{R}^N)$  of Radon measures such that  $\mu_n \rightharpoonup \mu$  in the weak\* topology. If we assume that  $u$  is a positive moderate solution, or equivalently that the initial measure  $\mu$  is positive, then the previous sequence can be constructed as being increasing and particularly  $\{\mu_n\} \subset W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ , where  $\mathfrak{M}_+^b(\mathbb{R}^N)$  is the set of all positive bounded Radon measures in  $\mathbb{R}^N$ .

If  $\nu \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ , then we have for some constant  $C > 0$  independent on  $\nu$  (see Lemma 3.2-[22])

$$C^{-1} \|\nu\|_{W^{-\frac{2}{q}, q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\nu]\|_{L^q(Q_T)} \leq C \|\nu\|_{W^{-\frac{2}{q}, q}(\mathbb{R}^N)}, \quad (5.17)$$

where we recall that  $\mathbb{H}[\nu]$  denotes the heat potential of  $\nu$  in  $Q$ .

**Lemma 5.6** *Let  $u$  be a moderate positive solution with initial data  $\mu$ . Then for any  $T > 0$  and bounded  $\mathfrak{T}_q$ -open set we have*

$$\int_0^T \int_{\mathbb{R}^N} u^q(t, x) \mathbb{H}^{2q'}[\chi_O] dx dt < \infty.$$

*Proof.* Let  $0 \leq \eta \in C_0^\infty(\mathbb{R}^N)$  and  $\eta = 1$  on  $O$  and  $s < T$ . We define here  $h = \mathbb{H}[\eta](x, t)$ ,  $h_s = \mathbb{H}[\eta](x, t - s)$  and  $\phi(r) = |r|^{2q'}$ . Then we have

$$\int_s^T \int_{\mathbb{R}^N} u(x, t) (\partial_t \phi(h_s) + \Delta \phi(h_s)) + |u|^q \phi(h_s) dx dt + \int_{\mathbb{R}^N} u \phi(h_s)(\cdot, T) dx = \int_{\mathbb{R}^N} u(x, s) \phi(\eta) dx.$$

In view of Proposition 5.2, (5.8) and Hölder's inequality, there exists a constant  $c = c(q, N)$  such that

$$\int_s^T \int_{\mathbb{R}^N} |u|^q \phi(h_s) dx dt + \int_{\mathbb{R}^N} u \phi(h_s)(\cdot, T) dx \leq c \left( \int_{\mathbb{R}^N} u(x, s) \phi(\eta) dx + \|\eta\|_{L^\infty}^{2q'} \|\eta\|_{W^{\frac{2}{q}, q'}}^{2q'} \right).$$

Using Fatou's lemma and the fact that, for any bounded Borel set  $E$

$$\limsup_{s \rightarrow 0} \int_E u(x, s) dx < \infty,$$

we conclude the proof.  $\square$



**Theorem 5.7** *Let  $u$  be a positive moderate solution with  $\mu$  as initial data, then*

(i)  $\mu$  is regular relative to the  $\mathfrak{T}_q$ -topology.

(ii) For each quasi continuous function  $\phi \in L^\infty(\mathbb{R}^N)$  with bounded  $\mathfrak{T}_q$ -support in  $\mathbb{R}^N$ , we have

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \phi(x) dx = \int_{\mathbb{R}^N} \phi(x) d\mu.$$

*Proof.* The proof is similar to the one given [23].

(i) Every Radon measure on  $\mathbb{R}^N$  is regular in the usual Euclidean topology, i.e.

$$\begin{aligned} \mu(E) &= \inf\{\mu(D) : E \subset D, D \text{ open}\} \\ &= \inf\{\mu(K) : K \subset E, K \text{ compact}\}, \end{aligned}$$

for any Borel set  $E$ . But if  $D$  is open and contains  $E$ , it is  $\mathfrak{T}_q$ -open, hence

$$\mu(E) \leq \inf\{\mu(D) : E \subset D, D \text{ } \mathfrak{T}_q\text{-open}\} \leq \inf\{\mu(D) : E \subset D, D \text{ open}\} = \mu(E),$$

and the result follows.

(ii) Since the measure  $\mu_t = u(t, x)dx \rightarrow \mu$  in the weak\* topology we have

$$\limsup_{t \rightarrow 0} \mu_t(E) \leq \mu(E), \quad \liminf_{t \rightarrow 0} \mu_t(A) \geq \mu(A),$$

for any compact set  $E$ , respectively, open set  $A$ . This extends to any bounded  $\mathfrak{T}_q$ -closed set  $E$  (resp.  $\mathfrak{T}_q$ -open set  $A$ ).

Indeed, let  $E$  be a  $\mathfrak{T}_q$ -closed set and  $\{K_n\}$  be an increasing sequence of closed sets such that  $C_{\frac{2}{q}, q'}(E \setminus K_n) \rightarrow 0$ . Then for any  $m \in \mathbb{N}$  and any open set  $E \subset O$  we have

$$\limsup_{t \rightarrow 0} \mu_t(E) \leq \limsup_{t \rightarrow 0} \mu_t(K_m) + \limsup_{t \rightarrow 0} \mu_t(E \setminus K_m) \leq \mu(O) + \limsup_{t \rightarrow 0} \mu_t(E \setminus K_m).$$

Now we assert that

$$\lim_{m \rightarrow \infty} \limsup_{t \rightarrow 0} \mu_t(E \setminus K_m) = 0.$$

We will prove it by contradiction. We assume that  $\lim_{m \rightarrow \infty} \limsup_{t \rightarrow 0} \mu_t(E \setminus K_m) = \varepsilon > 0$ .

Let  $\{t_n\}$  be a decreasing sequence tending to 0 and  $\lim_{n \rightarrow \infty} \mu_{t_n}(E \setminus K_m) = \limsup_{t \rightarrow 0} \mu_t(E \setminus K_m)$ . Then there exists subsequence of positive solutions  $\{u_k^m\}_{k=1}^\infty$  with initial data  $\mu_{t_{n_k}} \chi_{E \setminus K_m}$  such that  $u_k^m \rightarrow u^m$  for any  $m \in \mathbb{N}$ . Since  $u$  is a moderate solution and  $u_k^m \leq u$ ,  $u^m$  is a moderate solution too. Also by construction, the sequence  $\{u^m\}$  is nonincreasing and  $u_m \leq U_{E \setminus K_m}$ . By proposition 5.17 we have  $U_{E \setminus K_m} \rightarrow 0$  which implies  $u_m \rightarrow 0$  and

$$\lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \mu_{t_{n_k}}(E \setminus K_m) = 0.$$

The proof follows in the case where  $E$  is  $\mathfrak{T}_q$ -closed. The proof is similar in the other case.

If  $A$  is  $\mathfrak{T}_q$ -open and

$$\mu(A) = \mu(\tilde{A}),$$

then

$$\lim_{t \rightarrow 0} \mu_t(A) = \mu(A).$$

Without loss of generality we may assume that  $\phi \geq 0$  (since otherwise we set  $\phi = \phi^+ - \phi^-$ ) and  $\phi \leq 1$ . Given  $k \in \mathbb{N}$  and  $m = 0, \dots, 2^k - 1$  choose a number  $a_{m,k}$  in the interval  $(m2^{-k}, (m+1)2^{-k})$  such that  $\mu(\phi^{-1}(\{a_{m,k}\})) = 0$ . Put

$$A_{m,k} = \phi^{-1}((a_{m,k}, (a_{m+1,k}]), \quad m = 1, \dots, 2^k - 1, \quad A_{0,k} = \phi^{-1}((a_{0,k}, (a_{1,k}]),$$

then we note that since  $\phi$  has compact support the above sets are bounded and

$$\lim_{t \rightarrow 0} \mu_t(A_{m,k}) = \mu(A_{m,k}), \quad \forall m \geq 0, k \in \mathbb{N}. \quad (5.18)$$

Consider the step function  $\phi_k = \sum_{\mu=0}^{2^k-1} m2^{-k} \chi_{A_{m,k}}$ , then  $\phi_k \uparrow \phi$  uniformly, and by (5.18),

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \phi_k dx = \int_{\mathbb{R}^N} \phi_k d\mu, \quad \forall \zeta \in C_0^\infty(\mathbb{R}^N).$$

This completes the proof of (ii).  $\square$

## 5.2 Vanishing properties

**Definition 5.8** A continuous function  $u \in \mathcal{U}_+(Q_T)$  vanishes on a  $\mathfrak{T}_q$ -open subset  $G \subset \mathbb{R}^N$ , if for any  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q\text{-supp}(\eta) \subset^q G$ , there holds

$$\lim_{t \rightarrow 0} \int_G u(x, t) \eta_+^{2q'}(x) dx dt = 0. \quad (5.19)$$

When this is case we write  $u \approx_G 0$ . We denote by  $\mathcal{U}_G(Q_T)$  the set of  $u \in \mathcal{U}_+(Q_T)$  which vanish on  $G$ .

We have the following simple result.

**Proposition 5.9** Let  $A$  be a  $\mathfrak{T}_q$ -open subset of  $\mathbb{R}^N$  and  $u_1, u_2 \in \mathcal{U}_+(Q_T)$ . If  $u_2 \approx_A 0$  and  $u_1 \leq u_2$  then  $u_1 \approx_A 0$ .

**Proposition 5.10** Let  $G, G'$  be  $\mathfrak{T}_q$ -open sets such that  $G \sim^q G'$ . If  $u \in \mathcal{U}_G(Q_T)$  then  $u \in \mathcal{U}_{G'}(Q_T)$

*Proof.* If  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q\text{-supp}(\zeta) \subset^q G$ , then  $\mathfrak{T}_q\text{-supp}(\zeta) \subset^q G'$ . Since  $|G \setminus G'| = |G' \setminus G| = 0$  the result follows.  $\square$

If  $G$  is an open subset, this notion coincides with the usual definition of vanishing, since we can take test function  $\eta \in C_0^\infty(G)$ . In that case  $u \in C(Q_T \cup \{G \times \{0\}\})$ .

**Lemma 5.11** Assume  $u \in \mathcal{U}_G(Q_T)$ . Then for any  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q\text{-supp}(\eta) \subset^q G$ , there holds

$$\int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\eta]_+^{2q'} dx dt + \int_{\mathbb{R}^N} u(x, T) \mathbb{H}[\eta]_+^{2q'}(x, T) dx \leq C_1 \|\eta\|_{L^\infty}^{q'} \|\eta\|_{W^{\frac{2}{q}, q'}}^{q'}. \quad (5.20)$$

*Proof.* If  $u \in \mathcal{U}_G(Q_T)$  and  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q\text{-supp}(\eta) \subset^q G$ , there holds, with  $h = \mathbb{H}[\eta]$  and  $\phi(r) = r_+^{2q'}$ .

$$\int_0^T \int_{\mathbb{R}^N} (-u(\partial_t \phi(h) + \Delta \phi(h))) + u^q \phi(h) dx d\tau + \int_{\mathbb{R}^N} u \phi(h)(\cdot, T) dx = 0. \quad (5.21)$$

Therefore (5.20) follows from (5.8).  $\square$

**Lemma 5.12** *Let  $G \subset \mathbb{R}^N$  be a  $\mathfrak{T}_q$ -open set. Then there exists a nondecreasing sequence  $\{u_n\} \subset \mathcal{U}_G(Q_T)$  which converges to  $\sup \mathcal{U}_G(Q_T)$ . Furthermore  $\sup \mathcal{U}_G(Q_T) \in \mathcal{U}_G(Q_T)$ .*

*Proof.* If  $u_1$  and  $u_2$  belongs to  $\mathcal{U}_G(Q_T)$ , then  $u_1 + u_2$  is a supersolution and it satisfies (5.19). Therefore  $u_1 \vee u_2$  is a solution which is smaller than  $u_1 + u_2$ , thus  $u_1 \vee u_2 \in \mathcal{U}_G(Q_T)$ . By Proposition 3.4 there exists a increasing sequence  $\{u_n\} \subset \mathcal{U}_G(Q_T)$  which converges to  $u := \sup \mathcal{U}_G(Q_T)$ . By (5.21),

$$\int_0^T \int_{\mathbb{R}^N} (-u_n(\partial_t \phi(h) + \Delta \phi(h))) + u_n^q \phi(h) dx d\tau + \int_{\mathbb{R}^N} u_n \phi(h)(\cdot, T) dx = 0. \quad (5.22)$$

Now,  $u_n^q \phi(h) \uparrow u^q \phi(h)$  in  $L^1(Q_T)$  and  $u_n \phi(h)(\cdot, T) \uparrow u \phi(h)(\cdot, T)$  in  $L^1(\mathbb{R}^N)$ . If  $E$  is any Borel subset of  $Q_T$ , there holds by Hölder's inequality, as in (5.8)

$$\left| \int_0^T \int_E u_n(\partial_t \phi(h) + \Delta \phi(h)) dx d\tau \right| \leq C \left( \int_0^T \int_E u_n^q \phi(h) dx d\tau \right)^{\frac{1}{q}} \|\eta\|_{L^\infty} \|\eta\|_{W^{\frac{2}{q}, q'}}. \quad (5.23)$$

The right-hand side tends to zero when  $|E| \rightarrow 0$ , thus by Vitali's convergence theorem, we derive

$$\int_0^T \int_{\mathbb{R}^N} (-u(\partial_t \phi(h) + \Delta \phi(h))) + u^q \phi(h) dx d\tau + \int_{\mathbb{R}^N} u \phi(h)(\cdot, T) dx = 0, \quad (5.24)$$

from (5.22). Thus  $u \in \mathcal{U}_G(Q_T)$ .  $\square$

**Definition 5.13** (a) *Let  $u \in \mathcal{U}_+(Q_T)$  and let  $A$  denote the union of all  $\mathfrak{T}_q$ -open sets on which  $u$  vanishes. Then  $A^c$  is called the fine initial support of  $u$ , to be denoted by  $\mathfrak{T}_q\text{-supp}(u)$ .*

(b) *Let  $F$  be a Borel subset of  $\mathbb{R}^N$ . We denote by  $U_F$  the maximal element of  $\mathcal{U}_{\tilde{F}^c}(Q_T)$ .*

### 5.3 Maximal solutions

**Definition 5.14** *Let  $\mathfrak{M}_+^b(\mathbb{R}^N)$  be the set of all positive bounded Radon measures in  $\mathbb{R}^N$ . Also let  $u_\mu \in \mathcal{U}_+(Q_T)$  be the moderate solution with initial data  $\mu$ .*

*For any Borel set  $E \subset \mathbb{R}^N$  of positive  $C_{\frac{2}{q}, q'}$ -capacity put*

$$\mathcal{V}_{mod}(E) = \{u_\mu : \mu \in W^{-\frac{2}{q}, q'}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \mu(E^c) = 0\}.$$

$$V_E = \sup \mathcal{V}_{mod}(E).$$

The following result due to Marcus and Véron [22] shows that the maximal solution which vanishes on an open set is indeed  $\sigma$ -moderate. This is obtained by proving a capacity quasi-representation of the solution via a Wiener type test.

**Proposition 5.15** *Let  $F$  be a closed subset of  $\mathbb{R}^N$  and  $q \geq 1 + \frac{2}{N}$ . Then there exist two positive constants  $C_1, C_2 > 0$ , depending only on  $N$  and  $q$  such that*

$$\begin{aligned} C_1 t^{-\frac{1}{q-1}} \sum_{k=0}^{\infty} (k+1)^{\frac{N}{2}} e^{-\frac{k}{4}} C_{\frac{2}{q}, q'} \left( \frac{F \cap F_k(x, t)}{\sqrt{(k+1)t}} \right) &\leq V_F(x, t) \leq U_F(x, t) \\ &\leq C_2 t^{-\frac{1}{q-1}} \sum_{k=0}^{\infty} (k+1)^{\frac{N}{2}} e^{-\frac{k}{4}} C_{\frac{2}{q}, q'} \left( \frac{F \cap F_k(x, t)}{\sqrt{(k+1)t}} \right) \quad \forall (x, t) \in Q, \end{aligned} \quad (5.25)$$

where  $F_k(x, t) = \{y \in \mathbb{R}^N : \sqrt{kt} \leq |x - y| \leq \sqrt{(k+1)t}\}$ . As a consequence  $U_F = V_F$ .

*Remark.* We recall that the main argument for proving uniqueness is the fact that

$$U_F \leq \frac{C_2}{C_1} V_F \quad \text{in } Q. \quad (5.26)$$

This argument introduced in [17] for elliptic equations has been extended to parabolic equations in [19], [22].

**Definition 5.16** *Let  $F$  be a Borel subset of  $\mathbb{R}^N$ . We denote by  $U_F$  the maximal element of  $\mathcal{U}_{\tilde{F}^c}(Q_T)$ .*

**Proposition 5.17** *If  $\{A_n\}$  is a sequence of Borel sets such that  $C_{\frac{2}{q}, q'}(A_n) \rightarrow 0$ , then  $U_{A_n} \rightarrow 0$ .*

*Proof.* Let  $O_n$  be an open set such that  $A_n \subset O_n$  and  $C_{\frac{2}{q}, q'}(O_n) \leq C_{\frac{2}{q}, q'}(A_n) + \frac{1}{n}$ . Now since  $O_n$  is open,  $C_{\frac{2}{q}, q'}$  is an outer measure, by (2.36) and (iv)-Proposition 2.3, we have

$$C_{\frac{2}{q}, q'}(\overline{O}_n) = C_{\frac{2}{q}, q'}\left(\left(\overline{O}_n \cap b_q(O_n)\right) \cup \left(\overline{O}_n \cap e_q(\overline{O}_n)\right)\right) \leq C_{\frac{2}{q}, q'}(\tilde{O}_n) \leq c C_{\frac{2}{q}, q'}(O_n).$$

Thus  $C_{\frac{2}{q}, q'}(\overline{O}_n) \rightarrow 0$ . The result follows by

$$U_{A_n} \leq U_{\overline{O}_n}$$

and by (5.25). □

**Corollary 5.18** *Let  $E$  be a Borel set such that  $C_{\frac{2}{q}, q'}(E) = 0$ . If  $u \in \mathcal{U}_{\tilde{E}^c}(Q_T)$  then  $u = 0$ . In particular  $U_E \equiv 0$ .*

**Proposition 5.19** *Let  $E, F$  be Borel sets.*

(i) *If  $E, F$  are  $\mathfrak{T}_q$ -closed, then  $U_E \wedge U_F = U_{E \cap F}$ .*

(ii) *If  $E, F$  are  $\mathfrak{T}_q$ -closed, then*

$$\begin{aligned} U_E < U_F &\Leftrightarrow [E \subset^q F \text{ and } C_{\frac{2}{q}, q'}(F \setminus E) > 0], \\ U_E = U_F &\Leftrightarrow E \sim^q F. \end{aligned} \quad (5.27)$$

(iii) If  $F_n$  is a decreasing sequence of  $\mathfrak{T}_q$ -closed sets, then

$$\lim_{n \rightarrow \infty} U_{F_n} = U_F \text{ where } F = \bigcap F_n.$$

(iv) Let  $A$  be a  $\mathfrak{T}_q$ -open set and  $u \in \mathcal{U}_+(Q_T)$ . Suppose that  $u$  vanishes  $\mathfrak{T}_q$ -locally in  $A$ , i.e. for every point  $\sigma \in A$  there exists a  $\mathfrak{T}_q$ -open set  $A_\sigma$  such that

$$\sigma \in A \subset A, \quad u \approx_{A_\sigma} 0.$$

Then  $u$  vanishes on  $A$ . In particular any  $u \in \mathcal{U}_+(Q_T)$  vanishes on the complement of  $\mathfrak{T}_q$ -supp( $u$ ).

*Proof.* The proof is similar to the one in [23] dealing with elliptic equations.

(i)  $U_E \wedge U_F$  is the largest solution under  $\inf(U_E, U_F)$  and therefore, by definition, it is the largest solution which vanishes outside  $E \cap F$ .

(ii) By (5.25)  $U_E$  and  $U_F$  satisfies the same capacity quasi-representation up to universal constants. By the Remark after Proposition 5.15 ,

$$E \sim^q F \Rightarrow \frac{C_1}{C_2} U_E \leq U_F \leq \frac{C_2}{C_1} U_E \Rightarrow U_E = U_F.$$

The proof of

$$E \subset^q F \Rightarrow U_E \leq U_F.$$

follows from Proposition 5.15 and the fact that  $U_E = V_E$  and  $U_F = V_F$  and  $V_E \leq V_F$ . In addition,

$$C_{q'}^2(F \setminus E) > 0 \Rightarrow U_E \neq U_F.$$

Indeed, if  $K$  is a compact subset of  $F \setminus E$  of positive capacity, then  $U_K > 0$  and  $U_K \leq U_F$  but  $U_K \not\leq U_E$ . Therefore  $U_E = U_F$  implies  $E \sim^q F$  and  $U_E \leq U_F$  implies  $E \subset^q F$ .

(iii) If  $V := \lim_{n \rightarrow \infty} U_{F_n}$  then  $U_F \leq V$ . But  $\mathfrak{T}_q$ -supp( $V$ )  $\subset F_n$  for each  $n \in \mathbb{N}$  and consequently  $V \leq U_F$ .

(iv) First assume that  $A$  is a countable union of  $\mathfrak{T}_q$ -open sets  $\{A_n\}$  such that  $u \approx_{A_n} 0$  for each  $n$ . Then  $u$  vanishes on  $\bigcup_{i=1}^k A_k$  for each  $k$ . Therefore we can assume that the sequence  $A_k$  is increasing. Put  $F_n = A_n^c$ . Then  $u \subset U_{F_n}$  and by (iii),  $U_{F_n} \downarrow U_F$  where  $F = A^c$ . Thus  $u \leq U_F$ , i.e., which is equivalent to  $u \approx_A 0$ .

We turn to the general case. It is known that the  $\mathfrak{T}_q$ -fine topology possesses the *quasi-Lindelöf property* (see [2, Sec. 6.5.11]) as any topology associated to a Bessel capacity  $C_{\alpha,p}$ . Therefore  $A$  is covered, up to a set of capacity zero, by a countable subcover of  $\{A_\sigma : \sigma \in A\}$ . Therefore the previous argument implies that  $u \approx_A 0$ .  $\square$

**Proposition 5.20** (i) Let  $E$  be a  $\mathfrak{T}_q$ -closed set. Then

$$\begin{aligned} U_E &= \inf\{U_D : E \subset D, D \text{ open}\} \\ &= \sup\{U_K : K \subset E, K \text{ closed}\}. \end{aligned} \tag{5.28}$$

(ii) If  $E, F$  are two Borel sets then

$$U_E = U_{F \cap E} \oplus U_{E \setminus F}.$$

(iii) Let  $E, F_n, n = 1, 2, \dots$  be Borel sets and let  $u$  be a positive solution of (3.1). If either  $C_{\frac{2}{q}, q'}(E \Delta F_n) \rightarrow 0$  or  $\tilde{F}_n \downarrow \tilde{E}$  then

$$U_{F_n} \rightarrow U_E.$$

*Proof.* (i) Let  $\{Q_j\}$  be the decreasing sequence of open sets of Lemma 2.8-(I) such that  $\cap Q_j = \cap \tilde{Q}_j = E' \sim^q E$ . Thus by Proposition 5.19 (iii) we have that  $U_{Q_j} \rightarrow U_E$ , this implies the first equality in (i).

Let  $\{F_n\}$  be a nondecreasing sequence of closed subsets of  $E$  such that  $C_{\frac{2}{q}, q'}(E \setminus F_n) \rightarrow 0$ . Let  $D_1$  and  $D_2$  be open sets such that  $F_n \subset D_1$  and  $E \setminus F_n \subset D_2$ . Also set  $D_3 = (\tilde{D}_1 \cup \tilde{D}_2)^c$ . Let  $u_\beta^{(i)}$  be the positive solution of

$$\begin{aligned} \partial_t u - \Delta u + u^q &= 0 && \text{in } \mathbb{R}^N \times (\beta, T] \\ u(\cdot, \beta) &= \chi_{\tilde{D}_i} U_E(\cdot) && \text{on } \mathbb{R}^N, \end{aligned} \quad (5.29)$$

where  $0 < \beta < T$ . For any  $(x, t) \in \mathbb{R}^N \times (\beta, T]$  we have

$$U_E \leq u_\beta^{(1)} + u_\beta^{(2)} + u_\beta^{(3)}.$$

Letting  $\beta \rightarrow 0$  (taking an subsequence if it is necessary) we have  $u_\beta^{(i)} \rightarrow u^{(i)}$  and

$$U_E \leq u^{(1)} + u^{(2)} + u^{(3)} \quad \text{in } Q_T,$$

But  $u^{(i)} \leq U_{D_i}$  thus

$$U_E \leq U_{D_1} + U_{D_2} + u^{(3)}.$$

Now  $u^{(3)} \leq U_{D_3}$  and  $u^{(3)} \leq U_E$  thus by Proposition 5.20-(i)  $u^{(3)} \leq U_{D_3 \cap E}$ . But  $D_1 \cup D_2$  is an open set and thus  $C_{\frac{2}{q}, q'}(D_3 \cap E) = 0$ , which implies by Corollary 5.18 that  $u^{(3)} = 0$ . Finally we have that

$$U_E \leq U_{D_1} + U_{D_2}.$$

Since  $D_i$  is arbitrary, we have by the first assertion of this Proposition

$$U_E \leq U_{F_n} + U_{E \setminus F_n}. \quad (5.30)$$

But  $C_{\frac{2}{q}, q'}(E \setminus F_n) \rightarrow 0$ , thus by Proposition 5.17, we have

$$U_E \leq \lim_{n \rightarrow \infty} U_{F_n} \Rightarrow U_E = \lim_{n \rightarrow \infty} U_{F_n},$$

since  $U_{F_n} \leq U_E$  for any  $n \in \mathbb{N}$ .

(ii) By similar argument as in the proof of (5.30) we can prove that

$$U_E \leq U_{F \cap E} + U_{E \setminus F} \Rightarrow U_E \leq U_{F \cap E} \oplus U_{E \setminus F}.$$

On the other hand both  $U_{F \cap E}$  and  $U_{E \setminus F}$  vanish outside of  $\tilde{E}$ . Consequently  $U_{F \cap E} \oplus U_{E \setminus F}$  vanishes outside  $\tilde{E}$  so that

$$U_E \geq U_{F \cap E} \oplus U_{E \setminus F},$$

and the result follows in this statement.

(iii) The previous statement implies,

$$U_E \leq U_{F_n \cap E} + U_{E \setminus F_n}, \quad U_{F_n} \leq U_{F_n \cap E} + U_{F_n \setminus E}. \quad (5.31)$$

If  $C_{\frac{2}{q}, q'}(E \triangle F_n) \rightarrow 0$  then Proposition 5.17 implies  $U_{E \triangle F_n} \rightarrow 0$ . And the result follows in this case by (5.31).

If  $\tilde{F}_n \downarrow \tilde{E}$  the result follows in this case by Proposition 5.19(iii).  $\square$

This implies the following extension of Proposition 5.15 to merely  $\mathfrak{T}_q$ -closed sets.

**Proposition 5.21** *If  $E$  is a  $\mathfrak{T}_q$ -closed set, then  $V_E$  and  $U_E$  satisfy the capacity estimates (5.15). Thus  $U_E = V_E$  and the maximal solution  $U_E$  is  $\sigma$ -moderate.*

*Remark.* Actually the estimates hold for any Borel set  $E$ . Indeed by definition,  $U_E = U_{\tilde{E}}$  and

$$C_{\frac{2}{q}, q'} \left( \frac{E \cap F_n(x, t)}{\sqrt{(n+1)t}} \right) \sim C_{\frac{2}{q}, q'} \left( \frac{\tilde{E} \cap F_n(x, t)}{\sqrt{(n+1)t}} \right).$$

*Proof.* The proof is same as in [23].

Let  $\{E_k\}$  be a  $\mathfrak{T}_q$ -stratification of  $E$ . If  $u \in \mathcal{V}_{mod}$  and  $\mu = \text{tru}$  then  $u_\mu = \sup u_{\mu_k}$  where  $\mu_k = \mu \chi_{E_k}$ . Hence  $V_E = \sup V_{E_k}$ . By proposition 5.25,  $U_{E_k} = V_{E_k}$ . These facts and Proposition 5.20(c) implies  $U_E = V_E$ . Since  $U_{E_k}$  satisfies the capacity estimates (5.15) and

$$C_{\frac{2}{q}, q'} \left( \frac{E_k \cap F_n(x, t)}{\sqrt{(n+1)t}} \right) \rightarrow C_{\frac{2}{q}, q'} \left( \frac{\tilde{E} \cap F_n(x, t)}{\sqrt{(n+1)t}} \right) \quad \text{as } n \rightarrow \infty.$$

it follows that  $U_E$  satisfies the corresponding capacity estimates.  $\square$

## 5.4 Localization

**Definition 5.22** *Let  $A$  be a Borel subset of  $\mathbb{R}^N$ , we denote by  $[u]_A$  the supremum of the  $v \in \mathcal{U}_+(Q_T)$  which are dominated by  $u$  and vanishes on  $A^c$ .*

We note here that  $[u]_A = u \wedge U_A$

**Lemma 5.23** *If  $G \subset \mathbb{R}^N$  is a  $\mathfrak{T}_q$ -open set and  $u \in \mathcal{U}_G(Q_T)$ , then*

$$u = \sup\{v \in \mathcal{U}_G(Q_T) : v \leq u, v \text{ vanishes on an open neighborhood of } G\}.$$

*Proof.* Set  $A = G^c$  and let  $\{A_n\}$  be a sequence of closed subsets of  $A$ , such that  $C_{\frac{2}{q}, q'}(A \setminus A_n) \rightarrow 0$ . By Proposition 5.20 we have

$$U_A \leq U_{A_n} + U_{A \setminus A_n},$$

thus

$$u = u \wedge U_A \leq u \wedge U_{A_n} + u \wedge (U_{A \setminus A_n}).$$

By Proposition 5.17, we have

$$U_{A \setminus A_n} \rightarrow 0.$$

Thus

$$u = \lim_{n \rightarrow \infty} u \wedge U_{A_n},$$

and the result follows.  $\square$

The next result points out the set-regularity of the correspondence  $E \mapsto [u]_E$ .

**Proposition 5.24** *Let  $u \in \mathcal{U}_+(Q_T)$ .*

(i) *If  $E$  is  $\mathfrak{T}_q$ -closed then,*

$$[u]_E = \inf\{[u]_D : E \subset D, D \text{ open}\}. \quad (5.32)$$

$$= \sup\{[u]_F : F \subset E, F \text{ closed}\}. \quad (5.33)$$

(ii) *If  $E, F$  are two Borel sets then*

$$[u]_E \leq [u]_{F \cap E} + [u]_{E \setminus F}, \quad (5.34)$$

and

$$[[u]_E]_F = [[u]_F]_E = [u]_{F \cap E}. \quad (5.35)$$

(iii) *Let  $E, F_n, n = 1, 2, \dots$  be Borel sets and let  $u$  be a positive solution of (3.1). If either  $C_{\frac{2}{q}, q'}(E \triangle F_n) \rightarrow 0$  or  $\tilde{F}_n \downarrow \tilde{E}$  then*

$$[u]_{F_n} \rightarrow [u]_E.$$

*Proof.* The proof uses a similar argument as in [23].

(i) Let  $\mathcal{D} = \{D\}$  be the family of sets in (5.32). By (5.28) (with respect to the family  $\mathcal{D}$ )

$$\inf(u, U_E) = \inf(u, \inf_{D \in \mathcal{D}} U_D) = \inf_{D \in \mathcal{D}} \inf(u, U_D) \geq \inf_{D \in \mathcal{D}} [u]_D. \quad (5.36)$$

Obviously

$$[u]_{D_1} \wedge [u]_{D_2} \geq [u]_{D_1 \cap D_2},$$

thus we can apply Proposition 3.4 and obtain that the function  $v := \inf_{D \in \mathcal{D}} [u]_D$  is a solution of (3.1). Hence (5.36) implies  $[u]_E \geq v$ . The opposite inequality is obvious.

For the equality (5.33), Firstly, we note that the set  $\{v \in \mathcal{U}_+(Q_T) : v \leq u, \mathfrak{T}_q\text{-supp}(v) \subset^q E\}$  is closed under  $\vee$ . Thus, by Proposition 3.4, there exists an increasing sequence  $\{v_n\}$  such that  $v_n \approx_{E^c} 0$  and  $\lim_{n \rightarrow \infty} v_n = [u]_E$ . Since  $v_n$  is an increasing sequence by Proposition 5.23 we can construct an increasing sequence  $\{w_n\}$  such that each  $w_n$  vanishes on an open neighborhood  $B_n$  of  $E$ ,  $B_n \subset B_{n+1}$  and  $\lim_{n \rightarrow \infty} w_n = [u]_E$ . Now set  $K_n = B_n^c$ , then

$$w_n \leq [u]_{K_n} \leq [u]_E.$$

Letting  $n$  tend to infinity, we obtain the desired result.

(ii) Let  $v \in \mathcal{U}_+(Q_T)$ ,  $v \leq u$  and  $\mathfrak{T}_q\text{-supp}(v) \subset E$ . Let  $D$  and  $D'$  be open sets such that



$\widetilde{E \cap F} \subset D$  and  $\widetilde{E \setminus F} \subset D'$ . By Lemma 2.8-[19], there exists a unique solution  $v_j^1$ , where  $\frac{1}{[T]} < j \in \mathbb{N}$ , of the problem

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= 0, & \text{in } \mathbb{R}^N \times (\frac{1}{j}, T] \\ u(\cdot, \frac{1}{j}) &= \chi_{D'}(\cdot)v(\cdot, \frac{1}{j}) & \text{in } \mathbb{R}^N. \end{aligned}$$

Also we consider  $v_j^2$  and  $v_j^3$  the unique solutions of the above problem with initial data  $\chi_{D'}(x)v(x, \frac{1}{j})$  and  $\chi_{(D_1 \cup D_2)^c}$ . In view of the proof of Proposition 5.20 we can prove that  $v \leq v_j^1 + v_j^2 + v_j^3$ . By standard arguments there exists a subsequence, say  $\{v_j^i\}$ ,  $i = 1, 2, 3$ , such that  $v_j^i \rightarrow v^i$  and  $v \leq v^1 + v^2 + v^3$ . Since  $v$  vanishes outside of  $E$ , it vanishes outside of  $(D_1 \cup D_2)$ , consequently  $v(x, \frac{1}{j})\chi_{(D_1 \cup D_2)^c} \rightarrow 0$ , as  $j \rightarrow \infty$ , which implies  $v_j^3 \rightarrow 0$ . Thus we have

$$v \leq v^1 + v^2 \leq [u]_D + [u]_{D'}.$$

By (5.32) we have

$$v \leq [u]_{F \cap E} + [u]_{E \setminus F},$$

since  $v \in \{w \in \mathcal{U}_+(Q_T) : w \leq u, \mathfrak{T}_q\text{-supp}(w) \subset^q E\}$  is arbitrary the result follows in the case where  $E$  is closed. In the general case the result follows by (5.33).

Put  $A = \widetilde{E}$  and  $B = \widetilde{F}$ . It follows directly from the definition that

$$[[u]_A]_B \leq \inf(u, U_A, U_B).$$

The largest solution dominated by  $u$  and vanishing on  $A^c \cup B^c$  is  $[u]_{A \cap B}$ . Thus

$$[[u]_A]_B \leq [u]_{A \cap B}.$$

On the other hand

$$[u]_{A \cap B} = [[u]_{A \cap B}]_B \leq [[u]_A]_B,$$

this proves (5.35).

(iii) By (5.34)

$$[u]_E \leq [u]_{F_n \cap E} + [u]_{E \setminus F_n}, \quad [u]_{F_n} \leq [u]_{F_n \cap E} + [u]_{F_n \setminus E}.$$

If  $C_{\frac{2}{q}, q'}(E \Delta F_n) \rightarrow 0$ , then by Proposition (5.17)(c) we have that  $U_{E \Delta F_n} \rightarrow 0$ . Since  $[u]_{E \setminus F_n}, [u]_{F_n \setminus E} \leq U_{E \Delta F_n}$ , the result follows by the above inequalities, if we let  $n$  go to infinite.

If  $\widetilde{F}_n \downarrow \widetilde{E}$ . By Proposition (5.17)(c) we have  $U_{E_n} \rightarrow U_E$ , thus

$$[u]_E \leq \lim_{n \rightarrow \infty} [u]_{F_n} = \lim_{n \rightarrow \infty} u \wedge U_{F_n} \leq \lim_{n \rightarrow \infty} \inf(u, U_{F_n}) \leq \inf(u, U_E).$$

And since  $[u]_E$  is the largest solution under  $\inf(u, U_E)$  and the function  $v = \lim_{n \rightarrow \infty} [u]_{F_n}$  is a solution of (3.1), we have that  $U_E \leq v$ , and the proof of (5.34) is complete.  $\square$

**Definition 5.25** Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^N$  which vanishes on compact sets of  $C_{\frac{2}{q}, q}$ -capacity zero.

(a) The  $\mathfrak{T}_q$ -support of  $\mu$ , denoted  $\mathfrak{T}_q\text{-supp}(\mu)$ , is the intersection of all  $\mathfrak{T}_q$ -closed sets  $F$  such that  $\mu(F^c) = 0$ .

(b) We say that  $\mu$  is concentrated on a Borel set  $E$  if  $\mu(E^c) = 0$ .

**Proposition 5.26** If  $\mu$  is a measure as in the previous definition then,

$$\mathfrak{T}_q\text{-supp}(\mu) \sim^q \mathfrak{T}_q\text{-supp}(u_\mu).$$

*Proof.* Put  $F = \text{supp}^q u_\mu$ . By Proposition 5.19(iv)  $u_\mu$  vanishes on  $F^c$  and by Proposition 5.23(c) there exists an increasing sequence of positive solutions  $u_n$  such that each function  $u_n$  vanishes outside a closed subset  $F$ , say  $F_n$ , and  $u_n \uparrow u_\mu$ . If  $S_n := \mathfrak{T}_q\text{-supp}(u_n)$  then  $S_n \subset F_n$  and  $\{S_n\}$  increases. Thus  $\{\overline{S}_n\}$  is an increasing sequence of closed subsets of  $F$  and, setting  $\mu_n = \mu \chi_{\overline{S}_n}$ , we find  $u_n \leq u_{\mu_n} \leq u_\mu$  so that  $u_{\mu_n} \uparrow u_\mu$ . This, in turn, implies

$$\mu_n \uparrow \mu, \quad \mathfrak{T}_q\text{-supp}(\mu) \subset^q \widetilde{\bigcup_{n=1}^{\infty} \overline{S}_n} \subset F.$$

If  $D$  is an open set and  $\mu(D) = 0$  it is clear that  $u_\mu$  vanishes on  $D$ . Therefore  $u_{\mu_n}$  vanishes outside of  $\overline{S}_n$ , thus outside  $\mathfrak{T}_q\text{-supp}(\mu)$ . Consequently  $u_\mu$  vanishes outside  $\mathfrak{T}_q\text{-supp}(\mu)$ , i.e.,  $F \subset^q \mathfrak{T}_q\text{-supp}(\mu)$ .

*Second proof.* The result follows by Proposition 5.7 and Definition 5.8 □

**Definition 5.27** Let  $u$  be a positive solution and  $A$  a Borel set. Put

$$[u]^A := \sup\{[u]_F : F \subset^q A, F \text{ } q\text{-closed}\}.$$

**Definition 5.28** Let  $\beta > 0$ ,  $u \in C(Q_T)$ . For any Borel set  $A$  we denote by  $u_\beta^A$  the positive solution of

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v &= 0 && \text{in } \mathbb{R}^N \times (\beta, \infty) \\ v(\cdot, \beta) &= \chi_A(\cdot)u(\cdot, \beta) && \text{in } \mathbb{R}^N. \end{aligned}$$

**Proposition 5.29** Let  $u$  be a positive solution of (3.1) and put  $E = \mathfrak{T}_q\text{-supp}(u)$ .

(i) If  $D$  is a  $\mathfrak{T}_q$ -open set such that  $E \subset^q D$ , then

$$[u]^D = \lim_{\beta \rightarrow 0} u_\beta^D = [u]_D = u. \quad (5.37)$$

(ii) If  $A$  is a  $\mathfrak{T}_q$ -open set, then

$$u \approx_A 0 \Leftrightarrow u^Q = \lim_{\beta \rightarrow 0} u_\beta^Q = 0, \quad \forall Q \text{ } \mathfrak{T}_q\text{-open} : \widetilde{Q} \subset^q A. \quad (5.38)$$

(iii) Finally,

$$u \approx_A 0 \Leftrightarrow [u]^A = 0. \quad (5.39)$$

*Proof.* The proof is similar as in the one as in [23]

*Case 1:*  $E$  is closed. Since  $u$  vanishes in  $E^c$ , it yields  $u \in C(Q_\infty \cup E^c)$  and  $u = 0$  on  $E^c$ . If, in addition,  $D$  is an open neighborhood of  $E$ , we have

$$\lim_{t \rightarrow 0} \int_{E^c} \phi(x) u(x, t) dx = 0, \quad \forall \phi \in C_0(E^c).$$

Thus,

$$\lim_{\beta \rightarrow 0} u_\beta^{D^c} = 0.$$

Since

$$u_\beta^D \leq u \leq u_\beta^D + u_\beta^{D^c}, \quad \forall t \geq \beta,$$

it follows

$$u = \lim_{\beta \rightarrow 0} u_\beta^D. \quad (5.40)$$

If we assume that  $D$  is  $\mathfrak{T}_q$ -open and  $E \subset^q D$  then, for every  $\varepsilon > 0$ , there exists an open set  $O_\varepsilon$  such that  $D \subset O_\varepsilon$ ,  $E \subset O_\varepsilon$  and  $C_{\frac{2}{q}, q'}(O'_\varepsilon) < \varepsilon$  where  $O'_\varepsilon = O_\varepsilon \setminus D$ . Therefore

$$u_\beta^{O_\varepsilon}(x, t) - u_\beta^D(x, t) \leq U_{O'_\varepsilon}(x, t - \beta), \quad \forall t \geq \beta.$$

We note here that  $\lim_{\varepsilon \rightarrow 0} U_{O'_\varepsilon}(x, t - \beta) = 0$  holds uniformly with respect to  $\beta$ . Since  $\lim_{\beta \rightarrow 0} u_\beta^{O_\varepsilon}(x, t) = u$  it follows that  $u = \lim_{\beta \rightarrow 0} u_\beta^D$ . The same arguments shows that  $\lim_{\beta \rightarrow 0} u_\beta^{D^c} = 0$ . Thus we have

$$u = \lim_{\beta \rightarrow 0} u_\beta^D \leq [u]_D \leq u.$$

Hence  $u = [u]_D$ . By Lemma 2.7, there exists a  $\mathfrak{T}_q$ -open set  $Q$  such that  $E \subset^q Q \subset \tilde{Q} \subset^q D$ , then  $u = [u]_Q \leq [u]^D$ . Hence  $u = [u]^D$ .

In addition, there holds  $E \subset^q A^c \subset^q \tilde{Q}^c$ . Thus the direction " $\Rightarrow$ " in (5.38) follows by the previous argument if we replace  $D$  by  $\tilde{Q}^c$ . For the opposite direction, by Proposition 2.38, for any  $\xi \in A$ , there exists a  $\mathfrak{T}_q$ -open set  $O_\xi$  such that  $\tilde{O}_\xi \subset^q A$ . Using (i) we infer  $u = \lim_{\beta \rightarrow 0} u_\beta^{\tilde{O}_\xi}$ . Finally, since  $u_\beta^{\tilde{O}_\xi} \approx_{Q_\xi} 0$  for all  $\beta > 0$ , it implies  $u \approx_{O_\xi} 0$  by Proposition 5.17(i), and the result follows in this case by Proposition 5.19(iv).

*Case 2.* Assume  $E$  is  $\mathfrak{T}_q$ -closed. Let  $\{E_n\}$  be a  $\mathfrak{T}_q$ -stratification of  $E$  such that  $C_{\frac{2}{q}, q'}(E \setminus E_n) \rightarrow 0$ . If  $D$  is a  $\mathfrak{T}_q$ -open such that  $E \subset^q D$  then, by the first case we have,

$$\lim_{\beta \rightarrow 0} ([u]_{E_n})_\beta^D = [u]_{E_n}. \quad (5.41)$$

By (5.34) and the definition of  $u_\beta^D$ , and since  $[u]_E = u$ ,

$$u_\beta^D = ([u]_E)_\beta^D \leq ([u]_{E \cap E_n})_\beta^D + ([u]_{E \setminus E_n})_\beta^D = ([u]_{E_n})_\beta^D + ([u]_{E \setminus E_n})_\beta^D. \quad (5.42)$$

Let  $\{\beta_k\}$  be a decreasing sequence converging to 0 such that the following limits exist

$$w := \lim_{k \rightarrow \infty} u_{\beta_k}^D, \quad w_n = \lim_{k \rightarrow \infty} ([u]_{E \setminus E_n})_{\beta_k}^D, \quad n = 1, 2, \dots$$

Then by (5.41) and (5.42),

$$[u]_{E_n} \leq w \leq [u]_{E_n} + w_n \leq [u]_{E_n} + U_{E \setminus E_n}.$$

Further, by (5.33) and Proposition 5.20(c)

$$[u]_{E_n} \rightarrow [u]_E = u, \quad U_{E \setminus E_n} \rightarrow 0.$$

Hence  $w = u$ . This implies (5.40), which in turn implies (5.37).

To verify (5.38) in the direction  $\Rightarrow$  we apply (5.42) with  $D$  replaced by  $Q$ . We obtain

$$([u]_E)_\beta^Q \leq ([u]_{E_n})_\beta^Q + ([u]_{E \setminus E_n})_\beta^Q.$$

By the first case we have

$$\lim_{\beta \rightarrow 0} ([u]_{E_n})_\beta^Q = 0.$$

There exists a decreasing sequence converging to 0, still denoted by  $\{\beta_k\}$ , such that the following limits exist

$$\lim_{k \rightarrow \infty} u_{\beta_k}^Q, \quad \lim_{k \rightarrow \infty} ([u]_{E \setminus E_n})_{\beta_k}^Q, \quad n = 1, 2, \dots.$$

Then

$$\lim_{k \rightarrow \infty} u_{\beta_k}^Q \leq \lim_{k \rightarrow \infty} ([u]_{E \setminus E_n})_{\beta_k}^Q \leq U_{E \setminus E_n},$$

since  $U_{E \setminus E_n} \rightarrow 0$  we obtain (5.38) in the direction  $\Rightarrow$ . The assertion in the opposite direction is proved as in Case 1. This complete the proofs of (i) and (ii).

Finally we prove (iii). First assume that  $u \approx_A 0$ . If  $F$  is a  $\mathfrak{T}_q$ -closed set such that  $F \subset^q A$ , then by Lemma 2.7 there exists a  $\mathfrak{T}_q$ -open set  $\mathfrak{T}_q$  such that  $F \subset^q Q \subset \tilde{Q} \subset^q A$ . Therefore, applying (5.37) to  $v := [u]_F$  and using (5.38) we obtain

$$v = \lim_{\beta \rightarrow 0} v_\beta^Q \leq \lim_{\beta \rightarrow 0} u_\beta^Q = 0.$$

By definition of  $[u]^A$ , this implies  $[u]^A = 0$ .

If  $[u]^A = 0$ , then for any  $\mathfrak{T}_q$ -open set  $Q \subset \tilde{Q} \subset^q A$  there holds  $[u]_Q = 0$ . Now since  $\mathfrak{T}_q\text{-supp}(u_\beta^Q) \subset^q \tilde{Q}$  we have for some subsequence  $\beta_k \downarrow 0$ ,  $\lim_{k \rightarrow \infty} u_{\beta_k}^Q \leq [u]_Q = 0$ . Thus  $u \approx_Q 0$  by (5.38). Applying once again Proposition 2.38 and Proposition 5.19(iv) we conclude  $u \approx_A 0$ .  $\square$

**Definition 5.30** Let  $u, v \in \mathcal{U}_+(Q_T)$  and let  $A$  be a  $\mathfrak{T}_q$ -open set. We say that  $u = v$  on  $A$  if  $u \ominus v$  and  $v \ominus u$  vanishes on  $A$ . This relation is denoted by  $u \approx_A v$ .

**Proposition 5.31** Let  $u, v \in \mathcal{U}_+(Q_T)$  and let  $A$  be a  $\mathfrak{T}_q$ -open set. Then,

(i)

$$u \approx_A v \Leftrightarrow \lim_{\beta \rightarrow 0} |u - v|_\beta^Q = 0. \quad (5.43)$$

for every  $\mathfrak{T}_q$ -open set  $Q$  such that  $\tilde{Q} \subset^q A$ .

(ii)

$$u \approx_A v \Leftrightarrow [u]_F = [v]_F, \quad (5.44)$$

for every  $\mathfrak{T}_q$ -closed set  $F$  such that  $F \subset^q A$ .

*Proof.* The proof is similar, but in a parabolic framework, to the elliptic one in [23]. By definition  $u \approx_A v$  is equivalent to  $u \ominus v \approx_A = 0$  and  $v \ominus u \approx_A = 0$ . Hence, by (5.38) we have  $w_\beta = (u \ominus v)_\beta^Q \xrightarrow{\beta \rightarrow 0} 0$ . Set  $f_\beta = ((u - v)_+)_\beta^Q$  and consider the problem

$$\begin{aligned} \partial_t w - \Delta w + |w|^q &= 0, & \text{in } B_j(0) \times (\beta, \infty) \\ w &= 0, & \text{on } \partial B_j(0) \times (\beta, \infty) \\ w(\cdot, \beta) &= \mu, & \text{in } B_j(0). \end{aligned}$$

Let  $w_j$  and  $f_j$  be solutions of the above problem, with initial data  $\chi_Q(u \ominus v)(x, \beta)$  and  $\chi_Q(u - v)_+(x, \beta)$ . By [19, Lemma 2.7], the sequences  $\{w_j\}$  and  $\{f_j\}$  are increasing. Also, we recall that  $u \ominus v$  is the smallest solution which dominates the subsolution  $(u - v)_+$ , thus  $w_j \geq v_j$ ,  $\forall j \in \mathbb{N}$ . Furthermore, in view of [19, Lemma 2.8], there holds  $\lim_{j \rightarrow \infty} w_j = w_\beta$  and  $\lim_{j \rightarrow \infty} f_j = f_\beta$ . Thus  $w_\beta \geq f_\beta$ , and letting  $\beta \rightarrow 0$  we derive

$$((u - v)_+)_\beta^Q \rightarrow 0.$$

By the same argument we have

$$((v - u)_+)_\beta^Q \rightarrow 0,$$

this implies (5.43) in the direction  $\Rightarrow$ .

For the opposite direction, we consider the problem

$$\begin{aligned} \partial_t w - \Delta w + |w|^q &= 0, & \text{in } B_j(0) \times (\beta, \infty) \\ w &= h, & \text{on } \partial B_j(0) \times (\beta, \infty) \\ w(\cdot, \beta) &= \mu, & \text{in } B_j(0). \end{aligned}$$

Let  $Q \subset \tilde{Q} \subset^q A$  be a  $\mathfrak{T}_q$ -open set and  $w_j$  be the solution of the above problem, with  $h = \chi_Q(|u - v|)$  and  $\mu = \chi_Q|u - v|dx$ . Also, let  $f_j$  be the solution of the above problem with  $h = \chi_{Q^c}|u - v|$  and  $\mu = \chi_{Q^c}|u - v|dx$ , then

$$|u - v| \leq w_j + f_j.$$

In view of [19, Lemma 2.8], there exist subsequences, say  $\{w_j\}$  and  $\{f_j\}$ , satisfying  $\lim_{j \rightarrow \infty} w_j = w$  and  $\lim_{j \rightarrow \infty} f_j = f$ , such that  $(w, f)$  solves the problem

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v &= 0, & \text{in } \mathbb{R}^N \times (\beta, \infty) \\ v(\cdot, \beta) &= \mu & \text{in } \mathbb{R}^N, \end{aligned}$$

with initial data  $\mu = \chi_Q|u - v|dx$  and  $\mu = \chi_{Q^c}|u - v|dx$  respectively. By uniqueness of the problem (see [19, Lemma 2.8]), we have  $w = |u - v|_\beta^Q$  and  $f = |u - v|_\beta^{Q^c}$ . Let  $\beta_k$  be a decreasing sequence such that the following limit exists

$$\lim_{k \rightarrow \infty} |u - v|_{\beta_k}^{Q^c}.$$

Since  $\lim_{\beta \rightarrow 0} |u - v|_\beta^Q = 0$ , we have

$$|u - v| \leq \lim_{k \rightarrow \infty} |u - v|_{\beta_k}^{Q^c}.$$

Now since  $|u - v|_{\beta_k}^{Q^c} \approx_Q 0$ , by Proposition 5.17(i) we have  $\lim_{k \rightarrow \infty} |u - v|_{\beta_k}^{Q^c} \approx_Q 0$ . Using the fact that  $u \ominus v$  is the smallest solution which dominates the subsolution  $(u - v)_+$ , there holds  $\max\{u \ominus v, v \ominus u\} \leq \lim_{k \rightarrow \infty} |u - v|_{\beta_k}^{Q^c}$  and the result follows in this case by Propositions 5.23 and 5.19(iv).

(ii) We assume that  $u \approx_A v$ .

For any two positive solutions  $u, v$  we have

$$u + (v - u)_+ \leq v + (u - v)_+ \leq v + u \ominus v \quad (5.45)$$

If  $F$  is a  $\mathfrak{T}_q$ -closed set and  $Q$  a  $\mathfrak{T}_q$ -open set such that  $F \subset^q Q$ , we claim that

$$[u]_F \leq [v]_Q + [u \ominus v]_Q. \quad (5.46)$$

To verify this inequality, we observe first that (see (5.34))

$$u = [u]_{\mathbb{R}^N} \leq [u]_Q + [u]_{Q^c},$$

thus by (5.45)

$$[u]_F \leq [u]_{\mathbb{R}^N} \leq v + u \ominus v \leq [v]_Q + [v]_{Q^c} + [u \ominus v]_Q + [u \ominus v]_{Q^c}.$$

The subsolution  $w := ([u]_F - ([v]_Q + [u \ominus v]_Q))_+$  is dominated by the supersolution  $[u \ominus v]_{Q^c} + [v]_{Q^c}$ . By definition we have

$$w \leq [w]_{\dagger} \leq [u \ominus v]_{Q^c} \oplus [v]_{Q^c} \leq [u \ominus v]_{Q^c} + [v]_{Q^c}.$$

Thus  $[w]_{\dagger} \approx_Q 0$ . But  $w \leq [u]_F$  which implies  $[w]_{\dagger} \leq [u]_F$ , that is  $\mathfrak{T}_q\text{-supp}([w]_{\dagger}) \subset^q F \subset^q Q$ . Taking into account that  $[w]_{\dagger} \approx_Q 0$  we have that  $w = [w]_{\dagger} = 0$  and the proof of (5.46) is completed.

If we choose a  $\mathfrak{T}_q$ -open set  $Q$  such that  $F \subset^q Q \subset \tilde{Q} \subset^q A$  (see Lemma 2.7), and using the fact that  $u \ominus v \approx_A 0 \Rightarrow [u \ominus v]_F = 0$  (see (5.39)) and (5.46), we infer

$$[u]_F \leq [v]_Q.$$

Now by Lemma 2.8(I), we can construct a decreasing sequence  $\{Q_j\}$  of open sets such that  $\cap Q_j \sim^q F$ , thus by Proposition 5.24(iii) we have

$$[u]_F \leq \lim_{n \rightarrow \infty} [v]_{Q_n} = [v]_F.$$

Similarly,  $[v]_F \leq [u]_F$  and hence the equality holds.

Next we assume that  $[u]_F = [v]_F$  for any  $\mathfrak{T}_q$ -closed set  $F \subset^q A$ . If  $Q$  is a  $\mathfrak{T}_q$ -open set such that  $F \subset^q Q \subset \tilde{Q} \subset^q A$  (see Lemma 2.7), we have

$$u \ominus v \leq ([u]_Q \oplus [u]_{Q^c}) \ominus [v]_Q,$$

where in the last inequality we have used the fact that

$$u = [u]_{\mathbb{R}^N} \leq [u]_Q + [u]_{Q^c} \Rightarrow u \leq [u]_Q \oplus [u]_{Q^c} \leq [u]_Q + [u]_{Q^c}.$$

Since  $([u]_Q \oplus [u]_{Q^c}) \ominus ([v]_Q)$  is the smallest solution dominating  $(([u]_Q \oplus [u]_{Q^c}) - [v]_Q)_+$ , we have

$$(([u]_Q \oplus [u]_{Q^c}) - [v]_Q)_+ \leq (([u]_Q + [u]_{Q^c}) - [v]_Q)_+ = [u]_Q + [u]_{Q^c} - [v]_Q = [u]_{Q^c},$$

since by assumption we have  $[u]_Q = [v]_Q$ . Thus we have

$$[u \ominus v]_F \leq u \ominus v \leq [u]_{Q^c},$$

This means  $\mathfrak{T}_q\text{-supp}([u \ominus v]_F) \subset^q F$  and  $[u \ominus v]_F \approx_Q 0$ , which in turn implies  $[u \ominus v]_F = 0$ , and by 5.39  $u \ominus v \approx_A 0$ . Similarly,  $v \ominus u \approx_A 0$ .  $\square$

**Corollary 5.32** *If  $A$  is a  $\mathfrak{T}_q$ -open set, the relation  $\approx_A$  is an equivalence relation in  $\mathcal{U}_+(Q_T)$ .*

*Proof.* This is an immediate consequence of (5.43).  $\square$

## 6 The precise initial trace

### 6.1 The regular initial set

**Lemma 6.1** *Let  $u \in \mathcal{U}_+(Q_T)$  and  $Q$  be a  $\mathfrak{T}_q$ -open set. Then for any  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in  $\tilde{Q}^c$ , we have*

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_Q)^q(t, x) \mathbb{H}^{2q'}[\eta]_+ dx dt < \infty.$$

*Proof.* By Proposition 5.9 and the properties of  $U_Q$ , there holds

$$\lim_{t \rightarrow 0} \int_Q u \wedge U_Q(x, t) \eta_+^{2q'}(x) dx = 0,$$

and the result follows by the estimates in Lemma 5.11.  $\square$

**Proposition 6.2** *Let  $u \in \mathcal{U}_+(Q_T)$  and  $Q$  be a  $\mathfrak{T}_q$ -open set. We assume that  $u \wedge U_Q$  is a moderate solution with initial data  $\mu$ . Then for any  $\xi \in Q$  there exists a  $\mathfrak{T}_q$ -open set  $O_\xi \subset Q$  such that*

$$\int_0^T \int_{\mathbb{R}^N} u^q(t, x) \mathbb{H}^{2q'}[\chi_{O_\xi}]_+ dx dt < \infty.$$

*Furthermore, for any  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in  $Q$ , we have*

$$\lim_{t \rightarrow 0} \int_Q u(x, t) \eta_+^{2q'}(x) dx = \int_Q \eta^{2q'} d\mu.$$

*Proof.* Let  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in  $Q$ . Since  $\eta_+^{2q'}$  is a quasi continuous function we have by Lemma 5.7 that

$$\lim_{t \rightarrow 0} \int_Q u \wedge U_Q(x, t) \eta_+^{2q'}(x) dx = \int_Q \eta^{2q'} d\mu.$$

Using the properties of  $U_{Q^c}$ ,

$$\lim_{t \rightarrow 0} \int_Q u \wedge U_{Q^c}(x, t) \eta_+^{2q'}(x) dx = 0.$$

Combining all above and using the fact that  $u \leq u \wedge U_Q + u \wedge U_{Q^c}$  we get

$$\begin{aligned} \int_Q \eta^{2q'} d\mu &= \lim_{t \rightarrow 0} \int_Q u \wedge U_Q(x, t) \eta_+^{2q'}(x) dx \leq \lim_{t \rightarrow 0} \int_Q u(x, t) \eta_+^{2q'}(x) dx \\ &\leq \lim_{t \rightarrow 0} \int_Q u \wedge U_Q(x, t) \eta_+^{2q'}(x) dx + \lim_{t \rightarrow 0} \int_Q u \wedge U_{Q^c}(x, t) \eta_+^{2q'}(x) dx \\ &= \int_Q \eta^{2q'} d\mu + 0. \end{aligned}$$

In view of the proof of Lemma 5.2 and by 5.3 there holds

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_Q)^q(t, x) \mathbb{H}^{2q'}[\eta]_+ dx dt < \infty, \quad (6.1)$$

for any  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in  $Q$ . By Lemma 4.2, there exists  $\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that  $0 \leq \eta \leq 1$ ,  $\eta = 1$  on  $O_\varepsilon \subset Q$  and  $\mathfrak{T}_q$ -supp( $\eta$ )  $\subset Q$ . Thus we have by (6.1) and the properties of  $\eta$ ,

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_{Q^c})^q(t, x) \mathbb{H}^{2q'}[\chi_{O_\varepsilon}] dx dt < \infty. \quad (6.2)$$

□

**Definition 6.3** (Section 10.1-[2]) Let  $Q$  be a Borel set. We denote  $W^{\frac{2}{q}, q'}(E^c)$  the closure of the space of  $C^\infty$  functions (with respect the norm  $\|\cdot\|_{W^{\frac{2}{q}, q'}}$ ) with compact support in  $E^c$ .

**Proposition 6.4** Let  $u$  be a positive solution of (3.1) and  $Q$  a bounded  $\mathfrak{T}_q$ -open sets such that

$$\int_0^T \int_{\mathbb{R}^N} u^q(t, x) \mathbb{H}^{2q'}[\chi_Q] dx dt < \infty. \quad (6.3)$$

(i) Then, there exists an increasing sequence of  $\mathfrak{T}_q$ -open set  $\{Q_n\}$  satisfying  $Q_n \subset Q$ ,  $\tilde{Q}_n \subset^q Q_{n+1}$  and  $Q_0 := \bigcup_{n=1}^\infty Q_n \sim^q Q$ , such that the solution  $v_n = u \wedge Q_n$  is moderate,  $v_n \uparrow [u]_Q$ ,  $\text{tr}(v_n) \rightarrow \mu_Q$ .

(ii) For any  $\eta \in W^{\frac{2}{q}, q'}(Q)$  we have

$$\lim_{t \rightarrow 0} \int_Q u(x, t) \eta_+^{2q'}(x) dx = \int_Q \eta_+^{2q'}(x) d\mu_Q.$$



*Proof.* We choose a point  $z \in Q$ . Then by Lemma 4.2 there exist a  $\mathfrak{T}_q$ -open set  $V$ , such that  $z \in V \subset \tilde{V} \subset Q$ , and a function  $\psi \in W^{\frac{2}{q}, q'}(\mathbb{R}^N)$  such that  $\psi = 1$  q.a.e. on  $V$  and  $\psi = 0$  outside  $Q$ . By Lemma 2.38, there exists a  $\mathfrak{T}_q$ -open set  $z \in O_z \subset \tilde{O}_z \subset V$ .

We assert that the function

$$v_z = u \wedge U_{O_z} \tag{6.4}$$

is a moderate solution.

Indeed, let  $B_R(0)$  be a ball with radius  $R$  large enough such that  $Q \subset\subset B_R(0)$ . Also, let  $0 \leq \eta \leq 1$  be a smooth function with compact support in  $B_{2R}(0)$  and  $\eta = 1$  on  $B_R(0)$ . Then the function  $\zeta = (1 - \psi)\eta \in W^{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with compact support in  $B_{2R}(0) \setminus \tilde{V}$ . Now

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} v_z^q(t, x) \mathbb{H}^{2q'}[\chi_{B_R(0)}] dx dt &\leq \int_0^T \int_{\mathbb{R}^N} v_z^q(t, x) \mathbb{H}^{2q'}[\psi] dx dt + \int_0^T \int_{\mathbb{R}^N} v_z^q(t, x) \mathbb{H}^{2q'}[1 - \psi] dx dt \\ &\leq \int_0^T \int_{\mathbb{R}^N} v_z^q(t, x) \mathbb{H}^{2q'}[\psi] dx dt + \int_0^T \int_{\mathbb{R}^N} v_z^q(t, x) \mathbb{H}^{2q'}[\zeta] dx dt < \infty, \end{aligned}$$

where the first integral in the last inequality is finite by assumption and the second integral is finite by Lemma 6.1. Thus since  $B_R(0)$  is arbitrary, the function  $u \wedge O_z$  is a moderate solution.

By the *quasi-Lindelöff property* there exists a non decreasing sequence of  $\mathfrak{T}_q$ -open set  $\{O_n\}$  such that  $Q \sim^q \cup O_n$  and (by the above arguments) the solution  $u \wedge U_{O_n}$  is moderate for any  $n \in \mathbb{N}$ . Now, by Lemma 2.8 (II)(i)-(ii), for any  $n \in \mathbb{N}$ , there exists an increasing sequence  $\{A_{n,j}\}$  of  $\mathfrak{T}_q$ -open sets such that  $\tilde{A}_{n,j} \subset^q A_{n,j+1} \subset^q E_n$  and  $\bigcup_{j=1}^\infty A_{n,j} \sim^q E_n$ . Put

$$Q_n = \bigcup_{k+j=n} A_{k,j}.$$

Then

$$\tilde{Q}_n \subset \bigcup_{k+j=n} \tilde{A}_{k,j} \subset^q \bigcup_{k+j=n} \tilde{A}_{k,j+1} = Q_{n+1}.$$

Hence,

$$Q_0 := \bigcup Q_n \sim^q Q.$$

Now, we will prove that  $v_n = u \wedge U_{Q_n} \rightarrow u \wedge U_Q$ . By Proposition 5.24(ii) we have

$$u \wedge U_Q \leq u \wedge U_{Q_n} + u \wedge U_{Q \setminus Q_n}.$$

Since  $Q \setminus Q_n \downarrow F$  with  $C_{\frac{2}{q}, q'}(F) = 0$ , we have by Proposition 5.24(iii) that

$$u \wedge U_{Q \setminus Q_n} \rightarrow 0.$$

The opposite inequality is obvious and the result follows in this assertion. By Lemma 5.24(ii)  $v_n = [v_{n+k}]_{Q_n}$ ,  $\forall k \in \mathbb{N}$ . Therefore

$$\mu_n(Q_n) = \mu_{n+k}(Q_n) = \mu_Q(Q_n). \tag{6.5}$$

(ii) First we assume that the function  $\eta \in W^{\frac{2}{q}, q'}(Q)$  has compact support in  $Q$ . Then by Lemma 4.3 there exists a function  $\eta_k$  such that  $\mathfrak{T}_q\text{-supp}(\eta_k) \subset Q_k$ , and

$$\|\eta - \eta_k\|_{W^{\frac{2}{q}, q'}} \leq \frac{1}{k}, \quad (6.6)$$

and  $|\eta_k| \leq |\eta|$ . By Lebesgue's dominated theorem, we can assume that  $\eta_k$  satisfies

$$\int_0^T \int_{\mathbb{R}^N} u^q(t, x) (\mathbb{H}[\eta - \eta_k])^{2q'} dx dt < \frac{1}{k}$$

Also in view of Proposition 5.2 and (5.7)-(5.11),

$$\lim_{t \rightarrow 0} \int_Q u(x, t) \eta^{2q'}(x) dx \leq C \|\eta\|_{L^\infty(\mathbb{R}^N)}^{q'} \|\eta\|_{W^{\frac{2}{q}, q'}}^{q'} + \int_0^T \int_{\mathbb{R}^N} u^q(t, x) (\mathbb{H}[\eta])^{2q'} dx dt,$$

But by (6.5) and Lemma 6.2 we have

$$\begin{aligned} \left( \int_Q \eta_k^{2q'}(x) d\mu_Q \right)^{\frac{1}{2q'}} &= \lim_{t \rightarrow 0} \left( \int_Q u(x, t) \eta_k^{2q'}(x) dx \right)^{\frac{1}{2q'}} \\ &\leq \lim_{t \rightarrow 0} \left( \int_Q u(x, t) \eta^{2q'}(x) dx \right)^{\frac{1}{2q'}} \\ &\leq \lim_{t \rightarrow 0} \left( \int_Q u(x, t) (\eta - \eta_k)^{2q'}(x) dx \right)^{\frac{1}{2q'}} + \lim_{t \rightarrow 0} \left( \int_Q u(x, t) \eta_k^{2q'}(x) dx \right)^{\frac{1}{2q'}} \\ &\leq \left( \int_Q \eta_k^{2q'}(x) d\mu_Q \right)^{\frac{1}{2q'}} + C \|\eta - \eta_k\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{2}} \|\eta - \eta_k\|_{W^{\frac{2}{q}, q'}}^{\frac{1}{2}} \\ &\quad + \left( \int_0^T \int_{\mathbb{R}^N} u^q(t, x) (\mathbb{H}[\eta - \eta_k])^{2q'} dx dt \right)^{\frac{1}{2q'}} \\ &\leq \left( \int_Q \eta_k^{2q'}(x) d\mu_Q \right)^{\frac{1}{2q'}} + C \frac{1}{\sqrt{k}} \|\eta\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{2}} + \left( \frac{1}{k} \right)^{\frac{1}{2q'}}. \end{aligned}$$

The result follows in this case by letting  $k \rightarrow \infty$ .

For the general case, by theorem 10.1.1 in [2], there exists a function  $\eta_k$  with compact support in  $Q$  such that

$$\|\eta - \eta_k\|_{W^{\frac{2}{q}, q'}} \leq \frac{1}{k}, \quad (6.7)$$

and  $|\eta_k| \leq |\eta|$ . The result follows as above.  $\square$

*Remark.* By Lemma 6.2 and (6.4), we have that the definition of the regular points in the elliptic case (see [23]) coincides with our definition of the regular points.

**Lemma 6.5** *Let  $Q$  be a  $\mathfrak{T}_q$ -open set and  $u \in \mathcal{U}_+(Q_T)$  satisfy (6.3). Then*

*i)*

$$[u]_Q = \sup\{[u]_F : F \subset^q Q, F \text{ } \mathfrak{T}_q\text{-closed}\}. \quad (6.8)$$

ii) For every  $\mathfrak{T}_q$ -open set  $O \subset \tilde{O} \subset^q Q$  such that  $[u]_O$  is a moderate solution we have

$$\mu_Q \chi_{\tilde{O}} = \text{tr}'([u]_O) = \text{tr}([u_Q]_O). \quad (6.9)$$

Finally,  $\mu_Q$  is  $\mathfrak{T}_q$ -locally finite on  $Q$  and  $\sigma$ -finite on  $Q' := \cup Q_n$ .

iii) If  $\{w_n\} \subset \mathcal{U}_+(Q_T)$  is a nondecreasing sequence of moderate solutions such that  $\mathfrak{T}_q\text{-supp}(w_n) \subset^q Q$  and  $w_n \uparrow [u]_Q$ , then  $\text{tr}(w_n) = \nu_n \uparrow \mu_Q$ .

*Proof.* i) Let  $u^*$  denote the right-hand side of (6.8). By Proposition 3.4 there exists a nondecreasing sequence  $\{[u]_{F_n}\}$  such that  $[u]_{F_n} \uparrow u^*$ . We consider the function  $[u]_{Q_n}$  of Proposition 6.4. Then by Proposition 5.24 we have

$$[u]_{F_n} \leq [u]_{F_n \cap Q_m} + [u]_{F_n \setminus Q_m}.$$

Now we note that  $F_n \setminus Q_m$  is a  $\mathfrak{T}_q$ -closed set and  $\cap_{m=1}^{\infty} F_n \setminus Q_m = A$  with  $C_{\frac{2}{q}, q'}(A) = 0$ . Thus by Proposition 5.19 we have that  $\lim_{m \rightarrow \infty} U_{F_n \setminus Q_m} = 0$  which implies  $\lim_{m \rightarrow \infty} [u]_{F_n \setminus Q_m} = 0$ . Thus  $[u]_{F_n} \leq \lim [u]_{Q_m} = u_Q$ . Letting  $n \rightarrow \infty$  we have  $u^* \leq u_Q$ . By definition of  $u^*$  we have that  $u_Q \leq u^*$ , thus  $u^* = u_Q$ .

ii) Put  $\mu_O = \text{tr}([u]_O)$ . If  $F$  is a  $\mathfrak{T}_q$ -closed set such that  $F \subset^q O$ , by Proposition 5.24-(ii) we have

$$\text{tr}([u]_F) = \text{tr}([u]_O|_F) = \mu_O \chi_F. \quad (6.10)$$

In particular the compatibility condition holds: if  $O' \subset \tilde{O}' \subset^q Q$  is  $\mathfrak{T}_q$ -open set such that  $[u]_{O'}$  is a moderate solution

$$\mu_{O \cap O'} = \mu_O \chi_{\tilde{O} \cap \tilde{O}'} = \mu_{O'} \chi_{\tilde{O} \cap \tilde{O}'}. \quad (6.11)$$

With the notation of (6.5),  $[v_{n+k}]_{Q_k} = v_k$  and hence  $\mu_{n+k} \chi_{\tilde{Q}_k} = \mu_k$  for every  $k \in \mathbb{N}$ .

Since  $[u]_F$  is moderate, we have by (6.11)

$$[v_n]_F = [u]_{F \cap \tilde{Q}_n} \uparrow [u]_F. \quad (6.12)$$

In addition,  $[u_Q]_F \geq \lim_{n \rightarrow \infty} [v_n]_F = [u]_F$ , jointly with  $u_Q \leq u$ , leads to,

$$[u]_F = [u_Q]_F. \quad (6.13)$$

By (6.10) and (6.12), if  $F$  is a  $\mathfrak{T}_q$ -closed subset of  $\mathcal{R}_q(u)$  and  $[u]_F$  is moderate,

$$\text{tr}([u]_F) = \lim_{n \rightarrow \infty} \text{tr}([v_n]_F) = \lim_{n \rightarrow \infty} \mu_n \chi_F = \mu_{\mathcal{R}_q} \chi_F, \quad (6.14)$$

which implies (6.9).

Since  $Q' := \cup Q_n \sim^q Q$ ,  $\mu_Q$  is  $\sigma$ -finite on  $Q'$ . The assertion that  $\mu_Q$  is  $\mathfrak{T}_q$ -locally finite on  $Q$  is a consequence of the fact that every point in  $Q$  is contained in a  $\mathfrak{T}_q$ -open set  $O \subset^q \tilde{O} \subset Q$  such that  $[u]_O$  is a moderate solution (see (6.4)).

iii) If  $w$  is a moderate solution and  $w \leq u_Q$  and  $\mathfrak{T}_q\text{-supp}(w) \subset^q Q$ , then  $\tau := \text{tr}(w) \leq \mu_Q$ .  
Indeed

$$[w]_{Q_n} \leq [u]_{Q_n} = v_n, [w]_{Q_n} \uparrow w \Rightarrow \text{tr}([w]_{Q_n}) \uparrow \tau \leq \lim_{n \rightarrow \infty} \text{tr}(v_n) = \mu_Q.$$

Now, let  $\{w_n\}$  be an increasing sequence of moderate solutions such that  $F_n := \mathfrak{T}_q\text{-supp}(w_n) \subset^q Q$  and  $w_n \uparrow u_Q$ . We must show that, if  $\nu_n := \text{tr}(w_n)$ , then

$$\nu := \lim_{n \rightarrow \infty} \nu_n = \mu_Q. \quad (6.15)$$

By the previous argument  $\nu \leq \mu_Q$ . The opposite inequality is obtained as follows. Let  $D$  be a  $\mathfrak{T}_q$ -open set such that  $[u]_D$  is moderate. Also, let  $K$  be a compact subset of  $D$  such that  $C_{\frac{2}{q}, q'}(K) > 0$ .

$$w_n \leq [w_n]_D + [w_n]_{D^c} \Rightarrow u_Q = \lim_{n \rightarrow \infty} w_n \leq \lim_{n \rightarrow \infty} [w_n]_D + U_{D^c}.$$

The sequence  $\{[w_n]_D\}$  is dominated by the moderate solution  $[u_Q]_D$ . In addition  $\text{tr}([w_n]_D) = \nu_n \chi_{\tilde{D}} \uparrow \nu \chi_{\tilde{D}}$ . Hence,  $\nu \chi_{\tilde{D}}$  is a Radon measure which vanishes on sets with  $C_{\frac{2}{q}, q'}$ -capacity zero. Also,  $[w_n]_D \uparrow u_{\nu \chi_{\tilde{D}}}$ , where  $u_{\nu \chi_{\tilde{D}}}$  is a moderate solution with initial trace  $\nu \chi_{\tilde{D}}$ . Consequently

$$u_Q = \lim_{n \rightarrow \infty} w_n \leq u_{\nu \chi_{\tilde{D}}} + U_{D^c}.$$

This in turn implies

$$\left( [u_Q]_K - u_{\nu \chi_{\tilde{D}}} \right)_+ \leq \inf(U_{D^c}, U_K),$$

the function on the left being a subsolution and the one on the right a supersolution. Therefore

$$\left( [u_Q]_K - u_{\nu \chi_{\tilde{D}}} \right)_+ \leq [[U]_{D^c}]_K = 0.$$

Thus,  $[u_Q]_K \leq u_{\nu \chi_{\tilde{D}}}$  and hence  $\mu_Q \chi_K \leq \nu \chi_{\tilde{D}}$ . Further, if  $O$  is a  $\mathfrak{T}_q$ -open set such that  $\tilde{O} \subset^q D$  then, in view of the fact that

$$\sup\{\mu_Q \chi_K : K \subset O, K \text{ compact}\} = \mu_Q \chi_O,$$

we obtain,

$$\mu_Q \chi_O \leq \nu \chi_{\tilde{D}}. \quad (6.16)$$

Applying this inequality to the sets  $Q_m, Q_{m+1}$  we finally obtain

$$\mu_Q \chi_{Q_m} \leq \nu \chi_{\tilde{Q}_{m+1}} \leq \nu \chi_{Q_{m+2}}.$$

Letting  $m \rightarrow \infty$  we conclude that  $\mu_{\mathcal{R}_q} \leq \nu$ . This completes the proof of (6.15).  $\square$

## 6.2 $\mathfrak{T}_q$ -perfect measures

**Definition 6.6** Let  $\mu$  be a positive Borel measure on  $\mathbb{R}^N$ .

(i) We say that  $\mu$  is **essentially absolutely continuous relative to**  $C_{\frac{2}{q}, q'}$  if the following condition holds:

If  $Q$  is a  $\mathfrak{T}_q$ -open set and  $A$  is a Borel set such that  $C_{\frac{2}{q}, q'}(A) = 0$  then

$$\mu(Q \setminus A) = \mu(Q).$$

This relation be denoted by

$$\mu \prec\prec_f C_{\frac{2}{q}, q'}.$$

(ii)  $\mu$  is **regular relative to  $\mathfrak{T}_q$ -topology** if, for every Borel set  $E$ ,

$$\begin{aligned} \mu(E) &= \inf\{\mu(D) : E \subset D, D \text{ } \mathfrak{T}_q\text{-open}\} \\ &= \inf\{\mu(K) : K \subset E, K \text{ compact}\}. \end{aligned} \quad (6.17)$$

$\mu$  is **outer regular relative to  $\mathfrak{T}_q$ -topology** if the first equality in (6.17) holds.

(iii) A positive Borel measure is called  **$\mathfrak{T}_q$ -perfect** if it is essentially absolutely continuous relative to  $C_{\frac{2}{q}, q'}$  and outer regular relative to  $\mathfrak{T}_q$ -topology. The space of  $\mathfrak{T}_q$ -perfect Borel measures is denoted by  $\mathfrak{M}_q(\mathbb{R}^N)$ .

**Proposition 6.7** If  $\mu \in \mathfrak{M}_q(\mathbb{R}^N)$  and  $A$  is a non-empty Borel set such that  $C_{\frac{2}{q}, q'}(A) = 0$ , then

$$\mu = \begin{cases} \infty & \text{if } \mu(Q \setminus A) = \infty \quad \forall Q \text{ } \mathfrak{T}_q\text{-open neighborhood of } A, \\ 0 & \text{otherwise.} \end{cases} \quad (6.18)$$

If  $\mu_0$  is an essentially absolutely continuous positive measure on  $\mathbb{R}^N$  and  $Q$  is  $\mathfrak{T}_q$ -open set such that  $\mu_0(Q) < \infty$  then  $\mu_0|_Q$  is **absolutely continuous with respect to  $C_{\frac{2}{q}, q'}$  in the strong sense**, i.e., if  $\{A_n\}$  is a sequence of Borel subsets of  $\mathbb{R}^N$

$$C_{\frac{2}{q}, q'}(A_n) \rightarrow 0 \Rightarrow \mu_0(Q \cap A_n) \rightarrow 0.$$

Let  $\mu_0$  is an essentially absolutely continuous positive Borel measure on  $\mathbb{R}^N$  and denote

$$\mu(E) = \inf\{\mu_0(D) : E \subset D, D \text{ } \mathfrak{T}_q\text{-open}\}, \quad (6.19)$$

for every Borel set  $E$ ; then

$$\begin{aligned} (a) \quad & \mu_0 \leq \mu \quad \mu_0(Q) = \mu(Q) \quad \forall Q \text{ } \mathfrak{T}_q\text{-open}, \\ (b) \quad & \mu|_Q = \mu_0|_Q \quad \text{for every } \mathfrak{T}_q\text{-open set } Q \text{ such that } \mu_0(Q) < \infty. \end{aligned} \quad (6.20)$$

Finally  $\mu$  is  $\mathfrak{T}_q$ -perfect; thus  $\mu$  is the smallest measure in  $\mathfrak{M}_q(\mathbb{R}^N)$  which dominates  $\mu_0$ .

*Proof.* The first assertion follows immediately from the definition  $\mathfrak{M}_q(\mathbb{R}^N)$ . We turn to the second assertion. If  $\mu_0$  is an essentially absolutely continuous positive Borel measure on  $\mathbb{R}^N$ , and  $Q$  is a  $\mathfrak{T}_q$ -open set such that  $\mu_0(Q) < \infty$  then  $\mu_0 \chi_Q$  is a bounded Borel measure which vanishes on sets of  $C_{\frac{2}{q}, q'}$ -capacity zero. If  $\{A_n\}$  is a sequence of Borel sets such that  $C_{\frac{2}{q}, q'}(A_n) \rightarrow 0$  and  $\mu_n = \chi_{Q \cap A_n}$ , then by Lemma 2.8-[19], there exists a unique moderate solution  $u_{\mu_n}$ . Also in view of Lemma 2.8-[19] we can prove that the sequence  $\{u_{\mu_n}\}$  is decreasing. Also by Proposition 5.17, we have  $u_{\mu_n} \leq U_{Q \cap A_n} \rightarrow 0$ , since  $C_{\frac{2}{q}, q'}(Q \cap A_n) \rightarrow 0$ . Thus we have that  $u_{\mu_n} \rightarrow 0$  locally uniformly and  $\mu_n \rightarrow 0$  weakly with respect to  $C_0(\mathbb{R}^N)$ . Hence  $\mu(Q \cap A_n) \rightarrow 0$ . Thus  $\mu_0|_Q$  is absolutely continuous with respect to  $C_{\frac{2}{q}, q'}$  in the strong sense.

Assertion (6.20)(a) follows from (6.19). It is clear that  $\mu$ , as defined by (6.19), is a measure. Now if  $Q$  is  $\mathfrak{T}_q$ -open set such that  $\mu_0(Q) < \infty$ , then  $\mu(Q) < \infty$  and both  $\mu_0|_Q$  and  $\mu|_Q$  are regular. Since they agree on open sets, the regularity implies (6.20) (b).

If  $A$  is a Borel set such that  $C_{\frac{2}{q}, q'}(A) = 0$  and  $Q$  is a  $\mathfrak{T}_q$ -open set then  $Q \setminus A$  is  $\mathfrak{T}_q$ -open and consequently

$$\mu(Q) = \mu_0(Q) = \mu_0(Q \setminus A) = \mu(Q \setminus A).$$

Thus  $\mu$  is essentially absolutely continuous. By (6.20) (a) and the definition of  $\mu$ , we have that  $\mu$  is outer regular with respect to  $C_{\frac{2}{q}, q'}$ . Thus  $\mu \in \mathfrak{M}_q(\mathbb{R}^N)$ .  $\square$

### 6.3 The initial trace on the regular set

**Proposition 6.8** *Let  $u \in \mathcal{U}_+(Q_T)$ .*

(i) *There exists an increasing sequence of  $\mathfrak{T}_q$ -open sets  $\{Q_n\}$  with the properties  $Q_n \subset \mathcal{R}_q(u)$ ,  $\tilde{Q}_n \subset^q Q_{n+1}$  and  $\mathcal{R}_{q,0}(u) := \bigcup_{n=1}^{\infty} Q_n \sim^q \mathcal{R}(u)$ , such that the solution*

$$v_n = u \wedge U_{Q_n} \text{ is moderate} \quad v_n \uparrow v_{\mathcal{R}_q}, \quad \text{tr}(v_n) \rightarrow \mu_{\mathcal{R}_q}. \quad (6.21)$$

(ii)

$$v_{\mathcal{R}_q} := \sup\{[u]_F : F \subset^q \mathcal{R}_q(u), F \text{ } \mathfrak{T}_q\text{-closed}\}. \quad (6.22)$$

*Thus  $v_{\mathcal{R}_q}$  is  $\sigma$ -moderate.*

(iii) *If  $[u]_F$  is moderate and  $F \subset^q \mathcal{R}_q(u)$ , there exists a  $\mathfrak{T}_q$ -open set  $Q$  such that  $F \subset^q Q$ ,  $[u]_Q$  is moderate solution and  $Q \subset \mathcal{R}_q(u)$*

(iv) *For every  $\mathfrak{T}_q$ -open set  $Q$ , such that  $[u]_Q$  is a moderate solution, we have*

$$\mu_{\mathcal{R}_q} \chi_{\tilde{Q}} = \text{tr}([u]_Q) = \text{tr}([v_{\mathcal{R}_q}]_Q). \quad (6.23)$$

*Finally,  $\mu_{\mathcal{R}_q}$  is  $\mathfrak{T}_q$ -locally finite on  $\mathcal{R}_q(u)$  and  $\sigma$ -finite on  $\mathcal{R}_{q,0}(u) := \bigcup Q_n$ .*

(v) *If  $\{w_n\}$  is a sequence of moderate solutions such that  $w_n \uparrow u_{\mathcal{R}_q}$  then,*

$$\mu_{\mathcal{R}_q} = \lim_{n \rightarrow \infty} \text{tr}(w_n) := \lim_{n \rightarrow \infty} \nu_n. \quad (6.24)$$

(vi) *The regularized measure  $\bar{\mu}_{\mathcal{R}_q}$  given by*

$$\bar{\mu}_{\mathcal{R}_q}(E) = \inf\{\mu_{\mathcal{R}_q}(Q) : E \subset Q, \quad Q \text{ } \mathfrak{T}_q\text{-open}, \quad E \text{ Borel}\}, \quad (6.25)$$

is  $\mathfrak{T}_q$ -perfect.

(vii)

$$u \approx_{\mathcal{R}_q(u)} v_{\mathcal{R}_q}.$$

(viii) For every  $\mathfrak{T}_q$ -closed set  $F \subset^q \mathcal{R}_q(u)$  :

$$[u]_F = [v_{\mathcal{R}_q}]_F. \quad (6.26)$$

If, in addition,  $\mu_{\mathcal{R}_q}(F \cap K) < \infty$  for any compact  $K \subset \mathbb{R}^N$ , then  $[u]_F$  is moderate and

$$\text{tr}([u]_F) = \mu_{\mathcal{R}_q} \chi_F. \quad (6.27)$$

(ix) If  $F$  is a  $\mathfrak{T}_q$ -closed set and  $C_{\frac{2}{q}, q'}(F) > 0$  then

$$\mu_{\mathcal{R}_q}(F \cap K) < \infty \quad \text{for any compact } K \subset \mathbb{R}^N \Leftrightarrow [u]_F \text{ is moderate.} \quad (6.28)$$

*Proof.* (i) For any  $z \in \mathcal{R}_q(u)$  there exist a bounded  $\mathfrak{T}_q$ -open set  $Q \subset \mathcal{R}_q(u)$  such that

$$\int_0^T \int_{\mathbb{R}^N} u^q(t, x) \mathbb{H}^{2q'}[\chi_Q] dx dt < \infty.$$

The result follows by similar arguments as in Lemma 6.4. Also, we recall that for any  $z \in \mathcal{R}_q(u)$  there exists a  $\mathfrak{T}_q$ -open set  $O_z \subset \mathcal{R}_q(u)$  such that

$$[u]_{O_z}, \quad (6.29)$$

is moderate.

Also we recall that  $v_n = [v_{n+k}]_{Q_n}$ ,  $\forall k \in \mathbb{N}$  and

$$\mu_n(Q_n) = \mu_{n+k}(Q_n) = \mu_{\mathcal{R}_q}(Q_n). \quad (6.30)$$

(ii) The proof is same as the one of Lemma 6.5-a)

(iii) First we assume that  $F$  is bounded. By definition and (6.29), every point in  $\mathcal{R}_q(u)$  possesses a  $\mathfrak{T}_q$ -open neighborhood  $A$  such that  $[u]_A$  is moderate. Then by Proposition 2.9, for any  $\varepsilon > 0$  there exists a  $\mathfrak{T}_q$ -open set  $Q_\varepsilon$  such that  $C_{\frac{2}{q}, q'}(F \setminus Q_\varepsilon) < \varepsilon$  and  $[u]_{Q_\varepsilon}$  is moderate. Since  $F$  is bounded, we can assume that so is  $Q_\varepsilon$ . Let  $O_\varepsilon$  be an open set containing  $F \setminus Q_\varepsilon$  such that  $C_{\frac{2}{q}, q'}(O_\varepsilon) < 2\varepsilon$ . Put

$$F_\varepsilon := F \setminus O_\varepsilon. \quad (6.31)$$

Then  $F_\varepsilon$  is a  $\mathfrak{T}_q$ -closed set,  $F_\varepsilon \subset F$ ,  $C_{\frac{2}{q}, q'}(F \setminus F_\varepsilon) < 2\varepsilon$  and  $F_\varepsilon \subset Q_\varepsilon$ .

*Assertion 1.* Let  $E$  be a  $\mathfrak{T}_q$ -closed set,  $D$  a  $\mathfrak{T}_q$ -open set such that  $[u]_D$  is moderate and  $E \subset^q D$ . Then there exists a decreasing sequence of  $\mathfrak{T}_q$ -open sets  $\{G_n\}$  such that

$$E \subset^q G_{n+1} \subset \tilde{G}_{n+1} \subset^q G_n \subset^q D, \quad (6.32)$$

and

$$[u]_{G_n} \rightarrow [u]_E \quad \text{in } L^q(K) \text{ for any compact } K \subset \overline{Q_T}. \quad (6.33)$$

By Lemma 2.8 and Proposition 5.24-(iii), there exists a decreasing sequence of  $\mathfrak{T}_q$ -open sets  $\{G_n\}$  satisfying (6.32) and, in addition, such that  $[u]_{G_n} \downarrow [u]_E$  locally uniformly in  $\mathbb{R}^N$ . Since  $[u]_{G_n} \leq [u]_D$  and the later is a moderate solution we obtain (6.33).

Now we assume that  $F$  is  $\mathfrak{T}_q$ -closed set (possibly unbounded). Let  $x \in F$  and  $B_n = B_n(x)$ ;  $n \in \mathbb{N}$ . Set

$$E_n = \bigcup_{m=1}^n (F \cap B_n)_{\frac{1}{2^m}},$$

where  $(F \cap B_n)_{\frac{1}{2^m}}$  is the set in (6.31), if we replace  $F$  by  $F \cap B_n$  and  $\varepsilon$  by  $\frac{1}{2^m}$ . Also we assume without loss of generality that  $\{E_n\}$  is an increasing sequence. Also set

$$Q_n = \bigcup_{m=1}^n Q_{\frac{1}{m}},$$

where  $Q_{\frac{1}{m}} = (F \cap B_n)_{\frac{1}{m}}$ . Also we may assume that the sequence of set  $\{Q_n\}$  is increasing. Therefore, we have that  $E_n \subset E$ ,  $Q_n$  is  $\mathfrak{T}_q$ -open,  $[u]_{Q_n}$  is moderate and  $E_n \subset^q Q_n$  and  $\cup E_n = E' \sim^q F$ , since

$$\begin{aligned} C_{\frac{2}{q}, q'}(F \setminus \bigcup_{n=1}^{\infty} E_n) &\leq \sum_{k=1}^n C_{\frac{2}{q}, q'} \left( (F \cap B_k) \setminus \bigcup_{j=1}^{\infty} E_j \right) + \sum_{k=n+1}^{\infty} C_{\frac{2}{q}, q'}((F \cap B_k) \setminus E_k) \\ &\leq \frac{1}{2^n} + \sum_{k=n+1}^{\infty} \frac{1}{2^k}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Thus by Assertion 1, it is possible to choose a sequence of  $\mathfrak{T}_q$ -open sets  $\{V_n\}$  such that

$$E_n \subset^q V_n \subset \tilde{V}_n \subset^q Q_n, \quad \|[u]_{V_n} - [u]_{E_n}\|_{L^q(B_n(0)) \times (0, T]} \leq 2^{-n}. \quad (6.34)$$

We note here that since  $E_n, Q_n$  are bounded sets, the function  $[u]_{V_n}, [u]_{E_n}$  have compact support with respect to variable "x" in  $\mathbb{R}^N$ , thus we can take the norm in (6.34) in whole space  $\mathbb{R}^N \times (0, T]$ .

Because  $[u]_F$  is moderate, there exists a Radon measure  $\mu_F$  where  $\mu_F = \text{tr}([u]_F)$ . Moreover,  $[u]_F = [u]_{E'}$  since  $F \sim^q E'$ . Finally, we have by (5.35) and the fact that  $E_n \subset^q F$ ,

$$[u]_{E_n} = [u]_{E_n \cap F} = [[u]_{E_n}]_F.$$

Using the above equality and the fact that  $[u]_F$  is moderate, we have that  $\text{tr}([u]_{E_n}) = \chi_{E_n} \mu_F$ . Now since  $E_n \uparrow E' \sim^q F$ , it implies that  $[u]_{E_n} \uparrow [u]_F$   $L^q(K \times [0, T])$ , for compact set  $K \subset \mathbb{R}^N$ . Hence, we derive from by (6.34) that  $[u]_{V_n} \rightarrow [u]_F$  in  $L^q(K \times [0, T])$  for each compact set



$K \subset \mathbb{R}^N$ .

Let  $\{V_{n_k}\}$  be a sequence such that

$$\left( \int_0^1 \int_{B_k(0)} |[u]_{V_{n_k}} - [u]_F|^q dx dt \right)^{\frac{1}{q}} \leq 2^{-k}. \quad (6.35)$$

If  $K$  is a compact set, there exist  $k_0 \in \mathbb{N}$  such that  $K \subset B_k(0)$ ,  $\forall k \geq k_0$ . Set  $W = \bigcup_{k=1}^{\infty} V_{n_k}$ , then

$$[u]_W \leq \sum_{k=1}^{\infty} [u]_{V_{n_k}}.$$

Thus we have

$$\begin{aligned} \left( \int_0^T \int_K |[u]_W - [u]_F|^q dx dt \right)^{\frac{1}{q}} &\leq \sum_{k=1}^{k_0} \left( \int_0^T \int_K |[u]_{V_{n_k}} - [u]_F|^q dx dt \right) \\ &\quad + \sum_{k=k_0+1}^{\infty} \left( \int_0^T \int_{B_k(0)} |[u]_{V_{n_k}} - [u]_F|^q dx dt \right)^{\frac{1}{q}} \\ &\leq \sum_{k=1}^{k_0} \left( \int_0^T \int_K |[u]_{V_{n_k}} - [u]_F|^q dx dt \right) + \sum_{k=k_0+1}^{\infty} 2^{-k} \\ &< \infty. \end{aligned}$$

We recall that  $F \subset^q W$  and  $W$  is a  $\mathfrak{T}_q$ -open set. Using the facts that  $[u]_F$  is moderate,  $K$  is an abstract compact domain and the above inequality, we obtain that  $[u]_W$  is moderate. Thus by Lemma 6.2 we have that  $W \subset \mathcal{R}_q(u)$ .

(iv) Let  $Q$  be a  $\mathfrak{T}_q$ -open set and  $[u]_Q$  be a moderate solution, put  $\mu_Q = \text{tr}([u]_Q)$ . If  $F$  is a  $\mathfrak{T}_q$ -closed set such that  $F \subset^q Q$  then, by Proposition 5.24-(ii),

$$\text{tr}[u]_F = \text{tr}([u]_Q|_F) = \mu_Q \chi_F. \quad (6.37)$$

In particular the compatibility condition holds: if  $Q, Q'$  are  $\mathfrak{T}_q$ -open regular sets then

$$\mu_{Q \cap Q'} = \mu_Q \chi_{\tilde{Q} \cap \tilde{Q}'} = \mu_{Q'} \chi_{\tilde{Q} \cap \tilde{Q}'}. \quad (6.38)$$

With the notation of (i),  $[v_{n+k}]_{Q_k} = v_k$  and hence  $\mu_{n+k} \chi_{\tilde{Q}_k} = \mu_k$  for every  $k \in \mathbb{N}$ .

Let  $F$  be an arbitrary  $\mathfrak{T}_q$ -closed regular subset of  $\mathcal{R}_q(u)$ . Since  $[u]_F$  is moderate, we have by (6.38)

$$[v_n]_F = [u]_{F \cap \tilde{Q}_n} \uparrow [u]_F. \quad (6.39)$$

In addition,  $[v_{\mathcal{R}_q}]_F \geq \lim_{n \rightarrow \infty} [v_n]_F = [u]_F$ , jointly with  $v_{\mathcal{R}_q} \leq u$ , leads to,

$$[u]_F = [v_{\mathcal{R}_q}]_F. \quad (6.40)$$

By (6.37) and (6.39), if  $F$  is a  $\mathfrak{T}_q$ -closed subset of  $\mathcal{R}_q(u)$  and  $[u]_F$  is moderate,

$$\text{tr}([u]_F) = \lim_{n \rightarrow \infty} \text{tr}([v_n]_F) = \lim_{n \rightarrow \infty} \mu_n \chi_F = \mu_{\mathcal{R}_q} \chi_F, \quad (6.41)$$

which implies (6.23).

Since  $\mathcal{R}_{q,0}(u)$  has a regular decomposition,  $\mu_{\mathcal{R}_q}$  is  $\sigma$ -finite on  $\mathcal{R}_{q,0}(u)$ . The assertion that  $\mu_{\mathcal{R}_q}$  is  $\mathfrak{T}_q$ -locally finite on  $\mathcal{R}_q(u)$  is a consequence of the fact that every point  $\xi \in \mathcal{R}_q(u)$  is contained in a  $\mathfrak{T}_q$ -open set  $O_\xi \subset \mathcal{R}_q(u)$  such that  $[u]_{O_\xi}$  is moderate and thus  $\mu_{\mathcal{R}_q} \chi_{O_\xi} < \infty$ .

(v) If  $w$  is a moderate solution and  $w \leq v_{\mathcal{R}_q}$  and  $\mathfrak{T}_q\text{-supp}(w) \subset^q \mathcal{R}_q(u)$  then  $\tau := \text{tr}(w) \leq \mu_{\mathcal{R}_q}$ . Indeed

$$[w]_{Q_n} \leq [v_{\mathcal{R}_q}] = v_n, [w]_{Q_n} \uparrow w \Rightarrow \text{tr}([w]_{Q_n}) \uparrow \tau \leq \lim_{n \rightarrow \infty} \text{tr}(v_n) = \mu_{\mathcal{R}_q}.$$

Now, let  $\{w_n\}$  be an increasing sequence of moderate solutions such that  $F_n := \mathfrak{T}_q\text{-supp}(w_n) \subset^q \mathcal{R}_q(u)$  and  $w_n \uparrow v_{\mathcal{R}_q}$ . If  $\nu_n := \text{tr}(w_n)$ , we have to prove that

$$\nu := \lim_{n \rightarrow \infty} \nu_n = \mu_{\mathcal{R}_q}. \quad (6.42)$$

By the previous argument  $\nu \leq \mu_{\mathcal{R}_q}$ . The opposite inequality is obtained as follows. Let  $D$  be a  $\mathfrak{T}_q$ -open set,  $[u]_D$  be moderate and let  $K$  be a compact subset of  $D$  such that  $C_{\frac{2}{q}, q'}(K) > 0$ .

$$w_n \leq [w_n]_D + [w_n]_{D^c} \Rightarrow v_{\mathcal{R}_q} = \lim_{n \rightarrow \infty} w_n \leq \lim_{n \rightarrow \infty} [w_n]_D + U_{D^c}.$$

The sequence  $\{[w_n]_D\}$  is dominated by the moderate function  $[v_{\mathcal{R}_q}]_D$ . In addition  $\text{tr}([w_n]_D) = \nu_n \chi_{\tilde{D}} \uparrow \nu \chi_{\tilde{D}}$ . Hence,  $\nu \chi_{\tilde{D}}$  is a Radon measure which vanishes on sets with  $C_{\frac{2}{q}, q'}$ -capacity zero. Also,  $[w_n]_D \uparrow u_{\nu \chi_{\tilde{D}}}$ , where the function  $u_{\nu \chi_{\tilde{D}}}$  on the right is the moderate solution with initial trace  $\nu \chi_{\tilde{D}}$ . Consequently

$$v_{\mathcal{R}_q} = \lim_{n \rightarrow \infty} w_n \leq u_{\nu \chi_{\tilde{D}}} + U_{D^c}.$$

This in turn implies

$$\left( [v_{\mathcal{R}_q}]_K - u_{\nu \chi_{\tilde{D}}} \right)_+ \leq \inf(U_{D^c}, U_K).$$

Note that, in the previous relation, the function on the left being a subsolution and the one on the right a supersolution, we obtain

$$\left( [v_{\mathcal{R}_q}]_K - u_{\nu \chi_{\tilde{D}}} \right)_+ \leq [[U]_{D^c}]_K = 0.$$

Thus,  $[v_{\mathcal{R}_q}]_K \leq u_{\nu \chi_{\tilde{D}}}$  and hence  $\mu_{\mathcal{R}_q} \chi_K \leq \nu \chi_{\tilde{D}}$ . Further, if  $Q$  is a  $\mathfrak{T}_q$ -open set such that  $\tilde{Q} \subset^q D$  then, in view of the fact that

$$\sup\{\mu_{\mathcal{R}_q} \chi_K : K \subset Q, K \text{ compact}\} = \mu_{\mathcal{R}_q} \chi_Q,$$

we obtain,

$$\mu_{\mathcal{R}_q} \chi_Q \leq \nu \chi_{\tilde{D}}. \quad (6.43)$$

Applying this inequality to the sets  $Q_m, Q_{m+1}$  we finally obtain

$$\mu_{\mathcal{R}_q} \chi_{Q_m} \leq \nu \chi_{\tilde{Q}_{m+1}} \leq \nu \chi_{Q_{m+2}}.$$

Letting  $m \rightarrow \infty$  we conclude that  $\mu_{\mathcal{R}_q} \leq \nu$ . This completes the proof of (6.42) and of assertion (v).

(vi) The measure  $\mu_{\mathcal{R}_q}$  is essentially absolutely continuous relative to  $C_{\frac{2}{q}, q'}$ . Clearly this assertion follows now from Proposition 6.7.

(vii) By (5.34)

$$u \leq [u]_{Q_n} + [u]_{Q_n^c}.$$

Now since  $Q_n^c$  is  $\mathfrak{T}_q$ -closed and  $Q_n^c \downarrow \mathcal{R}_{q,0}^c(u)$ , we have by Proposition 5.24-(iii) that

$$[u]_{Q_n^c} \downarrow [u]_{\mathcal{R}_{q,0}^c(u)}.$$

Hence

$$\lim_{n \rightarrow \infty} (u - [u]_{Q_n}) = u - v_{\mathcal{R}_q} \leq [u]_{\mathcal{R}_{q,0}^c(u)},$$

therefore  $u \ominus v_{\mathcal{R}_q} \approx_{\mathcal{R}_{q,0}(u)} 0$ . Since  $v_{\mathcal{R}_q} \leq u$  this is equivalent to the statement  $u \approx_{\mathcal{R}_{q,0}(u)} v_{\mathcal{R}_q}$ .

(viii) (6.26) follows by the previous statement. Now we assume that  $\mu_{\mathcal{R}_q}(F)\chi_K < \infty$  for any compact  $K \subset \mathbb{R}^N$ . Now set  $F_n = F \cap \tilde{Q}_n$ . By (5.34).

$$[u]_F \leq [u]_{F_n} + [u]_{F \setminus F_n} = [u]_{F_n} + [u]_{F \setminus \tilde{Q}_n} \leq [u]_{F_n} + [u]_{F \setminus Q_n}.$$

Now since  $F \setminus Q_n$  is a  $\mathfrak{T}_q$ -closed set and  $\cap F \setminus Q_n = G$  with  $C_{\frac{2}{q}, q'}(G) = 0$ , we have by Proposition 5.24-(iii) that  $[u]_{F \setminus Q_n} \rightarrow [u]_G = 0$ . Hence  $[u]_F = \lim_{n \rightarrow \infty} [u]_{F_n}$ , and  $\text{tr}([u]_{F_n}) = \mu_{\mathcal{R}_q} \chi_{F_n} \uparrow \mu_{\mathcal{R}_q} \chi_F = \mu_{\mathcal{R}_q} \chi_F$ . Since  $\mu_{\mathcal{R}_q} \chi_F$  is a Radon measure essentially absolutely continuous relative to  $C_{\frac{2}{q}, q'}$ ,  $[u]_F$  is moderate and (6.27) holds.

(ix) If  $\mu_{\mathcal{R}_q}(F)\chi_K < \infty$  for any compact  $K \subset \mathbb{R}^N$  then, by (viii),  $[u]_F$  is moderate. Conversely, if  $[u]_F$  is moderate, by (6.23),  $\mu_{\mathcal{R}_q}(F)\chi_K < \infty$  for any compact  $K \subset \mathbb{R}^N$ .  $\square$

**Example.** We give below an example which shows that there exists  $u \in \mathcal{U}_+(Q_T)$  such that  $\mathcal{R}_q(u) = \mathbb{R}^N$  but  $u$  is not a moderate solution. Let  $\eta : [0, \infty) \rightarrow [0, \infty)$  be a smooth function such that  $\eta(r) > 0$  for any  $r > 0$  and  $\lim_{r \rightarrow 0^+} \eta(r) = 0$ , ( $\eta$  tends to 0 very fast, for example  $\eta(r) = e^{-\frac{1}{r^2}}$ ). Let  $K$  be the close set

$$K = \{(x', x_n) \in \mathbb{R}^N : |x'| \leq \eta(x_n), x_n \geq 0\}.$$

Then  $K$  is thin at the origin 0.

Set  $f(x) = \frac{1}{\eta^n(x_n)}$  if  $x \in K$  and  $f = 0$  otherwise. We define the measure

$$\mu = f dx.$$

This measure possesses the following properties:

1.  $\mu$  is  $\mathfrak{T}_q$ -locally finite.
  2.  $\mu(Q_n) < \infty$  where  $Q_n = B_{2n}(0) \setminus \overline{B}_{\frac{1}{n}}(0)$  and  $\cup Q_n \sim^q \mathbb{R}^N$
  3.  $\mu(F) = 0$  for any  $F$  such that  $C_{\frac{2}{q}, q'}(F) = 0$ .
  4. There exists a non decreasing sequence of bounded Radon measures  $\mu_n$  absolutely continuous with  $C_{\frac{2}{q}, q'}$  such that
    - (a)  $\mathfrak{T}_q\text{-supp}(\mu_n) \subset \tilde{Q}_n$ ,  $\mu_n(A) = \mu_{n+k}(A)$  for any  $A \subset \tilde{Q}_n$  and any  $n, k \in \mathbb{N}$ .
    - (b)  $\lim_{n \rightarrow \infty} \mu_n = \mu$
  5. We can construct a solution  $u \in \mathcal{U}_+(Q_T)$  with respect to this measure.
- As we see later this solution is unique since it is  $\sigma$ -moderate (see Proposition 6.12).

**Lemma 6.9** *Let  $\mu$  satisfying the conditions 1-4 as above. Then there exists an open set  $\mathcal{R}_q \sim^q \mathbb{R}^N$  such that the measure  $\mu$  is a Radon measure in  $\mathcal{R}_q$ .*

*Proof.* We consider the ball  $B_R(0)$  with  $R > 1$ . From [23, Lemma 2.5] there exists a sequence of open sets  $\{O_{\frac{1}{m}}\}_{m=1}^{\infty}$  and  $n(m) \in \mathbb{N}$  such that  $C_{\frac{2}{q}, q'}(O_{\frac{1}{m}}) < \frac{1}{m}$ , and

$$\overline{B}_R(0) \setminus O_{\frac{1}{m}} \subset \bigcup_{i=1}^{n(m)} Q_i. \quad (6.44)$$

Now since  $O_{\frac{1}{m}}$  is open we have

$$C_{\frac{2}{q}, q'}(\overline{O}_{\frac{1}{m}}) = C_{\frac{2}{q}, q'}(\tilde{O}_{\frac{1}{m}} \cup (\overline{O}_{\frac{1}{m}} \cap e_q(O))) \leq C_{\frac{2}{q}, q'}(\tilde{O}_{\frac{1}{m}}) \leq cC_{\frac{2}{q}, q'}(O_{\frac{1}{m}}) \rightarrow 0,$$

where  $e_q(O)$  is the set of thin points of  $O$ .

Thus if  $x \in B_R(0) \setminus \bigcap_{m=1}^{\infty} \overline{O}_{\frac{1}{m}}$  there exist  $r > 0$  small enough and  $N \in \mathbb{N}$  such that

$$B_r(x) \subset B_R(0) \setminus \bigcap_{m=1}^N \overline{O}_{\frac{1}{m}}.$$

Thus by the properties of  $\mu$  and (6.44) we have

$$\mu(B_r(x)) < \infty.$$

We define

$$\mathcal{R}_q := \{x \in \mathbb{R}^N : \exists r > 0 \text{ such that } \mu(B_r(x)) < \infty\}.$$

Then the set  $\mathcal{R}_q$  is open and by the above argument, letting  $R$  go to infinity, we have that  $\mathcal{R}_q \sim^q \mathbb{R}^N$ . Also by the definition of  $\mathcal{R}_q$ , it is easy to see that  $\mu(K) < \infty$  for any compact  $K \subset \mathcal{R}_q$  and by the properties of  $\mu$  we can prove that  $\mu$  is Radon measure in  $\mathcal{R}_q$ .  $\square$

## 6.4 The precise initial trace

We are now in condition to define the *precise initial trace*.

**Definition 6.10** *Let  $q \geq 1 + \frac{2}{N}$  and  $u \in \mathcal{U}_+(Q_T)$ .*

**a:** *The solution  $v_{\mathcal{R}_q}$  defined by (6.22) is called **regular component** of  $u$  and will be denoted by  $u_{reg}$ .*

**b:** *Let  $\{v_n\}$  be an increasing sequence of moderate solutions satisfying condition (6.21) and put  $\mu_{\mathcal{R}_q} = \mu_{\mathcal{R}_q}(u) := \lim_{n \rightarrow \infty} \text{tr}(v_n)$ . Then, the regularized measure  $\overline{\mu}_{\mathcal{R}_q}$ , defined by (6.25), is called the **regular initial trace** of  $u$ . It will be denoted by  $\text{tr}_{\mathcal{R}_q}(u)$ .*

**c:** *The couple  $(\text{tr}_{\mathcal{R}_q}(u), \mathcal{S}_q(u))$  is called **the precise initial trace** of  $u$  and will be denoted by  $\text{tr}^c(u)$ .*

**d:** *Let  $\nu$  be the Borel measure on  $\mathbb{R}^N$  given by*

$$\nu = \begin{cases} (\text{tr}_{\mathcal{R}_q}(u))(E) & \text{if } E \subset \mathcal{R}_q(u), \\ \nu(E) = \infty & \text{if } E \cap \mathcal{S}_q(u) \neq \emptyset, \end{cases} \quad (6.45)$$

*for every Borel set  $E$ . Then  $\nu$  is the measure representation of the precise trace of  $u$ , to be denoted by  $\text{tr}(u)$ .*

Note that, by Proposition 6.8-(v), the measure  $\mu_{\mathcal{R}_q}$  is independent of the choice of the sequence  $\{v_n\}$ .

**Theorem 6.11** *Assume that  $u \in \mathcal{U}_+(Q_T)$  is a  $\sigma$ -moderate solution, i.e., there exists an increasing sequence  $\{u_n\}$  of positive moderate solutions such that  $u_n \uparrow u$ . Let  $\mu_n = \lim_{n \rightarrow \infty} \text{tr}(u_n)$ ,  $\mu_0 := \lim_{n \rightarrow \infty} \mu_n$  and set, for any Borel set  $E$ ,*

$$\mu(E) = \inf \{ \mu_0(Q) : E \subset Q, \quad Q \text{ } \mathfrak{T}_q\text{-open} \}. \quad (6.46)$$

*Then:*

(i)  $\mu$  is the precise initial trace of  $u$  and  $\mu$  is  $\mathfrak{T}_q$ -perfect. In particular  $\mu$  is independent of the sequence  $\{u_n\}$  which appears in its definition.

(ii) If  $A$  is a Borel set such that  $\mu(A) < \infty$ , then  $\mu(A) = \mu_0(A)$ .

(iii) A solution  $u \in \mathcal{U}_+(Q_T)$  is  $\sigma$ -moderate if and only if

$$u = \sup \{ v \in \mathcal{U}_+(Q_T) : v \text{ moderate, } v \leq u \}, \quad (6.47)$$

which is equivalent to

$$u = \sup \{ u_\tau \in \mathcal{U}_+(Q_T) : \tau \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \quad \tau \leq \text{tr}(u) \}. \quad (6.48)$$

(iv) If  $u, w$  are  $\sigma$ -moderate solutions,

$$\text{tr}(w) \leq \text{tr}(u) \Leftrightarrow w \leq u. \quad (6.49)$$

*Proof.* The proof is an adaptation of the one in [23].

(i) Let  $Q$  be a  $\mathfrak{T}_q$ -open set and  $A$  a Borel set such that  $C_{\frac{2}{q}, q'}(A) = 0$ . Then  $\mu_n(A) = 0$  so that  $\mu_0(A) = 0$ . Thus  $\mu_0$  is essentially absolutely continuous and, by Proposition 6.7,  $\mu$  is  $\mathfrak{T}_q$ -perfect.

Let  $\{D_n\}$  be the family of  $\mathfrak{T}_q$ -open sets as in Proposition 6.8-(i). Put  $D'_n = \mathcal{R}_q(u) \setminus D_n$  and observe that  $D'_n \downarrow E$  where  $C_{\frac{2}{q}, q'}(E) = 0$ . Therefore, for fixed  $n$ ,

$$u_{\mu_n \chi_{D'_m}} \downarrow 0 \quad \text{when } m \rightarrow \infty.$$

Thus there exist a subsequence, say  $\{D'_n\}$ , such that

$$\left( \int_0^T \int_{B_n(0)} |u_{\mu_n \chi_{D'_n}}|^q dx dt \right)^{\frac{1}{q}} \leq 2^{-n}.$$

Since,

$$\mu_n(\mathcal{R}_q(u)) = \mu_n \chi_{D_n} + \mu_n \chi_{D'_n},$$

it follows that

$$\lim_{n \rightarrow \infty} \left| u_{\mu_n \chi_{\mathcal{R}_q(u)}} - u_{\mu_n \chi_{D_n}} \right| \leq \lim_{n \rightarrow \infty} u_{\mu_n \chi_{D'_n}} = 0.$$

Thus

$$u_n \leq u_{\mu_n \chi_{D_n}} + [u]_{\mathcal{S}_q(u)}.$$

Hence

$$u - [u]_{\mathcal{S}_q(u)} \leq w := \lim_{n \rightarrow \infty} u_{\mu_n \chi_{\mathcal{R}_q(u)}} = \lim_{n \rightarrow \infty} u_{\mu_n \chi_{D_n}} \leq u_{reg}.$$

This implies  $u \ominus [u]_{\mathcal{S}_q(u)} \leq u_{reg}$ . For the opposite inequality, by Proposition 6.8-(iv) we get

$$[u]_{D_n} \uparrow u_{reg}.$$

But by (5.46) and using the facts that  $\tilde{D}_n \subset^q D_{n+1} \subset \tilde{D}_{n+1} \subset^q \mathcal{R}_q(u)$ ,  $C_{\frac{2}{q}, q'}(\tilde{D}_{n+1} \cap \mathcal{S}_q(u)) = 0$ ,

$$[u]_{D_n} \leq [[u]_{\mathcal{S}_q(u)}]_{D_{n+1}} + [u \ominus [u]_{\mathcal{S}_q(u)}]_{D_{n+1}} = [u \ominus [u]_{\mathcal{S}_q(u)}]_{D_{n+1}} \leq u \ominus [u]_{\mathcal{S}_q(u)}.$$

Letting  $n \rightarrow \infty$  we derive  $u_{reg} \leq u \ominus [u]_{\mathcal{S}_q(u)}$ . Therefore  $\lim_{n \rightarrow \infty} u_{\mu_n \chi_{D_n}} = u_{reg}$ . Thus the sequence  $\{u_{\mu_n \chi_{D_n}}\}$  satisfies condition (6.21) and consequently, by Proposition 6.8-(iv) and Definition 6.10,

$$\lim_{n \rightarrow \infty} \mu_n \chi_{D_n} = \mu_{\mathcal{R}_q}, \quad \text{tr}_{\mathcal{R}_q}(u) = \bar{\mu}_{\mathcal{R}_q}. \quad (6.50)$$

Next, we claim that, if  $\xi \in \mathcal{S}_q(u)$  then, for every  $\mathfrak{T}_q$ -open bounded neighborhood  $Q$  of  $\xi$   $\mu_n(\tilde{Q}) \rightarrow \infty$ . Indeed let  $\eta \in W_{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with  $\mathfrak{T}_q$ -support in  $Q$ . Put  $h = \mathbb{H}[\eta]$  and  $\phi(r) = r_+^{2q'}$ . Then by Proposition 5.7, Lemma 5.3 and in view of the proof of Lemma 5.2 we have

$$\int_0^T \int_{\mathbb{R}^N} (-u_n(\partial_t \phi(h) + \Delta \phi(h))) + u_n^q \phi(h) dx d\tau + \int_{\mathbb{R}^N} u_n \phi(h)(\cdot, T) dx = \int_Q \eta^{2q'} d\mu_n.$$

In view of Lemma 5.2, we can prove

$$\int_0^T \int_{\mathbb{R}^N} u_n^q \phi(h) dx d\tau \leq C(q) \left( \int_Q \eta^{2q'} d\mu_n + \|\eta\|_{W_{\frac{2}{q}, q'}}^{2q'} + \|\eta\|_{L^\infty} \right).$$

By Lemma 4.2 there exist  $\eta \in W_{\frac{2}{q}, q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and a  $\mathfrak{T}_q$ -open set  $D \subset Q$  such that  $\eta = 1$  on  $D$ ,  $\eta = 0$  outside of  $Q$  and  $0 \leq \eta \leq 1$ . Letting  $n \rightarrow \infty$  we have

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^N} u_n^q \mathbb{H}^{2q'}[\chi_D] dx d\tau \leq C(q) \left( \lim_{n \rightarrow \infty} \int_Q \eta^{2q'} d\mu_n + \|\eta\|_{W_{\frac{2}{q}, q'}}^{2q'} + \|\eta\|_{L^\infty} \right),$$

the assertion follows by Lemma 5.4.

In conclusion, if  $\xi \in \mathcal{S}_q(u)$  then  $\mu_0(\tilde{Q}) = \infty$  for every  $\mathfrak{T}_q$ -open neighborhood of  $\xi$ . Consequently  $\mu(\xi) = \infty$ . This fact and (6.50) imply that  $\mu$  is the precise trace of  $u$ .

(ii) If  $\mu(A) < \infty$  then  $A$  is contained in a  $\mathfrak{T}_q$ -open set  $D$  such that  $\mu_0(D) < \infty$  and, by Proposition 6.7,  $\mu(A) = \mu_0(A)$ .

(iii) Let  $u \in \mathcal{U}_+(Q_T)$  be  $\sigma$ -moderate and put

$$u^* := \sup\{v : v \text{ moderate}, \quad v \leq u\}. \quad (6.51)$$

By its definition  $u^* \leq u$ . On the other hand, since there exists an increasing sequence of moderate solutions  $\{u_n\}$  converging to  $u$ , it follows that  $u \leq u^*$ . Thus  $u = u^*$ .

Conversely, if  $u \in \mathcal{U}_+(Q_T)$  and  $u = u^*$  then by proposition 3.4, there exists an increasing sequence of moderate solutions  $\{u_n\}$  converging to  $u$ . Therefore  $u$  is  $\sigma$ -moderate.

Since  $u$  is  $\sigma$ -moderate there exist an increasing sequence of moderate solutions  $\{u_n\}$  converging to  $u$ . In view of the discussion at the beginning of subsection 5.1, for any  $u_n$  there exist an increasing sequence of  $\{w_m\}$  such that  $w_m \uparrow u_n$  and  $\text{tr}(w_m) \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ . Thus

$$u_n \leq \sup\{u_\tau : \tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \tau \leq \text{tr}(u)\} =: u_n^\ddagger.$$

Letting  $n \rightarrow \infty$ , we have  $u \leq u^\ddagger$ .

On the other hand, if  $u$  is  $\sigma$ -moderate,  $\tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  and  $\tau \leq \text{tr}(u)$  then (with  $\mu_n$  and  $u_n$  as in the statement of the Proposition),  $\text{tr}(u_\tau \ominus u_n) = (\tau - \mu_n)_+ \downarrow 0$ . Hence,  $u_\tau \ominus u_n \downarrow 0$ , which implies,  $u_\tau \leq u$ . Therefore  $u^\ddagger \leq u$ . Thus (6.47) implies (6.48) and each of them that  $u$  is  $\sigma$ -moderate. Therefore the two statements are equivalent.

(iv) The assertion  $\Rightarrow$  is a consequence of (6.48). To verify the assertion  $\Leftarrow$  it is sufficient to show that if  $w$  is moderate,  $u$  is  $\sigma$ -moderate and  $w \leq u$ , then  $\text{tr}(w) \leq u$ . Let  $\{u_n\}$  be an increasing sequence of positive moderate solutions converging to  $u$ . Then  $u_n \vee w \leq u$  and consequently  $u_n \leq u_n \vee w \uparrow u$ . Therefore  $\text{tr}(u_n \vee w) \uparrow \mu' \leq \text{tr}(u)$  so that  $\text{tr}(w) \leq \text{tr}(u)$ .  $\square$

**Theorem 6.12** *Let  $u \in \mathcal{U}_+(Q_T)$  and put  $\nu = \text{tr}(u)$ .*

(i)  $u_{\text{reg}}$  is  $\sigma$ -moderate and  $\text{tr}(u_{\text{reg}}) = \text{tr}_{\mathcal{R}_q}(u)$ .

(ii) If  $v \in \mathcal{U}_+(Q_T)$

$$v \leq u \Rightarrow \text{tr}(v) \leq \text{tr}(u). \quad (6.52)$$

If  $F$  is a  $\mathfrak{T}_q$ -closed set, then

$$\text{tr}([u]_F) \leq \nu \chi_F. \quad (6.53)$$

(iii) A singular point can be characterized in terms of the measure  $\nu$  as follows:

$$\xi \in \mathcal{S}_q(u) \Leftrightarrow \nu(Q) = \infty \quad \forall Q : \xi \in Q, Q \text{ } \mathfrak{T}_q\text{-open}. \quad (6.54)$$

(iv) If  $Q$  is a  $\mathfrak{T}_q$ -open set then:

$$[u]_Q \text{ is moderate} \Leftrightarrow \exists \text{ Borel set } A : C_{\frac{2}{q},q'}(A) = 0, \nu(K \cap \widetilde{Q} \setminus A) < \infty, \quad (6.55)$$

for any compact  $K \subset \mathbb{R}^N$ .

(v) The singular set of  $u_{\text{reg}}$  may not be empty. In fact

$$\mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_{\text{reg}}) \subset \mathcal{S}_q(u) \cap \widetilde{\mathcal{R}_q(u)}, \quad (6.56)$$

where  $b_q(\mathcal{S}_q(u))$  is the set of  $C_{\frac{2}{q},q'}$ -thick points of  $\mathcal{S}_q(u)$ .

(vi) Put

$$\mathcal{S}_{q,0}(u) := \{\xi \in \mathbb{R}^N : \nu(Q \setminus \mathcal{S}_q(u)) = \infty \quad \forall Q \text{ } \mathfrak{T}_q\text{-open neighborhood of } \xi\}. \quad (6.57)$$

Then

$$\mathcal{S}_q(u_{\text{reg}}) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_{q,0}(u) \subset \mathcal{S}_q(u_{\text{reg}}) \bigcup b_q(\mathcal{S}_q(u)). \quad (6.58)$$

*Remark.* This results extends Proposition 6.8 which deals with the regular initial trace.

*Proof.* (i) By proposition 6.8-(ii)  $u_{reg}$  is  $\sigma$ -moderate. The second part of the statement follows from Definition 6.10 and Proposition 6.11-(i).

(ii) If  $v \leq u$  then  $\mathcal{R}_q(u) \subset \mathcal{R}_q(v)$  and by definition  $v_{reg} \leq u_{reg}$ . By Proposition 6.11-(iv)  $\text{tr}(v_{reg}) \leq \text{tr}(u_{reg})$  and consequently  $\text{tr}(v) \leq \text{tr}(u)$ . Inequality (6.53) is an immediate consequence of (6.52).

(iii) If  $\xi \in \mathcal{R}_q(u)$  there exists a  $\mathfrak{T}_q$ -open neighborhood  $Q$  of  $\xi$  such that  $[u]_Q$  is moderate. Hence  $\nu(Q) = \text{tr}_{\mathcal{R}_q}(u)(Q) < \infty$ . If  $\xi \in \mathcal{S}_q(u)$ , it follows immediately from the definition of precise trace that  $\nu(Q) = \infty$  for every  $\mathfrak{T}_q$ -open neighborhood  $Q$  of  $\xi$ .

(iv) If  $[u]_Q$  is moderate then  $Q \subset \mathcal{R}_q(u)$ . Proposition 6.8-(ix) implies (6.55) in the direction  $\Rightarrow$ . On the other hand,

$$\nu(K \cap \tilde{Q} \setminus A) < \infty, \forall \text{ compact } K \subset \mathbb{R}^N \Rightarrow \tilde{Q} \subset^q \mathcal{R}_q(u),$$

and  $\mu_{\mathcal{R}_q}(K \cap \tilde{Q}) = \mu_{\mathcal{R}_q}(K \cap \tilde{Q} \setminus A) < \infty$ . Hence, by Proposition 6.8-(ix),  $[u]_Q$  is moderate.

(v) Since  $\mathfrak{T}_q\text{-supp}(u_{reg}) \subset \widetilde{\mathcal{R}_q(u)}$  and  $\mathcal{R}_q(u) \subset \mathcal{R}_q(u_{reg})$  we have

$$\mathcal{S}_q(u_{reg}) \subset \mathcal{S}_q(u) \cap \widetilde{\mathcal{R}_q(u)}.$$

Next we show that  $\mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_{reg})$ .

If  $\xi \in \mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u))$  then  $\mathcal{R}_q(u) \cup \{\xi\}$  is a  $\mathfrak{T}_q$ -open neighborhood of  $\xi$ . By (i)  $u_{reg}$  is  $\sigma$ -moderate and consequently (by Proposition 6.11-(i)) its trace is  $\mathfrak{T}_q$ -perfect. Therefore, if  $Q_0$  is a bounded  $\mathfrak{T}_q$ -open neighborhood of  $\xi$  and  $Q = Q_0 \cap (\{\xi\} \cup \mathcal{R}_q(u))$  then

$$\text{tr}(u_{reg})(Q) = \text{tr}(u_{reg})(Q \setminus \{\xi\}) = \text{tr}(u)(Q \setminus \{\xi\}),$$

where in the last equality we have used the fact that  $Q \setminus \{\xi\} \subset \mathcal{R}_q(u)$ . Let  $D$  be a  $\mathfrak{T}_q$ -open set such that  $\xi \in D \subset \tilde{D} \subset Q$ . If  $\text{tr}(u)(\tilde{D} \setminus \{\xi\}) < \infty$  then, by (iv),  $[u]_D$  is moderate and  $\xi \in \mathcal{R}_q(u)$ , contrary to our assumption. Therefore  $\text{tr}(u)(\tilde{D} \setminus \{\xi\}) = \infty$  so that  $\text{tr}(u_{reg})(Q_0 \setminus \{\xi\}) = \infty$  for every  $\mathfrak{T}_q$ -open bounded neighborhood  $Q_0$  of  $\xi$ , which implies  $\xi \in \mathcal{S}_q(u_{reg})$ . This completes the proof of (6.56).

(vi) If  $\xi \notin b_q(\mathcal{S}_q(u))$ , there exists a  $\mathfrak{T}_q$ -open neighborhood  $D$  of  $\xi$  such that  $(D \setminus \{\xi\}) \cap \mathcal{S}_q(u) = \emptyset$  and consequently

$$\text{tr}(u_{reg})(D \setminus \{\xi\}) = \text{tr}(u_{reg})(D \setminus \mathcal{S}_q(u)) = \text{tr}(u)(D \setminus \mathcal{S}_q(u)). \quad (6.59)$$

If, in addition  $\xi \in \mathcal{S}_{q,0}(u)$  then

$$\text{tr}(u)(D \setminus \mathcal{S}_q(u)) = \text{tr}(u_{reg})(D \setminus \mathcal{S}_q(u)) = \infty.$$

If  $Q$  is an arbitrary  $\mathfrak{T}_q$ -open neighborhood of  $\xi$  then the same holds if  $D$  is replaced by  $Q \cap D$ . Therefore  $\text{tr}(u_{reg})(Q \setminus \{\xi\}) = \infty$  for any such  $Q$ . Consequently  $\xi \in \mathcal{S}_q(u_{reg})$ , which proves that  $\mathcal{S}_{q,0}(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_{reg})$ .

On the other hand, if  $\xi \in \mathcal{S}_q(u_{reg}) \setminus b_q(\mathcal{S}_q(u))$  then there exists a  $\mathfrak{T}_q$ -open neighborhood  $D$  such that (6.59) holds and  $\text{tr}(u_{reg})(D) = \infty$ . Since  $u_{reg}$  is  $\sigma$ -moderate,  $\text{tr}(u_{reg})$  is  $\mathfrak{T}_q$ -perfect so that  $\text{tr}(u_{reg})(D) = \text{tr}(u_{reg})(D \setminus \{\xi\}) = \infty$ . Consequently, by (6.59),  $\text{tr}(u)(D \setminus \mathcal{S}_q(u)) = \infty$ . If  $Q$  is any  $\mathfrak{T}_q$ -open neighborhood of  $\xi$  then  $D$  can be replaced by  $D \cap Q$ . Consequently  $\text{tr}(u)(Q \setminus \mathcal{S}_q(u)) = \infty$  and we conclude that  $\xi \in \mathcal{S}_{q,0}(u)$ . This completes the proof of (6.58).  $\square$



**Proposition 6.13** *Let  $F$  be a  $\mathfrak{T}_q$ -closed set. Then  $\mathcal{S}_q(U_F) = b_q(F)$ .*

*Proof.* Let  $\xi \in \mathbb{R}^N$  such that  $F$  is  $C_{\frac{2}{q}, q'}$ -thin at  $\xi$ . Let  $Q$  be a  $\mathfrak{T}_q$ -open neighborhood of  $\xi$  such that  $\tilde{Q} \subset^q F^c$ . Then  $[U_F]_Q = U_{F \cap \tilde{Q}} = 0$ . Therefore  $\xi \in \mathcal{R}_q(U_F)$ .

Conversely, assume that  $\xi \in F \cap \mathcal{R}_q(U_F)$ , thus there exists a  $\mathfrak{T}_q$ -open neighborhood  $Q$  of  $\xi$  such that  $[U_F]_Q$  is moderate. But  $[U_F]_Q = U_{F \cap \tilde{Q}}$  which implies  $C_{\frac{2}{q}, q'}(F \cap \tilde{Q}) = 0$  and  $Q \subset \mathcal{R}(u)$ . Now, note that  $C_{\frac{2}{q}, q'}(F) \leq C_{\frac{2}{q}, q'}(F \cap Q) + C_{\frac{2}{q}, q'}(Q^c)$ . Thus  $F$  is  $\mathfrak{T}_q$ -thin at  $\xi$ .  $\square$

## 6.5 The initial value problem

The following notations will be used in the sequel.

**Notation 6.14** *a: We denote by  $\mathfrak{M}_+(\mathbb{R}^N)$  the space of positive outer regular Borel measure on  $\mathbb{R}^N$ .*

*b: We denote by  $\mathcal{C}_q(\mathbb{R}^N)$  the space of couples  $(\tau, F)$  such that  $F$  is  $\mathfrak{T}_q$ -closed,  $\tau \in \mathfrak{M}_+(\mathbb{R}^N)$ ,  $\mathfrak{T}_q$ -supp  $(\tau) \subset \tilde{F}^c$  and  $\tau \chi_{F^c}$  is  $\mathfrak{T}_q$ -locally finite.*

*c: We denote by  $\mathbb{T} : \mathcal{C}_q(\mathbb{R}^N) \rightarrow \mathfrak{M}_+(\mathbb{R}^N)$  the mapping given by  $\nu = \mathbb{T}(\tau, F)$  where  $\nu$  is defined as in (6.45) with  $\mathcal{R}_q(u)$ ,  $\mathcal{S}_q(u)$  replaced by  $F$ ,  $F^c$  respectively. In this setting  $\nu$  is the measure representation of the couple  $(\tau, F)$ .*

*d: If  $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$  the set*

$$F_\tau = \{\xi \in \mathbb{R}^N : \tau(Q \setminus F) = \infty \quad \forall Q \text{ } \mathfrak{T}_q\text{-open neighborhood of } \xi\} \quad (6.60)$$

*is called the set of explosion points of  $\tau$ .*

*Remark.* Note that  $F_\tau \subset F$  (because  $\tau \chi_{F^c}$  is  $\mathfrak{T}_q$ -locally finite) and  $F_\tau \subset \tilde{F}^c$  (because  $\tau$  vanishes outside this set). Thus

$$F_\tau \subset b_q(F^c) \cap F. \quad (6.61)$$

**Proposition 6.15** *Let  $\nu$  be a positive Borel measure on  $\mathbb{R}^N$ .*

*(i) The initial value problem*

$$\partial_t u - \Delta u + |u|^{q-1}u = 0, \quad u > 0 \text{ in } Q_\infty = \mathbb{R}^N \times (0, T), \quad \text{tr}(u) = \nu \text{ in } \mathbb{R}^N \times \{0\}. \quad (6.62)$$

*possesses a solution if and only if  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ .*

*(ii) Let  $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$  and put  $\nu := \mathbb{T}(\tau, F)$ . Then  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$  if and only if*

$$\tau \in \mathfrak{M}_q(\mathbb{R}^N), \quad F = b_q(F) \bigcup F_\tau. \quad (6.63)$$

*(iii) Let  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$  and denote*

$$\begin{aligned} \mathcal{E}_\nu &:= \{E : E \text{ } \mathfrak{T}_q\text{-quasi-closed, } \nu(E \cap K) < \infty, \forall \text{ compact } K \subset \mathbb{R}^N\} \\ \mathcal{D}_\nu &:= \{D : D \text{ } \mathfrak{T}_q\text{-open, } \tilde{D} \sim^q E \text{ for some } E \in \mathcal{E}_\nu\}. \end{aligned} \quad (6.64)$$

Then a solution of (6.62) is given by  $u = v \oplus U_F$  where

$$G := \bigcup_{D_\nu} D, \quad F := G^c, \quad v := \sup\{u_\nu \chi_E : E \in \mathcal{E}_\nu\}. \quad (6.65)$$

Note that if  $E \in \mathcal{E}_\nu$  then  $\nu \chi_E$  is locally bounded Borel measure which does not charge sets of  $C_{\frac{2}{q}, q'}$ -capacity zero. Recall that if  $\mu$  is a positive measure possessing these properties, then  $u_\mu$  denotes the moderate solution with initial trace  $\mu$ .

(iv) The solution  $u = v \oplus U_F$  is  $\sigma$ -moderate and it is the unique solution of problem (6.62) in the class of  $\sigma$ -moderate solutions. Furthermore,  $u$  is the largest solution of the problem.

*Proof.* The proof is similar to the one in [23].

(A) If  $u \in \mathcal{U}_+(Q_T)$

$$\mathrm{tr}(u) = \nu \Rightarrow \nu \in \mathfrak{M}_q(\mathbb{R}^N). \quad (6.66)$$

By Proposition 6.8,  $u_{reg}$  is  $\sigma$ -moderate and  $u \approx_{\mathcal{R}_q(u)} u_{reg}$ . Therefore

$$\mathrm{tr}(u) \chi_{\mathcal{R}_q(u)} = \mathrm{tr}(u_{reg}) \chi_{\mathcal{R}_q(u)}.$$

By Proposition 6.11,  $\bar{\mu}_{\mathcal{R}_q} := \mathrm{tr}(u_{reg}) \in \mathfrak{M}_q(\mathbb{R}^N)$ . If  $v$  is defined as in (6.65) then

$$v = \sup\{[u]_F : F \text{ } \mathfrak{T}_q\text{-closed } F \subset^q \mathcal{R}_q(u)\} = u_{reg}, \quad (6.67)$$

where the second equality holds by definition. Indeed, by Theorem 6.12, for every  $\mathfrak{T}_q$ -open set  $Q$ ,  $[u]_Q$  is moderate if and only if  $\nu(K \cap \tilde{Q} \setminus A) < \infty$  for some set  $A$  with capacity zero and for any compact  $K$  subset of  $\mathbb{R}^N$ . This means that  $[u]_Q$  is moderate if and only if there exists  $E \in \mathcal{E}_\nu$  such that  $\tilde{Q} \sim^q E$ . When this is the case,

$$\mathrm{tr}([u]_Q) = \mu_{\mathcal{R}_q}(u) \chi_{\tilde{Q}} = \mu_{\mathcal{R}_q}(u) \chi_E = \nu \chi_E.$$

Thus  $v \geq u_{reg}$ . On the other hand, if  $E \in \mathcal{E}_\nu$ , then  $\tilde{E} \subset^q \mathcal{R}_q(u)$  and  $\mu_{\mathcal{R}_q}(u)(K \cap \tilde{E}) = \mu_{\mathcal{R}_q}(u)(K \cap E) < \infty$  for any compact  $K$  subset of  $\mathbb{R}^N$ . Therefore by Proposition 6.8-(ix),  $\tilde{E}$  is regular, i.e., there exist a  $\mathfrak{T}_q$ -open regular set  $Q$  such that  $E \subset^q Q$ . Hence  $u_\nu \chi_E \leq [u]_Q$  and we conclude that  $v \leq u_{reg}$ . This proves (6.67). In addition, if  $E \cap \mathcal{S}_q(u) \neq \emptyset$  then  $\nu(E) = \infty$ , by Definition 6.10. Therefore  $\nu$  is outer regular with respect to  $\mathfrak{T}_q$ -topology.

Next we must prove that  $\nu$  is essentially absolutely continuous. Let  $Q$  be a  $\mathfrak{T}_q$ -open set and  $A$  a non-empty  $\mathfrak{T}_q$ -closed subset of  $Q$  such that  $C_{\frac{2}{q}, q'}(A) = 0$ . Either  $\nu(Q \setminus A) = \infty$ , in which case  $\nu(Q \setminus A) = \nu(Q)$ , or  $\nu(Q \setminus A) < \infty$ , in which case  $Q \setminus A \subset \mathcal{R}_q(u)$  and

$$\nu(Q \setminus A) = \bar{\mu}_{\mathcal{R}_q}(Q \setminus A) = \bar{\mu}_{\mathcal{R}_q}(Q) < \infty.$$

Let  $\xi \in A$  let  $D$  be a  $\mathfrak{T}_q$ -open subset of  $Q$  such that  $\xi \in D \subset \tilde{D} \subset^q Q$ . Let  $B_n$  be a  $\mathfrak{T}_q$ -open neighborhood of  $A \cap \tilde{D}$  such that  $C_{\frac{2}{q}, q'}(B_n) < 2^{-n}$  and  $B_n \subset^q D$ . Then

$$[u]_D \leq [u]_{E_n} + [u]_{B_n}, \quad E_n = \tilde{D} \setminus B_n.$$

Since  $\lim_{n \rightarrow \infty} [u]_{B_n} = 0$  it follows that  $[u]_D = [u]_{E_n}$ . Now since  $E_n \subset \mathcal{R}_q(u)$ ,  $\nu(E_n) \leq \nu(Q \setminus A) < \infty$ , we have by definition of  $\nu$  and Proposition 6.8-(ix) that  $[u]_{E_n}$  is moderate. Also in view of Lemma 2.8 and Lemma 2.7(ii)-[19], we have for some positive constant  $C$

$$\int_0^T \int_K [u]_{E_n}^q dx dt \leq C\nu(E_n) \leq C\nu(Q \setminus A) < \infty,$$

for any compact  $K \subset \mathbb{R}^N$ . Therefore

$$\int_0^T \int_K [u]_D^q dx dt < \infty \quad \forall K \subset \mathbb{R}^N, K \text{ compact.}$$

which implies that  $[u]_D$  is moderate and thus  $D \subset \mathcal{R}_q(u)$ . Since every point  $A$  has a neighborhood  $D$  as above we conclude that  $A \subset \mathcal{R}_q(u)$  and hence  $\nu(A) = \text{tr}_R(u)(A) = 0$ . If  $A$  is any a non-empty Borel subset of  $Q$  such that  $C_{\frac{2}{q}, q'}(A) = 0$ , by inequality  $C_{\frac{2}{q}, q'}(\tilde{A}) \leq cC_{\frac{2}{q}, q'}(A)$ , we have that  $\nu$  is absolutely continuous and  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ .

Secondly we prove:

**(B)** Suppose that  $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$  satisfies (6.63) and put  $\nu = \mathbb{T}(\tau, F)$ . Then the solution  $u = v \oplus U_F$ , with  $v$  as in (6.65), satisfies  $\text{tr}(u) = \nu$ .

By (6.66), this also implies that  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ .

Clearly  $v$  is a  $\sigma$ -moderate solution. The fact that  $\tau$  is  $\mathfrak{T}_q$ -locally finite in  $F^c$  and essentially absolutely continuous relative to  $C_{\frac{2}{q}, q'}$  implies that

$$G := F^c \subset \mathcal{R}_q(v), \quad \text{tr}(v)\chi_G = \tau_G. \quad (6.68)$$

It follows from the definition of  $v$  that  $F_\tau \subset \mathcal{S}_q(v)$ . Hence, by Proposition 6.13 and (6.56) we have

$$F = b_q(F) \bigcup F_\tau \subset \mathcal{S}_q(v) \bigcup \mathcal{S}_q(U_F) \subset \mathcal{S}_q(u) \subset F. \quad (6.69)$$

Thus,  $\mathcal{S}_q(u) = F$ ,  $v = u_{reg}$  and  $\tau = \text{tr}(u_{reg})$ . Thus  $\text{tr}^c(u) = (\tau, F)$  which is equivalent to  $\text{tr}(u) = \nu$ .

Next we show: **(C)** Suppose that  $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$  and that there exists a solution  $u$  such that  $\text{tr}^c(u) = (\tau, F)$ . Then

$$\tau = \text{tr}_{\mathcal{R}_q}(u) = \text{tr}(u_{reg}), \quad F = \mathcal{S}_q(u). \quad (6.70)$$

If  $U := u_{reg} \oplus U_F$  then  $\text{tr}(U) = \text{tr}(u)$  and  $u \leq U$ .  $U$  is the unique  $\sigma$ -moderate solution of (6.62) and  $(\tau, F)$  satisfies condition (6.63). Assertion (6.70) follows by Proposition 6.8-(i) and Definition 6.10. Since  $u_{reg}$  is  $\sigma$ -moderate, it follows, by Theorem 6.11, that  $\tau \in \mathfrak{M}_q(\mathbb{R}^N)$ .

By Proposition 6.8 (vi),  $u \approx_{\mathcal{R}_q(u)} u_{reg}$ . Therefore  $w := u \ominus u_{reg}$  vanishes on  $\mathcal{R}_q(u)$  so that  $w \leq U_F$ . Note that  $u - u_{reg} \leq w$  and therefore

$$u \leq u_{reg} \oplus w \leq U. \quad (6.71)$$

By their definitions  $\mathcal{S}_{q,0}(u) = F_\tau$  and by Theorem 6.12 (vi) and Proposition 6.13,

$$\begin{aligned} \mathcal{S}_q(U) &= \mathcal{S}_q(u_{reg}) \bigcup \mathcal{S}_q(U_F) = \mathcal{S}_q(u_{reg}) \bigcup b_q(U_F) \\ &= \mathcal{S}_{q,0}(u) \bigcup b_q(U_F) = F_\tau \bigcup b_q(U_F). \end{aligned} \quad (6.72)$$

On the other hand,  $\mathcal{R}_q(U) \supset \mathcal{R}_q(u_{\mathcal{R}_q}) = \mathcal{R}_q(u)$  and, as  $u \leq U$ ,  $\mathcal{R}_q(U) \subset \mathcal{R}_q(u)$ . Hence  $\mathcal{R}_q(U) = \mathcal{R}_q(u)$  and  $\mathcal{S}_q(U) = \mathcal{S}_q(u)$ . Therefore, by (6.70) and (6.72),  $F = \mathcal{S}_q(U) = F_\tau \cup b_q(U_F)$ . Thus  $(\tau, F)$  satisfies (6.63) and  $\text{tr}^c(U) = (\tau, F)$ . The fact that  $U$  is the maximal solution with this trace follows from (6.71).

The solution  $U$  is  $\sigma$ -moderate because both  $u_{reg}$  and  $U_F$  are  $\sigma$ -moderate solutions (concerning  $U_F$ , see Proposition 5.21).

The uniqueness of the solution in the class of  $\sigma$ -moderate solutions follows from Proposition 6.11-(iv).

Finally we prove:

**(D)** If  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$  then the couple  $(\tau, F)$  defined by

$$v := \sup\{u_\nu \chi_E : E \in \mathcal{E}_\nu\}, \quad \tau := \text{tr}(v), \quad F = \mathcal{R}_q^c(v), \quad (6.73)$$

satisfies (6.63). This is the unique couple in  $\mathcal{C}_q(\mathbb{R}^N)$  satisfying  $\nu = \mathbb{T}(\tau, F)$ . The solution  $v$  is  $\sigma$ -moderate so that  $\tau \in \mathfrak{M}_q(\mathbb{R}^N)$ .

We claim that  $u := v \oplus U_F$  is a solution with initial trace  $\text{tr}^c(u) = (\tau, F)$ . Indeed  $u \geq v$  so that  $\mathcal{R}_q(u) \subset \mathcal{R}_q(v)$ . On the other hand since  $\tau$  is  $\mathfrak{T}_q$ -locally finite in  $\mathcal{R}_q(v) = F^c$ , it follows that  $\mathcal{S}_q(u) \subset F$ . Thus  $\mathcal{R}_q(v) \subset \mathcal{R}_q(u)$  and we conclude that  $\mathcal{R}_q(u) = \mathcal{R}_q(v)$  and  $F = \mathcal{S}_q(u)$ . This also implies that  $v = u_{reg}$ .

Finally

$$\mathcal{S}_q(u) = \mathcal{S}_q(v) \bigcup b_q(\mathcal{S}_q(U_F)) = b_q(F) \bigcup F_\tau,$$

so that  $F$  satisfies (6.63).

The fact that, for  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ , the couple  $(\tau, F)$  defined by (6.73) is the only one in  $\mathcal{C}_q(\mathbb{R}^N)$  satisfying  $\nu = \mathbb{T}(\tau, F)$  follows immediately from the definition of these spaces.

At end, statements **A-D** imply (i)-(iv).  $\square$

*Remark.* If  $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$  then  $G$  and  $v$  as defined by (6.65) have the following alternative representation:

$$G := \bigcup_{\mathcal{E}_\nu} E = \bigcup_{\mathcal{F}_\nu} Q, \quad v := \sup\{u_\nu \chi_Q : Q \in \mathcal{F}_\nu\}, \quad (6.74)$$

$$\mathcal{E}_\nu := \{Q : E \text{ } \mathfrak{T}_q\text{-open, } \nu(Q \cap K) < \infty, \forall \text{ compact } K \subset \mathbb{R}^N\}. \quad (6.75)$$

To verify this remark we first observe that Lemma 2.8 implies that if  $A$  is a  $\mathfrak{T}_q$ -open set then there exists an increasing sequence of  $\mathfrak{T}_q$ -quasi closed sets  $\{E_n\}$  such that  $A = \bigcup_{n=1}^\infty E_n$ . In fact, in the notation of Lemma 2.8 (II)(i)-(ii), we may choose  $E_n = F_n \setminus L$  where  $L = A' \setminus A$ , is a set of capacity zero.

Therefore

$$\bigcup_{\mathcal{D}_\nu} D \subset \bigcup_{\mathcal{F}_\nu} Q \subset \bigcup_{\mathcal{E}_\nu} E := H.$$

On the other hand, if  $E \in \mathcal{E}_\nu$  then  $\mu_{\mathcal{R}_q}(u)(K \cap \tilde{E}) = \mu_{\mathcal{R}_q}(u)(K \cap E) = \nu(K \cap E) < \infty$ , for any compact  $K \subset \mathbb{R}^N$ . By Proposition 6.8-(ix),  $\tilde{E}$  is regular, i.e., there exists a  $\mathfrak{T}_q$ -open regular set  $Q$  such that  $E \subset^q Q$ . Thus  $H = \bigcup_{\mathcal{D}_\nu} D$ .

If  $D$  is a  $\mathfrak{T}_q$ -open regular set then  $D = \cup_{n=1}^{\infty} E_n$ , where  $\{E_n\}$  is an increasing sequence of  $\mathfrak{T}_q$ -quasi closed sets. We infer

$$u_{\nu\chi_D} = \lim_{n \rightarrow \infty} u_{\nu\chi_{E_n}}.$$

Therefore

$$\sup\{u_{\nu\chi_Q} : Q \in \mathcal{D}_\nu\} \leq \sup\{u_{\nu\chi_Q} : Q \in \mathcal{F}_\nu\} \leq \sup\{u_{\nu\chi_Q} : Q \in \mathcal{E}_\nu\}.$$

On the other hand, if  $E \in \mathcal{E}_\nu$ , there exists a  $\mathfrak{T}_q$ -open regular set  $Q$  such that  $E \subset^q Q$ . Consequently the equality follows.

## 7 The equation $\partial_t u - \Delta u + Vu = 0$

Let  $0 < T \leq \infty$ ,  $Q_T := \mathbb{R}^N \times (0, T)$ ,  $C > 0$  and  $V : Q_T \rightarrow [0, \infty)$  be a Borel function satisfying

$$0 \leq V(x, t) \leq \frac{C}{t} \quad \forall (x, t) \in Q_T. \quad (7.1)$$

In this section we prove a general representation theorem for positive solutions of

$$\partial_t u - \Delta u + Vu = 0 \quad \text{in } Q_T. \quad (7.2)$$

### 7.1 Preliminaries

We recall that  $\mathfrak{M}(\mathbb{R}^N)$  is the set of Radon measures on  $\mathbb{R}^N$  and  $\mathfrak{M}_+(\mathbb{R}^N)$  its positive cone.

**Definition 7.1** *Let  $\mu \in \mathfrak{M}(\mathbb{R}^N)$ . We say that  $u$  is a weak solution of problem*

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0 && \text{in } Q_T \\ u(\cdot, 0) &= \mu && \text{in } \mathbb{R}^N, \end{aligned} \quad (7.3)$$

if  $u \in L^1_{loc}(\overline{Q}_T)$ ,  $Vu \in L^1_{loc}(\overline{Q}_T)$  and there holds

$$\int \int_{Q_T} u(-\phi_t - \Delta\phi) dx dt + \int \int_{Q_T} Vu\phi dx dt = \int_{\mathbb{R}^N} \phi(x, 0) d\mu, \quad (7.4)$$

for all  $\phi \in X(Q_T)$ , where

$$X(Q_T) = \{\phi \in C_c(\mathbb{R}^N \times [0, T]), \phi_t + \Delta\phi \in L^\infty_{loc}(\overline{Q}_T)\}.$$

*Remark.* The definition implies that for any  $\zeta \in C_c^2(\mathbb{R}^N)$ , the function  $t \mapsto \int \zeta(x)u(x, t)dx$  can be extended by continuity on  $[0, T]$  as a continuous function and

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \zeta(x)u(x, t)dx = \int_{\mathbb{R}^N} \zeta d\mu. \quad (7.5)$$

Therefore  $\|u(\cdot, t)\|_{L^1(\Omega)}$  remains uniformly bounded on  $(0, T)$  for any bounded open set  $\Omega \subset \mathbb{R}^N$ .

**Lemma 7.2** *Let  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$  and assume that there exists a positive weak solution  $u$  of problem (7.3) where  $V$  satisfies (7.1). Then for any smooth bounded domain  $\Omega$  there exists a unique positive weak solution  $v$  of problem*

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } Q_T^\Omega = \Omega \times (0, T) \\ v &= 0 && \text{on } \partial_t Q_T^\Omega = \partial\Omega \times (0, T) \\ v(\cdot, 0) &= \chi_\Omega \mu && \text{in } \Omega, \end{aligned} \quad (7.6)$$

where  $\chi_\Omega$  is the characteristic function on  $\Omega$  and there holds  $v \leq u$  in  $\Omega \times (0, T)$ .

*Proof.* Let  $\{t_j\}$  be a decreasing sequence converging to 0, such that  $t_j < T$  for all  $j \in \mathbb{N}$ . We consider the following problem

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } \Omega \times (t_j, T) \\ v &= 0 && \text{on } \partial\Omega \times (t_j, T) \\ v(\cdot, t_j) &= u(\cdot, t_j) && \text{in } \Omega. \end{aligned}$$

Since  $u(\cdot, t_j) \in L^1(\Omega)$  and  $0 \leq V \in L^\infty(\Omega \times (t_j, T])$ , there exists a unique positive weak solution  $v_j$  of the above problem, smaller than the solution  $\mathbb{H}^\Omega[u(\cdot, t_j)\chi_\Omega]$ , where  $\mathbb{H}^\Omega$  is the heat operator in  $Q^\Omega := \Omega \times (0, \infty)$  with zero boundary condition furthermore  $v_j \leq u$  in  $\Omega \times (t_j, T)$  for all  $j \in \mathbb{N}$ . By standard parabolic estimates we may assume that the sequence  $\{v_j\}$  converges locally uniformly in  $\Omega \times (0, T]$  to a nonnegative function  $v$  smaller than  $u$ . If  $\phi \in C^{1,1;1}(\overline{Q_T^\Omega})$  vanishes on  $\partial_t Q_T^\Omega$  and satisfies  $\phi(x, T) = 0$ , the following identity holds

$$\int_{t_j}^T \int_\Omega v_j(-\phi_t - \Delta\phi) dx dt + \int_{t_j}^T \int_\Omega V v_j \phi dx dt + \int_\Omega \phi(x, T-t_j) u(x, T-t_j) dx = \int_\Omega \phi(x, 0) u(x, t_j) dx,$$

where, in the above equality, we have take as test function  $\phi(\cdot, \cdot - t_j)$ . It follows by the dominated convergence theorem, that  $v$  is a weak solution of problem (7.6).  $\square$

**Lemma 7.3** *Assume (7.1) holds and let  $u$  be a positive weak solution of problem (7.4) with  $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ . Then for any  $(x, t) \in \mathbb{R}^N \times (0, T]$ , we have*

$$\lim_{R \rightarrow \infty} u_R = u,$$

where  $\{u_R\}$  is the increasing sequence of the weak solutions of the problem (7.6) with  $\Omega = B_R(0)$ . Moreover, the convergence is uniform in any compact subset of  $\mathbb{R}^N \times (0, T]$ .

*Proof.* By the maximum principle (see [19, Remark 2.5]),

$$u_R \leq u_{R'} \leq u$$

for all  $0 < R \leq R'$ . Thus  $u_R \rightarrow w \leq u$ . Also by standard parabolic estimates, this convergence is locally uniformly. Now by dominated convergence theorem, we have that  $w$  is a weak solution of problem (7.3) with initial data  $\mu$ . We set  $\tilde{w} = u - w \geq 0$ . Then  $\tilde{w}$  satisfies in the weak sense

$$\begin{aligned} \tilde{w}_t - \Delta \tilde{w} + V\tilde{w} &= 0 && \text{in } \mathbb{R}^N \times (0, T), \\ w(x, t) &\geq 0 && \text{in } \mathbb{R}^N \times (0, T) \\ \tilde{w}(x, 0) &= 0 && \text{in } \mathbb{R}^N. \end{aligned}$$

Since  $\tilde{w}$  satisfies in the weak sense

$$\begin{aligned}\tilde{w}_t - \Delta \tilde{w} &\leq 0 && \text{in } \mathbb{R}^N \times (0, T), \\ w(x, t) &\geq 0 && \text{in } \mathbb{R}^N \times (0, T), \\ \tilde{w}(x, 0) &= 0 && \text{in } \mathbb{R}^N,\end{aligned}$$

We extend  $\tilde{w}$  by 0 for  $t \leq 0$ , with the same notation and set  $\tilde{w}_n := \tilde{w} * J_{\epsilon_n}$  where  $\{J_{\epsilon_n}\}$  is a sequence of mollifiers in  $\mathbb{R}^{N+1}$ . Then  $\tilde{w}_n \leq 0$ , therefore  $\tilde{w} = 0$ .  $\square$

**Lemma 7.4** *Let  $u \in C^{2,1}(\mathbb{R}^N \times (0, T])$  be a positive solution of*

$$\partial_t u - \Delta u + Vu = 0 \quad \text{in } \mathbb{R}^N \times (0, T).$$

*Assume that, for any  $x \in \mathbb{R}^N$ , there exists an open bounded neighborhood  $U$  of  $x$  such that*

$$\int_0^T \int_U u(y, t) V(y, t) dx dt < \infty$$

*Then  $u \in L^1(U \times (0, T))$  and there exists a unique positive Radon measure  $\mu$  such that*

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(y, t) \phi(x) dx = \int_{\mathbb{R}^N} \phi(x) d\mu \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

*Proof.* Since  $Vu \in L^1(U \times (0, T))$  the following problem has a weak solution  $v$  (see [19]).

$$\begin{aligned}\partial_t v - \Delta v &= Vu, && \text{in } U \times (0, T], \\ v(x, t) &= 0 && \text{on } \partial U \times (0, T] \\ v(x, 0) &= 0 && \text{in } U.\end{aligned}$$

Thus the function  $w = u + v$  is a positive solution of the heat equation, thus there exists a unique Radon measure  $\mu$  such that

$$\lim_{t \rightarrow 0} \int_U w(y, t) \phi(x) dx = \int_U \phi(x) d\mu, \quad \forall \phi \in C_0^\infty(U).$$

But the initial data of  $v$  is zero, thus the result follows by a partition of unity and Lemma 7.3.  $\square$

## 7.2 Representation formula for positive solutions

Assume  $V$  satisfies (7.1) in  $Q_T$  and let  $u$  be a positive solution of (7.2). If  $\psi \in C^{2,1}(\mathbb{R}^N \times (0, T])$ , we set  $u(x, t) = e^{\psi(x, t)} v(x, t)$ . Then  $v$  satisfies

$$\partial_t v - \Delta v - 2\nabla \psi \nabla v - |\nabla \psi|^2 v - 2\Delta \psi v + (\psi_t + \Delta \psi + V)v = 0 \quad \text{in } \mathbb{R}^N \times (0, T]. \quad (7.7)$$

We choose  $\psi$  to be the solution of the problem

$$\begin{aligned}-\psi_t - \Delta \psi &= V && \text{in } \mathbb{R}^N \times (0, T] \\ \psi(x, T) &= 0 && \text{in } \mathbb{R}^N.\end{aligned}$$

Then

$$\psi(t, x) = \int_t^T \int_{\mathbb{R}^N} \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}} V(x, s) dx ds. \quad (7.8)$$

By a straightforward calculation we verify that

1.  $0 \leq \psi \leq C \ln \frac{T}{t}$ ,
2.  $|\nabla \psi| \leq C_1(T) + C_2(\ln \frac{T}{t})$ .

Thus (7.7) becomes

$$\partial_t v - \Delta v - \sum_{i=1}^n (2\psi_{x_i} v)_{x_i} - |\nabla \psi|^2 v = 0.$$

Since  $\int_0^1 |\ln t|^p dt < \infty$  for all  $p \geq 1$ , we verify by 1 and 2 that

$$\int_0^T \sup_{x \in \mathbb{R}^N} |\psi|^q ds < M_1 < \infty \quad \forall q \geq 1$$

and

$$\int_0^T \sup_{x \in \mathbb{R}^N} |\nabla \psi|^q ds < M_2 < \infty \quad \forall q \geq 1.$$

For  $A_{i,j} = \delta_{ij}$ ,  $A_i = 2\psi_{x_i}$ ,  $B_i = 0$  and  $C = |\nabla \psi|^2$  we see that the above operator satisfies the condition  $H$  in [3] for  $R_0 = \infty$  and  $p = \infty$ . Thus there exists a kernel  $\Gamma(x, t; y, s)$ , defined in  $Q_T \times Q_T$  satisfying the estimates

$$C_1(T, n, M_2) \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-A_1 \frac{|x-y|^2}{4(t-s)}} \leq \Gamma(x, t; y, s) \leq C_2(T, n, M_2) \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-A_2 \frac{|x-y|^2}{4(t-s)}}, \quad (7.9)$$

where  $A_1, A_2 > 0$  depend on  $T, n, M_2$  with the property that  $v$  admits the following representation formula:

$$v(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu(y), \quad (7.10)$$

where  $\mu$  is a uniquely defined positive Radon measure on  $\mathbb{R}^N$ , and there holds

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) \phi(x) d\mu(y) dx = \int_{\mathbb{R}^N} \phi d\mu \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Furthermore, if  $e^{-\gamma|x|^2} u_0 \in L^2(\mathbb{R}^N)$  for some  $\gamma \geq 0$ , and if  $u_0$  is continuous at  $y$ , then

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) u_0(x) dx = u_0(y). \quad (7.11)$$

Finally we have

$$u(x, t) = e^{\psi(x,t)} \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu(y). \quad (7.12)$$



## 8 $\sigma$ -moderate solutions

### 8.1 Preliminaries

**Proposition 8.1** *Let  $u \in \mathcal{U}_+(Q_T)$ . Then*

$$\max(u_{\mathcal{R}_q}, [u]_{\mathcal{S}_q(u)}) \leq u \leq u_{reg} + [u]_{\mathcal{S}_q(u)}. \quad (8.1)$$

*Proof.* The principle of the proof is similar as the one in [16].

By Proposition 6.8-(vii), the function  $v = u \ominus u_{reg}$  vanishes on  $\mathcal{R}_q(u)$  i.e.,  $\mathfrak{T}_q\text{-supp}(v) \subset \mathcal{S}_q(u)$ . Thus  $v$  is a solution dominated by  $u$  and supported in  $\mathcal{S}_q(u)$ , which implies that  $v \leq [u]_{\mathcal{S}_q(u)}$  by Definition 5.27. Since  $u - u_{reg} \leq v$  this implies the inequality on the right hand side of (8.1). The inequality on the left hand side is obvious.  $\square$

**Proposition 8.2** *Let  $u \in \mathcal{U}_+(Q_T)$  and let  $A, B$  be two disjoint  $\mathfrak{T}_q$ -closed subsets of  $\mathbb{R}^N$ . If  $\mathfrak{T}_q\text{-supp}(u) \subset A \cup B$  and  $[u]_A, [u]_B$  are  $\sigma$ -moderate then  $u$  is  $\sigma$ -moderate. Furthermore*

$$u = [u]_A \oplus [u]_B = [u]_A \vee [u]_B. \quad (8.2)$$

*Proof.* The proof is same as in [16].

By Proposition 6.11-(iii) there exist two increasing sequences  $\{\tau_n\}, \{\tau'_n\} \subset W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  such that

$$u_{\tau_n} \uparrow [u]_A, \quad u_{\tau'_n} \uparrow [u]_B.$$

By proposition 5.26,  $\mathfrak{T}_q\text{-supp}(\tau_n) \subset^q A$  and  $\mathfrak{T}_q\text{-supp}(\tau'_n) \subset^q B$ . Thus  $C_{\frac{2}{q}, q}'(\mathfrak{T}_q\text{-supp}(\tau_n) \cap \mathfrak{T}_q\text{-supp}(\tau'_n)) = 0$ , and

$$u_{\tau_n} \vee u_{\tau'_n} = u_{\tau_n} \oplus u_{\tau'_n} = u_{\tau_n + \tau'_n}.$$

By (5.34) and Definition 5.27,

$$\max([u]_A, [u]_B) \leq u \leq [u]_A + [u]_B. \quad (8.3)$$

Therefore,

$$\max(u_{\tau_n}, u'_{\tau_n}) \leq u \Rightarrow u_{\tau_n + \tau'_n} \leq u.$$

On the other hand

$$u - u_{\tau_n + \tau'_n} \leq [u]_A - u_{\tau_n} + [u]_B - u_{\tau'_n} \downarrow 0.$$

Thus

$$\lim_{n \rightarrow \infty} u_{\tau_n + \tau'_n} = u, \quad (8.4)$$

so that  $u$  is  $\sigma$ -moderate.

The assertion (8.2) is equivalent to the statements: (a)  $u$  is the largest solution dominated by  $[u]_A + [u]_B$  and (b)  $u$  is the smallest solution dominating  $\max([u]_A, [u]_B)$ . Let

$$u \leq w := [u]_A \oplus [u]_B \leq [u]_A + [u]_B.$$

Thus we have  $[u]_A \leq [w]_A$ . But  $[w]_A \leq w \leq [u]_A + [u]_B \Rightarrow [w]_A - [u]_A \leq [u]_B$ . By Notation 3.3 we have

$$v = [[w]_A - [u]_A]_+ \dagger \leq [u]_B, \quad v \leq [w]_A,$$

that is

$$\mathfrak{T}_q\text{-supp}(v) \subset A \quad \text{and} \quad \mathfrak{T}_q\text{-supp}(v) \subset B.$$

But  $A \cap B = \emptyset$ , which implies  $v = 0$  and  $[w]_A \leq [u]_A$ . Similarly, we have  $[w]_B \leq [u]_B$ . Thus

$$[w]_A = [u]_A, \quad [w]_B \leq [u]_B.$$

By (8.3) and the fact that for any Borel  $E$   $[u]_E \leq [u]_{\tilde{E} \cap A} + [u]_{\tilde{E} \cap B}$ , there holds

$$\mathcal{S}_q(u) = \mathcal{S}_q(w).$$

Let  $Q$  be a  $\mathfrak{T}_q$ -open regular set in  $\mathcal{R}_q(w)$ , then  $Q \in \mathcal{R}_q(u)$ . Using (5.34), (5.35) and the fact that  $\mathfrak{T}_q\text{-supp}(w) \subset A \cap B$ , we derive

$$[w]_Q \leq [w]_{\tilde{Q} \cap A} + [w]_{\tilde{Q} \cap B} = [[w]_A]_{\tilde{Q}} + [[w]_B]_{\tilde{Q}} = [u]_{\tilde{Q} \cap A} + [u]_{\tilde{Q} \cap B}.$$

Since  $[w]_Q, [u]_Q$  are moderate solutions and  $A \cap B = \emptyset$ , we have  $[u]_{\tilde{Q} \cap A} \oplus [u]_{\tilde{Q} \cap B} \leq [u]_Q$ , which implies  $[w]_Q = [u]_Q$ . Thus by Proposition 6.8-(ii)  $w_{\mathcal{R}_q} = u_{\mathcal{R}_q}$ , and since  $u$  is  $\sigma$ -moderate by Proposition 6.15 and the remark below we get

$$u \leq w \leq u_{\mathcal{R}_q} \oplus U_F.$$

By the uniqueness of  $\sigma$ -moderate solutions (Theorem 6.11-(iv)),  $w$  and  $u$  coincide. This proves (a).

For the statement (b), we note that

$$u_{\tau_n + \tau'_n} = u_{\tau_n} \vee u_{\tau'_n} \leq [u]_A \vee [u]_B,$$

since  $u_{\tau_n} \leq [u]_A$  and  $u_{\tau'_n} \leq [u]_B$ . Thus the result follows by (8.4) and (8.3), by letting  $n$  tend to infinity.  $\square$

## 8.2 Characterization of positive solutions of $\partial_t u - \Delta u + u^q = 0$

The following notation is used throughout the subsection.

Let  $u \in \mathcal{U}_+(Q_T)$ . Set

$$V = u^{q-1},$$

then

$$V \leq \left( \frac{1}{q-1} \right)^{q-1} t^{-1}.$$

Thus  $u \in C^{2,1}(\mathbb{R}^N \times (0, T])$  and satisfies

$$\partial_t u - \Delta u + Vu = 0, \quad \text{in} \quad \mathbb{R}^N \times (0, 1]. \quad (8.5)$$

Hence, by the representation formula (7.12),  $u$  satisfies

$$u(x, t) = e^\psi \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu(y), \quad \forall t \leq T, \quad (8.6)$$

where  $\mu$  is Radon measure (see subsection 7.2). The measure  $\mu$  is called the *extended initial trace* of  $u$ .

For any Borel set  $E$  set

$$\mu_E = \mu \chi_E \quad \text{and} \quad (u)_E = e^\psi \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu_E, \quad \forall t \leq T.$$

**Lemma 8.3** *Let  $F$  be a compact subset of  $\mathbb{R}^N$ . Then*

$$(u)_F \leq [u]_F, \quad \forall t \leq T.$$

*Proof.* We follow the ideas of [16], adapted to the parabolic framework.

Let  $A$  be a Borel subset of  $\mathbb{R}^N$ . Let  $0 < \beta \leq \frac{T}{2}$  and let  $v_\beta^A$  be the positive solution of

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } \mathbb{R}^N \times (\beta, T] \\ v(\cdot, \beta) &= u(\cdot, \beta) \chi_A(\cdot) && \text{in } \mathbb{R}^N. \end{aligned} \quad (8.7)$$

Also let  $w_\beta^A$  be the positive solution of

$$\begin{aligned} \partial_t w - \Delta w + |w|^{q-1}w &= 0 && \text{in } \mathbb{R}^N \times (\beta, T] \\ w(\cdot, \beta) &= \chi_A(\cdot)u(\cdot, \beta) && \text{in } \mathbb{R}^N. \end{aligned}$$

Then by the maximum principle  $w_\beta^A \leq u$ , which implies

$$0 = \frac{dw_\beta^A}{dt} - \Delta w_\beta^A + (w_\beta^A)^q \leq \frac{dw_\beta^A}{dt} - \Delta w_\beta^A + Vw_\beta^A.$$

Thus  $w_\beta^A$  is a supersolution of (8.7), and by the maximum principle (see [3] or Lemma 7.3), we have

$$v_\beta^A \leq w_\beta^A \leq u.$$

For any sequence  $\{\beta_k\}$  decreasing to zero one can extract a subsequence  $\{\beta_{k_n}\}$  such that  $\{w_{\beta_{k_n}}^A\}$  and  $\{v_{\beta_{k_n}}^A\}$  converge locally uniformly; we denote the limits  $w^A$  and  $v^A$  respectively (the limits may depend on the sequence). Then  $w^A \in \mathcal{U}_+(Q_T)$  while  $v^A$  is a solution of (8.5), and

$$v^A \leq w^A \leq [u]_{\tilde{Q}}, \quad \forall Q \text{ open}, A \subset Q. \quad (8.8)$$

The second inequality follows from the fact that  $\mathfrak{T}_q\text{-supp}(w_\beta^A) \subset \tilde{Q}$  for any  $\beta$ .

Now we set  $v_{\beta_{k_n}}^A = e^\psi \tilde{v}_n$ , where  $\psi$  is the function in subsection 7.2. Then  $\tilde{v}_n$  is the solution of

$$\begin{aligned} \partial_t v - \Delta v - 2\nabla\psi\nabla v - |\nabla\psi|^2 v - 2\Delta\psi v + (\psi_t + \Delta\psi + V)v &= 0 \quad \text{in} \quad \mathbb{R}^N \times (\beta_{k_n}, T]. \\ v(\cdot, \beta_{k_n}) &= \chi_A(\cdot) \int_{\mathbb{R}^N} \Gamma(\cdot, \beta_{k_n}; y, 0) d\mu(y) \quad \text{in} \quad \mathbb{R}^N. \end{aligned}$$

Now using the representation formula in [3], we derive that for any open  $Q \supset A$ , there holds

$$\begin{aligned} \tilde{v}_n(x, t) &= \int_{\mathbb{R}^N} \chi_A(x) \Gamma(x, t - \beta_{k_n}; y, 0) \left( \int_{\mathbb{R}^N} \Gamma(x, \beta_{k_n}; y, 0) d\mu(y) \right) dx \\ &= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \chi_A(x) \Gamma(x, t - \beta_{k_n}; y, 0) \Gamma(x, \beta_{k_n}; y, 0) dx \right) dx d\mu(y) \\ &\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \chi_Q(x) \Gamma(x, t - \beta_{k_n}; y, 0) \Gamma(x, \beta_{k_n}; y, 0) dx \right) dx d\mu(y). \end{aligned}$$

Therefore, by (7.11), estimate (7.9) and using the fact that  $\Gamma(x, t - s; y, 0)$  is a continuous function for any  $s < t$  (see [3]), we can let  $k_n \rightarrow \infty$  in the above inequality and get

$$\lim_{n \rightarrow \infty} \tilde{v}_n \leq \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu_{\tilde{Q}}.$$

Hence

$$v^A \leq (u)_{\tilde{Q}}.$$

We apply the same procedure to the set  $A^c$  extracting a further subsequence of  $\{\beta_{k_n}\}$  in order to obtain the limits  $v^{A^c}$  and  $w^{A^c}$ . Thus

$$v^{A^c} \leq w^{A^c} \leq [u]_{\tilde{Q}'}, \quad \forall Q' \text{ open, } A^c \subset Q'.$$

Note that

$$v^A + v^{A^c} = u, \quad v^A \leq (u)_{\tilde{Q}}, \quad v^{A^c} \leq (u)_{\tilde{Q}'}$$

Therefore

$$v^A = u - v^{A^c} \geq (u)_{(\tilde{Q}')^c}. \quad (8.9)$$

Now, given  $F$  compact, let  $A$  be a closed set and  $O$  an open set such that  $F \subset O \subset A$ . Note that  $A^c \cap F = \emptyset$ . By (8.9) with  $Q' = A^c$

$$v^A \geq (u)_O.$$

By (8.8)

$$v^A \leq w^A \leq [u]_{\tilde{Q}} \quad \forall Q \text{ open, } A \subset Q,$$

and consequently

$$(u)_F \leq (u)_O \leq [u]_{\tilde{Q}}. \quad (8.10)$$

By Lemma 2.8, we can choose a sequence of open sets  $\{Q_n\}$  such that  $\cap \tilde{Q}_j = E' \sim^q F$ , thus by Proposition 5.24-(iii)  $[u]_{Q_j} \downarrow [u]_F$ . The result follows by (8.10).  $\square$

In the next lemma we prove that the extended initial trace of a positive solution of (3.1) is absolutely continuous with respect to the  $C_{\frac{2}{q}, q'}$ -capacity.

**Lemma 8.4** *Let  $u \in \mathcal{U}_+(Q_T)$ ,  $\mu$  its extended initial trace. If  $E$  is a Borel set and  $C_{\frac{2}{q}, q'}(E) = 0$  then  $\mu(E) = 0$ .*

*Proof.* The proof is similar as the one in [16]. If  $F$  is a compact subset of  $E$ , then  $C_{\frac{2}{q}, q'}(F) = 0$  and therefore by Proposition 5.17,  $U_F = 0$ . But  $[u]_F = u \wedge U_F = 0$ . Therefore, by Lemma 8.3  $(u)_F = 0$ . Consequently  $\mu(F) = 0$ . As this holds for every compact subset of  $E$  we conclude that  $\mu(E) = 0$ .  $\square$

We recall that, if  $\nu \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ , then for any  $T > 0$ , there exists a constant  $C > 0$  independent on  $\nu$  (see Lemma 2.11-[22]) such that

$$C^{-1} \|\nu\|_{W^{-\frac{2}{q}, q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\nu]\|_{L^q(Q_T)} \leq C \|\nu\|_{W^{-\frac{2}{q}, q}(\mathbb{R}^N)}, \quad (8.11)$$

where  $\mathbb{H}[\nu]$  is the solution of the heat equation in  $Q_\infty$  with  $\nu$  as initial data.

**Lemma 8.5** *Let  $u \in \mathcal{U}_+(Q_T)$ ,  $\mu$  its extended initial trace and  $\nu \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ . Suppose that there exists no positive solution of (3.1) dominated by the supersolution  $v = \inf\{u, \mathbb{H}[\nu]\}$ . Then  $\mu \perp \nu$ .*

*Proof.* Set  $V' = v^{q-1}$ , then  $v$  is a supersolution of

$$\partial_t w - \Delta w + V' w = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T]. \quad (8.12)$$

We first claim that there exists no positive solution of the above problem dominated by  $v$ . We proceed by contradiction in assuming that there exists a positive solution  $w \leq v$  of (8.12). Then  $w$  satisfies

$$\partial_t w - \Delta w + w^q \leq \partial_t w - \Delta w + V' w = 0.$$

Since

$$\|v\|_{L^q(Q_T)} \leq \|\mathbb{H}[\nu]\|_{L^q(Q_T)} \approx \|\nu\|_{W^{-\frac{2}{q}, q}(\mathbb{R}^N)}$$

this implies that  $w$  is a positive moderate solution of (3.1) dominated by  $v$ , contrary to assumption. Now for any  $t \leq T$ , we have by representation formula (7.12),

$$\begin{aligned} \inf\{u, \mathbb{H}[\nu]\} &= \inf \left\{ e^\psi \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu(y), \mathbb{H}[\nu] \right\} \\ &\geq \inf \left\{ \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) d\mu(y), \mathbb{H}[\nu] \right\} \\ &\geq C \inf \left\{ \mathbb{H}[\mu]\left(\frac{t}{A_2}, x\right), \mathbb{H}[\nu](t, x) \right\} \\ &\geq C \inf \left\{ \mathbb{H}[\mu]\left(\frac{t}{\max(A_2, 1)}, x\right), \mathbb{H}[\nu]\left(\frac{t}{\max(A_2, 1)}, x\right) \right\}, \end{aligned}$$

where, in the above inequalities, we have used estimates (7.9) and the constants  $C > 0$ ,  $A_2 > 0$  therein.

Now since  $\inf \left\{ \mathbb{H}[\mu]\left(\frac{t}{\max(A_2, 1)}, x\right), \mathbb{H}[\nu]\left(\frac{t}{\max(A_2, 1)}, x\right) \right\}$  is a supersolution of  $\partial_t w - \frac{1}{\max(A_2, 1)} \Delta w = 0$ , there exists a positive Radon measure  $\tilde{\nu}$  such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \phi(x) \inf \left\{ \mathbb{H}[\mu]\left(\frac{t}{\max(A_2, 1)}, x\right), \mathbb{H}[\nu]\left(\frac{t}{\max(A_2, 1)}, x\right) \right\} dx = \int_{\mathbb{R}^N} \phi(x) d\tilde{\nu} \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Thus in view of Lemmas 7.3 and 7.4, there exists a positive weak solution  $\tilde{v} \leq v$  of the problem

$$\begin{aligned} \partial_t w - \Delta w + V'w &= 0 & \text{in } & \mathbb{R}^N \times (0, T]. \\ w(\cdot, 0) &= \tilde{\nu} & \text{in } & \mathbb{R}^N, \end{aligned}$$

and by the first claim it yields  $\tilde{v} = 0$ .

By the Lebesgue-Radon-Nikodym Theorem we can write  $d\nu = \phi d\mu + d\sigma$ , where  $0 \leq \phi \in L_{loc}^1(\mathbb{R}^N, \mu)$  and  $\sigma \perp \mu$ . Thus we have

$$\begin{aligned} 0 &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \phi(x) \inf \left\{ \mathbb{H}[\mu]\left(\frac{t}{\max(A_2, 1)}, x\right), \mathbb{H}[\nu]\left(\frac{t}{\max(A_2, 1)}, x\right) \right\} dx \\ &\geq \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \phi(x) h\left(\frac{t}{\max(A_2, 1)}, x, y\right) \min\{f, 1\}(y) d\mu(y) dx \\ &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \phi(y) \min\{f, 1\}(y) d\mu(y) = 0, \end{aligned}$$

where, we recall it,  $h(t, x, y)$  in the heat kernel in  $Q_\infty$ . Hence  $f = 0$  and  $\nu \perp \mu$ .  $\square$

**Lemma 8.6** *Let  $u \in \mathcal{U}_+(Q_T)$ ,  $\mu$  its extended initial trace and suppose that for every  $\nu \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap W^{-\frac{2}{q}, q}(\mathbb{R}^N)$  there exists no positive solution of (3.1) dominated by  $v = \inf(u, \mathbb{H}[\nu])$ . Then  $u = 0$ .*

*Proof.* The proof is similar as the one in [16]. By Lemma 8.5,

$$\mu \perp \nu \quad \forall \nu \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N).$$

Suppose that  $\mu \neq 0$ . By Lemma 8.4,  $\mu$  vanishes on sets of  $C_{\frac{2}{q}, q}$ -capacity zero. Thus, there exists an increasing sequence  $\{\nu_k\} \subset W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  which converges to  $\mu$ . Thus  $\mu \perp \nu_k$  and for every  $k \in \mathbb{N}$  there exists a Borel set  $A_k \subset \mathbb{R}^N$  such that

$$\mu(A_k) = 0 \quad \text{and} \quad \nu_k(A_k^c) = 0.$$

Therefore, if  $A = \cup_k A_k$  then

$$\mu(A) = 0 \quad \text{and} \quad \nu_k(A^c) = 0 \quad \forall k.$$

Since  $\nu_k \leq \mu$  we have  $\nu_k(A) = 0$  and therefore  $\nu_k = 0$ . Contradiction.  $\square$

**Lemma 8.7** *Let  $u \in \mathcal{U}_+(Q_T)$ . Then  $[u]_{S_q(u)}$  is  $\sigma$ -moderate.*

*Proof.* To simplify the notations we put  $u_S = [u]_{\mathcal{S}_q(u)}$  and denote  $F := \mathfrak{T}_q\text{-supp}(u_S)$ . Incidentally,  $F \subset \mathcal{S}_q(u)$ ; since if  $\mathcal{S}_q(u)$  is thin at  $\xi$ , then  $\mathcal{S}_q(u)^c \cup \{\xi\}$  is  $\mathfrak{T}_q$ -open and  $\mathcal{S}_q(u)^c \cup \{\xi\} \sim^q \mathcal{S}_q(u)^c$ . Thus by definition of  $F$ , we see that  $F$  consists precisely of the  $C_{\frac{2}{q}, q'}$ -thick points of  $\mathcal{S}_q(u)$ . The set  $\mathcal{S}_q(u) \setminus F$  is contained in the singular set of  $u_{\mathcal{R}_q}$ .

For  $\nu \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  we denote by  $u_\nu$  the solution of (3.1) with initial trace  $\nu$ . Put

$$u^* := \sup\{u_\nu : \nu \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), u_\nu \leq u_S\}. \quad (8.13)$$

By Lemma 8.6 the family over which the supremum is taken is not empty. Therefore  $u^*$  is a positive solution of (3.1) and, by Proposition 6.11-(iii), it is  $\sigma$ -moderate, thus it is the largest  $\sigma$ -moderate solution dominated by  $u_S$ . We denote by  $\{\nu_n\} \subset W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  an increasing sequence such that  $u^* = \lim_n \nu_n \rightarrow \infty u_{\nu_n}$ .

Let  $F^* = \mathfrak{T}_q\text{-supp}(u^*)$ . Then  $F^*$  is  $\mathfrak{T}_q$ -closed and  $F^* \subset F$ . Suppose that

$$C_{\frac{2}{q}, q'}(F \setminus F^*) > 0.$$

Then there exists a compact set  $E \subset F \setminus F^*$  such that  $C_{\frac{2}{q}, q'}(E) > 0$  and  $(F^*)^c =: Q^*$  is a  $\mathfrak{T}_q$ -open set containing  $E$ . Furthermore by Lemma 2.7 there exists a  $\mathfrak{T}_q$ -open set  $Q'$  such that  $E \subset^q Q \subset \widetilde{Q}' \subset^q Q^*$ . Since  $Q' \subset^q \mathfrak{T}_q\text{-supp}(u_S)$ ,  $[u_S]_{Q'} > 0$  and therefore by Lemma 8.6, there exists a positive measure  $\tau \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  supported in  $\widetilde{Q}'$  such that  $u_\tau \leq u_S$ . As the  $\mathfrak{T}_q\text{-supp}(\tau)$  is a  $\mathfrak{T}_q$ -closed set disjoint from  $F^*$ , it follows that the inequality  $u^* \geq u_\tau$  does not hold. On the other hand, since  $\tau \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  and  $u_\tau \leq u_S$ , it follows that  $u_\tau \leq u^*$ . This contradiction shows that

$$C_{\frac{2}{q}, q'}(F \setminus F^*) = 0. \quad (8.14)$$

Since  $u_{\nu_n} \uparrow u^*$ ,  $\mathfrak{T}_q\text{-supp}(u_{\nu_n}) \subset \mathfrak{T}_q\text{-supp}(u^*) =: F^*$ . Therefore there exists a  $\mathfrak{T}_q$ -closed set  $F_0^* \subset F^*$  such that  $\mathcal{S}_q(u^*) = F_0^*$  and  $\mathcal{R}_q(u^*) = (F_0^*)^c$ . Suppose that

$$C_{\frac{2}{q}, q'}(F \setminus F_0^*) > 0.$$

Let  $Q'$  be a  $\mathfrak{T}_q$ -open subset of  $\mathcal{R}_q(u^*)$  such that  $[u_S]_{Q'}$  is a moderate solution, then  $\widetilde{Q}' \subset^q \mathcal{R}_q(u^*)$  and  $[u^*]_{\widetilde{Q}'}$  is a moderate solution of (3.1), i.e.,

$$\int_0^T \int_{\mathbb{R}^N} [u^*]_{\widetilde{Q}'}^q \phi(x) dx dt < \infty \quad \forall \phi \in C_0(\mathbb{R}^N), \phi \geq 0.$$

On the other hand  $Q'$  is a  $\mathfrak{T}_q$ -open subset of  $F = \mathfrak{T}_q\text{-supp}(u_S)$ ; therefore the initial trace of  $[u^*]_{\widetilde{Q}'}$  has no regular part, i.e.,

$$\mathcal{R}_q([u^*]_{\widetilde{Q}'}) = 0 \text{ and } \mathcal{S}_q([u^*]_{\widetilde{Q}'}) = \mathfrak{T}_q\text{-supp}([u^*]_{\widetilde{Q}'});$$

we say that  $[u^*]_{\widetilde{Q}'}$  is a *purely singular solution* of (3.1). It follows that  $v := \left[ [u_S]_{\widetilde{Q}'} - [u^*]_{\widetilde{Q}'} \right]_{\dagger}$  is a purely singular solution of (3.1).

Let  $v^*$  be defined as in (8.13) with  $u$  replaced by  $v$ . Then  $v^*$  is a singular  $\sigma$ -moderate solution of (3.1). Since  $v^*$  is smaller than  $u$  and since it is  $\sigma$ -moderate it is dominated by  $u^*$ . On the other hand, since  $v^*$  is singular and  $\mathfrak{T}_q\text{-supp}(v^*) \subset {}^q\widetilde{Q}' \subset {}^q\mathcal{R}_q(u^*)$  it follows that  $u^*$  is not larger or equal to  $v^*$ , i.e.  $(v^* - u^*)_+$  is not identically zero. Since both  $u^*$  and  $v^*$  are  $\sigma$ -moderate, it implies that there exists  $\tau \in W^{-\frac{2}{q}, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  such that  $u_\tau \leq v^*$ , and  $(u_\tau - u^*)$  is not identically zero. Therefore  $u^* \not\leq \max(u^*, u_\tau)$ . The function  $\max(u^*, u_\tau)$  is a subsolution of (3.1) and the smallest solution above it, denoted by  $Z$ , is strictly larger than  $u^*$ . However  $u_\tau \leq v^* \leq u^*$  and consequently  $Z = u^*$ .

This contradiction proves that  $C_{\frac{2}{q}, q'}(Q') = 0$ , for any set  $Q' \subset \mathcal{R}_q(u^*)$  such that  $[u]_{Q'}$  is moderate solution, that is  $C_{\frac{2}{q}, q'}(\mathcal{R}_q(u^*)) = 0$  which implies

$$C_{\frac{2}{q}, q'}(F \setminus F_0^*) = 0. \quad (8.15)$$

In conclusion,  $u^*$  is  $\sigma$ -moderate,  $\mathfrak{T}_q\text{-supp}(u^*) \subset F$  and  $F_0^* = \mathcal{S}_q(u^*) \sim {}^q F$ . Therefore, by Proposition 6.15 and the remark below,  $u^* = U_F$ . Since by definition  $u^* \leq u \leq U_F$ , it follows  $u^* = u$ .  $\square$

**Theorem 8.8** *Every positive solution of (3.1) is  $\sigma$ -moderate.*

*Proof.* We borrow the ideas of the proof to [16]. By Proposition 6.8-(i),  $\mathcal{R}_q(u)$  has regular decomposition  $\{Q_n\}$ . Furthermore

$$v_n = [u]_{Q_n} \uparrow u_{\mathcal{R}_q}.$$

Thus the solution  $u_{\mathcal{R}_q}$  is  $\sigma$ -moderate and

$$u \ominus u_{\mathcal{R}_q} \leq [u]_{\mathcal{S}_q(u)}.$$

Put

$$u_n = v_n \oplus [u]_{\mathcal{S}_q(u)}.$$

By Lemma 8.7 we have that  $[u]_{\mathcal{S}_q(u)}$  is  $\sigma$ -moderate solution, thus by Proposition 8.2, as  $\widetilde{Q}_n \cap \mathcal{S}_q(u) = \emptyset$ , it follows that  $u_n$  is  $\sigma$ -moderate. As  $\{u_n\}$  is increasing it follows that  $\bar{u} = \lim_{n \rightarrow \infty} u_n$  is a  $\sigma$ -moderate solution of (3.1). In addition

$$v_n \vee [u]_{\mathcal{S}_q(u)} = u_n = v_n \oplus [u]_{\mathcal{S}_q(u)} \Rightarrow \max(u_{\mathcal{R}_q}, [u]_{\mathcal{S}_q(u)}) \leq \bar{u} \leq u_{\mathcal{R}_q} + [u]_{\mathcal{S}_q(u)}.$$

This further implies that  $\mathcal{S}_q(u) = \mathcal{S}_q(\bar{u})$ . By construction we have

$$[u]_{Q_n} = v_n \leq [\bar{u}]_{Q_n}$$

Letting  $n \rightarrow \infty$  we have by Proposition 6.8

$$u_{\mathcal{R}_q} \leq \bar{u}_{\mathcal{R}_q} \Rightarrow u_{\mathcal{R}_q} = \bar{u}_{\mathcal{R}_q},$$

thus  $\text{tr}(u) = \text{tr}(\bar{u})$  and since  $\bar{u} \leq u$ , we have by Proposition 6.15 and the uniqueness of  $\sigma$ -moderate solutions that  $u = \bar{u}$ .  $\square$



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