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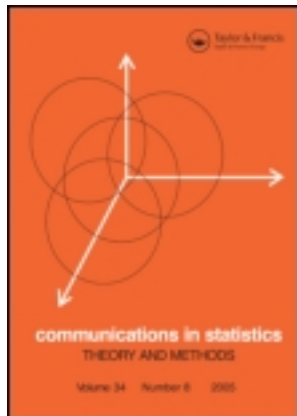
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### Bayesian Analysis of ARA Imperfect Repair Models

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# Bayesian Analysis of ARA Imperfect Repair Models

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*This article proposes a Bayesian analysis of a class of imperfect repair models, the ARA models. The choice of prior distributions and the computation of posterior distributions are discussed. The presentation is unified for all ARA models and many kinds of possible priors. A numerical study on the quality of the Bayesian estimators is presented, as well as a comparison with the maximum likelihood estimators. Finally, the approach is applied to a real data set.*

**Keywords** Bayesian inference; Imperfect repair; Maintenance efficiency; MCMC algorithms; Repairable systems reliability; Virtual age.

**Mathematics Subject Classification** Primary 62N05; Secondary 60K10.

## 1. Introduction

The reliability of a repairable system depends on both the aging process and the efficiency of repair. The basic assumptions on repair efficiency are known as minimal repair or As Bad As Old (ABAO) and perfect repair or As Good As New (AGAN). The corresponding stochastic models for the failure process are, respectively, the Non Homogeneous Poisson Processes (NHPP) and the Renewal Processes (RP). The reality is generally between these two extreme cases: standard repair is better than minimal but not necessarily perfect. This is known as imperfect repair.

Many imperfect repair models have been proposed. The most usual are Kijima's virtual age models (Kijima, 1989), where repair is assumed to rejuvenate the system. Brown and Proschan (1983) assumed that repair is perfect with probability  $p$  and minimal with probability  $1 - p$ . Doyen and Gaudoin (2004) proposed two classes of models, the ARA and ARI models, based on a reduction of virtual age or failure intensity.

Statistical inference is needed in order to estimate model parameters and compute reliability indicators from failure data. Many results exist in the NHPP and RP cases, but only a few articles deal with the statistical inference in imperfect repair models. For virtual age models, empirical studies on maximum likelihood

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estimators (MLE) have been presented by Shin et al. (1996), Lim (1998), Kaminskiy and Krivtsov (2000), Gasmi et al. (2003), Doyen and Gaudoin (2004), Veber et al. (2008), Dijoux (2009), and Syamsundar and Naikan (2011). Theoretical results on these estimators were derived by Peña et al. (2007) and Doyen (2010).

In practice, for reliable systems, only a few failures occur. So the asymptotic results on MLE cannot be used. Moreover, the engineers knowledge about the degradation and failure process could be very helpful to improve the estimations of the model parameters. Thus, a Bayesian analysis of these models is an interesting alternative to usual frequentist methods.

The Bayesian inference has been extensively studied in the NHPP case: Guida et al. (1989, 1990), Guida and Pulcini (2006), Bar-Lev et al. (1992), Campodonico and Singpurwalla (1995), Calabria et al. (1996), Kuo and Yang (1996), Beiser and Rigdon (1997), Sen (2002), Ryan (2003), and Pievatolo and Ruggeri (2004). Conversely, Bayesian analysis for imperfect repair models has been seldom studied: Lim et al. (1998) for the Brown-Proschan model, and Pan and Rigdon (2009) for the  $ARA_1$  and  $ARI_1$  models. With a different purpose, Sethuraman and Hollander (2006) derived theoretical results for general virtual age models when the virtual ages are known.

The aim of this article is to present a Bayesian analysis of the class of  $ARA$  imperfect repair models. This class involves the  $ARA_1$  model studied by Pan and Rigdon (2009) and also many others including the  $ARA_\infty$  model.

This article is organized as follows. The  $ARA$  models are defined in Sec. 2. The Bayesian analysis is developed in Sec. 3, with a discussion on the choice of prior distributions and the computation of posterior distributions. Section 4 studies the properties of the estimators by means of Monte-Carlo simulations. Finally, an application to real data is presented in Sec. 5.

## 2. The $ARA$ Imperfect Repair Models

Let  $\{T_i\}_{i \geq 1}$  be the successive failure times of a repairable system, starting from  $T_0 = 0$ . We assume that a repair task is performed after each failure and that repair durations are negligible. Let  $\{X_i\}_{i \geq 1}$  be the times between failures ( $X_i = T_i - T_{i-1}$ ) and  $N_t$  be the number of failures observed up to time  $t$ .

Then, the failure process is a random point process. Its distribution can be characterized by the failure intensity, defined as:

$$\forall t \geq 0, \quad \lambda_t = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(N_{(t+\Delta t)-} - N_{t-} = 1 | \mathcal{H}_{t-}), \quad (1)$$

where  $\mathcal{H}_{t-}$  is the history of the failure process just before time  $t$ , i.e., the set of all events occurred just before  $t$ .  $N_{t-}$  denotes the left-hand limit of  $N_t$ . In most cases, the failure process is a self-excited process, i.e.,  $\lambda_t$  is a function of the number of failures and the failure times before  $t$ :  $\lambda_t = \lambda_t(N_{t-}, T_1, \dots, T_{N_{t-}})$ . In this case, the distribution of the failure process is completely given by the intensity.

We assume that before the first failure, the intensity is a deterministic continuous function of time, denoted  $\lambda(t)$ , and called the initial intensity. The initial intensity characterizes the intrinsic behaviour of a new unrepaired system.  $\lambda$  is the hazard rate of the first failure time  $T_1 = X_1$ . In the following, we assume, as in Pan

and Rigdon (2009), that the initial intensity is the intensity of a Power Law Process (PLP):

$$\lambda(t) = \alpha\beta t^{\beta-1}, \quad \alpha > 0, \quad \beta > 0. \tag{2}$$

In general, the system is ageing intrinsically, so  $\lambda(t)$  is increasing and  $\beta > 1$ .

For the two basic models, the failure intensity of a NHPP (ABAO) is a function of time:

$$\lambda_t = \lambda(t) \tag{3}$$

and the failure intensity of a RP (AGAN) is a function of the time elapsed since last repair:

$$\lambda_t = \lambda(t - T_{N_{t-}}). \tag{4}$$

The idea of the virtual age models proposed by Kijima (1989) is to assume that there exists a sequence of random variables  $\{A_i\}_{i \geq 1}$ , with  $A_0 = 0$ , such that after the  $i$ th repair, the system behaves like a new one having survived without failure until  $A_i$ . This property can be written:

$$P(X_{i+1} > x | A_i, X_1, \dots, X_i) = P(Y > A_i + x | Y > A_i, A_i) \quad \text{for all } x \geq 0, \tag{5}$$

where  $Y$  denotes a random variable with the same distribution as the first failure time  $X_1$ . Then, it can easily be proved that the failure intensity is:

$$\lambda_t = \lambda(A_{N_{t-}} + t - T_{N_{t-}}). \tag{6}$$

The virtual age of the system at time  $t$  is  $A_{N_{t-}} + t - T_{N_{t-}}$ .  $A_i$  is the virtual age just after the  $i$ th repair and is called the  $i$ th effective age. The NHPP is a virtual age model with  $\forall i, A_i = T_i$  and the RP is a virtual age model with  $\forall i, A_i = 0$ .

This article is dedicated to the Arithmetic Reduction of Age (ARA) class of imperfect repair models, defined in Doyen and Gaudoin (2004), which are particular virtual age models. The Arithmetic Reduction of Age model with memory  $m$ , denoted  $ARA_m$ , is defined by its failure intensity:

$$\lambda_t = \lambda \left( t - \rho \sum_{j=0}^{\min(m-1, N_{t-}-1)} (1 - \rho)^j T_{N_{t-}-j} \right). \tag{7}$$

$m$  reflects a Markovian property: it is the maximal number of previous failure times involved in the failure intensity.  $\rho$  is a parameter which characterizes repair efficiency and will be described later.

Two values of  $m$  are particularly interesting.

- $m = 1$ . The failure intensity of the  $ARA_1$  model is:

$$\lambda_t = \lambda(t - \rho T_{N_{t-}}). \tag{8}$$

In this model, the effect of a repair is to reduce the virtual age just before repair of a quantity proportional to the time elapsed since last repair:

$$A_i = A_{i-1} + X_i - \rho X_i \implies A_i = (1 - \rho)T_i.$$

The  $ARA_1$  model is similar to Kijima's Type I model (Kijima, 1989) with deterministic repair effects, and is also the same as the Proportional Reduction of Age model proposed by Malik (1979).

- $m = +\infty$ . The failure intensity of the  $ARA_\infty$  model is:

$$\lambda_t = \lambda \left( t - \rho \sum_{j=0}^{N_t-1} (1 - \rho)^j T_{N_t-j} \right). \quad (9)$$

In this model, the effect of a repair is to reduce the virtual age just before repair of a quantity proportional to this virtual age:

$$A_i = (1 - \rho)(A_{i-1} + X_i).$$

The  $ARA_\infty$  model is similar to Kijima's Type II model (Kijima, 1989) with deterministic repair effects, and is also the same as the model proposed by Brown et al. (1983).

In these models, the repair efficiency is measured by the value of  $\rho$ . Particular values of  $\rho$  are:

- $\rho = 1$ : perfect repair (AGAN);
- $0 < \rho < 1$ : efficient repair;
- $\rho = 0$ : minimal repair (ABAO); and
- $\rho < 0$ : harmful repair.

For the sake of simplicity, we will assume that repair cannot be harmful, so  $\rho \in [0, 1]$ . Of course, in practice, it is doubtful that repair efficiency is constant. It is possible to take into account time-dependent (Fuqing and Kumar, 2012) or random (Doyen, 2011) repair effects, but the corresponding statistical inference is much more complex. So, we focus our analysis on the rather simple ARA models, in which  $\rho$  can be understood as a mean repair efficiency. Note that the Bayesian framework, in which a prior distribution is given on  $\rho$ , is a way of considering non constant repair efficiency.

In this article, we derive all our results under the general framework of the  $ARA_m$  model. The interest is that these results will be valid, among others, for both the  $ARA_1$  and  $ARA_\infty$  models. For the  $ARA_1$  model, our work extends the results obtained by Pan and Rigdon (2009). The results for the  $ARA_\infty$  model are new.

For a point process model with parameter  $\theta$  (here  $\theta = (\alpha, \beta, \rho)$ ), the general expression of the likelihood function associated with the observation of  $n$  failures up to time  $t$  of a single repairable system is (Cook and Lawless, 2007):

$$L_t(\theta; n, \mathbf{t}_n) = \left[ \prod_{i=1}^n \lambda_{t_i} \right] \exp \left( - \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} \lambda_s ds \right), \quad (10)$$

where  $\mathbf{t}_n = (t_1, \dots, t_n)$  denotes the vector of observed failure times,  $t_0 = 0$  and  $t_{n+1} = t$ . In this situation, the process is time-truncated at time  $t$ . For a process which is failure-truncated after  $n$  failures, (10) is still valid by taking  $t_{n+1} = t_n$ .

For the  $ARA_m$  model with PLP initial intensity, the likelihood becomes, with the same kind of notations as in Pulcini (2000):

$$L_t(\alpha, \beta, \rho; n, \mathbf{t}_n) = \alpha^n \beta^n V(\rho)^{\beta-1} \exp(-\alpha Z(\beta, \rho)), \tag{11}$$

where

$$V(\rho) = \prod_{i=1}^n [t_i - S_i(\rho)],$$

$$Z(\beta, \rho) = \sum_{i=1}^{n+1} [(t_i - S_i(\rho))^\beta - (t_{i-1} - S_i(\rho))^\beta],$$

$$S_1(\rho) = 0 \quad \text{and} \quad \forall i \geq 2, S_i(\rho) = \rho \sum_{j=0}^{\min(m-1, i-2)} (1 - \rho)^j t_{i-j-1}.$$

Let  $\hat{\alpha}_t, \hat{\beta}_t,$  and  $\hat{\rho}_t$  be the MLE of  $\alpha, \beta,$  and  $\rho,$  obtained by maximizing (11). It is easy to show that:

$$\hat{\alpha}_t = \frac{N_t}{Z(\hat{\beta}_t, \hat{\rho}_t)}. \tag{12}$$

However, numerical optimization procedures are necessary to compute  $\hat{\beta}_t$  and  $\hat{\rho}_t.$

### 3. Bayesian Inference

In this article, we consider a single repairable system, which is so reliable that only very few failures occur. Then, the quality of the MLE can be very poor. The Bayesian analysis improves the accuracy of parameter estimations by adding the expert knowledge to operation feedback data. In the Bayesian framework, the parameters are considered as random variables. Their prior distributions reflect the expert knowledge on the system aging and repair efficiency.

The ARA models have three parameters  $\alpha > 0, \beta > 0,$  and  $\rho \in [0, 1].$  For a given prior density  $\pi_{(\alpha, \beta, \rho)}(\alpha, \beta, \rho),$  the posterior density is obtained by:

$$\pi(\alpha, \beta, \rho | n, \mathbf{t}_n) \propto L_t(\alpha, \beta, \rho; n, \mathbf{t}_n) \pi_{(\alpha, \beta, \rho)}(\alpha, \beta, \rho). \tag{13}$$

The choice of prior distributions for this kind of parameters has been discussed in many articles. Non informative priors are used when no particular information is known on the failure and repair process. When such information is available, it is translated into informative priors. In practice, experts give an average behaviour and a degree of uncertainty on this information, which can be converted into prior mean and variance.

Since  $\alpha$  and  $\beta$  are the parameters of a PLP, the prior distributions used in the Bayesian analysis of the PLP can be chosen here. Non informative priors for

$\alpha$  and  $\beta$  are used by Guida et al. (1989), Bar-Lev et al. (1992), Calabria et al. (1996), and Sen (2002). Informative priors are used either by considering expert knowledge on the failure mechanism, or by using conjugate distributions. For  $\alpha$ , the gamma distribution has been used by Kuo and Yang (1996), Pivatolo and Ruggeri (2004), and Pan and Rigdon (2009). For  $\beta$ , the uniform distribution has been used by Guida et al. (1989), Sen (2002), and Pan and Rigdon (2009). The beta distribution has been used by Sen (2002).

For the repair efficiency parameter of the Brown-Proschan model, Lim et al. (1998) used the beta distribution. For the repair efficiency parameters of the  $ARA_1$  and  $ARI_1$  models, Pan and Rigdon (2009) also used the beta distribution.

In this article, we will choose informative, non informative, and semi informative priors on the model parameters. Informative priors on reliability indicators will also be considered.

### 3.1. Informative Priors on the Parameters

For the scale parameter  $\alpha$ , we choose the gamma distribution, denoted  $\Gamma(a, b)$ , with density function:

$$\pi_\alpha(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} \exp(-b\alpha), \quad (14)$$

where  $\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt$ . The expectation and variance of this distribution are, respectively,  $a/b$  and  $a/b^2$ . We will see later that this is a conjugate prior for  $\alpha$ .

For the shape parameter  $\beta$ , we choose the uniform distribution on  $[\beta_1, \beta_2]$ , denoted  $\mathcal{U}[\beta_1, \beta_2]$ :

$$\pi_\beta(\beta) = \frac{1}{\beta_2 - \beta_1} \mathbb{1}_{[\beta_1, \beta_2]}(\beta). \quad (15)$$

$\beta$  is linked to the system aging in a Weibull framework. So experts are likely to give plausible values for  $\beta$ .  $\beta_1$  and  $\beta_2$  can be understood as, respectively, the lower and upper bounds for these values. It is possible to take  $\beta_1 = 1$  in order to assume that, without repair, the system wears out.

For the repair efficiency parameter  $\rho$ , since we have assumed that  $\rho$  belongs to  $[0, 1]$ , we choose the beta distribution, denoted  $Be(c, d)$ , with density function:

$$\pi_\rho(\rho) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \rho^{c-1} (1-\rho)^{d-1}. \quad (16)$$

The prior expectation is  $\frac{c}{c+d}$  and the prior variance is  $\frac{cd}{(c+d)^2(c+d+1)}$ .

Finally, the priors are supposed to be independent, so the joint prior density is:

$$\begin{aligned} \pi_{(\alpha, \beta, \rho)}(\alpha, \beta, \rho) &= \pi_\alpha(\alpha) \pi_\beta(\beta) \pi_\rho(\rho) \\ &\propto \alpha^{a-1} \exp(-b\alpha) \rho^{c-1} (1-\rho)^{d-1} \mathbb{1}_{[\beta_1, \beta_2]}(\beta). \end{aligned} \quad (17)$$



**3.2. Non Informative Priors**

The usual non informative priors for  $\alpha$  and  $\beta$  in the PLP case are:

$$\pi_\alpha(\alpha) \propto \frac{1}{\alpha} \quad \text{and} \quad \pi_\beta(\beta) \propto \frac{1}{\beta}. \tag{18}$$

For non harmful repair,  $\rho$  belongs to  $[0, 1]$ . So a non informative prior in this case is the uniform distribution  $\mathcal{U}[0, 1]$ .

Then, with the independence of priors, the joint non informative prior is:

$$\pi_{(\alpha, \beta, \rho)}(\alpha, \beta, \rho) \propto \frac{1}{\alpha\beta}. \tag{19}$$

**3.3. Unified Priors**

Following Pulcini (2000), we can unify the informative and non informative cases by making the following remarks.

- For  $\alpha$ , the non informative prior is a limit case of the  $\Gamma(a, b)$  distribution, where  $a$  and  $b$  tend to zero.
- For  $\beta$ , the  $\mathcal{U}[\beta_1, \beta_2]$  density and the non informative prior can both be written:

$$\pi_\beta(\beta) \propto \beta^{-\gamma} \mathbb{I}[\beta_1, \beta_2](\beta), \tag{20}$$

where  $\gamma = 0$  corresponds to the uniform distribution, and  $\gamma = 1, \beta_1 = 0$ , and  $\beta_2 = +\infty$  correspond to the non informative prior.

- For  $\rho$ , the  $\mathcal{U}[0, 1]$  distribution is the same as the  $Be(1, 1)$  distribution, so it is a particular case of the  $Be(c, d)$  distribution.

So we can define a unified prior as:

$$\pi_{(\alpha, \beta, \rho)}(\alpha, \beta, \rho) \propto \alpha^{a-1} \exp(-b\alpha) \beta^{-\gamma} \rho^{c-1} (1 - \rho)^{d-1} \mathbb{I}[\beta_1, \beta_2](\beta). \tag{21}$$

The informative prior (17) is obtained for  $\gamma = 0$  and the non informative prior (19) is obtained for  $a = b = 0, \gamma = 1, c = d = 1, \beta_1 = 0$ , and  $\beta_2 = +\infty$ .

It is also possible to use semi-informative priors. Indeed, the three parameters of the ARA models have well-identified meanings.  $\beta$  is linked to the intrinsic aging,  $\rho$  characterizes repair efficiency, and  $\alpha$  is a scale parameter. In practice, it is much easier to obtain prior information on  $\beta$  and  $\rho$  than on  $\alpha$ . Therefore, one can consider, as in Pulcini (2000), to use the non informative prior for  $\alpha$  and the informative priors for  $\beta$  and  $\rho$ . This can be done by taking only  $a = b = 0$  in (21).

**3.4. Posterior Distributions**

For the unified prior, the joint posterior distribution is derived using (13), (11), and (21). The posterior density of  $(\alpha, \beta, \rho)$  is:

$$\begin{aligned} \pi(\alpha, \beta, \rho \mid n, \mathbf{t}_n) &\propto \alpha^{n+a-1} \exp(-[b + Z(\beta, \rho)]\alpha) \\ &\quad \beta^{n-\gamma} V(\rho)^{\beta-1} \rho^{c-1} (1 - \rho)^{d-1} \mathbb{I}[\beta_1, \beta_2](\beta). \end{aligned} \tag{22}$$

The conditional posterior densities of each parameter can also be derived as:

$$\pi(\alpha | \beta, \rho, n, \mathbf{t}_n) \propto \alpha^{n+a-1} \exp(-[b + Z(\beta, \rho)]\alpha) \quad (23)$$

$$\pi(\beta | \alpha, \rho, n, \mathbf{t}_n) \propto \beta^{n-\gamma} V(\rho)^{\beta-1} \exp(-\alpha Z(\beta, \rho)) \mathbb{I}[\beta_1, \beta_2](\beta) \quad (24)$$

$$\pi(\rho | \alpha, \beta, n, \mathbf{t}_n) \propto V(\rho)^{\beta-1} \exp(-\alpha Z(\beta, \rho)) \rho^{c-1} (1 - \rho)^{d-1}. \quad (25)$$

Equation (23) proves that the posterior distribution of  $\alpha$  given  $\beta$  and  $\rho$  is the gamma distribution  $\Gamma(n + a, b + Z(\beta, \rho))$ . Then the gamma distribution is a conjugate prior for  $\alpha$ .

Because the parameter  $(\alpha, \beta, \rho)$  is multidimensional and there are no conjugate priors for  $\beta$  and  $\rho$ , the posterior distributions will be derived by means of the Gibbs sampling algorithm, with Metropolis-Hastings steps (Robert and Casella, 2004). This Markov Chain Monte Carlo (MCMC) algorithm produces samples from the joint posterior distribution, from which it is possible to compute all the features of the posterior distributions. For instance, the posterior mean, median, and mode are point Bayesian estimators of the parameters. With the posterior distribution, it is also possible to give credibility intervals for the parameters.

We propose two versions of the algorithm, one for which a prior is proposed for  $\alpha$ , and one for which  $\alpha$  is derived from (12).

**Algorithm 1** (Algorithm with a Prior for  $\alpha$ ).

1.  $k \leftarrow 0$ . Choose initial values of  $\alpha_0$ ,  $\beta_0$ , and  $\rho_0$ .
2.  $k \leftarrow k + 1$ .
3. Sample  $\alpha_k$  given  $\beta_{k-1}$  and  $\rho_{k-1}$  from (23).
4. Sample  $\beta_k$  given  $\alpha_k$  and  $\rho_{k-1}$  from (24).
5. Sample  $\rho_k$  given  $\alpha_k$  and  $\beta_k$  from (25).
6. While  $k < K$ , go to step 2.

Since it is not possible to sample directly from (24) and (25), we use the Metropolis-Hastings algorithm in steps 4 and 5. For instance, in step 4, we sample  $\beta^*$  from an instrumental distribution easy to simulate, with density  $q$ . Let

$$r = \frac{\pi(\beta^* | \alpha_k, \rho_{k-1}, n, \mathbf{t}_n) q(\beta_{k-1} | \beta^*)}{\pi(\beta_{k-1} | \alpha_k, \rho_{k-1}, n, \mathbf{t}_n) q(\beta^* | \beta_{k-1})}. \quad (26)$$

Then,  $\beta_k$  will be equal to  $\beta^*$  with probability  $\min(r, 1)$  and to  $\beta_{k-1}$  otherwise. For a symmetric transition,  $q(\beta_{k-1} | \beta^*) = q(\beta^* | \beta_{k-1})$ , so (26) becomes

$$r = \frac{\pi(\beta^* | \alpha_k, \rho_{k-1}, n, \mathbf{t}_n)}{\pi(\beta_{k-1} | \alpha_k, \rho_{k-1}, n, \mathbf{t}_n)}. \quad (27)$$

Here, we take for  $q$  a Gaussian distribution centered on the given value of  $\beta$ .

As said before, it is not easy to give a prior on the scale parameter  $\alpha$ . A first possibility is to choose a non informative prior by taking  $a = b = 0$  in (21). Another possibility is to use the fact that, in (12), the MLE of  $\alpha$  is computed as a function of the MLE of  $\beta$  and  $\rho$ . Then, we propose a second version of the algorithm, for which priors are given only for  $\beta$  and  $\rho$ , and  $\alpha$  is computed as a function of  $\beta$  and  $\rho$ , according to (12).

**Algorithm 2** (Algorithm with no Prior for  $\alpha$ ).

1.  $k \leftarrow 0$ . Choose initial values of  $\beta_0$  and  $\rho_0$ .

2. Compute  $\alpha_0$  as a function of  $\beta_0$  and  $\rho_0$  from (12).
3.  $k \leftarrow k + 1$ .
4. Sample  $\beta_k$  given  $\alpha_{k-1}$  and  $\rho_{k-1}$  from (24).
5. Sample  $\rho_k$  given  $\alpha_{k-1}$  and  $\beta_k$  from (25).
6. Compute  $\alpha_k$  as a function of  $\beta_k$  and  $\rho_k$  from (12).
7. While  $k < K$ , go to step 3.

**3.5. Informative Prior on Reliability Indicators**

Giving a prior is facilitated if the considered quantity has a clear physical meaning, such as a reliability indicator. In the PLP case, Guida et al. (1989), Calabria et al. (1996), and Sen (2002) proposed a gamma prior for the expected number of failures  $E[N_i]$ . With an informative prior on  $\beta$ , it is possible to derive a prior for  $(\alpha, \beta)$ .

For the ARA model, the mean number of failures takes into account both the intrinsic aging (through  $\alpha$  and  $\beta$ ) and repair efficiency (through  $\rho$ ). Since these parameters can have compensating effects, it is not so easy to give a prior value for  $E[N_i]$ . Then, we propose to give a prior on an indicator which is linked only to the intrinsic aging, the mean time to the first failure:

$$\mu = E[T_1] = \frac{1}{\alpha^{1/\beta}} \Gamma\left(1 + \frac{1}{\beta}\right). \tag{28}$$

Since  $\alpha$  appears at the denominator of  $\mu$ , we choose an inverse gamma prior, denoted  $\Pi(A, B)$ , for  $\mu$ . Then, the prior density is:

$$\pi_\mu(\mu) = \frac{B^A}{\Gamma(A)} \mu^{-A-1} \exp\left(-\frac{B}{\mu}\right). \tag{29}$$

For  $A > 2$ , the expectation and variance of this distribution are, respectively,  $\frac{B}{A-1}$  and  $\frac{B^2}{(A-1)^2(A-2)}$ .

The conditional prior density of  $\alpha$  is derived thanks to a change of variables:

$$\pi(\alpha | \beta) = \frac{B^A \alpha^{A/\beta-1}}{\beta \Gamma(A) \Gamma(1 + \frac{1}{\beta})^A} \exp\left(-\frac{B \alpha^{1/\beta}}{\Gamma(1 + \frac{1}{\beta})}\right). \tag{30}$$

If we keep the informative priors  $\mathcal{U}[\beta_1, \beta_2]$  for  $\beta$  and  $Be(c, d)$  for  $\rho$ , the joint prior distribution is:

$$\begin{aligned} \pi_{\alpha, \beta, \rho}(\alpha, \beta, \rho) &= \pi(\alpha | \beta) \pi(\beta, \rho) \\ &\propto \frac{\alpha^{A/\beta-1}}{\beta \Gamma(1 + \frac{1}{\beta})^A} \exp\left(-\frac{B \alpha^{1/\beta}}{\Gamma(1 + \frac{1}{\beta})}\right) \rho^{c-1} (1 - \rho)^{d-1} \mathbb{I}[\beta_1, \beta_2](\beta), \end{aligned} \tag{31}$$

and the joint posterior distribution is:

$$\begin{aligned} \pi(\alpha, \beta, \rho | n, \mathbf{t}_n) &\propto \frac{\alpha^{n+A/\beta-1}}{\Gamma(1 + \frac{1}{\beta})^A} \beta^{n-1} \rho^{c-1} (1 - \rho)^{d-1} V(\rho)^{\beta-1} \\ &\quad \exp\left(-\alpha Z(\beta, \rho) - \frac{B \alpha^{1/\beta}}{\Gamma(1 + \frac{1}{\beta})}\right) \mathbb{I}[\beta_1, \beta_2](\beta). \end{aligned} \tag{32}$$

**3.6. Posterior Distributions of Reliability Indicators**

With the posterior distribution of the parameters, it is possible to determine the posterior distribution of any quantity of interest, function of these parameters, such as reliability indicators. First, we give the posterior distribution of the mean time to the first failure  $\mu = E[T_1]$  studied previously. Then, following Pulcini (2000), we derive the posterior distribution of the failure intensity and the expected number of failures. In this section, we do not consider the case where  $\alpha$  is computed as a function of  $\beta$  and  $\rho$ .

3.6.1. *Posterior distribution of the mean time to the first failure.* For the estimation of  $\mu = E[T_1]$  we consider two cases. In the first one, we use the prior distributions on the parameters, and in the second one we use the prior distribution on  $\mu$ .

In the first case, from Eqs. (22) and (28) with a change of variables, the posterior distribution of  $\mu$  is found to be:

$$\pi(\mu | n, \mathbf{t}_n) \propto \int_{\beta_1}^{\beta_2} \int_0^1 \beta^{n-\gamma+1} V(\rho)^{\beta-1} \Gamma(1 + 1/\beta)^{\beta(n+a)} \rho^{c-1} (1 - \rho)^{d-1} \mu^{-\beta(n+a)-1} \exp\left(-[b + Z(\beta, \rho)] \left[\frac{\Gamma(1 + 1/\beta)}{\mu}\right]^\beta\right) d\rho d\beta. \tag{33}$$

Thus, the Bayesian estimator of  $\mu$  is the posterior mean:

$$E[\mu | n, \mathbf{t}_n] = \frac{1}{C} \int_{\beta_1}^{\beta_2} \int_0^1 \beta^{n-\gamma} V(\rho)^{\beta-1} \Gamma(1 + 1/\beta) \rho^{c-1} (1 - \rho)^{d-1} \frac{\Gamma(n + a - 1/\beta)}{[b + Z(\beta, \rho)]^{n+a-1/\beta}} d\rho d\beta, \tag{34}$$

where

$$C = \int_{\beta_1}^{\beta_2} \int_0^1 \beta^{n-\gamma} V(\rho)^{\beta-1} \rho^{c-1} (1 - \rho)^{d-1} \frac{\Gamma(n + a)}{[b + Z(\beta, \rho)]^{n+a}} d\rho d\beta.$$

The integrals involved in these equations are computed by Monte Carlo simulations. It is also possible to obtain credibility intervals for  $\mu$  from Eq. (33).

Secondly, we consider the case of Sec. 3.5, where the inverse gamma prior (29) is chosen for  $\mu$ . With the informative priors  $\mathcal{U}[\beta_1, \beta_2]$  for  $\beta$  and  $Be(c, d)$  for  $\rho$ , the posterior distribution of  $\mu$  is:

$$\pi(\mu | n, \mathbf{t}_n) \propto \int_{\beta_1}^{\beta_2} \int_0^1 \beta^n V(\rho)^{\beta-1} \Gamma(1 + 1/\beta)^{\beta n} \rho^{c-1} (1 - \rho)^{d-1} \mu^{-\beta n - A - 1} \exp\left(-Z(\beta, \rho) \left(\frac{\Gamma(1 + 1/\beta)}{\mu}\right)^\beta - \frac{B}{\mu}\right) d\rho d\beta. \tag{35}$$

The Bayesian estimator of  $\mu$  is the posterior mean.

3.6.2. *Posterior distribution of the failure intensity.* The failure intensity at a time  $\tau \in ]t_i, t_{i+1}]$  is

$$\lambda_\tau = \alpha\beta[\tau - S_{i+1}(\rho)]^{\beta-1}.$$

A change of variables in (22) leads to the posterior distribution of  $\lambda_\tau$ :

$$\begin{aligned} \pi(\lambda_\tau | n, \mathbf{t}_n) &\propto \int_{\beta_1}^{\beta_2} \int_0^1 \frac{V(\rho)^{\beta-1} \rho^{c-1} (1-\rho)^{d-1}}{\beta^{a+\gamma} [\tau - S_{i+1}(\rho)]^{(n+a)(\beta-1)}} \lambda_\tau^{n+a-1} \\ &\exp\left(-\frac{b + Z(\beta, \rho)}{\beta[\tau - S_{i+1}(\rho)]^{\beta-1}} \lambda_\tau\right) d\rho d\beta. \end{aligned} \tag{36}$$

The Bayesian point estimator of  $\lambda_\tau$  is the posterior mean:

$$E[\lambda_\tau | n, \mathbf{t}_n] = \frac{1}{C} \int_{\beta_1}^{\beta_2} \int_0^1 \frac{V(\rho)^{\beta-1} \rho^{c-1} (1-\rho)^{d-1} \beta^{n-\gamma+1} [\tau - S_{i+1}(\rho)]^{\beta-1}}{[b + Z(\beta, \rho)]^{n+a+1}} d\rho d\beta, \tag{37}$$

where  $C$  is a normalizing constant. The integrals involved in (37) are computed by Monte Carlo simulations.

Credibility intervals for  $\lambda_\tau$  can also be obtained from (36). Then, computing these values for each  $\tau$ , it is possible to draw a plot of the Bayesian point and interval estimators of the failure intensity.

3.6.3. *Posterior distribution of the expected number of failures.* A usual result of the theory of point processes is that the expected number of failures at any time  $\tau$  is linked to the failure intensity by:

$$N_\tau = \int_0^\tau \lambda_s ds + M_\tau, \tag{38}$$

where  $\{M_\tau\}_{\tau \geq 0}$  is a martingale. Having observed failure times  $t_1 < \dots < t_n$ , for  $\tau \in ]t_i, t_{i+1}]$ ,  $1 \leq i \leq n$ , the cumulative failure intensity is:

$$\Lambda_\tau = \int_0^\tau \lambda_s ds = \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \lambda_s ds + \int_{t_i}^\tau \lambda_s ds. \tag{39}$$

For the ARA models, we obtain:

$$\Lambda_\tau = \alpha Z_{i,\tau}(\beta, \rho), \tag{40}$$

where

$$Z_{i,\tau}(\beta, \rho) = \sum_{j=1}^i [(t_j - S_j(\rho))^\beta - (t_{j-1} - S_j(\rho))^\beta] + (\tau - S_{i+1}(\rho))^\beta - (t_i - S_{i+1}(\rho))^\beta \tag{41}$$

Of course,  $Z_{n,i}(\beta, \rho) = Z(\beta, \rho)$ .

Then, it is possible to make a Bayesian prediction of the expected number of failures at time  $\tau$  by computing the posterior distribution of  $\Lambda_\tau$ . For a given data

set, comparing the prediction with the observed value of  $N_\tau$  will give an indication of the goodness-of-fit of the ARA model.

Note that the expression of  $\Lambda_\tau$  is similar to that of  $\lambda_\tau = \alpha\beta[\tau - S_{i+1}(\rho)]^{\beta-1}$ . In fact, the results for  $\Lambda_\tau$  can be derived from those on  $\lambda_\tau$  by replacing  $\beta[\tau - S_{i+1}(\rho)]^{\beta-1}$  by  $Z_{i,\tau}(\beta, \rho)$ .

Then, the posterior distribution of  $\Lambda_\tau$  is:

$$\pi(\Lambda_\tau | n, \mathbf{t}_n) \propto \int_{\beta_1}^{\beta_2} \int_0^1 \frac{V(\rho)^{\beta-1} \rho^{c-1} (1-\rho)^{d-1} \beta^{n-\gamma}}{Z_{i,\tau}(\beta, \rho)^{n+a}} \Lambda_\tau^{n+a-1} \exp\left(-\frac{b + Z(\beta, \rho)}{Z_{i,\tau}(\beta, \rho)} \Lambda_\tau\right) d\rho d\beta \quad (42)$$

and the Bayesian point estimator of  $\Lambda_\tau$  is the posterior mean:

$$E[\Lambda_\tau | n, \mathbf{t}_n] = \frac{1}{C} \int_{\beta_1}^{\beta_2} \int_0^1 \frac{V(\rho)^{\beta-1} \rho^{c-1} (1-\rho)^{d-1} \beta^{n-\gamma} Z_{i,\tau}(\beta, \rho)}{[b + Z(\beta, \rho)]^{n+a+1}} d\rho d\beta, \quad (43)$$

where  $C$  is a normalizing constant.

For  $\tau = t$  and  $i = n$ ,  $Z_{n,t}(\beta, \rho) = Z(\beta, \rho)$ . With the non informative prior for  $\alpha$  ( $a = b = 0$ ), it is easy to check that  $E[\Lambda_t | n, \mathbf{t}_n] = n$ .

## 4. Simulation Results

### 4.1. Comparison of MLE and Bayesian Estimators

Here, we compare the maximum likelihood and Bayesian estimators by means of Monte Carlo simulations. We simulate 200 samples of size  $n \in \{5, 10, 25\}$  of both  $ARA_1$  and  $ARA_\infty$  models, with parameters equal to  $\alpha = 0.01$ ,  $\beta = 3$ , and  $\rho = 0.6$ . These values are chosen in order to reflect the behaviour of a system with important aging ( $\beta = 3$ ), good repair efficiency ( $\rho = 0.6$ ) and a mean time of the first failure around 4 ( $E[T_1] = 4.14$ ).

For each sample, we compute the MLE and several Bayesian estimators of each parameter, corresponding to different choices of priors. We also compute the estimators of  $E[T_1]$ .

The prior for  $\beta$  is non informative ( $\gamma = 1$ ,  $\beta_1 = 0$ ,  $\beta_2 = +\infty$ ) or uniform on  $[1, 4]$  ( $\gamma = 0$ ,  $\beta_1 = 1$ ,  $\beta_2 = 4$ ). This is a rather good prior since the prior mean is 2.5 and the prior standard deviation is 0.87.

Three priors are chosen for  $\rho$ : the non informative ( $c = d = 1$ ), a good one ( $c = 1.652$ ,  $d = 0.708$  with a prior mean equal to 0.7), and a bad one ( $c = 0.708$ ,  $d = 1.652$  with a prior mean equal to 0.3). For both informative priors, the standard deviation is 0.25.

For  $\alpha$ , we consider several possibilities:

- The non informative prior:  $a = b = 0$ .
- A good informative prior:  $a = 2.25$ ,  $b = 150$ , with a prior mean equal to 0.015 and a prior standard deviation equal to 0.01.
- The estimator of  $\alpha$  computed as a function of the estimators of  $\beta$  and  $\rho$ , according to (12).
- The prior for  $\alpha$  given  $\beta$  computed from a prior on  $E[T_1]$ , according to (30). The prior for  $E[T_1]$  is inverse gamma with  $A = 8.25$  and  $B = 36.25$ , leading to a prior mean equal to 5 and a prior standard deviation equal to 2.

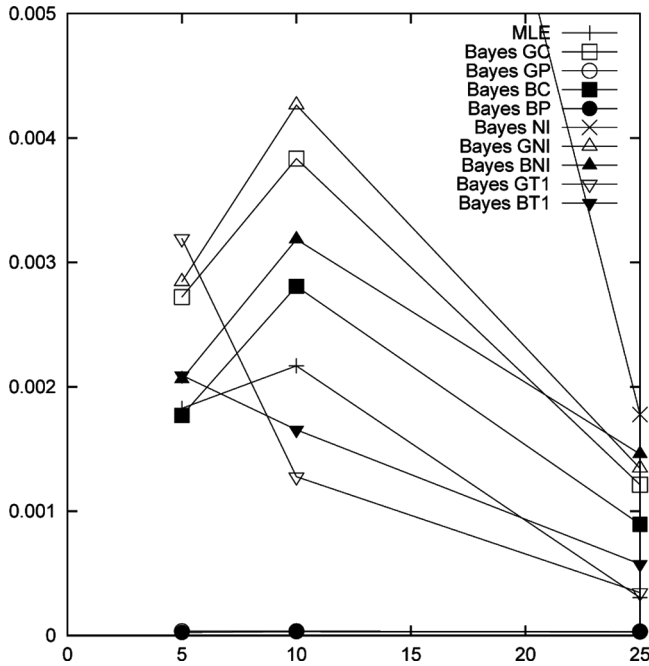


Figure 1.  $ARA_1$ -mse for  $\alpha$ .

The posterior distributions are derived by the MCMC algorithms with  $K = 10,000$  iterations. In the algorithms, the initial values of the parameters are the MLE. The Bayesian point estimators are the posterior means of the last 1,000 obtained values. The empirical means and variances of the 200 samples are computed for each estimator and for  $E[T_1]$ . Figures 1–8 give the corresponding mean squared errors (mse) as functions of the sample size.

In these figures, the following notations are adopted. The corresponding values of the hyperparameters are given in Table 1.

- MLE: Maximum Likelihood Estimator.
- NI: Non Informative prior on each parameter.
- GP: Good Prior for  $\rho$ , prior for  $\alpha$ .
- GC: Good prior for  $\rho, \alpha$  Computed.
- GNI: Good prior for  $\rho$ , non Informative for  $\alpha$ .
- GT1: Good prior for  $\rho$ , prior on  $E[T_1]$ .
- BP: Bad Prior for  $\rho$ , prior for  $\alpha$ .
- BC: Bad prior for  $\rho, \alpha$  Computed.
- BNI: Bad prior for  $\rho$ , Non Informative for  $\alpha$ .
- BT1: Bad prior for  $\rho$ , prior on  $E[T_1]$ .

The behavior of the estimators is essentially similar for the  $ARA_1$  and  $ARA_\infty$  models. The decrease of the MSE is not always true for so small samples. The worst estimators are the non informative and the MLE. For both estimators of  $\beta$ , the mse is so large that it does not appear in Figs. 3 and 4. Both estimators overestimate strongly  $\beta$ , but the MLE is correct for  $\alpha$  and  $\rho$ , while the non informative is correct

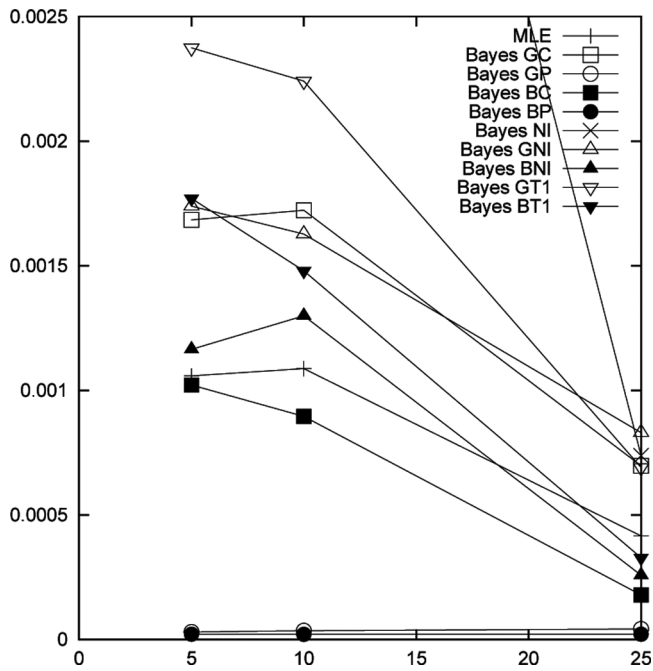


Figure 2.  $ARA_{\infty}$ -mse for  $\alpha$ .

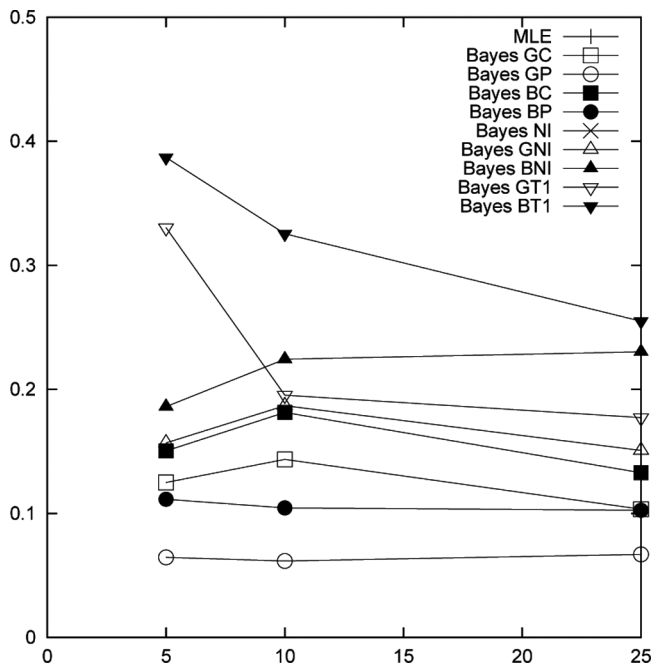


Figure 3.  $ARA_1$ -mse for  $\beta$ .



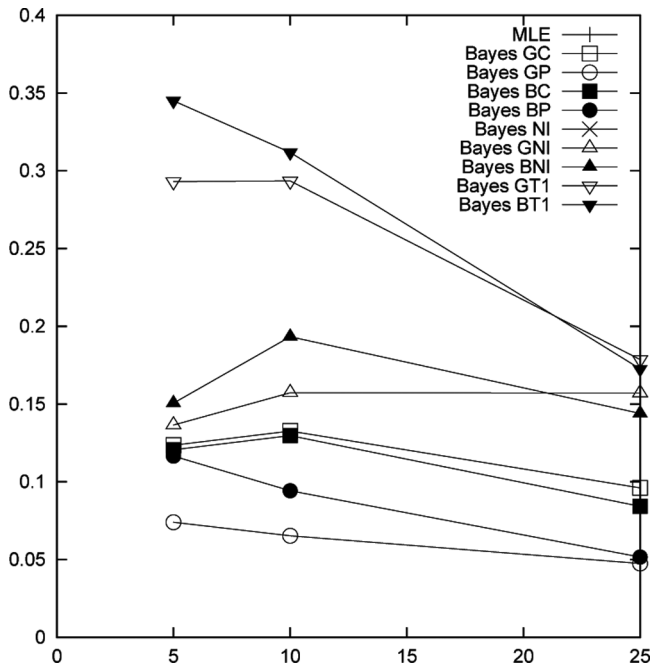


Figure 4.  $ARA_\infty$ -mse for  $\beta$ .

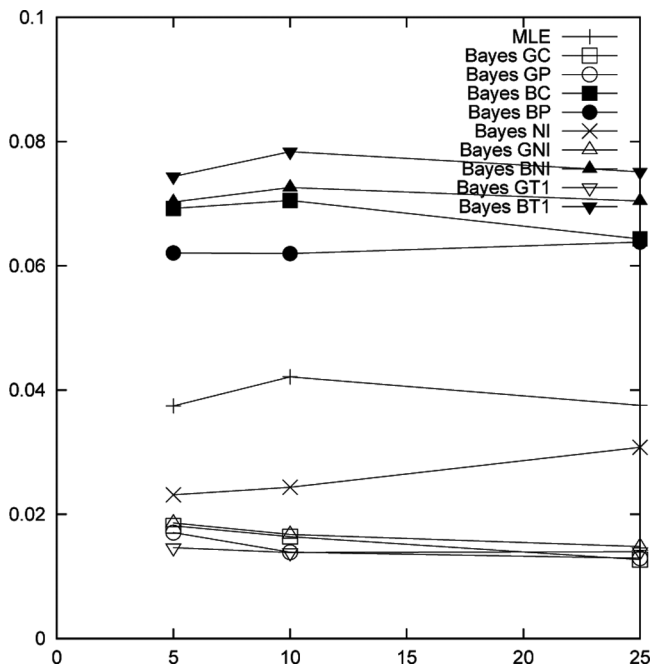


Figure 5.  $ARA_1$ -mse for  $\rho$ .

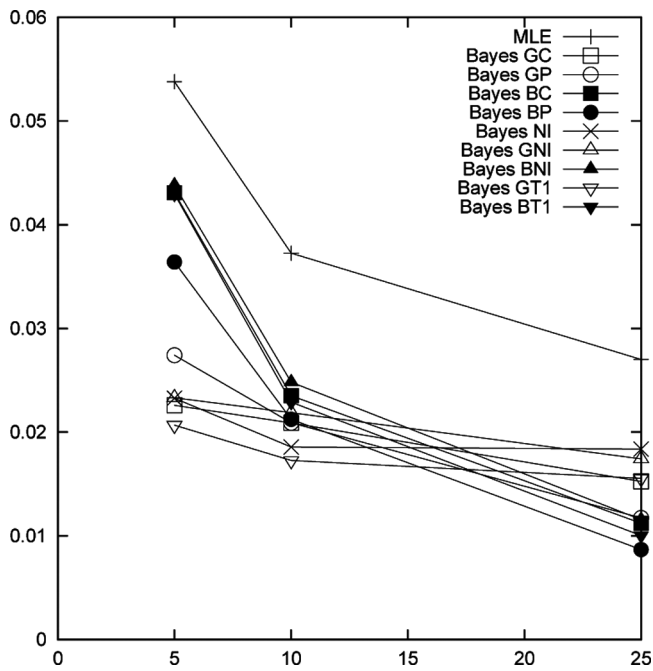


Figure 6.  $ARA_\infty$ -mse for  $\rho$ .

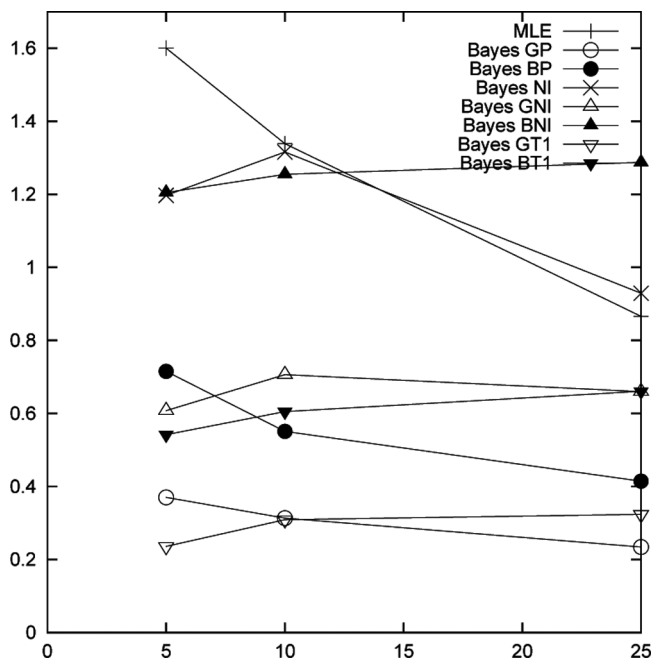


Figure 7.  $ARA_1$ -mse for  $E[T_1]$ .

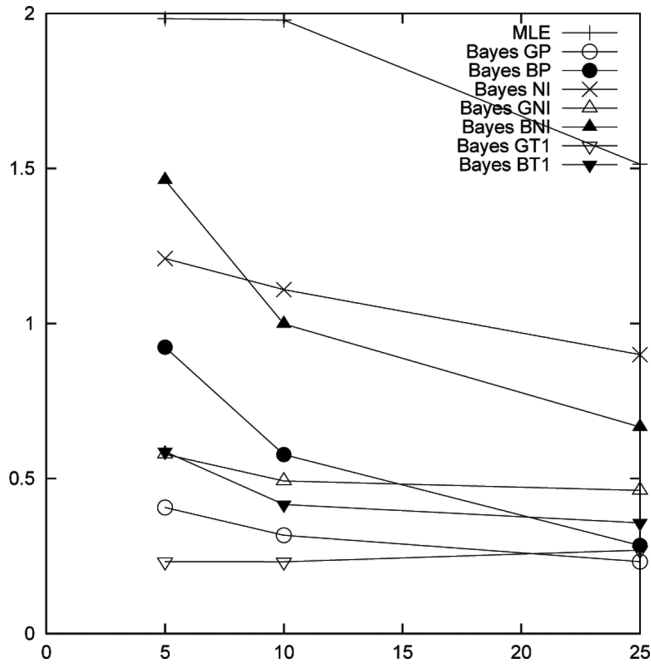


Figure 8. ARA<sub>∞</sub>-mse for E[T<sub>1</sub>].

Table 1  
Hyperparameter values for the Bayesian estimates

	<i>a</i>	<i>b</i>	$\gamma$	$\beta_1$	$\beta_2$	<i>c</i>	<i>d</i>	<i>A</i>	<i>B</i>
NI	0	0	1	0	$+\infty$	1	1		
GP	2.25	150	0	1	4	1.652	0.708		
GC			0	1	4	1.652	0.708		
GNI	0	0	0	1	4	1.652	0.708		
GT1			0	1	4	1.652	0.708	8.25	36.25
BP	2.25	150	0	1	4	0.708	1.652		
BC			0	1	4	0.708	1.652		
BNI	0	0	0	1	4	0.708	1.652		
BT1			0	1	4	0.708	1.652	8.25	36.25

only for  $\rho$ . As expected, the estimators are globally better when the prior on  $\rho$  is good (G) than when it is bad (B).

The comparison of the ways to estimate  $\alpha$  leads to the logic conclusion that the best estimators are obtained for a good informative prior (P). The three other estimators have similar behaviors, which slightly differ depending on the model and parameters. The estimator with a prior on  $E[T_1]$  (T1) can be used if a good knowledge on this value is available. The estimator with a non informative prior on  $\alpha$  (NI) and the estimator with  $\alpha$  computed (C) do not need any kind of knowledge on  $\alpha$ . Both need the MCMC algorithm. C is simpler to compute than NI because the MCMC algorithm involves only two parameters instead of three.

**Table 2**  
Relative mean squared error

$\alpha$		$\beta$		$\rho$		$E[T_1]$	
ARA <sub>1</sub>	ARA <sub>∞</sub>	ARA <sub>1</sub>	ARA <sub>∞</sub>	ARA <sub>1</sub>	ARA <sub>∞</sub>	ARA <sub>1</sub>	ARA <sub>∞</sub>
30	15	0.03	0.02	0.17	0.10	0.06	0.07

For the estimation of  $\rho$  in the ARA<sub>1</sub> model, the prior has a strong impact. We can see in Fig. 5 that the mse of all the estimators with good prior are much lower than the mse of all the estimators with bad prior. For the ARA<sub>∞</sub> model, this result is still true for small  $n$ , but much weaker.

For the estimation of  $E[T_1]$ , it is interesting to note that the best estimator is logically obtained when a prior on  $E[T_1]$  is used (GT1). The worst estimator is the MLE, followed by NI. Here again, the G estimators are better than the B estimators.

Table 2 gives the relative mean squared error for the model parameters and for  $E[T_1]$ . For a quantity  $\theta$ , the relative MSE is  $MSE/\theta^2$ , where mse is taken as an average value of the MSE previously computed. The less this value is, the better  $\theta$  is estimated. From this table, we see that  $\alpha$  is much less well estimated than the other parameters. But it doesn't prevent  $E[T_1]$  from being well estimated.

Finally, the Bayesian approach will be particularly profitable for the estimation of  $\beta$  and  $E[T_1]$ . Of course, the greater is the knowledge on the parameters, the better are the estimations. But Bayesian estimators are still satisfactory even with a bad prior.

#### 4.2. Study of One Data Set

In practice, one has to analyze one data set made of  $n$  successive failure times of a repairable system. The first objective is to estimate the parameters of ARA models and reliability indicators. Thanks to the Bayesian analysis, it is also possible to compute credibility intervals on these quantities (the equivalent is not possible with the MLE). We can also compare prior and posterior distributions. In this section, we apply this approach to data simulated from the ARA<sub>1</sub> model. In the next section, it will be applied to a real data set for both ARA<sub>1</sub> and ARA<sub>∞</sub> models.

Table 3 presents  $n = 5$  failure times simulated from the ARA<sub>1</sub> model with parameters  $\alpha = 0.01$ ,  $\beta = 3$ , and  $\rho = 0.6$ .

Tables 4–7 present point estimates of the parameters  $\alpha$ ,  $\beta$ ,  $\rho$ , and of  $E[T_1]$ . The considered estimators are the MLE and the Bayesian GP, GNI, BNI. For the Bayesian estimators, the 80% credibility intervals and posterior standard deviations are also given.

The MLE is naturally of bad quality since we have to estimate three parameters with five observations. Indeed, the MLE of  $\alpha$ ,  $\beta$ , and  $E[T_1]$  are very far from the

**Table 3**  
Simulated data set from the ARA<sub>1</sub> model

6.17	8.30	9.20	10.47	12.84
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**Table 4**  
One data set— $ARA_1$ —estimation of  $\alpha$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	0.00003			
GP	0.0123	0.00360	0.0236	0.0086
GNI	0.0277	0.00095	0.0718	0.0648
BNI	0.0122	0.00053	0.0357	0.0270

**Table 5**  
One data set— $ARA_1$ —estimation of  $\beta$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	5.869			
GP	2.921	2.344	3.572	0.465
GNI	2.954	1.826	3.825	0.755
BNI	3.070	2.030	3.829	0.655

true values, but the MLE of  $\rho$  is quite good. The GP estimates (good priors for  $\alpha$  and  $\rho$ ) are always good, in mean and variance, as well as for the length of credibility intervals. The GNI is a good estimate for  $\beta$  and  $\rho$ . The BNI is a good estimate for  $\alpha$  and  $\beta$ . For  $\rho$ , the BNI is highly biased, because it is close to the (bad) prior mean.

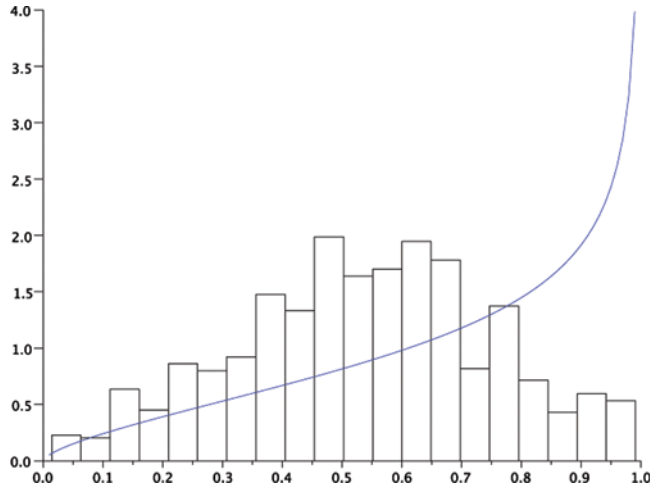
Figures 9 and 10 compare the prior density (solid line) and the posterior density (histogram from the MCMC samples) of the GNI and BNI estimators of  $\rho$ . In both cases, the posterior is brought closer to the MLE. The posterior distribution is

**Table 6**  
One data set— $ARA_1$ —estimation of  $\rho$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	0.554			
GP	0.605	0.339	0.821	0.192
GNI	0.538	0.242	0.818	0.214
BNI	0.317	0.040	0.629	0.215

**Table 7**  
One data set— $ARA_1$ —estimation of  $E[T_1]$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	5.424			
GP	4.619	3.5939	5.8197	0.9524
GNI	4.7645	3.2231	6.4155	1.2888
BNI	5.6575	3.8603	7.5476	1.4780

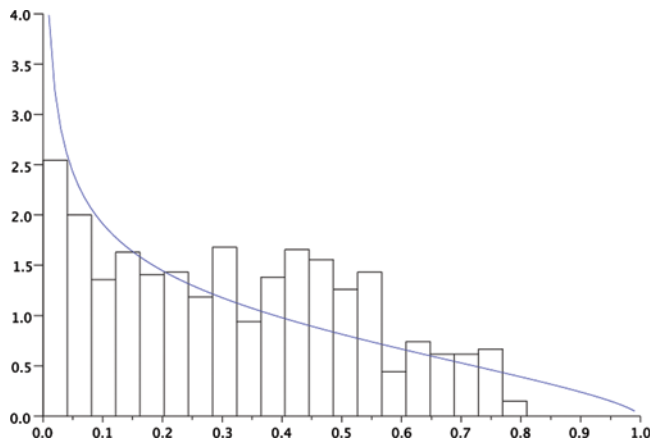


**Figure 9.** GNI—Comparison of prior and posterior densities. (color figure available online.)

naturally more concentrated around the true value  $\rho = 0.6$  for the good prior than for the bad prior.

Finally, the GNI Bayesian estimates of the failure intensity and expected number of failures are plotted in Figs. 11 and 12. The solid line represents the Bayesian point estimate, the credibility limits are in dashed (blue) lines. The true value and maximum likelihood estimates of the functions are, respectively, plotted in dash-dot (red) line and dotted (green).

These figures show that the Bayesian estimate provides a better fit than the MLE: the blue line is closer to the red line than the green line. The real intensity, cumulative intensity, and observed number of failures are always within the credibility limits. In Fig. 12, the ML and GNI estimates are equal to  $n$  for  $t = t_n$ . For GNI, this property has been stated in Sec. 3. For the MLE, this property comes from (12) and (40).



**Figure 10.** BNI—Comparison of prior and posterior densities. (color figure available online.)

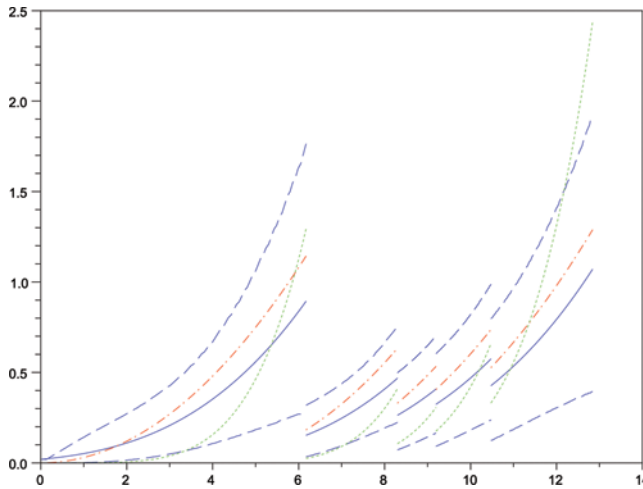


Figure 11.  $ARA_1$ —GNI estimate of the intensity. (color figure available online.)

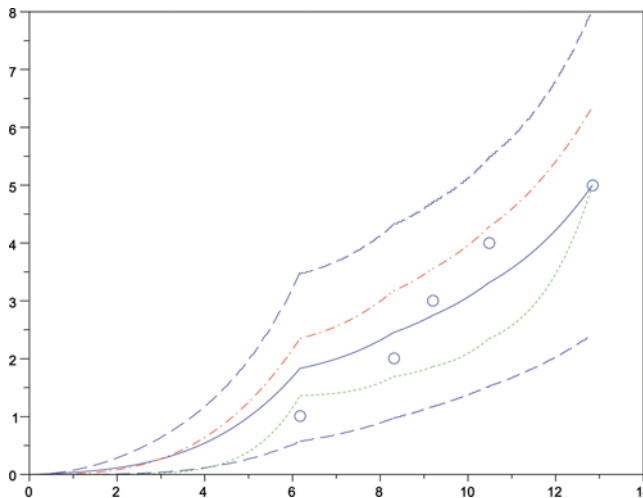


Figure 12.  $ARA_1$ —GNI estimate of the expected number of failures. (color figure available online.)

### 5. Application to Automobile Data

In this section, we present an application to automobile data studied by Ahn et al. (1998) and Guida and Pulcini (2006). The data given in Table 8 are  $n = 18$  failure times of an AMC Ambassador car owned by the Ohio state government.

**Table 8**  
AMC Ambassador data

202	265	363	508	571	755	770	818	868
999	1054	1068	1108	1230	1268	1330	1376	1447

**Table 9**  
AMC data—Maximum likelihood estimates and model selection criteria

	$\hat{\alpha}_t$	$\hat{\beta}_t$	$\hat{\rho}_t$	$\max \ln \mathcal{L}$	AIC	AICc	BIC
PLP	$1.32 \cdot 10^{-4}$	1.625		-95.15	194.29	195.09	201.86
ARA <sub>1</sub>	$1.30 \cdot 10^{-7}$	3.102	0.898	-92.00	189.99	191.71	201.33
ARA <sub>∞</sub>	$2.12 \cdot 10^{-9}$	3.583	0.246	-92.68	191.36	193.07	202.70

The MLEs of the parameters for the PLP, ARA<sub>1</sub>, and ARA<sub>∞</sub> models are given in Table 9. Usual model selection criteria are also given. For a model with  $k$  estimated parameters and  $n$  observed data, these criteria are defined as:

- $\max \ln \mathcal{L}$  is the maximal log-likelihood;
- $\text{AIC} = -2 \max \ln \mathcal{L} + 2k$ ;
- $\text{AICc} = -2 \max \ln \mathcal{L} + \frac{2kn}{n-k-1}$ ; and
- $\text{BIC} = -2 \max \ln \mathcal{L} + 2k \ln n$ .

The “best” model has the smallest AIC/AICc/BIC. For all the criteria, the ARA<sub>1</sub> model is found to be better than the PLP. For all the criteria except the BIC, the ARA<sub>∞</sub> model is also better than the PLP. The best model is ARA<sub>1</sub>. The estimates of  $\beta$  are close to 3, indicating a strong intrinsic wear-out. The estimates of  $\alpha$  and  $\rho$  in the two models are very different. It is not surprising since the models have different meanings (considerations of that kind are given in Doyen and Gaudoin, 2004). Anyway, the estimates of  $\rho$  indicate a good repair efficiency.

Now we perform the Bayesian analysis of these data. Since we have no prior knowledge on this system, the first idea is to use the completely non informative Bayesian estimators. But it makes sense to assume that this automobile is wearing. So we will also use the uniform prior on  $[1, 4]$  for  $\beta$ . For  $\rho$ , it is interesting to note that the MLE are relatively close to some values used as prior means in the simulation study: for the ARA<sub>1</sub> model,  $\hat{\rho}_t = 0.898$  is not far from 0.7 (the GNI of Sec. 4.1) and for the ARA<sub>∞</sub> model,  $\hat{\rho}_t = 0.246$  is close to 0.3 (the BNI of Sec. 4.1). Then, we will also use the corresponding informative priors for  $\rho$ .

Tables 11–16 present point estimates, the 80% credibility intervals and posterior standard deviations of the parameters  $\alpha$ ,  $\beta$ , and  $\rho$  for ARA<sub>1</sub> and ARA<sub>∞</sub> models. In these tables, the following estimates are used. The corresponding values of the hyperparameters are given in Table 10.

- MLE: Maximum Likelihood Estimator.
- NI: Non Informative prior on each parameter.
- NIU: Non Informative prior for  $\alpha$  and  $\rho$ , uniform prior for  $\beta$ .
- GNI: Non Informative prior for  $\alpha$ , uniform prior for  $\beta$ , beta prior for  $\rho$  for the ARA<sub>1</sub> model.
- BNI: Non Informative prior for  $\alpha$ , uniform prior for  $\beta$ , beta prior for  $\rho$  for the ARA<sub>∞</sub> model.

For the ARA<sub>1</sub> model, the NI estimates are very different from the others, especially for  $\beta$ . They also have rather high standard deviation. The GNI estimates are globally closer to the MLE than the NIU, with smaller standard deviation and narrower credibility intervals. This is due to the fact that we have chosen a prior close to the MLE. For the ARA<sub>∞</sub> model, these properties are no longer true. The NIU estimates are closer to the MLE than the BNI.



**Table 10**  
Hyperparameter values for the Bayesian estimates

	$a$	$b$	$\gamma$	$\beta_1$	$\beta_2$	$c$	$d$
NI	0	0	1	0	$+\infty$	1	1
NIU	0	0	0	1	4	1	1
GNI	0	0	0	1	4	1.652	0.708
BNI	0	0	0	1	4	0.708	1.652

**Table 11**  
AMC data—ARA<sub>1</sub>—estimation of  $\alpha$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	$1.30 \cdot 10^{-7}$			
NI	$4.87 \cdot 10^{-4}$	$1.20 \cdot 10^{-5}$	$1.44 \cdot 10^{-3}$	$7.07 \cdot 10^{-4}$
NIU	$1.03 \cdot 10^{-5}$	$6.09 \cdot 10^{-7}$	$3.19 \cdot 10^{-5}$	$1.99 \cdot 10^{-5}$
GNI	$3.05 \cdot 10^{-7}$	$1.21 \cdot 10^{-9}$	$6.73 \cdot 10^{-7}$	$2.78 \cdot 10^{-7}$

**Table 12**  
AMC data—ARA<sub>1</sub>—estimation of  $\beta$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	3.102			
NI	1.736	1.362	2.184	0.331
NIU	2.430	1.989	2.696	0.254
GNI	3.104	2.718	3.894	0.418

**Table 13**  
AMC data—ARA<sub>1</sub>—estimation of  $\rho$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	0.898			
NI	0.680	0.284	0.939	0.256
NIU	0.801	0.626	0.938	0.159
GNI	0.866	0.800	0.928	0.054

**Table 14**  
AMC data—ARA <sub>$\infty$</sub> —estimation of  $\alpha$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	$2.12 \cdot 10^{-9}$			
NI	$2.60 \cdot 10^{-6}$	$2.00 \cdot 10^{-7}$	$8.20 \cdot 10^{-6}$	$4.41 \cdot 10^{-6}$
NIU	$2.73 \cdot 10^{-7}$	$5.41 \cdot 10^{-9}$	$8.41 \cdot 10^{-7}$	$6.79 \cdot 10^{-7}$
BNI	$4.12 \cdot 10^{-5}$	$4.83 \cdot 10^{-6}$	$1.09 \cdot 10^{-4}$	$4.79 \cdot 10^{-5}$

**Table 15**  
AMC data— $ARA_{\infty}$ —estimation of  $\beta$

	Point estimate	Lower limit	Upper limit	Standard deviation
MLE	3.583			
NI	2.621	2.312	2.819	0.201
NIU	3.089	2.636	3.418	0.291
BNI	2.075	1.824	2.313	0.200

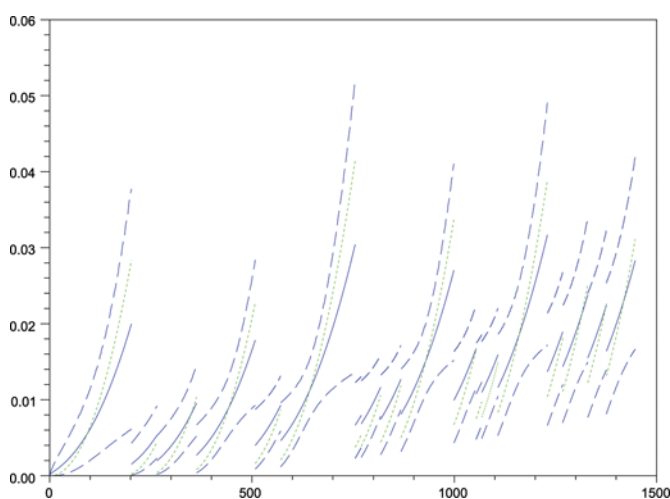
**Table 16**  
AMC data— $ARA_{\infty}$ —estimation of  $\rho$

	Point estimate	Lower limit	Upper limit	Standard deviation
LE	0.246			
NI	0.352	0.176	0.567	0.148
NIU	0.281	0.164	0.428	0.114
BNI	0.356	0.109	0.674	0.212

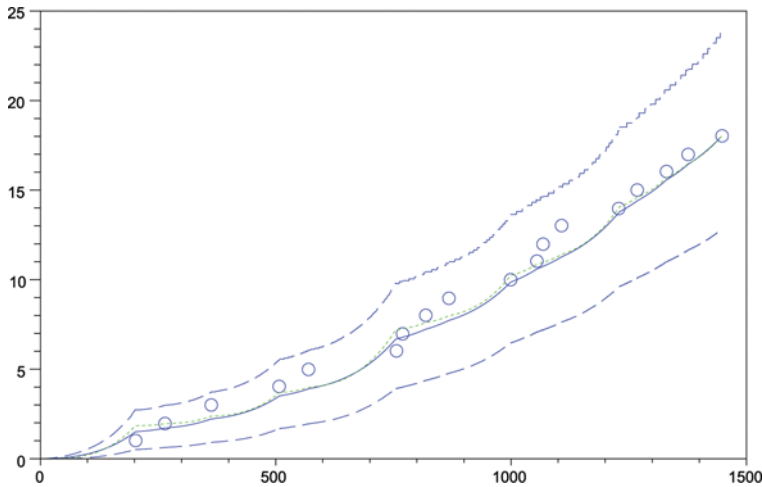
The NIU estimates give coherent results and are not submitted to the subjective choice of a prior for  $\rho$ , so we recommend them in this case.

Figures 13 and 14 present, respectively, the NIU Bayesian estimates of the intensity and expected number of failures, for the  $ARA_1$  model. As before, the Bayesian point estimate is in solid line, the credibility limits are in (blue)/dashed lines and the MLE is in (blue)/dotted (green) line.

In these figures, the Bayesian estimates are very close to the ML estimates, even if the estimates of the parameters are not close. In fact, this result is also true with the NI and GNI estimates. Moreover, we can see in Fig. 14 that the estimates are



**Figure 13.** AMC data— $ARA_1$ —NIU estimate of the intensity. (color figure available online.)



**Figure 14.** AMC data— $ARA_1$ —NIU estimate of the expected number of failures. (color figure available online.)

also very close to the observed number of failures. On the one hand, this means that all the estimation methods manage to catch the behavior of the failure process and to estimate correctly reliability indicators. On the other hand, the  $ARA_1$  model provides a good fit to these data.

## 6. Conclusion and Future Work

In this article, we presented a unified approach of the Bayesian analysis of ARA imperfect repair models, including all ARA models and many kinds of possible priors. It allows to compute meaningful reliability indicators (with credibility intervals) even when very few data are available. Other ideas can be explored in the same framework: computing other reliability indicators such as predictive MTTF, performing Bayesian tests, analyzing several data sets in parallel, etc.

Another promising area is the Bayesian inference of imperfect maintenance models, where both preventive and corrective maintenance are performed. Until now, this case has been studied by Pulcini (2000) and Sheu et al. (2001), with only ABAO CM effects, and by Yu et al. (2008) with  $ARA_1$  CM and PM effects.

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