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Functional properties of Hörmander's space of distributions having a specified wavefront set

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Abstract: The space $\mathcal{D}'_\Gamma$ of distributions having their wavefront sets in a closed cone Γ has become important in physics because of its role in the formulation of quantum field theory in curved spacetime. In this paper, the topological and bornological properties of $\mathcal{D}'_\Gamma$ and its dual $\mathcal{E}'_\Lambda$ are investigated. It is found that $\mathcal{D}'_\Gamma$ is a nuclear, semi-reflexive and semi-Montel complete normal space of distributions. Its strong dual $\mathcal{E}'_\Lambda$ is a nuclear, barrelled and (ultra)bornological normal space of distributions which, however, is not even sequentially complete. Concrete rules are given to determine whether a distribution belongs to $\mathcal{D}'_\Gamma$, whether a sequence converges in $\mathcal{D}'_\Gamma$ and whether a set of distributions is bounded in $\mathcal{D}'_\Gamma$.

1. Introduction

Standard quantum field theory uses Feynman diagrams in the momentum space. However, this framework is not suitable for quantum field theory in arbitrary spacetimes because of the absence of translation invariance. In 1992, Radzikowski [69,70] showed the wavefront set of distributions to be a key concept to describe quantum fields in curved spacetime. This idea was developed into a rigorous renormalized scalar field theory in curved spacetime by Brunetti and Fredenhagen [8], followed by Hollands and Wald [42]. This approach was rapidly extended to deal with Dirac fields [54,40,16,17,75,73], gauge fields [41,26,27] and even the quantization of gravitation [9].

This tremendous progress was made possible by a complete reformulation of quantum field theory, where the wavefront set of distributions plays a central role, for example to determine the algebra of microcausal functionals, to define a spectral condition for time-ordered products and quantum states and to give a rigorous description of renormalization.

In other words, the natural space where quantum field theory takes place is not the space of distributions $\mathcal{D}'$, but the space $\mathcal{D}'_\Gamma$ of distributions having their

wavefront set in a specified closed cone Γ. This space and its simplest properties were described by Hörmander in 1971 [43]. Since $\mathcal{D}'_\Gamma$ is now a crucial tool of quantum field theory, it is important to investigate its topological and functional properties. For example, renormalized time-ordered products are determined as an extension of a distribution to the thin diagonal. Since this extension is defined as the limit of a sequence, we need simple criteria to determine the convergence of a sequence in $\mathcal{D}'_\Gamma$. The ambiguity of renormalization is determined, among other things, by the way this distribution varies under scaling. Scaled distributions are defined with respect to a bounded set in $\mathcal{D}'_\Gamma$. Thus, we need simple tests to know when a set of distributions is bounded. The purpose of this paper is to provide tools to answer these questions in a simple way.

The wavefront set of distributions plays also a key role in microlocal analysis, to determine whether a distribution can be pulled back, restricted to a submanifold or multiplied by another distribution [44, Chapter 8]. Therefore, the wavefront set has become a standard subject in textbooks of distribution theory and microlocal analysis [44,20,35,13,72,28,80,31,81,22,86]. However, to the best of our knowledge, no detailed study was published on the functional properties of $\mathcal{D}'_\Gamma$.

Many properties of $\mathcal{D}'_\Gamma$ will be deduced from properties of its dual. Thus, we shall first calculate the dual of $\mathcal{D}'_\Gamma$, denoted by $\mathcal{E}'_\Lambda$, which turns out to be the space of compactly supported distributions having their wavefront set included in an *open cone* Λ which is the complement of Γ up to a change of sign. Such a space $\mathcal{E}'_\Lambda$ is used in quantum field theory to define microcausal functionals [26].

We now summarize our main results. Although they are both nuclear and normal spaces of distributions, $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$ have very contrasted properties; (i) $\mathcal{D}'_\Gamma$ is semi-reflexive and complete while $\mathcal{E}'_\Lambda$ is not even sequentially complete; (ii) $\mathcal{E}'_\Lambda$ is barrelled, and ultrabornological, while $\mathcal{D}'_\Gamma$ is neither barrelled nor bornological. For applications, the most significant property of $\mathcal{D}'_\Gamma$ is to be semi-Montel. Indeed, two steps involving $\mathcal{D}'_\Gamma$ are particularly important in the renormalization process described by Brunetti and Fredenhagen [8]. The first step is a control of the divergence of the relevant distributions near the diagonal: there must be a real number s such that the family $\{\lambda^{-s}u_\lambda\}_{0<\lambda\leq 1}$ is a bounded set of distributions, where u_λ is a scaled distribution. This proof is facilitated by our determination of bounded sets:

Proposition 1 *A set B of distributions in $\mathcal{D}'_\Gamma$ is bounded if and only if, for every $v \in \mathcal{E}'_\Lambda$, there is a constant C_v such that $|\langle u, v\rangle| < C_v$ for all $u \in B$. Such a weakly bounded set is also strongly bounded and equicontinuous. Moreover, the closed bounded sets of $\mathcal{D}'_\Gamma$ are compact, complete and metrizable.*

The second step is the proof that the extension of a distribution can be defined as the limit of a sequence of distributions in $\mathcal{D}'_\Gamma$. For this we derive the following convergence test:

Proposition 2 *If u_i is a sequence of elements of $\mathcal{D}'_\Gamma$ such that, for any $v \in \mathcal{E}'_\Lambda$, the sequence $\langle u_i, v\rangle$ converges in $\mathbb{C}$ to a number λ_v, then u_i converges to a distribution u in $\mathcal{D}'_\Gamma$ and $\langle u, v\rangle = \lambda_v$ for all $v \in \mathcal{E}'_\Lambda$.*

We now describe the organization of the paper. After this introduction, we determine a pairing between $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$ and we show that this pairing is compatible with duality. Then, we prove that $\mathcal{D}'_\Gamma$ is a normal space of distributions.

The next section investigates several topologies on $\mathcal{E}'_\Lambda$ and shows their equivalence. Then, the nuclear and bornological properties of $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$ are discussed. Bornology enables us to prove that $\mathcal{D}'_\Gamma$ is complete and it is relevant to the problem of quantum field theory on curved spacetime because some isomorphisms of the space of sections of a vector bundle over a manifold are stronger in the bornological setting than in the topological one (see section 5). These results are put together to determine the main functional properties of $\mathcal{D}'_\Gamma$ and its dual. Finally, a counter-example is constructed to show that $\mathcal{E}'_\Lambda$ is not sequentially complete. This will imply that $\mathcal{D}'_\Gamma$ and its dual do not enjoy all the nice properties of $\mathcal{D}'$.

2. The dual of $\mathcal{D}'_\Gamma$

In this section, we review what is known about the topology of $\mathcal{D}'_\Gamma$ and we describe the functional analytic tools (duality pairing and normal spaces of distributions) that enable us to investigate the dual of $\mathcal{D}'_\Gamma$.

2.1. What is known about $\mathcal{D}'_\Gamma$. Let us fix the notation. Let Ω be an open set in $\mathbb{R}^n$, we denote by $T^*\Omega$ the cotangent bundle over Ω, by $UT^*\Omega = \{(x;k) \in T^*\Omega\,; |k| = 1\}$ (where $|k|$ is the standard Euclidian norm on $\mathbb{R}^n$) the sphere bundle over Ω and by $\dot{T}^*\Omega = T^*\Omega \backslash \{(x;0)\,; x \in \Omega\}$ the cotangent bundle without the zero section. We say that a subset Γ of $\dot{T}^*\Omega$ is a cone if $(x;\lambda k) \in \Gamma$ whenever $(x;k) \in \Gamma$ and $\lambda > 0$ and such a cone is said to be closed if it is closed in $\dot{T}^*\Omega$. For any closed cone Γ, Hörmander defined [43, p. 125] the space $\mathcal{D}'_\Gamma$ to be the set of distributions in $\mathcal{D}'(\Omega)$ having their wavefront set in Γ. He also described what he called a *pseudo-topology* on $\mathcal{D}'_\Gamma$, which means that he defined a concept of convergence in $\mathcal{D}'_\Gamma$ but not a topology (as a family of open sets). His definition was equivalent to the following one [44, p. 262]: a sequence $u_j \in \mathcal{D}'_\Gamma$ converges to $u \in \mathcal{D}'_\Gamma$ if

(i) The sequence of numbers $\langle u_j, f\rangle$ converges to $\langle u, f\rangle$ in the ground field $\mathbb{K}$ (i.e. $\mathbb{R}$ or $\mathbb{C}$) for all $f \in \mathcal{D}(\Omega)$.
(ii) If V is a closed cone in $\mathbb{R}^n$ and χ is an element of $\mathcal{D}(\Omega)$ that satisfy $(\operatorname{supp}\chi \times V) \cap \Gamma = \emptyset$, then $\sup_{k\in V}(1+|k|)^N |\widehat{u_j\chi}(k) - \widehat{u\chi}(k)| \to 0$ for all integers N,

where a hat over a distribution (e.g. $\widehat{u\chi}$) denotes its Fourier transform (the Fourier transform of $f \in \mathcal{D}(\Omega)$ being defined by $\hat{f}(k) = \int_\Omega e^{ik\cdot x} f(x)dx$).

Hörmander then showed that $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'_\Gamma$. More precisely, for every $u \in \mathcal{D}'_\Gamma$ there is a sequence of functions $u_j \in \mathcal{D}(\Omega)$ such that u_j converges to u in the above sense [44, p. 262]. This concept of convergence is compatible with different topologies. The topology of $\mathcal{D}'_\Gamma$ used in the literature [31,11,20,34,35, 1], which is usually called the Hörmander topology [8,81,79]), is that of a locally convex topological vector space defined by the following seminorms:

(i) $p_f(u) = |\langle u, f\rangle|$ for all $f \in \mathcal{D}(\Omega)$.
(ii) $||u||_{N,V,\chi} = \sup_{k\in V}(1+|k|)^N |\widehat{u\chi}(k)|$, for all integers N, all closed cones V and all $\chi \in \mathcal{D}(\Omega)$ such that $(\operatorname{supp}\chi \times V) \cap \Gamma = \emptyset$.

We immediately observe that $\mathcal{D}'_\Gamma$ is a Hausdorff locally convex space because $u = 0$ if $p_i(u) = 0$ for all its seminorms p_i [46, p. 96]. Indeed, if $p_f(u) = |\langle u, f\rangle| = 0$

for all $f \in \mathcal{D}(\Omega)$, then $u = 0$. When we speak of "all the seminorms" of a locally convex space E, we mean all the seminorms of a family of seminorms defining the topology of E [83, p. 63].

2.2. Duality pairing. Mackey's duality theory [58,57,59,60] is a powerful technique to investigate the topological properties of locally convex spaces [3,46]. The first step of this method is to find a duality pairing between two spaces.

Let us take the example of the duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Any test function $u \in \mathcal{D}(\Omega)$ can be paired to any $f \in \mathcal{D}(\Omega)$ by $\langle u, f\rangle = \int_\Omega u(x)f(x)dx$. The density of $\mathcal{D}(\Omega)$ in $\mathcal{D}'(\Omega)$ implies that this pairing can be uniquely extended to a pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$, also denoted by $\langle u, f\rangle$, that can be written

$$\langle u, f\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{u\varphi}(k)\hat{f}(-k)dk, \tag{1}$$

where the function $\varphi \in \mathcal{D}(\Omega)$ is equal to 1 on a compact neighborhood of the support of f. Indeed, $\langle u, f\rangle = \langle \varphi u, f\rangle$ [78, p. 90] and φu has a Fourier transform because it is a compactly supported distribution [44, p. 165]. This pairing is compatible with duality, in the sense that any element α in the topological dual of $\mathcal{D}(\Omega)$ can be written $\alpha(f) = \langle u, f\rangle$ for one element u of $\mathcal{D}'(\Omega)$, by definition of the space of distributions.

We would like to find a similar pairing between $\mathcal{D}'_\Gamma$ and another space to be determined. Grigis and Sjöstrand [31, p. 80] showed that the pairing $\langle u, v\rangle = \int_\Omega u(x)v(x)dx$ between $C^\infty(\Omega)$ and $\mathcal{D}(\Omega)$ extends uniquely to the pairing defined by eq. (1) between $\mathcal{D}'_\Gamma$ and every space $\mathcal{E}'_\Xi$ of compactly supported distributions whose wavefront set is contained in Ξ, where Ξ is any closed cone such that $\Gamma' \cap \Xi = \emptyset$, where $\Gamma' = \{(x;k) \in \dot{T}^*\Omega\,; (x;-k) \in \Gamma\}$ (see also [13, p. 512] for a similar result).

We need to slightly extend their definition by pairing $\mathcal{D}'_\Gamma$ with the space $\mathcal{E}'_\Lambda$, where Λ is now the *open* cone $\Lambda = (\Gamma')^c$. Note that this space is the union of the ones considered by Grigis and Sjöstrand. The next lemma does not contain more information than their result, but, for the reader's convenience, we first show that this extended pairing is well defined.

Lemma 3 *If Γ is a closed cone in $\dot{T}^*\Omega$ and $\Lambda = (\Gamma')^c = \{(x;k) \in \dot{T}^*\Omega\,; (x,-k) \notin \Gamma\}$, then the following pairing between $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda = \{v \in \mathcal{E}'(\Omega)\,; \mathrm{WF}(v) \subset \Lambda\}$ is well defined:*

$$\langle u, v\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{u\varphi}(k)\hat{v}(-k)dk,$$

where $u \in \mathcal{D}'_\Gamma$, $v \in \mathcal{E}'_\Lambda$ and φ is any function in $\mathcal{D}(\Omega)$ equal to 1 on a compact neighborhood of the support of v. This pairing is separating and, for any $v \in \mathcal{E}'_\Lambda$, the map $\lambda : \mathcal{D}'_\Gamma \to \mathbb{K}$ defined by $\lambda(u) = \langle u, v\rangle$ is continuous.

Proof. We first consider the case where Γ is neither empty nor $\dot{T}^*\Omega$. A distribution $v \in \mathcal{E}'_\Lambda$ is compactly supported and its wavefront set is a closed cone contained in Λ, which implies $\mathrm{WF}(v) \cap \Gamma' = \emptyset$. The product of distributions uv is then a well-defined distribution by Hörmander's theorem [44, p. 267]. We estimate now $\langle u, v\rangle = (2\pi)^{-n}\int \widehat{u\varphi}(k)\hat{v}(-k)dk$.

By a classical construction [22, p. 61], there is a finite set of non-negative smooth functions ψ_j such that $\sum_j \psi_j^2 = 1$ on a compact neighborhood K of the support of v and there are closed cones V_{uj} and V_{vj} that satisfy the three conditions: (i) $V_{uj} \cap (-V_{vj}) = \emptyset$, (ii) $\operatorname{supp}\psi_j \times V_{uj}^c \cap \Gamma = \emptyset$ and (iii) $\operatorname{supp}\psi_j \times V_{vj}^c \cap \mathrm{WF}(v) = \emptyset$. As a consequence of these conditions, we have $\Gamma|_K \subset \cup_j (\operatorname{supp}\psi_j \times V_{uj})$ and $\mathrm{WF}(v) \subset \cup_j (\operatorname{supp}\psi_j \times V_{vj})$. If we choose $\varphi = \sum_j \psi_j^2$ we can write $\langle u, v\rangle = \sum_j I_j$, where $I_j = (2\pi)^{-n} \int \widehat{u\psi_j}(k)\widehat{v\psi_j}(-k)dk$.

Following again Eskin [22, p. 62], we can define homogeneous functions of degree zero α_j and β_j on $\mathbb{R}^n$, which are smooth except at the origin, measurable, non-negative and bounded by 1 on $\mathbb{R}^n$ and such that $\operatorname{supp}\alpha_j$ and $\operatorname{supp}\beta_j$ are closed cones satisfying the three conditions (i), (ii) and (iii) stated above, with $\alpha_j = 1$ on V_{uj} and $\beta_j = 1$ on V_{vj}. Then we insert $1 = \big(\alpha_j + (1-\alpha_j)\big)\big(\beta_j + (1-\beta_j)\big)$ in the integral defining I_j and we obtain $I_j = I_{1j} + I_{2j} + I_{3j} + I_{4j}$, where

$$
\begin{aligned}
I_{1j} &= (2\pi)^{-n} \int_{\mathbb{R}^n} \alpha_j(-k)\widehat{\psi_j u}(-k)\, \beta_j(k)\widehat{\psi_j v}(k)\, dk,\\
I_{2j} &= (2\pi)^{-n} \int_{\mathbb{R}^n} \alpha_j(-k)\widehat{\psi_j u}(-k)\, \big(1-\beta_j(k)\big)\widehat{\psi_j v}(k)\, dk,\\
I_{3j} &= (2\pi)^{-n} \int_{\mathbb{R}^n} \big(1-\alpha_j(-k)\big)\widehat{\psi_j u}(-k)\, \beta_j(k)\widehat{\psi_j v}(k)\, dk,\\
I_{4j} &= (2\pi)^{-n} \int_{\mathbb{R}^n} \big(1-\alpha_j(-k)\big)\widehat{\psi_j u}(-k)\, \big(1-\beta_j(k)\big)\widehat{\psi_j v}(k)\, dk.
\end{aligned}
$$

We first notice that $I_{1j} = 0$ because $(-\operatorname{supp}\alpha_j) \cap \operatorname{supp}\beta_j = \emptyset$. We estimate I_{4j}. The function β_j was built so that $(1-\beta_j) = 0$ on V_{vj} and $\operatorname{supp}\psi_j \times \operatorname{supp}(1-\beta_j) \cap \mathrm{WF}(v) = \emptyset$. Then, for any integer N,

$$\big|\big(1-\beta_j(k)\big)\widehat{\psi_j v}(k)\big| \le ||v||_{N,U_{\beta j},\psi_j}(1+|k|)^{-N},$$

where $U_{\beta j} = \operatorname{supp}(1-\beta_j)$. Similarly

$$\big|\big(1-\alpha_j(k)\big)\widehat{\psi_j u}(k)\big| \le ||u||_{M,U_{\alpha j},\psi_j}(1+|k|)^{-M}, \tag{2}$$

where $U_{\alpha j} = \operatorname{supp}(1-\alpha_j)$. Thus, for $N + M > n$,

$$|I_{4j}| \le ||u||_{M,U_{\alpha j},\psi_j}||v||_{N,U_{\beta j},\psi_j} I_n^{N+M},$$

where $I_n^N = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|k|)^{-N}dk$.

For I_{3j} we use the fact that, $\psi_j v$ being a compactly supported distribution, there is an integer m and a constant C such that $|\widehat{\psi_j v}(k)| \le C(1+|k|)^m$ [44, p. 181]. When this estimate is combined with eq. (2) we obtain for $M > n+m$,

$$|I_{3j}| \le ||u||_{M,U_{\alpha j},\psi_j} C I_n^{M-m}.$$

For the integral I_{2j} we proceed differently because we want to recover a seminorm of $\mathcal{D}'_\Gamma$. If we define $\hat{f}_j(k) = \alpha_j(-k)(1-\beta_j(k))\widehat{\psi_j v}(k)$, then

$$I_{2j} = (2\pi)^{-n} \int \widehat{\psi_j u}(-k)\hat{f}_j(k)dk.$$

We call *fast decreasing* a function $f(k)$ such that, for every integer N, $|f(k)| \leq C_N(1+|k|)^{-N}$ for some constant C_N. Note that our fast decreasing functions are different from Schwartz rapidly decreasing functions. The function $\hat{f}_j(k)$ is fast decreasing because α_j and β_j are bounded by 1, $\widehat{\psi_j v}(k)$ is fast decreasing outside the wavefront set of v and $(1-\beta_j(k))$ cancels $\widehat{\psi_j v}(k)$ on this wavefront set. The function $\hat{f}_j$ is also measurable because it is the product of measurable functions. Thus, by a standard result in the spirit of [28, p. 145], its inverse Fourier transform f_j exists and is smooth. We can now rewrite $I_{2j} = \langle \psi_j u, f_j\rangle = \langle u, \psi_j f_j\rangle$, which is well defined because $\psi_j f_j$ is smooth and compactly supported. Finally $|I_{2j}| \leq p_{\psi_j f_j}(u)$, where $p_{\psi_j f_j}(u) = |\langle u, \psi_j f_j\rangle|$, and we obtain

$$|\langle u, v\rangle| \leq \sum_j \Big(p_{\psi_j f_j}(u) + ||u||_{M,U_{\alpha j},\psi_j} C I_n^{M-m} + ||u||_{M,U_{\alpha j},\psi_j} ||v||_{N,U_{\beta j},\psi_j} I_n^{N+M} \Big). \tag{3}$$

Thus, $\langle u, v\rangle$ is well defined because all the terms in the right hand side are finite and the sum is over a finite number of j. Note that $p_{\psi_j f_j}(u)$ and $||u||_{M,U_{\alpha j},\psi_j}$ are seminorms of $\mathcal{D}'_\Gamma$ because $\psi_j f_j \in \mathcal{D}(\Omega)$ and, by construction, $U_{\alpha j}$ is a closed cone and $\operatorname{supp}\psi_j \times U_{\alpha j} \cap \Gamma = \emptyset$. Equation (3) shows that, for any $v \in \mathcal{E}'_\Lambda$, the map $u \mapsto \langle u, v\rangle$ is continuous.

The second case is $\Gamma = \dot{T}^*\Omega$ and $\Lambda = \emptyset$, so that $\mathcal{D}'_\Gamma = \mathcal{D}'(\Omega)$ and $\mathcal{E}'_\Lambda = \mathcal{D}(\Omega)$. The seminorm $|\langle u, v\rangle| = p_v(u)$ is then a seminorm of $\mathcal{D}'_\Gamma$ since $v \in \mathcal{D}(\Omega)$. The last case is when $\Gamma = \emptyset$ and $\Lambda = \dot{T}^*\Omega$, so that $\mathcal{D}'_\Gamma = C^\infty(\Omega)$ and $\mathcal{E}'_\Lambda = \mathcal{E}'(\Omega)$. If we use the fact that the usual topology of $C^\infty(\Omega)$ is equivalent with the topology defined by $||\cdot||_{N,V,\chi}$ for all closed cones V and all $\chi \in \mathcal{D}(\Omega)$ [6], then we see that the elements of $\mathcal{E}'(\Omega)$ are continuous maps from $C^\infty(\Omega)$ to $\mathbb{K}$ [78, p. 89].

Finally, the pairing is separating because, if $\langle u, v\rangle = 0$ for all $v \in \mathcal{E}'_\Lambda$, then $\langle u, f\rangle = 0$ for all $f \in \mathcal{D}(\Omega)$ because $\mathcal{D}(\Omega) \subset \mathcal{E}'_\Lambda$ and a distribution u which is zero on $\mathcal{D}(\Omega)$ is the zero distribution. Similarly, $v = 0$ if $\langle u, v\rangle = 0$ for all $u \in \mathcal{D}'_\Gamma$ because $\mathcal{D}(\Omega) \subset \mathcal{D}'_\Gamma$.

To simplify the discussion, we used Eskin's α_j and β_j functions to build maps from $v \in \mathcal{E}'_\Lambda$ to $f_j \in C^\infty(\Omega)$. This can be improved by defining maps from $\mathcal{E}'_\Lambda$ to the Schwartz space $\mathcal{S}$ of rapidly decreasing functions (see section 4).

2.3. Normal space of distributions. The usual spaces of distribution theory (e.g. $\mathcal{D}$, $\mathcal{S}$, C^∞, $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{E}'$), are *normal spaces of distributions* [77, p. 10], which enjoy useful properties with respect to duality. They are defined as follows:

Definition 4 *A Hausdorff locally convex space E is said to be a* normal space of distributions *if there are continuous injective linear maps $i : \mathcal{D}(\Omega) \to E$ and $j : E \to \mathcal{D}'(\Omega)$, where $\mathcal{D}'(\Omega)$ is equipped with its strong topology, such that: (i) The image of i is dense in E, (ii) for any f and g in $\mathcal{D}(\Omega)$ $\langle j \circ i(f), g\rangle = \int_\Omega f(x)g(x)dx$ [46, p. 319].*

To transform $\mathcal{D}'_\Gamma$ into a normal space of distributions we need to refine its topology. In the case of $\mathcal{D}'_\Gamma$ condition (ii) is obviously satisfied because the injections i and j are the identity. The fact that j is a continuous injection

means that the topology of $\mathcal{D}'_\Gamma$ must be finer than the topology induced on it by the strong topology of $\mathcal{D}'(\Omega)$ [83, p. 302]. Therefore, we now equip $\mathcal{D}'_\Gamma$ with the topology defined by the seminorms $p_B(u) = \sup_{f\in B} |\langle u, f\rangle|$ of uniform convergence on the bounded sets B of $\mathcal{D}(\Omega)$ (instead of only the seminorms $p_f = |\langle u, f\rangle|$) and we keep the seminorms $||u||_{N,V,\chi}$ defined in section 2.1. Since p_B are the seminorms of $\mathcal{D}'(\Omega)$, $\mathcal{D}'_\Gamma$ has more seminorms than $\mathcal{D}'(\Omega)$, the identity is a continuous injection and its topology is finer than that of $\mathcal{D}'(\Omega)$ [46, p. 98]. We call this topology the *normal topology* of $\mathcal{D}'_\Gamma$, while the usual topology will be called the *Hörmander topology* of $\mathcal{D}'_\Gamma$. Note that $\mathcal{D}'_\Gamma$ is Hausdorff for the normal topology because it is Hausdorff for the coarser Hörmander topology. It remains to show that

Lemma 5 *The injection of $\mathcal{D}(\Omega)$ in $\mathcal{D}'_\Gamma$ is continuous.*

Proof. We have to prove that the identity map $\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}'_\Gamma$ is continuous. Because of the inductive limit topology of $\mathcal{D}(\Omega)$, we must show that, for any compact subset K of Ω, the map $\mathcal{D}(K) \hookrightarrow \mathcal{D}'_\Gamma$ is continuous for the topology of $\mathcal{D}(K)$ [78, p. 66]. Recall that $\mathcal{D}(K)$ is the set of elements of $\mathcal{D}(\Omega)$ whose support is contained in K. Its topology is defined by the seminorms $\pi_{m,K}(f) = \sup_{|\alpha|\le m} \sup_{x\in K} |\partial^\alpha f(x)|$.

Continuity is proved by showing that all the seminorms of $\mathcal{D}'_\Gamma$ are bounded by seminorms of $\mathcal{D}(K)$ [46, p. 98]. Let B be a bounded set of $\mathcal{D}(\Omega)$ and $p_B(f) = \sup_{g\in B} |\langle f, g\rangle|$ with $\langle f, g\rangle = \int_K f(x)g(x)dx$. The function $f(x)$ is bounded by $\pi_{0,K}(f)$ and all the $g(x)$ in B are bounded by a common number M_0 because B is bounded [78, p. 69]. Thus, $p_B(f) \le |K|M_0\pi_{0,K}(f)$, where $|K|$ is the volume of K.

We still must estimate the seminorms $||f||_{N,V,\chi} = \sup_{k\in V}(1 + |k|)^N|\widehat{f\chi}(k)|$. By using $(1 + |k|) \le \beta(1 + |k|^2)$, with $\beta = (1 + \sqrt{2})/2$, we find

$$\begin{aligned}(1 + |k|)^N|\widehat{f\chi}(k)| &\le \beta^N\Big|(1 + |k|^2)^N \int e^{ik\cdot x} f(x)\chi(x)dx\Big| \\ &\le \beta^N\Big| \int e^{ik\cdot x}(1 - \Delta)^N(f\chi)(x)dx\Big|\end{aligned}$$

We expand $(1 - \Delta)^N = \sum_{i=0}^N \binom{N}{i}(-\Delta)^i$ and we estimate each $|\Delta^i(f\chi)(x)| \le n^i\pi_{2N,K}(f\chi)$. This gives us $(1 + |k|)^N|\widehat{f\chi}(k)| \le ((1 + n)\beta)^N|K|\pi_{2N,K}(f\chi)$. To calculate $\pi_{2N,K}(f\chi)$ we notice that, for any multi-index α such that $|\alpha| \le m$, we have

$$\begin{aligned}|\partial^\alpha(f\chi)| &\le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}|\partial^\beta f||\partial^{\alpha-\beta}\chi| \le \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\pi_{m,K}(f)\pi_{m,K}(\chi) \\ &\le 2^m\pi_{m,K}(f)\pi_{m,K}(\chi). \qquad (4)\end{aligned}$$

Thus,

$$(1 + |k|)^N|\widehat{f\chi}(k)| \le (4(n + 1)\beta)^N|K|\pi_{2N,K}(\chi)\pi_{2N,K}(f), \qquad (5)$$

with a bound independent of k and $||f||_{N,V,\chi} \le C\pi_{2N,K}(f)$, where $C = (4(n + 1)\beta)^N|K|\pi_{2N,K}(\chi)$. The proof that the identity is continuous is complete.

It is now clear that $\mathcal{D}'_\Gamma$ with its normal topology is a normal space of distribution because $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'_\Gamma$ (since sequential convergence for the weak and strong topologies of $\mathcal{D}'(\Omega)$ are equivalent [78, p. 70] and from Hörmander's density result [44, p. 262]). From the general properties of normal spaces of distributions we obtain:

Proposition 6 *If we (temporarily) denote by $\mathcal{D}_\Gamma$ the dual of $\mathcal{D}'_\Gamma$, then*

(i) The restriction map induces an injection $\mathcal{D}_\Gamma \hookrightarrow \mathcal{D}'(\Omega)$ [46, p. 259]
(ii) If $\mathcal{D}_\Gamma$ is equipped with the strong topology $\beta(\mathcal{D}_\Gamma, \mathcal{D}'_\Gamma)$, then the injection $\mathcal{D}_\Gamma \hookrightarrow \mathcal{D}'(\Omega)$ is continuous [46, p. 259]
(iii) If $\mathcal{D}_\Gamma$ is equipped with the topology $\kappa(\mathcal{D}_\Gamma, \mathcal{D}'_\Gamma)$ of uniform convergence on the balanced, convex, compact sets for the normal topology of $\mathcal{D}'_\Gamma$ (also called Arens topology [2]), then $\mathcal{D}_\Gamma$ is a normal space of distributions [46, p. 259] and the dual of $\mathcal{D}_\Gamma$ is $\mathcal{D}'_\Gamma$ [46, p. 235]
(iv) A distribution $v \in \mathcal{D}'(\Omega)$ belongs to $\mathcal{D}_\Gamma$ if and only if it is continuous on $\mathcal{D}(\Omega)$ for the topology induced by $\mathcal{D}'_\Gamma$ [46, p. 319]
(v) $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}_\Gamma$ equipped with any topology compatible with duality [77, p. 10]

We are now ready to prove

Proposition 7 *The dual of $\mathcal{D}'_\Gamma$ for its normal topology is $\mathcal{E}'_\Lambda$.*

Proof. We already proved that $\mathcal{E}'_\Lambda \hookrightarrow \mathcal{D}_\Gamma$ because, by lemma 3, any $v \in \mathcal{E}'_\Lambda$ defines a continuous map $\mathcal{D}'_\Gamma \to \mathbb{K}$ (for the Hörmander and thus for the normal topology) and the injectivity is obvious by density of $\mathcal{D}(\Omega)$ in $\mathcal{D}'_\Gamma$. It remains to show that any continuous linear map $\lambda : \mathcal{D}'_\Gamma \to \mathbb{K}$ defines a distribution in $\mathcal{E}'_\Lambda$. By item (i) of proposition 6, we know that λ is a distribution. We first show that this distribution is compactly supported, then that its wavefront set is included in Λ.

Since the map λ is continuous for the normal topology of $\mathcal{D}'_\Gamma$, there exists a finite number of seminorms p_i and a constant M such that $|\lambda(u)| \leq M \sup_i p_i(u)$ for all u in $\mathcal{D}'_\Gamma$ [46, p. 98]. In other words, there is a bounded set B in $\mathcal{D}(\Omega)$ (one is enough because $\sup_i p_{B_i} \leq p_B$ where $B = \cup_i B_i$), and there are r integers N_i, r functions χ_i in $\mathcal{D}(\Omega)$ and r closed cones V_i such that $\operatorname{supp}\chi_i \times V_i \cap \Gamma = \emptyset$ and

$$|\lambda(u)| \leq M \sup(p_B(u), ||u||_{N_1,V_1,\chi_1}, \ldots, ||u||_{N_r,V_r,\chi_r}).$$

We first show that λ is a compactly supported distribution. Indeed, B is a bounded set of $\mathcal{D}(\Omega)$ if and only if there is a compact subset K of Ω and constants M_m such that all $g \in B$ are supported on K and $\pi_{m,K}(g) \leq M_m$ [78, p. 68]. According to the definition of the support of a distribution [44, p. 42], $\langle u, g\rangle = 0$ if $\operatorname{supp} u \cap \operatorname{supp} g = \emptyset$. Thus $p_B(u) = \sup_{g\in B} |\langle u, g\rangle| = 0$ if $\operatorname{supp} u \cap K = \emptyset$. Similarly, $||u||_{N_i,V_i,\chi_i} = 0$ if $\operatorname{supp} u \cap \operatorname{supp}\chi_i = \emptyset$. Finally, for any $f \in \mathcal{D}(\Omega)$ whose support does not meet $K_\lambda = \cup_i \operatorname{supp}\chi_i \cup K$, we have $|\lambda(f)| = 0$. This implies that the support of λ is included in the compact set K_λ [44, p. 42].

Then we show that $\mathrm{WF}(\lambda) \subset \Lambda_M = \cup_{i=1}^M \operatorname{supp}\chi_i \times (-V_i)$. We fix an integer N, a function $\psi \in \mathcal{D}(\Omega)$ and a closed cone W such that $\operatorname{supp}\psi \times W \cap \Lambda_M = \emptyset$ and we define $f_k = (1 + |k|)^N \psi e_k$, where $e_k(x) = e^{ik\cdot x}$. Hence,

$$||\lambda||_{N,W,\psi} = \sup_{k\in W} (1 + |k|)^N |\widehat{\lambda\psi}(k)| = \sup_{k\in W} |\lambda(f_k)|,$$

where we used the fact that the Fourier transform of the compactly supported distribution $\lambda\psi$ is $\lambda(\psi e_k)$ [44, p. 165]. Since, by continuity, $|\lambda(f_k)| \leq M \sup_i p_i(f_k)$, where $p_0 = p_B$ and $p_i = ||\cdot||_{N_i,V_i,\chi_i}$, it suffices to bound each $\sup_{k\in W} p_i(f_k)$.

We first estimate $p_B(f_k)$. Since B is a bounded set in $\mathcal{D}(\Omega)$, the support of all $g \in B$ is contained in a common compact set K [78, p. 88] and

$$\begin{aligned}|\langle f_k, g\rangle| &= (1+|k|)^N |\langle \psi e_k, g\rangle| = (1+|k|)^N |\widehat{\psi g}(k)| \\ &\leq (4(n+1)\beta)^N |K| \pi_{2N,K}(g)\pi_{2N,K}(\psi),\end{aligned}$$

where we used eq. (5). Moreover, all the seminorms of elements of B are bounded [78, p. 88]. Thus, there is a number M_{2N} such that $\pi_{2N,K}(g) \leq M_{2N}$ for all $g \in B$ and we obtain $|\langle f_k, g\rangle| \leq (8\beta)^N |K| \pi_{2N,K}(\psi) M_{2N}$. Since this bound is independent of k, we obtain our first bound $\sup_{k\in\mathbb{R}^n} p_B(f_k) < \infty$.

Consider now the second type of seminorms and calculate $p_i(f_k) = ||f_k||_{N_i,V_i,\chi_i}$. We have two cases:

(i) If $(\operatorname{supp}\psi \cap \operatorname{supp}\chi_i) = \emptyset$, then $\sup_{k\in\mathbb{R}^n} p_i(f_k) = 0$ and we are done.

(ii) If $\operatorname{supp}\psi \cap \operatorname{supp}\chi_i \neq \emptyset$, we want to estimate

$$||f_k||_{N_i,V_i,\chi_i} = \sup_{q\in V_i} (1+|q|)^{N_i} |\widehat{f_k\chi_i}(q)| = \sup_{q\in V_i} (1+|q|)^{N_i} (1+|k|)^N |\widehat{e_k\psi\chi_i}(q)|.$$

We have $\widehat{e_k\psi\chi_i}(q) = \langle e_k e_q, \psi\chi_i\rangle = \widehat{\psi\chi_i}(k+q)$. Since we chose W such that $(-V_i) \cap W = \emptyset$, by compactness of the intersection of V_i and W with the unit sphere, there is a $1 \geq c > 0$ such that $|k-q|/|k| > c$ and $|q-k|/|q| > c$ for all $k \in W$ and $q \in -V_i$. We thus deduce:

$$||f_k||_{N_i,V_i,\chi_i} \leq c^{-N-N_i} \sup_{q\in V_i} (1+|k+q|)^{N+N_i} \widehat{\psi\chi_i}(k+q).$$

The function $\psi\chi_i$ is smooth and compactly supported. We can use eq. (5) again to show that the right hand side of this inequality is bounded uniformly in k.

This concludes the proof of $\mathrm{WF}(\lambda) \subset \Lambda_M$. Finally, $\operatorname{supp}\chi_i \times V_i \cap \Gamma = \emptyset$ implies $\operatorname{supp}\chi_i \times (-V_i) \subset \Lambda$ and $\Lambda_M \subset \Lambda$. Thus, $\mathrm{WF}(\lambda) \subset \Lambda$ and since λ is compactly supported we have $\lambda \in \mathcal{E}'_\Lambda$.

In the following, we shall use $\mathcal{E}'_\Lambda$ (instead of $\mathcal{D}_\Gamma$) to denote the dual of $\mathcal{D}'_\Gamma$. Note that a similar proof shows that $\mathcal{E}'_\Lambda$ is the topological dual of $\mathcal{D}'_\Gamma$ equipped with the Hörmander topology. Indeed, lemma 3 shows in fact that the pairing is continuous for the Hörmander topology because $p_{\psi_i f_i}$ in Eq. (3) is a seminorm of the weak topology of $\mathcal{D}'(\Omega)$, and the proof of the reverse inclusion just requires to replace p_B by a finite set of p_{f_i}.

3. Topologies on $\mathcal{E}'_\Lambda$

Our purpose in this section is to show that, if $(\mathcal{E}'_\Lambda, \beta)$ denotes the space $\mathcal{E}'_\Lambda$ equipped with the strong $\beta(\mathcal{E}'_\Lambda, \mathcal{D}'_\Gamma)$ topology, then the topological dual of $(\mathcal{E}'_\Lambda, \beta)$ is $\mathcal{D}'_\Gamma$. This implies immediately that $\mathcal{D}'_\Gamma$ is semi-reflexive and $\mathcal{E}'_\Lambda$ is barrelled. However, we shall not work directly with the strong topology $\beta(\mathcal{E}'_\Lambda, \mathcal{D}'_\Gamma)$. It will be convenient (especially to show that $\mathcal{E}'_\Lambda$ is nuclear and $\mathcal{D}'_\Gamma$ is complete) to define a topology on $\mathcal{E}'_\Lambda$ as an inductive limit. Then, we prove that the inductive topology is compatible with duality and we conclude by showing that this inductive topology is equivalent to the strong topology.

3.1. Inductive limit topology on $\mathcal{E}'_\Lambda$. We want to define a topology on $\mathcal{E}'_\Lambda$ as the topological inductive limit of some topological spaces E_ℓ. We shall first determine the vector spaces E_ℓ, then we equip them with a topology.

Let us express $\mathcal{E}'_\Lambda$ as the union of increasing spaces E_ℓ. Inspired by the work of Brunetti and coll. [7], we take E_ℓ to be a set of distributions whose wavefront set is contained in some closed cone, that we denote by Λ_ℓ. To determine Λ_ℓ we notice that Λ is an open set and the projection π_i of a product space into each of its coordinate spaces is open [51, p. 90]. Thus, $\pi_1(\Lambda)$ is an open subset of Ω. On the other hand, the singular support of $v \in \mathcal{E}'_\Lambda$ (i.e. $\Sigma(v) = \pi_1(\mathrm{WF}(v))$ [44, p. 254]) is closed [28, p. 108]. It is even compact because it is a closed subset of the support of v, which is compact. Hence, if we exhaust $\pi_1(\Lambda)$ by an increasing sequence of compact sets K_ℓ we know that, for any $v \in \mathcal{E}'_\Lambda$, $\Sigma(v)$ will be contained in K_ℓ for ℓ large enough (because $\Sigma(v) \subset \pi_1(\Lambda)$ implies that the distance between the compact set $\Sigma(v)$ and the closed set $\pi_1(\Lambda)^c$ is strictly positive). Let us define K_ℓ to be the set of points that are at a distance smaller than ℓ from the origin and at a distance larger than $1/\ell$ from the boundary of Ω and from the boundary of $\pi_1(\Lambda)$: $K_\ell = \{x \in \Omega\,; |x| \leq \ell, d(x, \Omega^c) \geq 1/\ell, d(x, \partial\pi_1(\Lambda)) \geq 1/\ell\}$, where $\partial\pi_1(\Lambda)$ is the boundary of $\pi_1(\Lambda)$ and $d(x, A) = \inf\{|x-y|, y \in A\}$ is the distance between a point x and a subset A of Ω. If A is empty, we consider that $d(x, A) = +\infty$. The sets K_ℓ are obviously compact (they are intersections of closed sets with a compact ball), $K_\ell \subset K_{\ell+1}$ and $\pi_1(\Lambda) = \cup_{\ell=1}^{\infty} K_\ell$. Indeed, Ω^c is closed because Ω is open and $\partial\pi_1(\Lambda)$ is a closed set disjoint from $\pi_1(\Lambda)$ because $\pi_1(\Lambda)$ is open [51, p. 46]. Thus, any point of $\pi_1(\Lambda)$ is at a finite distance ϵ_1 from Ω^c, ϵ_2 from $\partial\pi_1(\Lambda)$ and M from zero. Then $x \in K_\ell$ for all integers ℓ greater than $1/\epsilon_1$, $1/\epsilon_2$ and M.

We can now build the closed cones Λ_ℓ, that will be subsets of $\pi_1^{-1}(K_\ell)$ at a finite distance from Γ': $\Lambda_\ell = \{(x;k) \in \dot{T}^*\Omega\,; x \in K_\ell, d\big((x;k/|k|), \Gamma'\big) \geq 1/\ell\}$. This set is clearly a cone because it is defined in terms of $k/|k|$ and it is closed in $\dot{T}^*\Omega$ because it is the intersection of two close sets: $\pi_1^{-1}(K_\ell)$ and $\{(x;k) \in \dot{T}^*\Omega\,; d\big((x;k/|k|), \Gamma'\big) \geq 1/\ell\}$. The first set is closed because K_ℓ is compact and π_1 is continuous and the second set is closed because the function $(x;k) \mapsto d\big((x;k/|k|), \Gamma'\big)$ is continuous on $\dot{T}^*\Omega$.

For some proofs, it will be useful for the support of the distributions to be contained in a fixed compact set. Therefore, we also consider an increasing sequence of compact sets $\{L_\ell\}_{\ell\in\mathbb{N}}$ exhausting Ω and such that L_ℓ is a compact neighborhood of $K_\ell \cup L_{\ell-1}$ ($L_0 = \emptyset$). Finally, we define $E_\ell = \mathcal{E}'_{\Lambda_\ell}(L_\ell)$ to be the set of distributions in $\mathcal{E}'(\Omega)$ whose support is contained in L_ℓ and whose wavefront set is contained in Λ_ℓ. Note that E_ℓ will be equipped with the topology induced by $\mathcal{D}'_{\Lambda_\ell}$ as a closed subset (it is closed because, by definition of the support of a distribution, E_ℓ is the intersection of the kernel of all continuous maps $u \mapsto \langle u, \phi\rangle$ where $\operatorname{supp}\phi \subset L_\ell^c$).

This is an increasing sequence of spaces exhausting $\mathcal{E}'_\Lambda$. It is increasing because $L_\ell \subset L_{\ell+1}$ and $\Lambda_\ell \subset \Lambda_{\ell+1}$ imply $\mathcal{E}'_{\Lambda_\ell}(L_\ell) \subset \mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$. To show that it is exhausting, consider any $v \in \mathcal{E}'_\Lambda$. Since the support of v is compact, it is contained in some L_{ℓ_0} and then in L_ℓ for all $\ell \geq \ell_0$. To show that $\mathrm{WF}(v) \subset \Lambda_{\ell_1}$ for some ℓ_1, consider the set $S_v = \{(x;k)\,; |k| = 1 \text{ and } (x;k) \in WF(v)\}$. It is compact because it is closed and bounded (the support of v being compact). Since $\mathrm{WF}(v) \subset \Lambda$ and $\Lambda \cap \Gamma' = \emptyset$, we have $S_v \cap \Gamma' = \emptyset$. There is a number $\delta > 0$ such that $d\big((x;k), \Gamma'\big) > \delta$ for all $(x;k) \in S_v$ because S_v is compact and

Γ' is closed. Thus, $S_v \subset \Lambda_\ell$ for $\ell > 1/\delta$. Since both S_v and Λ_ℓ are cones we have $\mathrm{WF}(v) \subset \Lambda_\ell$. Finally, $v \in E_\ell$ for all ℓ larger than ℓ_0 and $1/\delta$.

We obtained the first part of

Lemma 8 *If Λ is an open cone in $\dot{T}^*\Omega$, then*

$$\mathcal{E}'_\Lambda = \bigcup_{\ell=1}^{\infty} E_\ell,$$

where $E_\ell = \mathcal{E}'_{\Lambda_\ell}(L_\ell)$ is the set of distributions in $\mathcal{E}'(\Omega)$ with a wavefront set contained in Λ_ℓ and a support contained in L_ℓ. If E_ℓ is equipped with the topology induced by $\mathcal{D}'_{\Lambda_\ell}$ (with its normal topology) we define on $\mathcal{E}'_\Lambda$ the topological inductive limit

$$\mathcal{E}'_\Lambda = \lim_{\rightarrow} E_\ell.$$

This topology will be called the inductive topology *on $\mathcal{E}'_\Lambda$.*

Proof. The inductive limit of E_ℓ defines a topology on $\mathcal{E}'_\Lambda$ iff the injections $E_\ell \hookrightarrow E_{\ell+1}$ are continuous [52, p. 221]. Since $E_\ell \subset \mathcal{D}'_{\Lambda_\ell}$, we can equip E_ℓ with the topology induced by $\mathcal{D}'_{\Lambda_\ell}$, which is defined by the seminorms $p_B(v)$ for all bounded sets B of $\mathcal{D}(\Omega)$ and $||\cdot||_{N,V\chi}$, where $\operatorname{supp}\chi \times V \cap \Lambda_\ell = \emptyset$. We prove that $E_\ell \hookrightarrow E_{\ell+1}$ is continuous by showing that E_ℓ has more seminorms than $E_{\ell+1}$. We have $\Lambda_\ell \subset \Lambda_{\ell+1}$. Thus, $\Lambda_\ell^c \supset \Lambda_{\ell+1}^c$, $\operatorname{supp}\chi \times V \cap \Lambda_\ell = \emptyset$ if $\operatorname{supp}\chi \times V \cap \Lambda_{\ell+1} = \emptyset$ and all the seminorms $||v||_{N,V,\chi}$ on $\mathcal{E}'_{\Lambda_{\ell+1}}$ are also seminorms on $\mathcal{E}'_{\Lambda_\ell}$. The seminorms p_B are the same for $\mathcal{E}'_{\Lambda_{\ell+1}}$ and $\mathcal{E}'_{\Lambda_\ell}$ because the sets B are identical (i.e. the bounded sets of $\mathcal{D}(\Omega)$).

This inductive limit is not strict if the open cone Λ is not closed. Indeed, if the inductive limit were strict, then the Dieudonné-Schwartz theorem [46, p. 161] would imply that each bounded set of $\mathcal{E}'_\Lambda$ is included and bounded in an E_ℓ, which is wrong when Λ is not both open and closed, as we shall prove in section 5.3.

3.2. Duality of the inductive limit. In this section, we show that the inductive topology on $\mathcal{E}'_\Lambda$ is compatible with the pairing:

Proposition 9 *The topological dual of $\mathcal{E}'_\Lambda$ equipped with its inductive topology is $\mathcal{D}'_\Gamma$.*

Proof. We first show that $\mathcal{D}'_\Gamma \hookrightarrow (\mathcal{E}'_\Lambda)'$. We already know that, for any $u \in \mathcal{D}'_\Gamma$, $\langle u, v\rangle$ is well defined for all $v \in E_\ell$ because $E_\ell \subset \mathcal{E}'_\Lambda$. Note that injectivity is obvious since smooth compactly supported functions, which form a separating set for distributions, are in $\mathcal{E}'_\Lambda$. A linear map from an inductive limit into a locally convex space is continuous if and only if its restriction to all E_ℓ is continuous [52, p. 217]. Therefore, we must show that, for any ℓ, the map $\lambda : v \mapsto \langle u, v\rangle$ is continuous from E_ℓ to $\mathbb{K}$. The proof is so close to the derivation of lemma 3 that it suffices to list the differences. We define a finite number of compactly supported smooth functions ψ_j such that $\sum_j \psi_j^2 = 1$ on a compact neighborhood of L_ℓ (here we use the fact that the support of all $v \in E_\ell$ is contained in a common

compact set) and closed cones V_{uj} and V_{vj} satisfying the three conditions (i) $V_{uj} \cap (-V_{vj}) = \emptyset$, (ii) $\operatorname{supp} \psi_j \times V_{uj}^c \cap \mathrm{WF}(u) = \emptyset$ and (iii) $\operatorname{supp} \psi_j \times V_{vj}^c \cap \Lambda_\ell = \emptyset$. The integral I_{2j} is calculated as I_{3j} in lemma 3 if we interchange u and v, α and β: $|I_{2j}| \leq ||v||_{N,U_{\beta j},\psi_j} C I_n^{N-m}$, where m is the order of v, and I_{3j} is bounded as I_{2j} in lemma 3: $|I_{3j}| \leq p_{\psi_j g_j}(v)$, where $\hat{g}_j(k) = \beta_j(k)(1-\alpha_j(-k))\widehat{\psi_j u}(-k)$. We obtain

$$|\langle u, v\rangle| \leq \sum_j \Big(p_{\psi_j g_j}(v) + ||v||_{N,U_{\beta j},\psi_j} C I_n^{N-m} \\ + ||u||_{M,U_{\alpha j},\psi_j} ||v||_{N,U_{\beta j},\psi_j} I_n^{N+M} \Big),$$

for any $N > m + n$ (the condition $N + M > n$ being then satisfied for any nonnegative integer M). This shows the continuity of λ because the right hand side is a finite sum of terms involving seminorms of $\mathcal{D}'_{\Lambda_\ell}$, which induce the topology of E_ℓ.

Conversely, to prove that $(\mathcal{E}'_\Lambda)' \hookrightarrow \mathcal{D}'_\Gamma$, we show that any element λ of $(\mathcal{E}'_\Lambda)'$ defines by restriction to $\mathcal{D}(\Omega)$ a distribution and then that its wavefront set is contained in Γ. This will be enough since by density of $\mathcal{D}(\Omega)$ in $\mathcal{E}'_\Lambda$ the restriction then extends uniquely to $\mathcal{E}'_\Lambda$ and is thus the inverse of the reverse embedding. A linear map $\lambda : \mathcal{E}'_\Lambda \to \mathbb{K}$ is continuous if its restriction to all E_ℓ is continuous. In other words, for each E_ℓ there is a bounded set B in $\mathcal{D}(\Omega)$ and there are smooth functions χ_i and closed cones V_i such that $\operatorname{supp} \chi_i \times V_i \cap \Lambda_\ell = \emptyset$ and

$$|\lambda(v)| \leq M \sup(p_B(v), ||v||_{N_1,V_1,\chi_1}, \ldots, ||v||_{N_r,V_r,\chi_r}). \tag{6}$$

We first prove that λ is a distribution, i.e. a continuous linear map from $\mathcal{D}(\Omega)$ to $\mathbb{K}$. Recall that the space $\mathcal{D}(\Omega)$ is the inductive limit of $\mathcal{D}(L_\ell)$ because L_ℓ is an increasing sequence of compact sets exhausting Ω [78, p. 66]. Thus, a map λ is a distribution if the restriction of λ to each $\mathcal{D}(K_\ell)$ is continuous. For any $f \in \mathcal{D}(K_\ell)$, we must show that all the seminorms on the right hand side of eq. (6) can be bounded by some $\pi_m(f)$. But this is a consequence of the fact that $\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}'_{\Lambda_\ell}$ is continuous, which was established in lemma 5.

Since λ is a distribution, it has a wavefront set. To prove that $\mathrm{WF}(\lambda) \subset \Gamma$ consider a smooth compactly supported function ψ and a closed cone W such that $\operatorname{supp} \psi \times W \cap \Gamma = \emptyset$, i.e. $\operatorname{supp} \psi \times (-W) \subset \Lambda$. Since the restriction of $\operatorname{supp} \psi \times (-W)$ to the unit sphere is compact, there is an ℓ such that $\operatorname{supp} \psi \times (-W) \subset \Lambda_\ell$. Note also that $\operatorname{supp} \psi \subset \pi_1(\Lambda_\ell) \subset L_\ell$ so that $f_k = (1+|k|)^N \psi e_k$ is in E_ℓ. We can now repeat the same reasoning as for the proof of proposition 7 to show that $||\lambda||_{N,W,\psi} = \sup_{k \in W} |\lambda(f_k)|$ is bounded. This shows that $\mathrm{WF}(\lambda) \subset \Gamma$, which implies $\lambda \in \mathcal{D}'_\Gamma$ and $(\mathcal{E}'_\Lambda)' \subset \mathcal{D}'_\Gamma$.

This completes the proof that $(\mathcal{E}'_\Lambda)' = \mathcal{D}'_\Gamma$.

3.3. The strong topology on $\mathcal{E}'_\Lambda$. We showed that the coupling between $\mathcal{E}'_\Lambda$ and $\mathcal{D}'_\Gamma$ is compatible with duality. Thus, the inductive topology on $\mathcal{E}'_\Lambda$ is coarser than the Mackey topology [3, p. IV.4]. The strong topology $\beta(\mathcal{E}'_\Lambda, \mathcal{D}'_\Gamma)$ is always finer than the Mackey topology [3, p. IV.4]. Therefore, if we can show that the inductive topology is finer than the strong topology, we prove the identity of the inductive, Mackey and strong topologies.

Lemma 10 *The inductive, Mackey and strong topologies on $\mathcal{E}'_\Lambda$ are equivalent.*

Proof. To show that the identity map, from $\mathcal{E}'_\Lambda$ with the inductive topology to $\mathcal{E}'_\Lambda$ with the strong topology, is continuous, we must prove that the identity map is continuous from all E_ℓ to $\mathcal{E}'_\Lambda$ with the strong topology. In other words, for any bounded set B' of $\mathcal{D}'_\Gamma$, we must show that $p_{B'}(v) = \sup_{u\in B'} |\langle u, v\rangle|$ is bounded on E_ℓ by some seminorms of E_ℓ.

We proceed as in the proof of lemma 3. From the fact that $\Gamma' \cap \Lambda_\ell = \emptyset$ and $\operatorname{supp} v \subset L_\ell$ we can build a finite number of smooth compactly supported functions ψ_j such that $\sum_j \psi_j^2 = 1$ on a compact neighborhood K' of L_ℓ, and closed cones V_{uj} and V_{vj} satisfying the three conditions (i) $V_{uj} \cap (-V_{vj}) = \emptyset$, (ii) $\operatorname{supp}\psi_j \times V_{uj}^c \cap \Gamma = \emptyset$ and (iii) $\operatorname{supp}\psi_j \times V_{vj}^c \cap \Lambda_\ell = \emptyset$. The support of all ψ_j is assumed to be contained in a common compact neighborhood K of K'. Then, we define again homogeneous functions α_j and β_j of degree 0, measurable, smooth except at the origin, non-negative and bounded by 1 on $\mathbb{R}^n$, such that the closed cones $\operatorname{supp}\alpha_j$ and $\operatorname{supp}\beta_j$ satisfy the three conditions (i), (ii) and (iii), with $\alpha_j = 1$ on V_{uj} and $\beta_j = 1$ on V_{vj} and, as in the proof of lemma 3, we write $\langle u, v\rangle = \sum_j (I_{1j} + I_{2j} + I_{3j} + I_{4j})$. We have again $I_{1j} = 0$ because the supports of α_j and β_j are disjoint, and $|I_{4j}| \le ||u||_{M,U_{\alpha j},\psi_j} ||v||_{N,U_{\beta j},\psi_j} I_n^{N+M}$ for any integers N and M such that $N + M > n$. It is important to remark that ψ_j, α_j and β_j depend only on Γ, L_ℓ and Λ_ℓ and not on u and v.

To estimate I_{2j} and I_{3j}, we need to establish some properties of the bounded sets of $\mathcal{D}'_\Gamma$. The continuity of the injection $\mathcal{D}'_\Gamma \hookrightarrow \mathcal{D}'(\Omega)$ implies that a set B' which is bounded in $\mathcal{D}'_\Gamma$ is also bounded in $\mathcal{D}'(\Omega)$ [46, p. 109]. According to Schwartz [78, p. 86], a subset B' is bounded in $\mathcal{D}'(\Omega)$ iff, for any relatively compact open set $U \subset \Omega$, there is an integer m such that every $u \in B'$ can be expressed in U as $u = \partial^\alpha f_u$ for $|\alpha| \le m$, where f_u a continuous function. Moreover, there is a number M such that $|f_u(x)| \le M$ for all $x \in U$ and $u \in B'$. The elements of E_ℓ are supported on L_ℓ and we need only consider bounded sets of $\mathcal{D}'_\Gamma$ that are defined on the compact neighborhood K of L_ℓ. Thus, we can take for U any relatively compact open set containing K.

To calculate I_{2j}, as in the proof of lemma 3, we define $\hat{g}_j(k) = \alpha_j(-k)(1 - \beta_j(k))\widehat{\psi_j v}(k)$ and we obtain $I_{2j} = (2\pi)^{-n} \int \widehat{\psi_j u}(-k)\hat{g}_j(k)dk = \langle u, \psi_j g_j\rangle$. At this stage one might apply the Banach-Steinhaus theorem but we shall use an equivalent method using $u = \partial^\alpha f_u$:

$$\langle u, \psi_j g_j\rangle = \langle \partial^\alpha f_u, \psi_j g_j\rangle = (-1)^{|\alpha|}\langle f_u, \partial^\alpha(\psi_j g_j)\rangle = (-1)^{|\alpha|}\langle \varphi f_u, \partial^\alpha(\psi_j g_j)\rangle,$$

where φ is a smooth function, equal to 1 on K and supported on U. Thus

$$\begin{aligned}\langle u, \psi_j g_j\rangle &= (-1)^{|\alpha|}(2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\varphi f_u}(-k)\widehat{\partial^\alpha(\psi_j g_j)}(k)dk \\ &= i^{|\alpha|}(2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\varphi f_u}(-k)k^\alpha \widehat{\psi_j g_j}(k)dk.\end{aligned}$$

We must estimate $\widehat{\psi_j g_j}(k) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\psi_j}(k-q)\widehat{g_j}(q)dq$. The functions α_j and $(1 - \beta_j)$ are bounded by 1 and $\widehat{\psi_j v}$ is fast decreasing on $U_{\beta j} = \operatorname{supp}(1 - \beta_j)$. Thus, $|\widehat{g_j}(q)| \le ||v||_{N,U_{\beta j},\psi_j}(1+|q|)^{-N}$ for all integers N. In the proof of lemma 5, we estimated the Fourier transform of a smooth compactly supported function:

$|\psi_j(k-q)| \leq C_j^{N'}(1+|k-q|)^{-N'}$ for all integers N', where $C_j^{N'} = ((1+n)\beta)^{N'}|K|\pi_{2N',K}(\psi_j)$. If we take $N = n+m+1$, where $m = |\alpha|$ is the degree of ∂^α, and $N' = 2N$ we obtain

$$\begin{aligned}|\widehat{\psi_j g_j}(k)| &\leq (2\pi)^{-n}||v||_{N,U_{\beta j},\psi_j} C_j^{2N} \int_{\mathbb{R}^n} (1+|k-q|)^{-2N}(1+|q|)^{-N} dq \\ &\leq ||v||_{N,U_{\beta j},\psi_j} C_j^{2N} I_n^N (1+|k|)^{-N},\end{aligned}$$

where we used $(1+|q|)^{-N} \leq (1+|k-q|)^N(1+|k|)^{-N}$ [22, p. 50]. This estimate enables us to calculate

$$\begin{aligned}|I_{2j}| = |\langle u, \psi_j g_j\rangle| &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{\varphi f_u}(-k)||k|^m |\widehat{\psi_j g_j}(k)| dk \\ &\leq (2\pi)^{-n}|U|M||v||_{N,U_{\beta j},\psi_j} C_j^{2N} I_n^N \int_{\mathbb{R}^n} \frac{|k|^m}{(1+|k|)^{n+m+1}} dk \\ &\leq |U|M||v||_{N,U_{\beta j},\psi_j} C_j^{2N} I_n^N I_n^{n+1},\end{aligned}$$

where $N = n+m+1$, $|U|$ is the volume of U and we used the obvious bound $|\widehat{\varphi f_u}(-k)| \leq |U|M$.

For the estimate of I_{3j} we start from $I_{3j} = (2\pi)^{-n} \int \widehat{g_j^u}(-k)\widehat{\psi_j v}(k)dk$, where $\widehat{g_j^u}(-k) = (1-\alpha_j(-k))\beta_j(k)\widehat{\psi_j u}(-k)$. Thus $|I_{3j}| = |\langle \psi_j g_j^u, v\rangle|$ can be bounded by $p_{B_j}(v) = \sup_{f\in B_j} |\langle f, v\rangle|$ if the set $B_j = \{\psi_j g_j^u\,; u \in B'\}$ is a bounded set in $\mathcal{D}(\Omega)$. It is clear that all $f \in B_j$ are supported on $K = \operatorname{supp}\psi_j$ and that all $\psi_j g_j^u$ are smooth because ψ_j is smooth and the Fourier transform of g_j^u is fast decreasing. It remains to show that all the derivatives of $\psi_j g_j^u$ are bounded by a constant independent of u. For this we write

$$\partial^\gamma(\psi_j g_j^u)(x) = (2\pi)^{-n}(-i)^{|\gamma|} \int_{\mathbb{R}^n} e^{-ik\cdot x} k^\gamma \widehat{\psi_j g_j^u}(k) dk.$$

If $|\gamma| \leq m$, we use the estimate of $\widehat{\psi_j g_j}$ obtained in the previous section and we interchange u and v, α_j and β_j

$$|\widehat{\psi_j g_j^u}(k)| \leq ||u||_{N,U_{\alpha j},\psi_j} C_j^{2N} I_n^N (1+|k|)^{-N},$$

where $N = n+m+1$. A set B' is bounded in $\mathcal{D}'_\Gamma$ iff it is bounded for all the seminorms of $\mathcal{D}'_\Gamma$ [46, p. 109]. In particular, there is a constant $M_{N,U_{\alpha j},\psi_j}$ such that $||u||_{N,U_{\alpha j},\psi_j} \leq M_{N,U_{\alpha j},\psi_j}$ for all $u \in B'$. Thus, for all $f \in B_j$, $|\widehat{f}(k)| \leq M_{N,U_{\alpha j},\psi_j} C_j^{2N} I_n^N (1+|k|)^{-N}$ and

$$|\partial^\gamma f(x)| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |k|^m |\widehat{f}(k)| dk \leq M_{N,U_{\alpha j},\psi_j} C_j^{2N} I_n^N I_n^{n+1},$$

where $N = n+m+1$ as for the estimate of I_{3j}. In other words, for any γ there is a constant $C_{|\gamma|}$ such that $|\partial^\gamma f| \leq C_{|\gamma|}$ for all $f \in B_j$. Thus, $\pi_m(f) \leq \sup_{0\leq k\leq m} C_k$ is bounded independently of f, and we proved that B_j is a bounded set of $\mathcal{D}(\Omega)$. Hence, $|I_{3j}| \leq p_{B_j}(v)$ where p_{B_j} is a seminorm of $\mathcal{D}'_\Gamma$.

If we gather our results we obtain

$$p_{B'}(v) \leq \sum_j \Big(M_{n,U_{\alpha j},\psi_j} ||v||_{n,U_{\beta j},\psi_j} I_n^{2n} + M||v||_{N,U_{\beta j},\psi_j} C_j^{2N} I_n^N I_n^{n+1} + p_{B_j}(v) \Big), \tag{7}$$

where the sum over j is finite and $N = n + m + 1$ where m is the maximum order of the distributions of B'. The proof is complete.

4. Nuclearity

In this section we investigate the nuclear properties of the spaces studied in this paper. To prove that $\mathcal{D}'_\Gamma$ with the normal topology is nuclear, we use a theorem due to Grothendieck [33, Ch. II, p. 48] that can be expressed as follows [66, p. 92]:

Theorem 11 *Let E be a locally convex space and $(f_i)_{i\in I}$ a family of continuous linear maps from E to nuclear locally convex spaces F_i. If the topology of E is the initial topology for the maps f_i, then E is nuclear.*

We recall that, if the topology of F_i is defined by the seminorms $(p^i_\alpha)_{\alpha\in J_i}$, then the initial topology of E is defined by the seminorms $(p^i_{\alpha_i} \circ f_i)_{i\in I,\alpha_i\in J_i}$ [46, p. 152].

The simplest case to prove is

Proposition 12 *The space $\mathcal{D}'_\Gamma$ with the normal topology is nuclear.*

Proof. We first construct the spaces F_i and the linear maps f_i. For $i = 0$ we take $F_0 = \mathcal{D}'(\Omega)$ and f_0 the continuous inclusion $\mathcal{D}'_\Gamma \hookrightarrow \mathcal{D}'(\Omega)$ where $\mathcal{D}'(\Omega)$, equipped with its strong topology, is nuclear [83, p. 53]. For each $i = (V, \chi)$ where $(\operatorname{supp}\chi \times V) \cap \Gamma = \emptyset$, the target space F_i will be the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}^n$ equipped with the family of seminorms [46, p. 90].

$$||f||_{N,m} = \sup_{|\alpha|\leq m} \sup_{k\in\mathbb{R}^n} (1 + |k|)^N |\partial^\alpha f(k)|.$$

The space $\mathcal{S}$ is nuclear [83, p. 430]. To build the linear maps, we choose a real function $h \in \mathcal{D}(\mathbb{R}^n)$ such that $h(k) = 1$ for $|k| \leq 1$, $h(k) = 0$ for $|k| > 2$ and $0 \leq h(k) \leq 1$ for all k, and a nonnegative function $\gamma \in \mathcal{D}(\mathbb{R}^n)$ which is bounded by 1, equal to 1 on $V \cap S^{n-1}$ and such that $(\operatorname{supp}\chi \times \operatorname{supp}\gamma) \cap \Gamma = \emptyset$. We define the homogeneous function $\zeta(k) = \gamma(k/|k|)$, which is smooth outside the origin and bounded by 1. The function $g = (1 - h)\zeta$ is smooth on $\mathbb{R}^n$. By using the homogeneity of ζ and the fact that h and γ are in $\mathcal{D}(\mathbb{R}^n)$, we see that for any integer m there is a constant C_m such that $|\partial^\alpha g(k)| \leq C_m$ for all $|\alpha| \leq m$.

We can now define $f_i : \mathcal{D}'_\Gamma \to \mathcal{S}$ by $f_i(u) = g\,\widehat{u\chi}$. The functions $f_i(u)$ are in $\mathcal{S}$ because $\widehat{u\chi}$ is in $\mathcal{S}$ by definition of the wavefront set and g is supported on the cone $W = \{\lambda k\,; k \in \operatorname{supp}\gamma, |k| = 1, \lambda > 0\}$ and $(\operatorname{supp}\chi \times W) \cap \Gamma = \emptyset$. To show

that f_i is continuous, we must estimate $||f_i(u)||_{N,m}$ in terms of the seminorms of u in $\mathcal{D}'_\Gamma$. By noticing that $\partial^\alpha \widehat{u\chi} = \widehat{(ix)^\alpha u\chi}$ we obtain

$$||f_i(u)||_{N,m} \leq \sup_{k\in W}(1+|k|)^N \sup_{|\alpha|\leq m} \sum_\beta \binom{\alpha}{\beta} C_m |\widehat{x^\beta u\chi}(k)| \leq C \sup_{|\alpha|\leq m} ||u||_{N,W,x^\alpha\chi}.$$

We have shown that all f_i are continuous. Thus, the topology of $\mathcal{D}'_\Gamma$ is finer than the initial topology defined by the family f_i. To show that the two topologies are equivalent, it remains to prove that every seminorm defining the topology of $\mathcal{D}'_\Gamma$ can be bounded with seminorms of the initial topology.

This is obvious for the seminorms of $\mathcal{D}'(\Omega)$ because they are the same in $\mathcal{D}'_\Gamma$. For the seminorms $||\cdot||_{N,V,\chi}$ we note that $\widehat{u\chi} = h\widehat{u\chi} + (1-h)\widehat{u\chi}$. The function $g = (1-h)\zeta$ corresponding to $i = (V,\chi)$ enables us to write $\widehat{u\chi} = h\widehat{u\chi} + g\widehat{u\chi} = h\widehat{u\chi} + f_i(u)$ on V and we obtain

$$||u||_{N,V,\chi} \leq ||f_i(u)||_{N,0} + \sup_{k\in V}(1+|k|)^N |h(k)\widehat{u\chi}(k)|. \tag{8}$$

We just need a bound for the last term. We notice that $\widehat{u\chi}(k) = \langle u, \chi e_k\rangle$, where $e_k(x) = e^{ik\cdot x}$, so that $\sup_{k\in V}(1+|k|)^N|h(k)\widehat{u\chi}(k)| \leq p_B(u)$, where $B = \{(1+|k|)^N\chi e_k\,;k\in V\cap \mathrm{supp}(h)\}$. Thus, the equivalence is proved if p_B is a seminorm of the strong topology of $\mathcal{D}'(\Omega)$, i.e. if B is bounded in $\mathcal{D}(\Omega)$. All the elements of B are supported on $K = \mathrm{supp}\,\chi$. It remains to show that they are bounded for all seminorms $\pi_{m,K}$ but this is obvious by Eq. (4) and $\pi_{m,K}(e_k) \leq |k|^m$.

We emphasize an interesting structural consequence of the proof above for $\mathcal{D}'_\Gamma$. Recall that the class of (PLS)-spaces is the smallest class stable by countable projective limits and containing strong duals of Fréchet-Schwartz spaces. Since such strong duals are known to be inductive limits of Banach spaces with compact linking maps, they are also called (LS)-spaces and since they are bornological, their associated convex bornological space is sometimes called a Silva space [38, 39]. This class appeared recently as useful in applications of homological algebra to functional analysis (see e.g. [87]) having applications to parameter dependence of PDE's [19]. It is known that any Fréchet-Schwartz space is a (PLS)-space. See more generally [18] for a review. It is also known that the strong dual of a (PLS)-space is an (LFS)-space (see below), i.e. a countable inductive limit of Fréchet-Schwartz spaces. Moreover, both are well-known to be strictly webbed spaces in the sense of De Wilde (using general stability properties of these spaces, see e.g. [53, §35]) and thus they satisfy corresponding open-mapping and closed graph theorems. Recall also that the classical sequence space $\mathfrak{s}$ is known to be isomorphic to the Fréchet nuclear space $\mathcal{S}$ (see e.g. [85, pp. 325 and 413]) having universal properties for nuclear spaces in the sense that any nuclear locally convex space is a linear subspace of $\mathfrak{s}^I$ for some set I.

Corollary 13 *$\mathcal{D}'_\Gamma$ with its normal topology is isomorphic to a closed subspace of the countable product $(\mathfrak{s}')^{\mathbb{N}} \times (\mathfrak{s})^{\mathbb{N}}$, and thus it is a (PLS)-space and its strong dual $\mathcal{E}'_\Lambda$ is an (LFS)-space.*

Proof. By [85, p. 385] $\mathcal{D}'(\Omega)$ is known to be isomorphic to $(\mathfrak{s}')^{\mathbb{N}}$. Moreover, we showed in ref. [6] (see alternatively [31, p. 80]) that the additional seminorms of $\mathcal{D}'_\Gamma(\Omega)$ could be chosen in a countable set $\{p_n\,;n\in\mathbb{N}\}$. Thus, the proof of our

previous lemma gives an embedding of $\mathcal{D}'_\Gamma(\Omega)$ in $(\mathfrak{s}')^{\mathbb{N}} \times (\mathfrak{s})^{\mathbb{N}}$. Finally, one can either prove directly that this subspace is closed (and deduce in this way that $\mathcal{D}'_\Gamma(\Omega)$ is complete) or merely use the completeness of $\mathcal{D}'_\Gamma(\Omega)$ proved below in Corollary 25 to deduce that it is necessarily closed as any complete subspace of a Hausdorff space. Finally, it is known (see e.g. [87, p. 96]) that a closed subspace of a (PLS)-space is again a (PLS)-space. The fact that the dual is an (LFS) space is also well-known but we recall the argument by lack of an explicit reference. Since a complete Schwartz space is semi-Montel, its strong dual is also its Mackey dual, since closed subspaces of (LS) spaces are still (LS) spaces, we can assume the projective limit of (LS)-spaces to be reduced, so that one can apply [52, §22.7.(9) p 294] to get its Mackey dual as an inductive limit of Mackey duals. But an (LS) space is known to be a Montel space thus this Mackey dual is also its strong dual which is known to be a Fréchet-Schwartz space (see e.g. [12, Prop 8.5.26 p 293] or [39, p 28]).

The fact that $\mathcal{D}'_\Gamma$ is also nuclear for the Hörmander topology was stated by Fredenhagen and Rejzner [26]. However, since the proof was only sketched, we demonstrate it for completeness.

Proposition 14 *The space $\mathcal{D}'_\Gamma$ with the Hörmander topology is nuclear.*

Proof. The map $f_0 : \mathcal{D}'_\Gamma \to \mathcal{D}'(\Omega)$ goes now from $\mathcal{D}'_\Gamma$ with the Hörmander topology to $\mathcal{D}'(\Omega)$ with the weak topology, which is also nuclear (every locally convex space being nuclear for its weak topology [39, p. 202]). The end of the previous proof cannot be used because the seminorm p_B is not available in the weak topology. Instead we define, for each $j = (K, \chi)$ where K is the image of $[0,1]^n$ by an invertible linear map L such that $(\operatorname{supp}\chi \times K) \cap \Gamma = \emptyset$, the additional map $g_j : \mathcal{D}'_\Gamma \to C^\infty(K)$, where $C^\infty(K)$ is the space of functions $f \in C^\infty(\mathring{K})$ such that f and all its derivatives have continuous extensions to K. The space $C^\infty(K)$, equipped with the seminorms $\pi_{m,K}$, is a nuclear space because $K = L([0,1]^n)$, where L is a linear change of variable, and $C^\infty([0,1]^n)$ is nuclear [85, pp. 325 and 378] or [65].

We define $g_j(u) = h\,\widehat{u\chi}|_K$ (i.e. the restriction to K of the smooth function $h\widehat{u\chi}$). The maps g_j are continuous because $\pi_{m,K}(h\widehat{u\chi}) \leq 2^m \pi_{m,K}(h)\pi_{m,K}(\widehat{u\chi})$ and $\pi_{m,K}(\widehat{u\chi}) \leq \sup_{|\alpha|\leq m} ||u||_{0,V,x^\alpha\chi}$ with $V = \mathbb{R}_+K$. Conversely, for V a closed cone such that $(\operatorname{supp}\chi \times V) \cap \Gamma = \emptyset$, there is a finite set of $K_\ell = L_\ell([0,1]^n)$ such that $(\operatorname{supp} h \cap V) \subset \cup_{\ell=1}^p \mathring{K}_\ell$ and $(\operatorname{supp}\chi \times K_\ell) \cap \Gamma = \emptyset$. Indeed, for every $k \in (\operatorname{supp} h \cap V)$, there is parallelepiped $K_k = L_k([0,1]^n)$, with one vertex at zero, such that $k \in \mathring{K}_k$ and $(\operatorname{supp}\chi \times K_k) \cap \Gamma = \emptyset$. Thus, $(\operatorname{supp} h \cap V) \subset \cup_k \mathring{K}_k$ and we can extract a finite covering because $\operatorname{supp} h \cap V$ is compact. To estimate $(1+|k|)^N |h(k)\widehat{u\chi}(k)|$ in the right hand side of inequality (8), we can take $|k| \leq 2$ because $h(k) = 0$ for $|k| > 2$ and, for every $k \in \operatorname{supp} h \cap V$, we have $|h\widehat{u\chi}(k)| = g_\ell(u)(k)$ if $k \in K_\ell$ and $|h\widehat{u\chi}(k)| = 0$ if $k \notin K_\ell$, where $g_\ell(u) = |h\widehat{u\chi}|_{K_\ell}$. Thus, for all $k \in V$,

$$(1+|k|)^N |h(k)\widehat{u\chi}(k)| \leq 3^N \max_{\ell=1,\dots,p} \pi_{0,K_\ell}(g_{j_\ell}(u)),$$

and

$$||u||_{N,V,\chi} \leq ||f_i(u)||_{N,0} + 3^N \max_{\ell=1,\dots,p} \left(\pi_{0,K_\ell}(g_{j_\ell}(u))\right).$$

Thus, the Hörmander topology is nuclear because it is the initial topology of (f_i) and (g_j).

To complete this section, we show that

Proposition 15 *The space $\mathcal{E}'_\Lambda$ with the strong topology is nuclear.*

Proof. Each E_ℓ is nuclear because it is a vector subspace of the nuclear space $\mathcal{D}'_{\Lambda_\ell}$ with the normal topology [83, p. 514]. Thus, $\mathcal{E}'_\Lambda$ is nuclear since it is the countable inductive limit of the nuclear spaces E_ℓ [83, p. 514].

5. Bornological properties

We study the bornological properties of $\mathcal{D}'_\Gamma$ because they enable us to prove that $\mathcal{D}'_\Gamma$ is complete and because they have a better behaviour than the topological properties with respect to the tensor product of sections. More precisely, if $\Gamma_c(E)$ is the space of compactly supported sections of a vector bundle E over M, then there is a bornological isomorphism between $\Gamma_c(E\otimes F)$ and $\Gamma_c(E)\otimes^\beta_{C^\infty(M)}\Gamma(F)$, where F is another vector bundle over M [63]. As a consequence, there is also a bornological isomorphism between the distribution spaces $\Gamma_c(E\otimes F)'$ and $\Gamma(E^*)\otimes^\beta_{C^\infty(M)}\Gamma_c(F)'$ [63].

5.1. Bornological concepts. We start by recalling some elementary concepts of bornology theory [38].

Definition 16 *A* bornology *on a set X is a family $\mathcal{B}$ of subsets of X satisfying the following axioms:*

B.1: $\mathcal{B}$ is a covering of X, i.e. $X=\cup_{B\in\mathcal{B}}B$.
B.2: $\mathcal{B}$ is hereditary under inclusion: if $A\in\mathcal{B}$ and $B\subset A$, then $B\in\mathcal{B}$.
B.3: $\mathcal{B}$ is stable under finite union.

A pair $(X,\mathcal{B})$ is called a *bornological set* and the elements of $\mathcal{B}$ are called the *bounded subsets* (or the bounded sets) of X.

To define a convex bornological space we need the concept of a *disked hull* [38, p. 6]. We recall that a subset A of a vector space is a *disk* if it is convex and balanced (i.e. if $x\in A$ and $\lambda\in\mathbb{K}$ with $|\lambda|\leq 1$, then $\lambda x\in A$) [38, p. 4].

Definition 17 *If E is a vector space, the disked hull of a subset A of E, denoted by $\Gamma(A)$, is the smallest disk containing A.*

Definition 18 *Let E be a vector space on $\mathbb{K}$. A bornology $\mathcal{B}$ on E is said to be a* convex bornology *if, for every A and B in $\mathcal{B}$ and every $t\in\mathbb{K}$, the sets $A+B$, tA and $\Gamma(A)$ belong to $\mathcal{B}$. Then E or $(E,\mathcal{B})$ is called a convex bornological space.*

We shall also need to define the convergence of a sequence in a convex bornological space [55, p. 12]:

Definition 19 *Let E be a convex bornological space. A sequence x_n in E is said to* Mackey-converge *to x if there exist a disked bounded subset B of E and a sequence α_n of positive real numbers tending to zero, such that $(x_n-x)\in\alpha_n B$ for every integer n.*

One writes $x_n \overset{M}{\to} x$ to express the fact that the sequence x_n Mackey-converges to x. Note that we could equivalently define Mackey convergence in terms of a bounded subset B which is not disked, because the disked hull of a bounded set is bounded by definition of convex bornological spaces.

A convex bornological space is called *separated* if the only vector subspace of $\mathcal{B}$ is $\{0\}$. A convex bornological space is separated iff every Mackey-convergent sequence has a unique limit [38, p. 28].

5.2. Completeness of $\mathcal{D}'_\Gamma$. The set of bounded maps from a convex bornological space E to $\mathbb{K}$ is called the *bornological dual* of E and is denoted by $E^\times$.

A powerful theorem of bornology states [38, p. 77].

Theorem 20 *If a convex bornological space E is regular (i.e. if $E^\times$ separates points in E [38, p. 66]), then its bornological dual $E^\times$, endowed with its natural topology, is a complete locally convex space.*

We are now going to build a bornological space E such that $E^\times$ with its natural topology is equal to $\mathcal{D}'_\Gamma$ with its normal topology. This implies the completeness of $\mathcal{D}'_\Gamma$.

Recall that E_ℓ is the space $\mathcal{E}'_{\Lambda_\ell}(L_\ell)$ of the distributions compactly supported on L_ℓ whose wavefront set is included in Λ_ℓ, where the family (L_ℓ) exhausts Ω and the family (Λ_ℓ) exhausts Λ. To every locally convex space E_ℓ we associate the convex bornological space ${}^bE_\ell$ which is the vector space E_ℓ equipped with its von Neumann bornology (i.e. the bornology defined by the bounded sets of the locally convex space E_ℓ) [38, p. 48]. Let E be the bornological inductive limit of ${}^bE_\ell$, which is the vector space $\mathcal{E}'_\Lambda$ equipped with the bornology defined by the bounded sets of E_ℓ for all integers ℓ [38, p. 33].

The bornological dual $E^\times$ of a convex bornological space E is a locally convex space for the natural topology defined by the bounded sets of E. In other words, the seminorms of $E^\times$ are of the form $p_{B'}(u) = \sup_{v\in B'} |\langle u, v\rangle|$, where B' runs over the bounded sets of E.

We start by three lemmas, undoubtedly well-known to experts :

Lemma 21 *If E is a quasi-complete, Hausdorff locally convex space whose strong dual is a Schwartz space, then the Mackey-convergence of a sequence in E is equivalent to its topological convergence. In particular, this is the case for $\mathcal{D}'(\Omega)$ and $\mathcal{D}'_\Gamma$.*

Proof. In a locally convex space, every Mackey-convergent sequence (for the von Neumann bornology) is topologically convergent [38, p. 26]. We have to prove that, conversely, any topologically convergent sequence is also Mackey convergent. Grothendieck [32] showed that this holds if the *strict Mackey convergence condition* is satisfied: In a Hausdorff topological vector space E, the strict Mackey convergence condition holds if, for every compact subset K of E, there is a bounded disk B in E such that K is compact in $E_B = Span(B)$ (normed with the gauge of B, see [38, p. 26], [12, p. 158],[46, p. 285], [30]).

To show that this condition is satisfied with the hypotheses of the lemma, we use the following theorem due to Randtke [71]: *Let E be a locally convex Hausdorff space whose strong dual is a Schwartz space. Then, for each precompact set A of E, there is a balanced, convex, bounded subset C of E such that C*

absorbs A and A is a precompact subset of E_C. Thus, there is an $\alpha > 0$ such that $A \subset \alpha C$ and, if we denote αC by B, we have a balanced, convex and bounded subset B of E such that $A \subset B$ and A is precompact in $E_B = E_C$.

Consider a compact set K in a locally convex space E that satisfies the hypotheses of the lemma. According to Randtke's theorem, there is a balanced convex and bounded subset B containing K for which K is precompact in E_B. The closure $\bar{B}$ of B is a balanced, convex, bounded and closed subset of E such that the injection $E_B \hookrightarrow E_{\bar{B}}$ is continuous [3, p. II.26]. Moreover, K is also precompact in $E_{\bar{B}}$. Indeed, K is precompact in E_B iff it is totally bounded, i.e. for every neighborhood V of zero, equivalently $V = \epsilon B, \epsilon > 0$, there is a finite number of points $(x_i)_{1\leq i\leq m}$ of E_B such that $K \subset \cup_{i=1}^m (x_i + V)$ [46, p. 145]. Since $E_B \subset E_{\bar{B}}$, the points x_i also belong to $E_{\bar{B}}$ and $K \subset \cup_{i=1}^m (x_i + \epsilon\bar{B})$ is precompact in $E_{\bar{B}}$. The closed bounded set $\bar{B}$ is complete because E is a quasi-complete Hausdorff locally convex space [46, p. 128]. As a consequence, $E_{\bar{B}}$ is complete [46, p. 207] and K is compact in $E_{\bar{B}}$ because every precompact set is relatively compact in a complete space [46, p. 235] and K is closed in $E_{\bar{B}}$ (it is the inverse image of K under the continuous injection $E_{\bar{B}} \hookrightarrow E$ [76, p. 97]). Therefore, E satisfies the strict Mackey convergence condition and the first part of the lemma is proved.

It remains to show that the conditions of the lemma are fulfilled for $\mathcal{D}'(\Omega)$ and $\mathcal{D}'_\Gamma$. We know that $\mathcal{D}'(\Omega)$ is quasi-complete for the weak topology and complete for the strong topology. Its strong dual is $\mathcal{D}(\Omega)$, which is a Schwartz space [46, p. 282]. Therefore, the Mackey and topological sequential convergence coincide in $\mathcal{D}'(\Omega)$ with the weak and strong topologies.

We proved that $\mathcal{D}'_\Gamma$ is quasi-complete with the Hörmander topology (prop. 29) and is complete with the normal topology (cor. 25). Its strong dual $\mathcal{E}'_\Lambda$ is a Schwartz space because it is nuclear [55, p. 581]. Therefore, the Mackey and topological sequential convergence coincide in $\mathcal{D}'_\Gamma$ with the Hörmander and normal topologies.

Lemma 22 *$\mathcal{D}(\Omega)$ is Mackey-sequentially-dense in E.*

Proof. Take $u \in {}^bE_\ell = \mathcal{E}'_{\Lambda_\ell}(L_\ell)$. It suffices to find $u_n \in \mathcal{D}(\Omega)$ such that $u_n - u$ tends bornologically to 0 in $\mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$.

From the proof of Hörmander's density Theorem [44, p. 262] we see that there exists a sequence $u_n \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(u_n) \subset L_{\ell+1}$ such that $u_n \to u$ in $\mathcal{D}'_{\Lambda_{\ell+1}}$ and thus in $\mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1}) = E_{\ell+1}$. The (topological) convergence of u_n in $E_{\ell+1}$ implies its convergence in $\mathcal{D}'(\Omega)$ and, by lemma 21, its bornological convergence in $\mathcal{D}'(\Omega)$. Thus, there exists a sequence α_n of positive real numbers tending to zero and a disked bounded set B in $\mathcal{D}'(\Omega)$ such that $(u_n - u) \in \alpha_n B$ for every integer n.

However, we only know that B is bounded in $\mathcal{D}'(\Omega)$, while we need to find a set which is bounded in $E_{\ell+1}$ to show that u is the bornological limit of a sequence of test functions in $E_{\ell+1}$. In other words, we still have to show that B is bounded for the additional seminorms $||\cdot||_{N,V,\chi}$.

We already used in the proof of corollary 13 that these additional seminorms could be chosen in a countable set $\{p_n\,; n \in \mathbb{N}^*\}$. We can extract a subsequence v_n from u_n such that, for all $k \leq n$, $p_k(v_n - u) \leq 1/n$. Hence, for every seminorm p_k, we have $p_k(v_n - u) \leq M_k/n$ for all positive integers n, where $M_k = \sup_{n<k}\{np_k(v_n - u), 1\}$ is finite. If we define the sequence

$\beta_n = \max(\alpha_{N_n}, 1/n)$ of positive real numbers tending to zero, the Mackey convergence of u_n in $\mathcal{D}'(\Omega)$ implies that, for every integer n, there is an element b_n of B such that $v_n - u = \alpha_{N_n} b_n = \beta_n(\alpha_{N_n}/\beta_n)b_n = \beta_n c_n$ where $c_n = (\alpha_{N_n}/\beta_n)b_n \in B$ because $\alpha_{N_n}/\beta_n \leq 1$ and B is balanced. Moreover, $p_k(v_n - u)/\beta_n \leq 1/(n\beta_n)M_k \leq M_k$. Thus, for every n, $(v_n - u)/\beta_n$ belongs to the set $C = \{x \in B \cap E_{\ell+1}\,; p_k(x) \leq M_k$ for every integer $k\}$, which is balanced and bounded in $E_{\ell+1}$.

Finally, we have showed that any distribution $u \in E_\ell$ is the Mackey-limit in $E_{\ell+1}$ of a sequence of elements of $\mathcal{D}(\Omega)$ and the lemma is proved.

Lemma 23 *Let B be a bounded set in $\mathcal{D}'(\Omega)$, then for every $f \in \mathcal{D}(\Omega)$ there exists M such that*

$$\sup_{u\in B}\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{-M}|\widehat{fu}(\xi)| < \infty.$$

Proof. This lemma is an obvious consequence of uniform boundedness principle. Consider $(T_u)_{u\in B}$ the family of maps $T_u : C^\infty(\Omega) \to \mathbb{C}$ defined on the Fréchet space $C^\infty(\Omega)$ by $T_u(g) = u(fg)$. Since $fg \in \mathcal{D}(\Omega)$ and B is weakly bounded, $\forall g \in C^\infty(\Omega), \exists C_g < \infty, \forall u \in B, |T_u(g)| \leq C_g$. Thus by the uniform boundedness principle, there exists a seminorm p_l of $C^\infty(\Omega)$ such that

$$\sup_{u\in B}|T_u(g)| \leq Cp_l(g).$$

Since $\forall \xi \in \mathbb{R}^n, p_l(e_\xi) \leq c(1+|\xi|)^M$ for some constants c and M, this concludes.

Proposition 24 *If E is the bornological inductive limit of the spaces ${}^bE_\ell$ as above, then $E^\times = \mathcal{D}'_\Gamma$ and its natural topology is equivalent to the normal topology of $\mathcal{D}'_\Gamma$.*

Proof. From lemma 3 and proposition 9, any $u \in \mathcal{D}'_\Gamma$ defines a continuous linear form on each E_ℓ and thus a bounded linear form of ${}^bE_\ell$, i.e. an element of $(E)^\times$. This gives an embedding $\mathcal{D}'_\Gamma \hookrightarrow (E)^\times$ since injectivity comes from the fact $\mathcal{D}(\Omega) \subset E$.

Conversely, we want to prove that each bounded linear form λ on E: (i) defines a distribution when restricted to $\mathcal{D}(\Omega) \subset E$; (ii) has a wavefront set contained in Γ.

This will be enough to conclude the computation of the bornological dual since, from lemma 22 and the fact that a bounded linear functional is Mackey-continuous [36, p. 10], the restriction of a bounded linear functional to $\mathcal{D}(\Omega)$ has a unique extension to E, proving that the second map above is injective.

To prove that λ restricts to a distribution, we notice that the injection $\mathcal{D}(L_\ell) \hookrightarrow E_\ell$ is continuous because E_ℓ is a normal space of distributions. Any bounded set B of $\mathcal{D}(\Omega)$, which is actually in some E_ℓ, is bounded in E_ℓ thus in E because the image of a bounded set by a continuous linear map is a bounded set [46, p. 109]. Thus, λ is also a bounded map from $\mathcal{D}(\Omega)$ to $\mathbb{K}$. It is well-known that $\mathcal{D}(\Omega)$ is bornological [46, p. 222]. Hence, λ is a continuous map from $\mathcal{D}(\Omega)$ to $\mathbb{K}$ because any bounded map from a bornological locally convex space to $\mathbb{K}$ is continuous [46, p. 220]. In other words, λ is an element of $\mathcal{D}'(\Omega)$.

We still have to show that $\lambda \in \mathcal{D}'_\Gamma$, i.e. that for any $\chi \in \mathcal{D}(\Omega)$ and any closed convex neighborhood V such that $\operatorname{supp}\chi \times V \cap \Gamma = \emptyset$, the seminorm $||\lambda||_{N,V,\chi}$ is finite for all integers N. For this we use again the remark made in the proof of proposition 7 that $||\lambda||_{N,V,\chi} = \sup_{k\in V} |\lambda(f_k)|$, where $f_k = (1+|k|)^N \chi e_k$. Thus, if $B' = \{f_k\,; k \in V\}$ is a bounded set in E, then we know that $p_{B'}(\lambda) = \sup_{k\in V} |\lambda(f_k)| < +\infty$ because the image of the bounded set B' by the bounded map λ is bounded. It remains to show that B' is a bounded set of some E_ℓ. We proceed as in the proof of lemma 10.

First, $\operatorname{supp}\chi$ is a compact subset of the open set $\pi_1(\Lambda)$. Therefore, there is an integer ℓ such that L_ℓ is a compact neighborhood of $\operatorname{supp}\chi$ and $U^*\Omega \cap \Lambda_\ell$ is a compact neighborhood of $U^*\Omega \cap (\operatorname{supp}\chi \times (-V))$ because L_ℓ exhausts Ω and Λ_ℓ exhausts Λ. This space E_ℓ contains B' because each f_k is smooth and compactly supported and we want to show that B' is bounded in this E_ℓ.

Consider $||f_k||_{N',W,\psi}$ where $\operatorname{supp}\psi \times W \cap \Lambda_\ell = \emptyset$. If $\operatorname{supp}\psi \cap \operatorname{supp}\chi = \emptyset$, then $||f_k||_{N',W,\psi} = 0$. If $\operatorname{supp}\psi \cap \operatorname{supp}\chi \neq \emptyset$, then $W \cap (-V) = \emptyset$ and thus, by compactness of the intersections of these cones with the unit sphere, there is a $c > 0$ such that $|k+q|/|q| > c$ and $|k+q|/|k| > c$ for all $k \in V, q \in W$. We follow the proof of proposition 7 to show that

$$||f_k||_{N',W,\psi} \leq c^{-N-N'} \sup_{q\in W} (1+|k+q|)^{N+N'} |\widehat{\psi\chi}(k+q)|.$$

According to eq. (5), there is a constant $C_{N+N',\psi\chi}$ such that $||f_k||_{N',W,\psi} \leq c^{-N-N'} C_{N+N',\psi\chi}$. Therefore, $||f_k||_{N',W,\psi}$ is uniformly bounded for all values of $k \in V$.

To conclude the proof of the boundedness of B' in E_ℓ, we show that $p_B(f_k)$ is bounded for all bounded sets $B \subset \mathcal{D}(\Omega)$. We know that $\mathcal{D}(\Omega)$ is a Montel space [83, p. 357]. Thus, it is barrelled and it is enough to show that B' is weakly bounded: i.e. that, for any $g \in \mathcal{D}(\Omega)$, $\langle f_k, g\rangle$ is bounded. Indeed we have $|\langle f_k, g\rangle| = (1+|k|)^N |\langle e_k, \chi g\rangle| = (1+|k|)^N |\widehat{\chi g}(k)|$, which is bounded uniformly in $k \in \mathbb{R}^n$, as seen from eq. (5).

Finally, we have shown that B' is bounded in E_ℓ, which implies that B' is bounded in E and that $||\lambda||_{N,V,\chi} = p_{B'}(\lambda) < +\infty$ for all integers N and all V, χ such that $\operatorname{supp}\chi \times V \cap \Gamma = \emptyset$. This concludes our proof of $\mathrm{WF}(\lambda) \subset \Gamma$.

Moreover, this also shows that the natural topology of $E^\times$ is finer than the normal topology of $\mathcal{D}'_\Gamma$. Indeed, we proved that, for any seminorm $||\cdot||_{N,V,\chi}$ of $\mathcal{D}'_\Gamma$, there is a bounded set B' in E such that $||\cdot||_{N,V,\chi} = p_{B'}$ and the seminorms p_B, where B is bounded in $\mathcal{D}(\Omega)$ are both in $\mathcal{D}'_\Gamma$ and $E^\times$. In other words, $E^\times$ has more seminorms than $\mathcal{D}'_\Gamma$.

It remains to show the converse, i.e. the continuity of the injection $\mathcal{D}'_\Gamma \mapsto E^\times$. For this we have to describe more precisely $E^\times$, which is the bornological dual of a bornological inductive limit. In the topological case, it is well known that the dual of an inductive limit is a projective limit [74, p. 85][52, p. 290]. We have a similar result for the bornological case. Indeed, ${}^bE_\ell$ is the vector space $E_\ell = \mathcal{E}'_{\Lambda_\ell}(L_\ell)$ equipped with the bornology $\mathcal{B}_\ell$ whose elements are the subsets of E_ℓ which are bounded in $\mathcal{D}'_{\Lambda_\ell}$. The injection $j_\ell : {}^bE_\ell \to {}^bE_{\ell+1}$ is bounded because it is continuous. Thus, $E = \cup_\ell E_\ell$ is a convex bornological space whose bornology is $\mathcal{B} = \cup_\ell \mathcal{B}_\ell$ [38, p. 33] [88, p. 195].

By duality, $E^\times = \cap_\ell ({}^bE_\ell)^\times$ as a vector space. The (algebraic) dual map $j_\ell^* : {}^bE_{\ell+1}^\times \to {}^bE_\ell^\times$ is the inclusion. It is continuous if every $({}^bE_\ell)^\times$ is equipped

with its natural topology (since $\mathcal{B}_\ell \subset \mathcal{B}_{\ell+1}$, every seminorm p_B of ${}^bE_\ell^\times$ where $B \in \mathcal{B}_\ell$ is also a seminorm of ${}^bE_{\ell+1}^\times$). Therefore, $E^\times$ is the topological projective limit of the locally convex spaces $({}^bE_\ell)^\times$ [52, p. 230]. To show that the injection $\mathcal{D}'_\Gamma \hookrightarrow E^\times$ is continuous, we just have to prove that each injection $\mathcal{D}'_\Gamma \hookrightarrow ({}^bE_\ell)^\times$ is continuous [88, p. 149].

Said otherwise, we have to show that the bound (3) we obtained in lemma 3 can be made uniform in $v \in B$ for some bounded set B in E_ℓ. First note that the choices of functions ψ, α, β can be made uniformly for $v \in B$, B a bounded set in E_ℓ. Second, using lemma 23, one sees that the constants m, C used in the proof of the bound (3) can be made uniform in $v \in B$ so that $\sup_{v\in B} |\widehat{v\psi_j}(k)| \leq C(1+|k|)^m$. Moreover, by definition of boundedness $\sup_{v\in B} ||v||_{N,U_{\beta j},\psi_j} \leq M_{N,U_{\beta j},\psi_j}$.

We thus obtain:

$$p_B(u) = \sup_{v\in B} |\langle u, v\rangle| \leq \sum_j \Big(p_{B'_j}(u) + ||u||_{M,U_{\alpha j},\psi_j} C I_n^{M-m} + ||u||_{M,U_{\alpha j},\psi_j} M_{N,U_{\beta j},\psi_j} I_n^{N+M} \Big), \tag{9}$$

where $B'_j := \{\psi_j f_j^v \,;\, v \in B\}$ with $\widehat{f_j^v}(k) = \alpha_j(-k)(1-\beta_j(k))\widehat{\psi_j v}(k)$. To prove the expected continuity, it thus only remains to show that B' is bounded in $\mathcal{D}(\Omega)$ so that $p_{B'}$ is a seminorm of $\mathcal{D}'_\Gamma$.

But, let $K_j = \operatorname{supp} \psi_j$, we deduce:

$$\begin{aligned}
\pi_{N,K_j}(\psi_j f_j^v) &\leq 2^N \pi_{N,K_j}(\psi_j)\pi_{N,K_j}(f_j^v) \\
&\leq 2^N \pi_{N,K_j}(\psi_j)(2\pi)^{-n} \sup_{|\gamma|\leq N} \Big| \int_{\operatorname{supp}\beta'_j} dk(k^\gamma)\alpha_j(-k)(1-\beta_j(k))\widehat{\psi_j v}(k)\Big| \\
&\leq 2^N \pi_{N,K_j}(\psi_j) I_n^{n+1} ||v||_{N+n+1,U_{\beta j},\psi_j},
\end{aligned}$$

and the last seminorm is a seminorm in E_ℓ since $U_{\beta j} = \operatorname{supp}(1-\beta_j)$ has been chosen (in the process of choosing ψ, α, β) independent of $v \in B$, so that $\operatorname{supp}(\psi_j) \times \operatorname{supp}(1-\beta_j) \cap \Lambda_\ell = \emptyset$. The above estimate thus concludes.

Corollary 25 *$\mathcal{D}'_\Gamma$ with its normal topology is complete.*

Proof. From theorem 20, it remains to check that E, as a convex bornological space, is regular. From our computation of the dual, it was already proved in lemma 3 that $E^\times$ separates points in E. Thus, E is a regular convex bornological space and its dual $\mathcal{D}'_\Gamma$ is complete with its normal topology, because it is equivalent to the natural topology.

5.3. $\mathcal{E}'_\Lambda$ is ultrabornological. A locally convex space is *bornological* if its balanced, convex and bornivorous subsets are neighborhoods of zero [46, p. 220]. Bornological spaces have very convenient properties. For example, every linear map f from a bornological locally convex space E to a locally convex space F is continuous iff it is bounded (i.e. if f sends every bounded set of E to a bounded set of F) [46, p. 220].

Proposition 26 *$\mathcal{E}'_\Lambda$ is a bornological locally convex space.*

Proof. By a standard theorem [46, p. 221], a locally convex Hausdorff space E is bornological iff the topology of E is the Mackey topology and any bounded linear map from E to $\mathbb{K}$ is continuous. We already know from lemma 10 that the inductive topology on $\mathcal{E}'_\Lambda$ is equivalent to the Mackey topology. Thus, it remains to show that a linear map $\lambda : \mathcal{E}'_\Lambda \to \mathbb{K}$ is continuous if $\sup_{v\in B'} |\lambda(v)| < \infty$ for every bounded subset B' of $\mathcal{E}'_\Lambda$. Since λ is a fortiori bounded for the coarser bornology of E, we know from proposition 24 that it defines by restriction on $\mathcal{D}(\Omega)$ an element of $\mathcal{D}'_\Gamma$. Then this element extends to a continuous linear form on $\mathcal{E}'_\Lambda$ and since, by lemma 22, $\mathcal{D}(\Omega)$ is Mackey dense in E and a fortiori in $\mathcal{E}'_\Lambda$, the extension has to coincide with the original λ (which is bounded thus Mackey sequentially continuous). Therefore, λ is continuous.

Note that the previous argument says $\mathcal{E}'_\Lambda$ has the same bornological dual as E, but not necessarily with the same natural topology. Indeed, the natural topology of $(\mathcal{E}'_\Lambda)^\times$ is the strong $\beta(\mathcal{D}'_\Gamma, \mathcal{E}'_\Lambda)$ topology on $\mathcal{D}'_\Lambda$. If the normal topology of $\mathcal{D}'_\Lambda$ were the strong topology, then $\mathcal{E}'_\Lambda$ would be semi-reflexive because the dual of $\mathcal{D}'_\Gamma$ for the normal topology is $\mathcal{E}'_\Lambda$. Thus, $\mathcal{E}'_\Lambda$ would be quasi-complete and we shall prove in section 6.4 that this is not the case when the open cone Λ is not closed.

This implies another consequence regarding the regularity of the inductive limit. Recall that an inductive limit of locally convex spaces is said to be *regular* if each bounded set of E is contained and bounded in some E_ℓ [56,68]. If the inductive limit defining the topology of $\mathcal{E}'_\Lambda$ were regular, then the bornology of $\mathcal{E}'_\Lambda$ would be the bornology of E (because we already know that every bounded set of E is bounded in $\mathcal{E}'_\Lambda$). In that case, the natural topologies of their bornological dual $\mathcal{D}'_\Gamma$ would be identical and the normal topology on $\mathcal{D}'_\Gamma$ would be the strong topology. Thus, the inductive limit is not regular when Λ is not both open and closed.

Let us see how this bornological property also follows from a general theorem, even giving us a stronger result:

Proposition 27 *$\mathcal{E}'_\Lambda$ is an ultrabornological locally convex space.*

Proof. $\mathcal{D}'_\Gamma$ is complete and nuclear. Therefore, by noticing that any nuclear locally convex space is Schwartz [55, p. 581], we see that $\mathcal{E}'_\Lambda$ is ultrabornological because it is the strong dual of a complete Schwartz locally convex space [46, p. 287] [39, p. 15].

Note that ultrabornological spaces are also called completely bornological [38, p. 53] or fast-bornological [88, p. 203]. A locally convex space is ultrabornological iff it is the topologification of a complete convex bornological space [38, p. 53]. An ultrabornological space is the inductive limit of a family of separable Banach spaces [48, p. 274]. Further characterizations are known [84],[39, p. 207-210], [12, Ch. 6], [61, p. 283], [29, p. 54]. The relation between boundedness and continuity is: A linear map from an ultrabornological space E to a locally convex space F is continuous iff it is bounded on each compact disk of E [38, p. 54].

6. Functional properties of $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$

In this section, we put together the results derived up to now to determine the main functional properties of $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$.

6.1. General functional properties.

Proposition 28 *The space $\mathcal{D}'_\Gamma$ is a normal space of distributions. It is Hausdorff, nuclear and semi-reflexive. Its topological dual is $\mathcal{E}'_\Lambda$ which is Hausdorff, nuclear, and barrelled.*

Proof. We saw that $\mathcal{D}'_\Gamma$ is Hausdorff. Its dual $\mathcal{E}'_\Lambda$ is also Hausdorff because the pairing $\langle\cdot,\cdot\rangle$ is separating (see lemma 3) and the topology of $\mathcal{E}'_\Lambda$ is finer than the weak topology $\sigma(\mathcal{E}'_\Lambda, \mathcal{D}'_\Gamma)$ [46, p. 185]. We proved that $\mathcal{D}'_\Gamma$ is the dual of $\mathcal{E}'_\Lambda$ for the inductive topology and that the inductive topology of $\mathcal{E}'_\Lambda$ is equivalent to the strong topology $\beta(\mathcal{E}'_\Lambda, \mathcal{D}'_\Gamma)$. Therefore, $\mathcal{D}'_\Gamma$ is the topological dual of $\mathcal{E}'_\Lambda$, which is the strong dual of $\mathcal{D}'_\Gamma$. This implies that $\mathcal{D}'_\Gamma$ is semi-reflexive [46, p. 227].

The space $\mathcal{E}'_\Lambda$ is barrelled because it is the strong dual of a semi-reflexive space [46, p. 228]. This can also be deduced from the fact that the inductive topology of $\mathcal{E}'_\Lambda$ is equal to its strong topology [3, p. IV.5].

In fact, $\mathcal{D}'_\Gamma$ is even a completely reflexive locally convex space, because it is complete and Schwartz [38, p. 95]. Recall that a locally convex space E is completely reflexive (or ultra-semi-reflexive [88, p. 243]) if $E = (E')^\times$ algebraically and topologically [38, p. 89], where E' is the dual of E with the equicontinuous bornology and $(E')^\times$ is the bornological dual of E' with its natural topology. This has two useful consequences: (i) $\mathcal{E}'_\Lambda$ equipped with the equicontinuous bornology is a reflexive convex bornological space [38, p. 136]; (ii) the strong and ultra-strong topologies on $\mathcal{E}'_\Lambda$ are equivalent [38, p. 90].

6.2. Completeness properties of $\mathcal{D}'_\Gamma$. We state the results concerning the completeness of $\mathcal{D}'_\Gamma$:

Proposition 29 *In $\mathcal{D}'_\Gamma$:*

- *$\mathcal{D}'_\Gamma$ is complete for all topologies finer than the normal topology and coarser than the Mackey topology.*
- *$\mathcal{D}'_\Gamma$ is quasi-complete for all topologies compatible with the duality between $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$: all the bounded closed subsets are complete for these topologies. In particular, $\mathcal{D}'_\Gamma$ is quasi-complete for the Hörmander topology.*

Proof. We have proved that $\mathcal{D}'_\Gamma$ is complete for the normal topology. Thus, it is complete for all topologies that are finer than the normal topology and that are compatible with duality [3, p. IV.5]. We have also showed that $\mathcal{D}'_\Gamma$ is semi-reflexive. As a consequence, it is quasi-complete for the weak topology $\sigma(\mathcal{D}'_\Gamma, \mathcal{E}'_\Lambda)$ [46, p. 228]. This implies that $\mathcal{D}'_\Gamma$ is quasi-complete for every topology compatible with the duality between $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$, in particular for the normal topology [3, p. IV.5]. Since Bourbaki's proof is rather sketchy, we give it in more detail. Assume that E is quasi-complete for the weak topology $\sigma(E, E')$ and consider a topology $\mathcal{T}$ compatible with duality. The space E is quasi-complete for $\mathcal{T}$ iff every $\mathcal{T}$-closed $\mathcal{T}$-bounded subset of E is complete [46, p. 128]. Consider a subset C of E which is closed and bounded for $\mathcal{T}$. By the theorem of the bipolars, the bipolar $C^{\circ\circ}$ of C is a balanced, convex, $\sigma(E, E')$-closed set containing C. We also know that C is bounded for $\mathcal{T}$ iff it is bounded for $\sigma(E, E')$ because $\mathcal{T}$ is compatible with duality [46, p. 209]. Then, we use the fact that C is bounded for $\sigma(E, E')$ iff C° is absorbing [46, p. 191]. But $C^\circ = (C^{\circ\circ})^\circ$ so that $C^{\circ\circ}$ is

weakly bounded if and only if C is weakly bounded. Therefore, $C^{\circ\circ}$ is bounded, convex and closed for $\sigma(E, E')$, and also for the other topologies compatible with duality by the first two items of the proposition. Consider now a Cauchy filter on $C^{\circ\circ}$ for the topology $\mathcal{T}$. It is also a Cauchy filter for the weak topology. Indeed a filter $\mathfrak{F}$ is Cauchy if and only if, for any neighborhood V of zero, there is an $F \in \mathfrak{F}$ such that $F - F \subset V$. The topology $\mathcal{T}$ being compatible with duality, it is finer than the weak topology. Thus, any weak neighborhood V is also a neighborhood of $\mathcal{T}$ and $\mathfrak{F}$ is a Cauchy filter for the weak topology. This Cauchy filter converges to a point x because E is quasi-complete for the weak topology. Moreover, x is in $C^{\circ\circ}$ because $C^{\circ\circ}$ is weakly closed. Therefore, the Cauchy filter converges in $C^{\circ\circ}$ and $C^{\circ\circ}$ is complete for $\mathcal{T}$. As a consequence, C itself is also complete because it is a closed subset of a complete set [46, p. 128].

This brings us to the following result

Proposition 30 *The space $\mathcal{D}'_\Gamma$ with its normal topology is semi-Montel. The space $\mathcal{E}'_\Lambda$ is a normal space of distributions on which the strong, Mackey, inductive limit and Arens topologies are equivalent.*

Proof. We saw that $\mathcal{D}'_\Gamma$ is quasi-complete and nuclear for its normal topology. Thus, its bounded subsets are relatively compact [83, p. 520] and $\mathcal{D}'_\Gamma$ is semi-Montel by definition of semi-Montel spaces [46, p. 231]. We already know that the strong, Mackey and inductive limit topologies are equivalent. It is known that on the dual of a semi-Montel space, the Arens topology is equivalent to the strong and Mackey ones [46, p. 235]. By item (iii) of proposition 6, we obtain that $\mathcal{E}'_\Lambda$ is a normal space of distributions.

Semi-Montel spaces have interesting stability properties [46, § 3.9], [48, § 11.5] (for example, a closed subspace of a semi-Montel space is semi-Montel [46, p. 232], as well as a strict inductive limit of semi-Montel spaces [46, p. 240]). Moreover, if B is a bounded subset of $\mathcal{D}'_\Gamma$, then the topology induced on B by the normal topology is the same as that induced by the weak $\sigma(\mathcal{D}'_\Gamma, \mathcal{E}'_\Lambda)$ topology [46, p. 231] and B is metrizable (because $\mathcal{E}'_\Lambda$, the strong dual of $\mathcal{D}'_\Gamma$, is nuclear [39, p. 217]).

The following properties of semi-Montel spaces are a characterization of convergence [46, p. 232] which is useful in renormalization theory:

Proposition 31 *If u_i is a sequence of elements of $\mathcal{D}'_\Gamma$ such that $\langle u_i, v\rangle$ converges to some number $\lambda(v)$ in $\mathbb{K}$ for all $v \in \mathcal{E}'_\Lambda$, then the map $u : v \mapsto \lambda(v)$ belongs to $\mathcal{D}'_\Gamma$ and u_i converges to u in $\mathcal{D}'_\Gamma$.*

Proposition 32 *If $(u_\epsilon)_{0<\epsilon<\alpha}$ is a family of elements of $\mathcal{D}'_\Gamma$ such that $\langle u_\epsilon, v\rangle$ converges to some number $\lambda(v)$ in $\mathbb{K}$ as $\epsilon \to 0$ for all $v \in \mathcal{E}'_\Lambda$, then the map $u : v \mapsto \lambda(v)$ belongs to $\mathcal{D}'_\Gamma$ and $u_\epsilon \to u$ in $\mathcal{D}'_\Gamma$ as $\epsilon \to 0$.*

By proposition 29, we see that $\mathcal{D}'_\Gamma$ is quasi-complete for the Hörmander topology. However, it is generally not complete because $\mathcal{D}'(\Omega)$ is not complete for the weak topology (otherwise, every linear map from $\mathcal{D}(\Omega)$ to $\mathbb{K}$ would be continuous, whereas it is well known that the algebraic dual of $\mathcal{D}(\Omega)$ is larger than $\mathcal{D}'(\Omega)$ [64]).

6.3. Bounded sets. The bounded sets of $\mathcal{D}'_\Gamma$ are important in renormalization theory because they are used to define the scaling degree [8] of a distribution and the weakly homogeneous distributions [62].

The bounded sets of $\mathcal{D}'_\Gamma$ were characterized in the proof of lemma 10: a subset B' of $\mathcal{D}'_\Gamma$ is bounded if B' is a bounded set of $\mathcal{D}'(\Omega)$ and for every integer N, every $\psi \in \mathcal{D}(\Omega)$ and every closed cone V such that $\operatorname{supp}\psi \times V \cap \Gamma = \emptyset$, there is a constant $M_{N,V,\chi}$ such that $||u||_{N,V\chi} \leq M_{N,V,\chi}$ for all $u \in B'$. The bounded sets of $\mathcal{D}'(\Omega)$ have several characterizations (see [78, pp. 86 and 195] and [21, pp. 330 and 493]).

We can now give a list of the main properties of the bounded sets of $\mathcal{D}'_\Gamma$, which correspond to a Banach-Steinhaus theorem for $\mathcal{D}'_\Gamma$:

Theorem 33 *In $\mathcal{D}'_\Gamma$:*

- *The bounded subsets are the same for all topologies finer than the weak topology $\sigma(\mathcal{D}'_\Gamma, \mathcal{E}'_\Lambda)$ and coarser than the strong topology $\beta(\mathcal{D}'_\Gamma, \mathcal{E}'_\Lambda)$. In particular, they are the same for the normal and the Hörmander topologies.*
- *The bounded sets are equicontinuous.*
- *The closed bounded sets are compact, identical and topologically equivalent for the weak, Hörmander and normal topologies.*

Proof. In general, the bounded subsets of a topological vector space E are the same for all locally convex Hausdorff topologies on E compatible with the duality between E and E' [83, p. 371], i.e. for all topologies finer than the weak topology and coarser than the Mackey topology [83, p. 369]. The barrelledness of $\mathcal{E}'_\Lambda$ implies that these bounded sets are also identical with the strongly bounded sets [46, p. 212]. In the dual $\mathcal{D}'_\Gamma$ of the barrelled space $\mathcal{E}'_\Lambda$, a set is bounded if and only if it is equicontinuous [46, p. 212]. In a quasi-complete nuclear space, every closed bounded subset is compact [83, p. 520]. Especially, using propositions 28 and 29, this implies that bounded subsets closed for the Hörmander and normal topologies are compact for these topologies. In the dual of a barrelled space, the weakly closed bounded sets are weakly compact [46, p. 212]. After the proof of prop. 7, we showed that the Hörmander topology is compatible with the pairing [46, p. 198]. Thus, by the Mackey-Arens theorem [46, p. 205], it is finer than the weak topology and coarser than the Mackey one.

In the remarks following Proposition 30, we showed that the weak and normal topologies are equivalent on the bounded sets. Therefore, the Hörmander topology is equivalent to those since it is finer than the weak topology and coarser than the normal one. As a consequence, the closed and bounded sets are the same for the three topologies. Indeed, it suffices to remember that the bounded sets closed for one of these topologies are compact for the corresponding induced topology, and compactness is an internal topological property so that they are compact for all the induced topologies since they coincide. Finally, compactness implies in a Hausdorff space that they are closed for the three topologies.

In concrete terms, this means that a subset B' is bounded in $\mathcal{D}'_\Gamma$ if and only if one (and then all) of the following conditions is satisfied:

(i) For every $v \in \mathcal{E}'_\Lambda$, there is a constant M_v such that $|\langle u, v\rangle| \leq M_v$ for all $u \in B'$. This defines weakly bounded sets.
(ii) For every bounded set B of $\mathcal{E}'_\Lambda$, there is a constant M_B such that $|\langle u, v\rangle| \leq M_B$ for all $u \in B'$ and all $v \in B$. This defines strongly bounded sets.

(iii) There is a constant C and a finite set of seminorms p_i of $\mathcal{E}'_\Lambda$ such that $|\langle u, v\rangle| \leq C \max_i p_i(v)$. This defines equicontinuous sets [46, p. 200].

With respect to item (ii) recall that, the inductive limit being not regular, there are bounded sets in $\mathcal{E}'_\Lambda$ that are not contained and bounded in any E_ℓ. However, of course, as we already used, the bounded sets of every E_ℓ are bounded in $\mathcal{E}'_\Lambda$.

Note also that the closed *convex* subsets are the same for all topologies compatible with the duality between $\mathcal{D}'_\Gamma$ and $\mathcal{E}'_\Lambda$ [83, p. 370].

6.4. Completeness properties of $\mathcal{E}'_\Lambda$. By contrast with $\mathcal{D}'_\Gamma$, the completeness properties of $\mathcal{E}'_\Lambda$ are very poor. More precisely, we have

Theorem 34 *Assume that Λ is an open cone which is not closed, then $\mathcal{E}'_\Lambda$ with its strong topology is not (even weakly) sequentially complete. In particular, if Ω is connected and the dimension of spacetime is $n > 1$, then $\mathcal{E}'_\Lambda$ is not sequentially complete when Λ is any open conical nonempty proper subset of $\dot{T}^*\Omega$.*

Proof. In fact, if Λ is an open cone which is not closed in $\dot{T}^*\Omega$, we exhibit an explicit counterexample showing that $\mathcal{E}'_\Lambda$ is not sequentially complete. Since the construction of this counterexample is a bit elaborate, we first describe its main ideas. Consider a point $(x;\eta)$ in the boundary of Λ. There is a sequence of points $(x_m;\eta_m) \in \Lambda$ such that $(x_m;\eta_m) \to (x;\eta)$. By using an example due to Hörmander, we construct a distribution v_m whose wavefront set is exactly the line $\{(x_m;\lambda\eta_m)\,;\lambda > 0\}$. Then we show that the sum $v = \sum_m v_m/m!$ is a well-defined distribution which does not belong to $\mathcal{E}'_\Lambda$ because the point $(x;\eta)$ belongs to its wavefront set. Since the series defining v is a Cauchy sequence, we have defined a Cauchy sequence in $\mathcal{E}'_\Lambda$ whose limit is not in $\mathcal{E}'_\Lambda$.

The proof consists of several steps: (i) description of Hörmander's example, (ii) construction of the counter-example $v = \sum v_m/m!$, (iii) choice of the sequence $(x_m;\eta_m)$ and of the closed cones Γ_M, (iv) calculation of the seminorms of v_m in $\mathcal{D}'_{\Gamma_M}$, (v) determination of the wavefront set of v, (vi) proof that the series is Cauchy in $\mathcal{E}'_\Lambda$, (vii) discussion of the case where Λ is both open and closed.

Step 1: Hörmander's distribution

To build this counterexample we start from a family of distributions, defined by Hörmander [45, p. 188], whose wavefront sets are made of a single point x and a single direction λk and whose order is arbitrary: Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be equal to 1 in $(-\infty, 1/2)$ and to 0 in $(1, +\infty)$, with $0 \leq \chi \leq 1$. Fix $0 < \rho < 1$, let $\eta \in \mathbb{R}^n$ be a unit vector, take an orthonormal basis $(e_1 = \eta, e_2, ..., e_n)$ and write coordinates in this coordinate system.

Define $u_{\eta,s} \in \mathcal{S}'(\mathbb{R}^n)$, for $s \in \mathbb{R}$, by

$$\widehat{u_{\eta,s}}(\xi) = (1 - \chi(\xi_1))\xi_1^{-s}\chi((\xi_2^2 + \cdots + \xi_n^2)/\xi_1^{2\rho}).$$

Then $\mathrm{WF}(u_{\eta,s}) = \{(0;\xi); \xi_2 = \cdots = \xi_n = 0, \xi_1 > 0\} = \{0\} \times \mathbb{R}_+^*\eta$ and $u_{\eta,s}$ coincides with a function in $\mathcal{S}(\mathbb{R}^n)$ outside a neighborhood of the origin [45, p. 188]. It is clear that, if $\xi = \lambda\eta$ and $\lambda > 1$, then $|\widehat{u}_{\eta,s}(\xi)| = \lambda^{-s}$ for any $\lambda > 1$,

where s is an arbitrary real number. Thus, the degree of growth can be an arbitrary polynomial degree. Moreover, Hörmander actually proves that for any real number t and any integer m, there is a constant $C(t,m)$, such that if α,β are multi-indexes, and $|\alpha|\geq C(t,m)$ then $\{x^\alpha\partial^\beta u_{\eta,s}, s\geq t, |\beta|\leq m, \eta\in S^{n-1}\}$ are bounded continuous functions on $\mathbb{R}^n$, uniformly bounded by a constant $D(t,m)$.

One should also note that when the last factor in the definition does not vanish, we have $\xi_1^{2\rho}\geq(\xi_2^2+...+\xi_n^2)$ so that $|\xi_1|^2\geq\frac{|\xi_1|^2+(\xi_2^2+...+\xi_n^2)^{1/\rho}}{2}\geq|\xi|^2/2$ as soon as $\xi_2^2+...+\xi_n^2\geq 1$, and otherwise $|\xi|^2\leq|\xi_1|^2+1\leq 5|\xi_1|^2$ when $(1-\chi)(\xi_1)\neq 0$ (which implies $\xi_1\geq 1/2$). Moreover, when the first factor does not vanish $|\xi|\geq 1/2$ so that $|\xi|\geq(1+2|\xi|)/4\geq(1+|\xi|)/4$. As a consequence, we note for $s\geq 0$:

$$|\widehat{u_{\eta,s}}(\xi)|\leq(1-\chi(\xi_1))80^{s/2}(1+|\xi|)^{-s}\chi((\xi_2^2+\cdots+\xi_n^2)/\xi_1^{2\rho})\leq 10^s(1+|\xi|)^{-s}. \tag{10}$$

Step 2: Construction of the counterexample

Since Λ is open and not closed, its boundary $\partial\Lambda=\overline{\Lambda}\backslash\Lambda$ is not empty and $\partial\Lambda\cap\Lambda=\emptyset$ [51, p. 46]. Moreover, any point $(x;\eta)$ of $\partial\Lambda$ is the limit of a sequence of points $(x_m;\eta_m)$ in Λ [5, p. 9].

By starting from Hörmander's example, we build a family of distributions v_m such that the wavefront set of v_m is $\{(x_m;\lambda\eta_m);\lambda>0\}$ and $|\widehat{v_m}(\lambda\eta_m)|=(\lambda|\eta_m|)^{-m}$. For this we use the translation operator T_x acting on test functions by $(T_xf)(y)=f(y-x)$ and extend it to distributions by $\langle T_xu,f\rangle=\langle u,T_{-x}f\rangle$. Thus $T_{x_m}u_{\eta_m,m}$ has the desired properties. However, we want all distributions v_m to be compactly supported on Ω. Thus, we define the compact set $X=\cup_{m=1}^\infty\{x_m\}\cup\{x\}\subset\Omega$, so that $\delta=d(X,\Omega^c)>0$, and χ a smooth function compactly supported on $B(0,\delta/2)$ and equal to 1 on a neighborhood of the origin. Then $v_m=T_{x_m}(\chi u_{\eta_m,m})$ is a distribution in $\mathcal{E}'(\Omega)$ with the desired properties.

It is easy to show that the series $v=\sum_{m=1}^\infty v_m/m!$ converges to a distribution in $\mathcal{E}'(\Omega)$. Indeed, it is enough to prove that, for any $f\in\mathcal{D}(\Omega)$, the numerical series $\sum_m\langle v_m,f\rangle/m!$ converges in $\mathbb{K}$ [28, p. 13]. We have

$$\langle v_m,f\rangle=\langle T_{x_m}\chi u_{\eta_m,m},f\rangle=\langle u_{\eta_m,m},\chi T_{-x_m}f\rangle=(2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{u_{\eta_m,m}}(k)\widehat{\chi f_{-x_m}}(-k).$$

where $f_{-x_m}=T_{-x_m}f$. For every integer N we have by Eq.(5)

$$|\widehat{\chi f_{-x_m}}(k)|\leq(1+|k|)^{-N}(4(n+1)\beta)^N|K|\pi_{2N,K}(\chi)\pi_{2N,K}(f_{-x_m}),$$

where K is a compact neighborhood of $\operatorname{supp}\chi$ and $|K|$ its volume.

Now, $\pi_{2N,K}(f_{-x_m})\leq\pi_{2N,K'}(f)$, where K' is a compact neighborhood of $\operatorname{supp}f$. Thus, there is a constant $C_N=(4(n+1)\beta)^N|K|\pi_{2N,K}(\chi)\pi_{2N,K'}(f)$, independent of m, such that $|\widehat{\chi f_{-x_m}}(k)|\leq C_N(1+|k|)^{-N}$. The estimate (10) gives us, for $N=n$,

$$\begin{aligned}|\langle v_m,f\rangle|&\leq C_n(2\pi)^{-n}10^m\int_{\mathbb{R}^n}(1+|k|)^{-n-m}dk\\&\leq C_n(2\pi)^{-n}10^m\int_{\mathbb{R}^n}(1+|k|)^{-n-1}dk\leq C_n10^mI_n^{n+1},\end{aligned}$$

because $m \geq 1$, and the series defining v is absolutely convergent with $|\langle v, f\rangle| \leq C_n I_n^{n+1} e^{10}$.

We know that the distribution v is well defined but we have no control of its wavefront set. Indeed, the wavefront set of v can contain points that are not in any $WF(v_m)$ and there can be points that are in the wavefront set of some v_m but not in $WF(v)$ (see refs. [49,42] for concrete examples). Therefore, we must carefully choose the sequence $(x_m; \eta_m)$ so that $(x; \eta)$ is indeed in the wavefront set of v. This is done in the next step.

Step 3: Choice of the sequence and construction of the cones

We want to ensure that all points $(x_m; \eta_m)$ actually belong to $\mathrm{WF}(v)$. Thus, we choose the elements $(x_m; \eta_m)$ so that each direction η_m is at a finite distance from the other ones (except when $n = 1$, in which case we will choose x_m at a finite distance from one another), to avoid that their overlap concurs to remove $(x; \eta)$ from the wavefront set of v. Since Λ is a cone, we can choose $|\eta| = |\eta_m| = 1$ and, up to extraction and since Λ is open, it is possible to shift the points $(x_m; \eta_m)$ so that if $n = 1$, $x_m \neq x$ and $\eta_m = \eta$, $|x_{m+1} - x| < |x_m - x|/2, |x_m - x| < 1$ and if $n \neq 1$ $\eta_m \neq \eta$, $|\eta_{m+1} - \eta| < \min(|\eta_m - \eta|, d(\eta_m, -\Gamma_{x_m}))/2$, where $\Gamma_{x_m} = \{\xi\,; (x_m; \xi) \in \Gamma\}$, and $|\eta_m - \eta| < 1$ for all m. Let $\rho_m = \min\big(|\eta_m - \eta|, d(\eta_m, -\Gamma_{x_m})\big) < 1$ if $n \neq 1$ and set $\rho_m = 1/3^m$ if $n = 1$, and note that if $n \neq 1$, $\rho_{m+1} < \rho_m/2$ implies $|\eta_m - \eta_k| > \rho_m/2$ for all $k > m$. Indeed, if $|\eta_m - \eta_k| \leq \rho_m/2$ were true, $\rho_m/2 \geq \rho_m/2^{k-m} > \rho_k \geq |\eta_k - \eta|$ would imply that $\rho_m \leq |\eta_m - \eta| \leq |\eta_m - \eta_k| + |\eta_k - \eta| < \rho_m$, yielding a contradiction. Recall that $v_m = T_{x_m}(\chi u_{\eta_m, m})$ so that $v_m \in \mathcal{E}'_\Lambda$ and $\mathrm{WF}(v_m) = \{x_m\} \times \mathbb{R}_+^* \eta_m$.

To control the wavefront set, we define partial sums $S_m = \sum_{i=1}^m v_i/i!$, and we show that the cotangent directions of the wavefront set of $v - S_m$ do not meet $(x_i; \eta_i)$ for $i \leq m$. Thus, we have the finite sum $v = (v - S_m) + \sum_{i=1}^m v_i/i!$ and, since the cotangent directions of the wavefront set of the terms do not overlap, there can be no cancellation and all $(x_i; \eta_i)$ belong to the wavefront set for $i \leq m$. Then, we have indeed $(x_m; \eta_m) \subset \mathrm{WF}(v)$ for all m because this procedure can be applied for all values of m.

It remains to show that the wavefront set of $v - S_m$ belongs to a closed conical set Γ_m which does not meet $(x_i; \eta_i)$ for $i \leq m$. We first build these Γ_m as follows: Let $X_m = \cup_{l>m}^\infty \{x_l\} \cup \{x\} \subset \Omega$ and $\gamma_{m,i} = X_m \times \big(\mathbb{R}_+^* \overline{B(\eta_i, \rho_i/4)}\big)$. It is clear that if $n \neq 1$, $\gamma_{m,i} \cap \gamma_{m,j} = \emptyset$ because, for $j > i$, we have $|\eta_i - \eta_j| > \rho_i/2$ and $\rho_j < \rho_i$. Thus, $|\eta_i - \eta_j| > (\rho_i + \rho_j)/4$ and since this expression is symmetric in i and j, it holds for all $i \neq j$. This shows that the balls $\overline{B(\eta_i, \rho_i/4)}$ and $\overline{B(\eta_j, \rho_j/4)}$ do not meet and the result follows. The closed cones $\gamma_{m,i}$ are then used to define $\Gamma_m = \big(\bigcup_{i>m} \gamma_{m,i}\big) \cup (X_m \times \mathbb{R}_+^* \eta)$.

To show that the wavefront set of $v - S_m$ belongs to Γ_m, we prove that the series $\sum_{i=m+1}^\infty v_i/i!$ converges in $\mathcal{D}'_{\Gamma_m}$.

Step 4: Estimates on seminorms of v_m in $\mathcal{D}'_{\Gamma_M}$, $m > M$.

Fix $\psi \in \mathcal{D}(\Omega)$ and any closed cone W such that $\operatorname{supp} \psi \times W \cap \Gamma_M = \emptyset$. For convenience we define the distance $||x - y||_\infty = \sup_{i=1,\ldots,n} |x^i - y^i|$, where

x^i is the ith coordinate of x in a given orthonormal basis. Then, we define the distance between two sets to be $d_\infty(A,B) = \inf_{x\in A, y\in B} ||x-y||_\infty$.

We first consider the case when $X_M \cap \operatorname{supp}\psi = \emptyset$. Then, $v_m\psi$ is smooth, and we want to show that $\{v_m\psi, m\in\mathbb{N}\}$ is bounded in $\mathcal{D}(\Omega)$, since W above can be taken arbitrary. This is equivalent to prove that $\{\chi\psi_{-x_m}u_{\eta_m,m}, m\in\mathbb{N}\}$ is bounded, where $\psi_{-x_m} = T_{-x_m}\psi$. Let $\epsilon = d_\infty(X_M, \operatorname{supp}\psi) > 0$. Since ψ vanishes in a neighborhood of x_m on the ball $B_\infty(x_m,\epsilon)$ with $\epsilon > 0$, we deduce that $\chi\psi_{-x_m}(y)$ vanishes when $||y||_\infty \le \epsilon$. Thus, we can consider that $||y||_\infty/\epsilon \ge 1$.

Then, using the properties of Hörmander's construction, we bound uniformly in m. Fix y and choose y^i such that $|y^i| = ||y||_\infty$. Then,

$$\begin{aligned}|\partial^\alpha\chi\psi_{-x_m}u_{\eta_m,m}(y)| &\le \frac{1}{\epsilon^{C(0,|\alpha|)}}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}|\partial^\beta\chi\psi_{-x_m}|\,|(y^i)^{C(0,|\alpha|)}\partial^{\alpha-\beta}u_{\eta_m,m}|\\ &\le \frac{1}{\epsilon^{C(0,|\alpha|)}}2^{|\alpha|}\pi_{|\alpha|,K_m}(\chi\psi_{-x_m})D(0,|\alpha|),\end{aligned}$$

where $K_m = \operatorname{supp}(\psi_{-x_m})$. To establish Eq. (5) we showed that

$$\pi_{|\alpha|,K_m}(\chi\psi_{-x_m}) \le 2^{|\alpha|}\pi_{|\alpha|,K_m}(\chi)\pi_{|\alpha|,K_m}(\psi_{-x_m}).$$

But $\pi_{|\alpha|,K_m}(\chi) \le \pi_{|\alpha|,\operatorname{supp}\chi}(\chi)$ and $\pi_{|\alpha|,K_m}(\psi_{-x_m}) = \pi_{|\alpha|,\operatorname{supp}\psi}(\psi)$. Thus,

$$|\partial^\alpha\chi\psi_{-x_m}u_{\eta_m,m}(y)| \le \frac{1}{\epsilon^{C(0,|\alpha|)}}2^{2|\alpha|}\pi_{|\alpha|,\operatorname{supp}\chi}(\chi)\pi_{|\alpha|,\operatorname{supp}\psi}(\psi)D(0,|\alpha|)$$

is bounded independently of m.

In the case $X_M\cap\operatorname{supp}\psi \ne \emptyset$, we have $y\in\operatorname{supp}\psi$ for some $y\in X_M$ and $\{y\}\times W\cap\gamma_{M,m} = \emptyset$ for all $m > M$ by our assumption. Thus, $W\cap\mathbb{R}_+^*\overline{B(\eta_m,\rho_m/4)} = \emptyset$ for all $m > M$. Arguing as usual by a compactness argument, one can prove that there is a constant $1 > c > 0$ (independent of m) such that for all k satisfying both $k\in[\mathbb{R}_+^*B(\eta_m,\rho_m/4)]^c$ and $(k-q)\in\mathbb{R}_+^*B(\eta_m,\rho_m/8)$, we have $|q|\ge c\rho_m|k-q|$. We will deduce from this and our previous estimates a bound on:

$$||v_m||_{N,W,\psi} \le \sup_{k\in W}(1+|k|)^N\int_{\mathbb{R}^n}dq|\widehat{u_{\eta_m,m}}(k-q)\widehat{\chi\psi_{-x_m}}(q)| = I_1 + I_2,$$

where I_1 corresponds to the integral over $\Omega_1 = \{q\,;\,\frac{k-q}{|k-q|}\in B(\eta_m,\rho_m/8)\}$ and I_2 over $\Omega_2 = \mathbb{R}^n\backslash\Omega_1$. To estimate I_1, we use $|\widehat{u_{\eta_m,m}}(k-q)| \le 10^m(1+|k-q|)^{-m}$ (see Eq. (10)) and $(1+|k|)^N \le (1+|q|)^N(1+|k-q|)^N$ to obtain

$$I_1 \le 10^m\sup_{k\in W}\int_{\Omega_1}(1+|k-q|)^{N-m}|\widehat{\chi\psi_{-x_m}}(q)|(1+|q|)^N dq.$$

We bound $|\widehat{\chi\psi_{-x_m}}(q)|$ with $||\chi_{x_m}||_{n+1+N,\mathbb{R}^n,\psi}(1+|q|)^{-N-n-1}$. Then, if $N-m\le 0$ we bound $(1+|k-q|)^{N-m}$ with 1 and we obtain $I_1 \le 10^m I_n^{n+1}||\chi_{x_m}||_{n+1+N,\mathbb{R}^n,\psi}$, and if $N-m > 0$, then we bound $(1+|k-q|)^{N-m}$ with $(c\rho_m)^{m-N}(1+|q|)^{N-m}$ and we find

$$I_1 \le 10^m(c\rho_m)^{m-N}I_n^{n+1}||\chi_{x_m}||_{n+1+2N,\mathbb{R}^n,\psi}.$$

To estimate I_2, we start as for I_1 except that we use the first inequality of Eq. (10), where we replace $80^{m/2}$ by 10^m:

$$I_2 \leq 10^m \sup_{k\in W} \int_{\Omega_2} (1-\chi(\langle k-q,\eta_m\rangle))\chi(\frac{|k-q|^2-|\langle k-q,\eta_m\rangle|^2}{|\langle k-q,\eta_m\rangle|^{2\rho}})$$
$$(1+|k-q|)^{N-m}|\widehat{\chi\psi_{-x_m}}(q)|(1+|q|)^N dq.$$

By considering the support of χ, we see that the integrand is zero except (possibly) if (i) $\langle k-q,\eta_m\rangle \geq 1/2$ and (ii) $|\langle k-q,\eta_m\rangle|^{2\rho} \geq |k-q|^2-|\langle k-q,\eta_m\rangle|^2$. Now we show that the three conditions (i), (ii) and $q\notin \Omega_1$ imply $q\in B(k,r_m)$, where $r_m=(\rho_m^2/64-(\rho_m^2/128)^2)^{-1/(2-2\rho)}$. Indeed, $q\notin\Omega_1$ means that $|(k-q)/|k-q|-\eta_m|^2 = 2(|k-q|-\langle k-q,\eta_m\rangle)/|k-q| \geq (\rho_m/8)^2$, so that $|k-q|(1-\rho_m^2/128) \geq \langle k-q,\eta_m\rangle$. This implies with (i) and (ii): $|k-q|^2(1-(1-\rho_m^2/128)^2) \leq |k-q|^{2\rho}(1-\rho_m^2/128)^{2\rho} \leq |k-q|^{2\rho}$. Thus, $r_m^{2\rho-2}\leq |k-q|^{2\rho-2}$ and the result follows because $\rho<1$. By using $0\leq\chi\leq 1$ we find

$$I_2 \leq 10^m \sup_{k\in W} \int_{B(k,r_m)} (1+|k-q|)^{N-m}|\widehat{\chi\psi_{-x_m}}(q)|(1+|q|)^N dq.$$

We proceed now as for I_1 and obtain $I_2\leq 10^m I_n^{n+1}||\chi_{x_m}||_{n+1+N,\mathbb{R}^n,\psi}$ if $N-m\leq 0$ and $I_1 \leq 10^m(1+r_m)^{N-m}I_n^{n+1}||\chi_{x_m}||_{n+1+N,\mathbb{R}^n,\psi}$, if $N-m>0$. We have showed that $||\chi_{x_m}||_{n+1+N,\mathbb{R}^n,\psi}$ can be bounded independently of m. Thus, for $m>N$, there is a constant $C_{n,N}$ such that $||v_m||_{N,W,\psi}\leq 10^m C_{n,N}$. Since the set of $m\leq N$ is finite, we see that $10^{-m}||v_m||_{N,W,\psi}$ is bounded for all values of m.

Thus, we showed that, for any W and ψ such that $\operatorname{supp}\psi\times W\cap\Gamma_M=\emptyset$ and any integer N, the set $\{10^{-m}||v_m||_{N,W,\psi}\,;m>M\}$ is bounded in $\mathbb{R}$. To show that the set $A=\{10^{-m}v_m\,;m>M\}$ is bounded in $\mathcal{D}'_{\Gamma_M}$, we still have to show that it is bounded for the seminorms p_B with B bounded in $\mathcal{D}(\Omega)$. In the course of step 2, we showed that, for any $f\in\mathcal{D}$, the set $p_f(A)$ is bounded in $\mathbb{R}$. This means that A is bounded in $\mathcal{D}'_{\Gamma_M}$ equipped with the Hörmander topology. But we proved that this is equivalent to being bounded for the normal topology. Thus, A is bounded in $\mathcal{D}'_{\Gamma_M}$ with its normal topology.

Step 5: Let $S_m:=\sum_{k=1}^m \frac{1}{k!}v_k$ ($S_0=0$). Then for any $M\geq 0$, the sequence $(S_m-S_M)_{m\geq M}$ is a Cauchy sequence in $\mathcal{D}'_{\Gamma_M}$. As a consequence, S_m-S_M converges to $v-S_M$ in $\mathcal{D}'_{\Gamma_M}$ and $WF(v)\supset\{(x_m;\eta_m), m\in\mathbb{N}^*\}$.

In the previous step we showed that the set $A=\{10^{-m}v_m\,;m>M\}$ is bounded in $\mathcal{D}'_{\Gamma_M}$. Thus, for every seminorm p_i of $\mathcal{D}'_{\Gamma_M}$ and any $p\geq q>M$, we have $p_i(S_p-S_q)\leq C_i\sum_{m=q}^p 10^m/m!$, and each $p_i(S_m-S_M)$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathcal{D}'_{\Gamma_M}$, it implies that S_m-S_M converges to $v-S_M$ in $\mathcal{D}'_{\Gamma_M}$.

Since the wavefront set is known for each v_m ($WF(v_m)=\{x_m\}\times\mathbb{R}_+^*\eta_m$), v_M is the only one among the distributions $v-S_M, v_M, ..., v_1$ which is singular in direction $\mathbb{R}_+^*\eta_M$ at x_M (because $(x_M;\eta_M)\notin\Gamma_M$ by construction either because $x_m\neq x_M$ if $n=1$ or because $\eta_m\neq\eta_M$ if $n\neq 1$, for $m>M$), one deduces $\{x_M\}\times$

$\mathbb{R}_+^*\eta_M \subset WF(v)$. Indeed, by choosing a test function ψ such that $\psi(x_M) \neq 0$ and a closed cone $V \subset \mathbb{R}_+^* B(\eta_M, \rho_M/4)$, we have $\operatorname{supp}\psi \times V \cap \mathrm{WF}(v_m) = \emptyset$ for $m < M$ and $\operatorname{supp}\psi \times V \cap \mathrm{WF}(v - S_M) = \emptyset$. Therefore, $||v - v_M||_{N,V,\psi}$ is finite for all N and $\widehat{\psi(v - v_M)}(\lambda\eta_M)$ cannot compensate for the slow decrease of $\widehat{\psi v_M}(\lambda\eta_M)$, which is ensured by the fact that $\mathrm{WF}(\psi v_M) = \mathrm{WF}(v_M)$ when $\psi(x_M) \neq 0$ [43, p. 121]. Since this is valid for any M, this proves the wavefront set statement.

It remains to show that the sequence is also Cauchy in $\mathcal{E}'_\Lambda$.

Step 6: $S_m := \sum_{k=1}^m \frac{1}{k!} v_k$ is Cauchy in $\mathcal{E}'_\Lambda$ for the strong topology coming from its duality with $\mathcal{D}'_\Gamma$ (and even Mackey-Cauchy for the corresponding von Neumann bornology). Especially, $\mathcal{E}'_\Lambda$ is not sequentially complete (and not even Mackey-complete).

By construction $WF(S_m) \subset \Lambda$. Assume proved the statement about its Cauchy nature, then the last step enables to show that if it were (even weakly) convergent in $\mathcal{E}'_\Lambda$, then the limit would be v (since it would be weakly convergent in $\mathcal{D}'_{\overline{\Lambda}}$ where the limit is v) as a distribution, but since the wavefront set is closed, $(x;\eta) \in WF(v)$ and since $(x;\eta) \notin \Lambda$ this gives a contradiction, implying that S_m is a Cauchy sequence not (weakly) converging in $\mathcal{E}'_\Lambda$.

Thus it remains to show that S_m is Cauchy. Take $B \subset \mathcal{D}'_\Gamma$ bounded, we want to show that $p_B(S_m) = \sum_{k=1}^m p_B(v_k)/k!$ is a Cauchy sequence. First choose $\tilde{\chi} \in D(\Omega)$ which is identically one on the compact set $\cup_{y\in X}\operatorname{supp}\chi_y = X + \operatorname{supp}\chi$ (the sum of two compact sets is compact), where $\chi_y = T_y\chi$. Using lemma 23, since B is bounded in $\mathcal{D}'(\Omega)$, fix $M = M_B$ such that $\sup_{u\in B}\sup_{k\in\mathbb{R}^n}(1+|k|)^{-M}|\widehat{\tilde{\chi}u}(k)| = D < \infty$. Then, for $y \in X$, we bound:

$$\begin{aligned}
\sup_{u\in B} ||u||_{M,\mathbb{R}^n,\chi_y} &= \sup_{u\in B}\sup_{k\in\mathbb{R}^n}(1+|k|)^{-M}|\widehat{\chi_y u}(k)| = \sup_{u\in B}\sup_{k\in\mathbb{R}^n}(1+|k|)^{-M}|\widehat{\chi_y\tilde{\chi}u}(k)| \\
&\leq \sup_{u\in B}\sup_{k\in\mathbb{R}^n}\int_{\mathbb{R}^n} dq(1+|k-q|)^M|\widehat{\chi_y}(k-q)\widehat{\tilde{\chi}u}(q)|(1+|q|)^{-M} \\
&\leq DI_n^{n+1}((1+n)\beta)^{(M+n+1)}\pi_{2(M+n+1),\operatorname{supp}(\chi)}(\chi) = C_B < \infty.
\end{aligned}$$

It now suffices to estimate $p_B(v_m)$ for $m \geq M+n+1$. Thus, using this inequality and (10), we deduce for $m \geq M+n+1$:

$$\begin{aligned}
\sup_{u\in B}|\langle u, v_m\rangle| &= \sup_{u\in B}\frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} dk\chi\widehat{T_{-x_m}u}(k)\widehat{u_{\eta_m,m}}(-k)\right| \\
&\leq C_B 10^m\int_{\mathbb{R}^n} dk(1+|k|)^M(1+|k|)^{-m} \leq C_B 10^m I_n^{n+1}.
\end{aligned}$$

Thus, for $p \geq q \geq M+n+1$, $p_B(S_p - S_q) \leq C_B I_n^{n+1}\sum_{k=q+1}^p \frac{10^k}{k!}$, and $p_B(S_m)$ is Cauchy as we wanted.

More precisely, let us define the following bounded set for the strong topology of $\mathcal{E}'_\Lambda$

$$A' = \{v \in \mathcal{E}'_\Lambda : p_B(v) \leq \max(C_B I_n^{n+1}, \max_{m\leq M_B+n+1}(p_B(v_m)))\ \forall B \text{ bounded in } \mathcal{D}'_\Gamma(\Omega)\}.$$

Note that if $q \leq M_B + n + 1 \leq p$, or $q \leq p \leq M_B + n + 1$, we still have

$$p_B(S_p - S_q) \leq \sum_{k=q+1}^{p} \frac{10^k}{k!} \max(C_B I_n^{n+1}, \max_{m \leq M_B+n+1}(p_B(v_m))).$$

Thus, if $\lambda_{p,q} = \lambda_{q,p} = \sum_{k=q+1}^{p} \frac{10^m}{m!}$ for $p \geq q$, we showed $S_p - S_q \in \lambda_{p,q} A'$ and since $\lambda_{p,q} \underset{p,q\to\infty}{\to} 0$, we even deduce that S_p is Mackey-Cauchy. This concludes.

Step 7: Characterization of closed Λ.

To complete the proof we give some information on the case when Λ is open and closed. A subset of a topological space X is called *clopen* if it is both open and closed in X [5, p. 10]. A topological space X is connected if and only if its only clopen subsets are X and $\emptyset$ [5, p. 10]. Now, if Ω is connected, its cotangent bundle $T^*\Omega$ is connected. If the dimension of Ω is $n > 1$ the set $\dot{T}^*\Omega$, which is $T^*\Omega$ with the zero section removed, is also connected. In that case Λ is clopen if and only if it is either empty (so that $\mathcal{E}'_\Lambda = \mathcal{D}(\Omega)$) or $\dot{T}^*\Omega$ (so that $\mathcal{E}'_\Lambda = \mathcal{E}'(\Omega)$). Since both $\mathcal{D}(\Omega)$ and $\mathcal{E}'(\Omega)$ are complete, our theorem is optimal for connected $\dot{T}^*\Omega$.

Corollary 35 *If Λ is an open cone which is not closed, $\mathcal{E}'_\Lambda$ is not sequentially complete for any topology that is coarser than the normal topology and finer than the weak topology of distributions induced by $\mathcal{D}'(\Omega)$. In particular, the inductive limit of E_ℓ equipped with the Hörmander topology is also not sequentially complete.*

Proof. This result is a consequence of the proof above rather than of the statement. A sequence which is Cauchy for the normal topology remains Cauchy for topologies that are coarser than it, thus our counterexample above is Cauchy for the topologies considered. Therefore, it converges weakly in $\mathcal{D}'(\Omega)$ and we showed that the limit cannot be in $\mathcal{E}'_\Lambda$ so that $\mathcal{E}'_\Lambda$ is not sequentially complete.

Corollary 36 *If Λ is an open cone which is not closed, then $\mathcal{E}'_\Lambda$ is not a regular inductive limit for the inductive topology (which is equivalent to the strong topology) and it is not semi-reflexive. If $(\Gamma')^c = \Lambda$, $\mathcal{D}'_\Gamma$ is neither bornological nor barrelled in its normal topology*

Proof. If $\mathcal{E}'_\Lambda$ were semi-reflexive it would be weakly sequentially complete [46, p. 228]. If the inductive limit were regular, it would be semi-reflexive as explained at the end of section 4.3. Alternatively, one can see that the set of the Cauchy sequence $\{S_m, m \geq 1\}$ we built is bounded in $\mathcal{E}'_\Lambda$ and not in any E_ℓ.

The space $\mathcal{D}'_\Gamma$ is not bornological because the strong dual of a separated bornological space is complete [38, p. 77]. If $\mathcal{D}'_\Gamma$ were barrelled in its normal topology so that, since it is semi-Montel, it would be a Montel space [46, p. 231], then its strong dual $\mathcal{E}'_\Lambda$ would also be a Montel space [46, p. 234] and thus again semi-reflexive. Note that Bourbaki states that a space that is semi-reflexive and semi-barrelled is complete [3, p. IV.60], but this is wrong [4].

Remark that $\mathcal{D}'_\Gamma$ provides a concrete and natural example of a complete nuclear space whose strong dual is not sequentially-complete. Grothendieck constructed other examples by using sophisticated techniques of topological tensor products [33, Ch. II, p. 83 and p. 92] (see also [37, p. 96]).

7. Conclusion

This paper determined the main functional properties of Hörmander's space of distributions $\mathcal{D}'_\Gamma$ and its dual. In view of applications to the causal approach of quantum field theory, we derived simple rules to determine whether a distribution belongs to $\mathcal{D}'_\Gamma$, whether a sequence converges in $\mathcal{D}'_\Gamma$ and whether a subset of $\mathcal{D}'_\Gamma$ is bounded. The properties of $\mathcal{D}'_\Gamma$ can also be useful to other physical applications where the wavefront set played a crucial role [47,23,50,24,82,25,67].

By using the functional properties of $\mathcal{D}'_\Gamma$, the proof of renormalizability of scalar quantum field theory in curved spacetime can be considerably simplified and streamlined with respect to the original derivation given by Brunetti and Fredenhagen [8].

The results of the present paper will be extended in two directions: i) The continuity properties of the main operations with distributions in $\mathcal{D}'_\Gamma$ (tensor product, pull-back, push-forward, multiplication of distributions) [15]; ii) A detailed investigation of the microcausal functionals discussed by Brunetti, Dütsch, Fredenhagen, Rejzner and Ribeiro [7,26,10,9], which are the basis of a new and powerful formulation of quantum field theory. As noticed in ref. [10], the space of microcausal functionals is based on spaces of the type $\mathcal{E}'_\Lambda$ which have very poor completeness properties. This problem can be solved by using the completion of $\mathcal{E}'_\Lambda$, which is, because of the nuclearity of $\mathcal{E}'_\Lambda$, also the bornological dual of $\mathcal{D}'_\Gamma$ [39, p. 140]. The topological and bornological properties of this completion will be discussed in a forthcoming publication by the first author [14].

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