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FUNCTIONAL PROPERTIES OF HÖRMANDER'S SPACE OF DISTRIBUTIONS HAVING A SPECIFIED WAVEFRONT SET

YOANN DABROWSKI AND CHRISTIAN BROUDER

ABSTRACT. The space \mathcal{D}'_{Γ} of distributions having their wavefront sets in a closed cone Γ has become important in physics because of its role in the formulation of quantum field theory in curved space time. In this paper, the topological and bornological properties of \mathcal{D}'_{Γ} and its dual \mathcal{E}'_{Λ} are investigated. It is found that \mathcal{D}'_{Γ} is a nuclear, semi-reflexive and semi-Montel complete normal space of distributions. Its strong dual \mathcal{E}'_{Λ} is a nuclear, barrelled and bornological normal space of distributions which, however, is not even sequentially complete. Concrete rules are given to determine whether a distribution belongs to \mathcal{D}'_{Γ} , whether a sequence converges in \mathcal{D}'_{Γ} and whether a set of distributions is bounded in \mathcal{D}'_{Γ} .

1. INTRODUCTION

Standard quantum field theory uses Feynman diagrams in the momentum space. However, this framework was not able to deal with quantum field theory in curved space time because of the absence of translation invariance in general spacetimes. In 1992, Radzikowski [1, 2] showed the wavefront set of distributions to be a key concept to define quantum fields in curved spacetime. This idea was fully developed into a renormalized scalar field theory in curved spacetimes by Brunetti and Fredenhagen [3], followed by Hollands and Wald [4]. This approach was rapidly extended to deal with Dirac fields [5, 6, 7, 8, 9, 10], gauge fields [11, 12, 13] and even the quantization of gravitation [14].

This tremendous progress was made possible by a complete reformulation of quantum field theory, where the wavefront set of distributions plays a central role, for example to determine the algebra of microcausal functionals, to define a spectral condition for time-ordered products and quantum states and to give a rigorous description of renormalization.

In other words, the natural space where quantum field theory takes place is not the space of distributions $\mathcal{D}'_{}$, but the space \mathcal{D}'_{Γ} of distributions having their wavefront set in a specified closed cone Γ . This space and its simplest properties were described by Hörmander in 1971 [15]. Since \mathcal{D}'_{Γ} is now a crucial tool of quantum field theory, it is important to investigate its topological and functional properties. For example, renormalized time-ordered products are determined as an extension of a distribution to the thin diagonal. Since this extension is defined as the limit of a sequence, we need simple criteria to determine the convergence of a sequence in \mathcal{D}'_{Γ} . The ambiguity of renormalization is determined, among other things, by the way this distribution varies under scaling. The scaled distributions must form a bounded set in \mathcal{D}'_{Γ} (i.e. every seminorm of \mathcal{D}'_{Γ} must be bounded over B). Thus, we need simple tests to know when a set of distributions is bounded. The purpose of this paper is to provide tools to answer these questions in a simple way.

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The wavefront set of distributions plays also a key role in microlocal analysis, to determine whether a distribution can be pulled back, restricted to a submanifold or multiplied by another distribution [16, Chapter 8]. Therefore, the wavefront set has become a standard subject in textbooks of distribution theory and microlocal analysis [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. However, to the best of our knowledge, no detailed study was published on the functional properties of \mathcal{D}'_{Γ} .

Many properties of \mathcal{D}'_{Γ} will be deduced from properties of its dual. Thus, we shall first calculate the dual of \mathcal{D}'_{Γ} , denoted by \mathcal{E}'_{Λ} , which turns out to be the space of compactly supported distributions having their wavefront set included in an *open* cone Λ which is the complement of Γ up to a change of sign. Such a space \mathcal{E}'_{Λ} is used in quantum field theory to define microcausal functionals [12].

We now summarize our main results. Although they are both nuclear and normal spaces of distributions, \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} have very contrasted properties; (i) \mathcal{D}'_{Γ} is semi-reflexive and complete while \mathcal{E}'_{Λ} is not even sequentially complete; (ii) \mathcal{E}'_{Λ} is barrelled, and bornological, while \mathcal{D}'_{Γ} is neither barrelled nor bornological. For applications, the most significant property of \mathcal{D}'_{Γ} is to be semi-Montel. Indeed, two steps involving \mathcal{D}'_{Γ} are particularly important in the renormalization process described by Brunetti and Fredenhagen [3]. The first step is a control of the divergence of the relevant distributions near the diagonal: there must be a real number s such that the family $\{\lambda^{-s}u_{\lambda}\}_{0<\lambda\leq 1}$ is a bounded set of distributions, where u_{λ} is a scaled distribution. This proof is facilitated by our determination of bounded sets:

Proposition 1. A set B of distributions in \mathcal{D}'_{Γ} is bounded if and only if, for every $v \in \mathcal{E}'_{\Lambda}$, there is a constant C_v such that $|\langle u, v \rangle| < C_v$ for all $u \in B$. Such a weakly bounded set is also strongly bounded and equicontinuous. Moreover, the closed bounded sets of \mathcal{D}'_{Γ} are compact, complete and metrizable.

The second step is the proof that the extension of a distribution can be defined as the limit of a sequence of distributions in \mathcal{D}'_{Γ} . For this we derive the following convergence test:

Proposition 2. If u_i is a sequence of elements of \mathcal{D}'_{Γ} such that, for any $v \in \mathcal{E}'_{\Lambda}$, the sequence $\langle u_i, v \rangle$ converges in \mathbb{C} to a number λ_v , then u_i converges to a distribution u in \mathcal{D}'_{Γ} and $\langle u, v \rangle = \lambda_v$ for all $v \in \mathcal{E}'_{\Lambda}$.

We now describe the organization of the paper. After this introduction, we determine a pairing between \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} and we show that this pairing is compatible with duality. Then, we prove that \mathcal{D}'_{Γ} is a normal space of distributions (in the sense of Schwartz). The next section investigates several topologies on \mathcal{E}'_{Λ} and shows their equivalence. Then, the bornological of \mathcal{D}'_{Γ} are discussed. Bornology enables to prove that \mathcal{D}'_{Γ} is complete and it is relevant to the problem of quantum field theory on curved spacetime because some isomorphisms of the space of sections of a vector bundle over a manifold are stronger in the bornological setting than in the topological one (see section 4). These results are put together to determine the main functional properties of \mathcal{D}'_{Γ} and its dual. Finally, a counter-example is constructed to show that \mathcal{E}'_{Λ} is not sequentially complete. This will imply that \mathcal{D}'_{Γ} and its dual do not enjoy all the nice properties of \mathcal{D}' .

2. The dual of \mathcal{D}'_{Γ}

In this section, we review what is known about the topology of \mathcal{D}'_{Γ} and we describe the functional analytic tools (duality pairing and normal spaces of distributions) that enable us to calculate the dual of \mathcal{D}'_{Γ} .

2.1. What is known about \mathcal{D}'_{Γ} . Let us fix the notation. Let Ω be an open set in \mathbb{R}^n , we denote by $T^*\Omega$ the cotangent bundle over Ω , by $UT^*\Omega = \{(x;k) \in T^*\Omega; |k| = 1\}$ (where |k| is the standard Euclidian norm on \mathbb{R}^n) the sphere bundle over Ω and by $\dot{T}^*\Omega = T^*\Omega \setminus \{(x;0); x \in \Omega\}$ the cotangent bundle without the zero section. We say that a subset Γ of $\dot{T}^*\Omega$ is a cone if $(x;\lambda k) \in \Gamma$ whenever $(x;k) \in \Gamma$ and $\lambda > 0$. For any closed cone Γ , Hörmander defined [15, p. 125] the space \mathcal{D}'_{Γ} to be the set of distributions in $\mathcal{D}'(\Omega)$ having their wavefront set in Γ . He also described what he called a *pseudo-topology* on \mathcal{D}'_{Γ} , which means that he defined a concept of convergence in \mathcal{D}'_{Γ} but not a topology (as a family of open sets). His definition was equivalent to the following one [16, p. 262]: a sequence $u_j \in \mathcal{D}'_{\Gamma}$ converges to $u \in \mathcal{D}'_{\Gamma}$ if

- (i) The sequence of numbers $\langle u_j, f \rangle$ converges to $\langle u, f \rangle$ in the ground field \mathbb{K} (i.e. \mathbb{R} or \mathbb{C}) for all $f \in \mathcal{D}(\Omega)$.
- (ii) If V is a closed conical neighborhood in \mathbb{R}^n and χ is an element of $\mathcal{D}(\Omega)$ that satisfy $(\operatorname{supp} \chi \times V) \cap \Gamma = \emptyset$, then $\operatorname{sup}_{k \in V} (1 + |k|)^N |\widehat{u_j \chi}(k) \widehat{u\chi}(k)| \to 0$ for all integers N,

where \hat{v} denotes the Fourier transform of the distribution v. The Fourier transform of a function $f \in \mathcal{D}(\Omega)$ is defined by $\hat{f}(k) = \int_{\Omega} e^{ik \cdot x} f(x) dx$. Hörmander then showed that $\mathcal{D}(\Omega)$ is dense in \mathcal{D}'_{Γ} . More precisely, for every $u \in \mathcal{D}'_{\Gamma}$ there is a sequence of functions $u_j \in \mathcal{D}(\Omega)$ such that u_j converges to u in the above sense [16, p. 262].

This concept of convergence is compatible with different topologies. The topology of \mathcal{D}'_{Γ} used in the literature (see [22, p. 80], which is usually called the Hörmander topology [3, 23]), is that of a locally convex topological vector space defined by the following seminorms:

- (i) $p_f(u) = |\langle u, f \rangle|$ for all $f \in \mathcal{D}(\Omega)$.
- (ii) $||u||_{N,V,\chi} = \sup_{k \in V} (1 + |k|)^N |\widehat{u\chi}(k)| \to 0$, for all integers N, all closed conical neighborhoods V and all $\chi \in \mathcal{D}(\Omega)$ such that $(\operatorname{supp} \chi \times V) \cap \Gamma = \emptyset$.

An equivalent topology can be defined as the initial topology of a set of maps [26], from which it can be deduced that \mathcal{D}'_{Γ} is a nuclear space [12]. We immediately observe that \mathcal{D}'_{Γ} is a Hausdorff locally convex space because u = 0 if $p_i(u) = 0$ for all its seminorms p_i [27, p. 96]. Indeed, if $p_f(u) = |\langle u, f \rangle| = 0$ for all $f \in \mathcal{D}(\Omega)$, then u = 0.

2.2. **Duality pairing.** Mackey's duality theory [28, 29, 30, 31] is a powerful technique to investigate the topological properties of locally convex spaces [32, 27]. The first step of this method is to find a duality pairing between two spaces.

Let us take the example of the duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Any test function $u \in \mathcal{D}(\Omega)$ can be paired to any $f \in \mathcal{D}(\Omega)$ by $\langle u, f \rangle = \int_{\Omega} u(x)f(x)dx$. The density of $\mathcal{D}(\Omega)$ in $\mathcal{D}'(\Omega)$ implies that this pairing can be uniquely extended to a pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$, also denoted by $\langle u, f \rangle$, that can be written

(1)
$$\langle u, f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{u\varphi}(k) \widehat{f}(-k) dk,$$

where the function $\varphi \in \mathcal{D}(\Omega)$ is equal to 1 on a compact neighborhood of the support of f. Indeed, $\langle u, f \rangle = \langle \varphi u, f \rangle$ [33, p. 90] and φu has a Fourier transform because it is a compactly supported distribution [16, p. 165]. This pairing is compatible with duality, in the sense that any element α in the topological dual of $\mathcal{D}(\Omega)$ can be written $\alpha(f) = \langle u, f \rangle$ for one element u of $\mathcal{D}'(\Omega)$, by definition of the space of distributions.

We would like to find a similar pairing between \mathcal{D}'_{Γ} and another space to be determined. Grigis and Sjöstrand [22, p. 80] showed that the pairing $\langle u, v \rangle =$

 $\int_{\Omega} u(x)v(x)dx \text{ between } C^{\infty}(\Omega) \text{ and } \mathcal{D}(\Omega) \text{ extends uniquely to the pairing defined by eq. (1) between } \mathcal{D}'_{\Gamma} \text{ and every space } \mathcal{E}'_{\Xi} \text{ of compactly supported distributions whose wavefront set is contained in } \Xi, where \Xi is any closed cone such that <math>\Gamma' \cap \Xi = \emptyset$, where $\Gamma' = \{(x;k) \in \dot{T}^*\Omega; (x;-k) \in \Gamma\}$ (see also [18, p. 512] for a similar result).

We need to slightly extend their definition by pairing \mathcal{D}'_{Γ} with the space \mathcal{E}'_{Λ} , where Λ is now the *open* cone $\Lambda = (\Gamma')^c$. Note that this space is the union of the ones considered by Grigis and Sjöstrand. The next lemma does not contain more information than their result, but, for the reader's convenience, we still first show that this extended pairing is well defined.

Lemma 3. If Γ is a closed cone in $\dot{T}^*\Omega$ and $\Lambda = (\Gamma')^c = \{(x;k) \in \dot{T}^*\Omega; (x,-k) \notin \Gamma\}$, then the following pairing between \mathcal{D}'_{Γ} and $\mathcal{E}'_{\Lambda} = \{v \in \mathcal{E}'(\Omega); WF(v) \subset \Lambda\}$ is well defined:

$$\langle u, v \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{u\varphi}(k) \hat{v}(-k) dk$$

where $u \in \mathcal{D}'_{\Gamma}$, $v \in \mathcal{E}'_{\Lambda}$ and φ is any function in $\mathcal{D}(\Omega)$ equal to 1 on a compact neighborhood of the support of v. This pairing is separating and, for any $v \in \mathcal{E}'_{\Lambda}$, the map $\lambda : \mathcal{D}'_{\Gamma} \to \mathbb{K}$ defined by $\lambda(u) = \langle u, v \rangle$ is continuous.

Proof. We first consider the case where Γ is neither empty nor $\dot{T}^*\Omega$. A distribution $v \in \mathcal{E}'_{\Lambda}$ is compactly supported and its wavefront set is a closed cone contained in Λ , which implies WF $(v) \cap \Gamma' = \emptyset$. The product of distributions uv is then a well-defined distribution by Hörmander's theorem [16, p. 267]. We estimate now $\langle u, v \rangle = (2\pi)^{-n} \int \widehat{u\varphi}(k) \hat{v}(-k) dk$.

By a classical construction [24, p. 61], there is a finite set of non-negative smooth functions ψ_j such that $\sum_j \psi_j^2 = 1$ on a compact neighborhood K of the support of v and there are closed conical neighborhoods V_{uj} and V_{vj} that satisfy the three conditions: (i) $V_{uj} \cap (-V_{vj}) = \emptyset$, (ii) $\sup \psi_j \times V_{uj}^c \cap \Gamma = \emptyset$ and (iii) $\sup \psi_j \times V_{vj}^c \cap WF(v) = \emptyset$. As a consequence of these conditions, we have $\Gamma|_K \subset \bigcup_j (\sup \psi_j \times V_{uj})$ and $WF(v) \subset \bigcup_j (\sup \psi_j \times V_{vj})$. If we choose $\varphi = \sum_j \psi_j^2$ we can write $\langle u, v \rangle = \sum_j I_j$, where $I_j = (2\pi)^{-n} \int \widehat{u\psi_j}(k)\widehat{v\psi_j}(-k)dk$.

Following again Eskin [24, p. 62], we can define homogeneous functions of degree zero α_j and β_j on $\dot{T}^*\Omega$, which are smooth except at the origin, measurable, nonnegative and bounded by 1 on \mathbb{R}^n and such that $\operatorname{supp} \alpha_j$ and $\operatorname{supp} \beta_j$ are closed conical neighborhoods satisfying the three conditions (i), (ii) and (iii) stated above, with $\alpha_j = 1$ on V_{uj} and $\beta_j = 1$ on V_{vj} . Then we insert $1 = (\alpha_j + (1 - \alpha_j))(\beta_j + (1 - \beta_j))$ in the integral defining I_j and we obtain $I_j = I_{1j} + I_{2j} + I_{3j} + I_{4j}$, where

$$I_{1j} = (2\pi)^{-n} \int_{\mathbb{R}^n} \alpha_j(-k)\widehat{\psi_j u}(-k) \beta_j(k)\widehat{\psi_j v}(k) dk,$$

$$I_{2j} = (2\pi)^{-n} \int_{\mathbb{R}^n} \alpha_j(-k)\widehat{\psi_j u}(-k) \left(1 - \beta_j(k)\right)\widehat{\psi_j v}(k) dk,$$

$$I_{3j} = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(1 - \alpha_j(-k)\right)\widehat{\psi_j u}(-k) \beta_j(k)\widehat{\psi_j v}(k) dk,$$

$$I_{4j} = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(1 - \alpha_j(-k)\right)\widehat{\psi_j u}(-k) \left(1 - \beta_j(k)\right)\widehat{\psi_j v}(k) dk.$$

We first notice that $I_{1j} = 0$ because $(-\operatorname{supp} \alpha_j) \cap \operatorname{supp} \beta_j = \emptyset$. We estimate I_{4j} . The function β_j was built so that $(1 - \beta_j) = 0$ on V_{vj} and $\operatorname{supp} \psi_j \times \operatorname{supp} (1 - \beta_j) \cap WF(v) = \emptyset$. Then, for any integer N,

$$\left| \left(1 - \beta_j(k) \right) \widehat{\psi_j v}(k) \right| \leq ||v||_{N, \operatorname{supp} \beta'_j, \psi_j} (1 + |k|)^{-N},$$

where $\beta'_j = 1 - \beta_j$. Similarly

(2)
$$\left| \left(1 - \alpha_j(k) \right) \widehat{\psi_j u}(k) \right| \leq \left| |u| \right|_{M, \operatorname{supp} \alpha'_j, \psi_j} (1 + |k|)^{-M}$$

where $\alpha'_j = 1 - \alpha_j$. Thus, for N + M > n,

$$|I_{4j}| \leq ||u||_{M,\operatorname{supp}\alpha'_j,\psi_j}||v||_{N,\operatorname{supp}\beta'_j,\psi_j}I_n^{N+M},$$

where $I_n^N = \int_{\mathbb{R}^n} (1+|k|)^{-N} dk$. For I_{3j} we use the fact that, $\psi_j v$ being a compactly supported distribution, there is an integer m and a constant C such that $|\psi_i v(k)| \leq C(1+|k|)^m$ [16, p. 181]. When this estimate is combined with eq. (2) we obtain for M > n + m,

$$|I_{3j}| \leq ||u||_{M, \operatorname{supp} \alpha'_{j}, \psi_{j}} C I_{n}^{M-m}$$

For the integral I_{2j} we proceed differently because we want to recover a seminorm of \mathcal{D}'_{Γ} . If we define $\hat{f}_j(k) = \alpha_j(-k)(1-\beta_j(k))\widehat{\psi}_j v(k)$, then $I_{2j} = (2\pi)^{-n} \int \widehat{\psi}_j u(-k)\widehat{f}_j(k)dk$. We call fast decreasing a function f(k) such that, for every integer N, $|f(k)| \leq N$ $C_N(1+|k|)^{-N}$ for some constant C_N . Note that our fast decreasing functions are different from Schwartz rapidly decreasing functions. The function $f_j(k)$ is fast decreasing because α_i and β_i are bounded by 1, $\psi_i v(k)$ is fast decreasing outside the wavefront set of v and $(1 - \beta_j(k))$ cancels $\psi_j v(k)$ on this wavefront set. The function f_i is also measurable because it is the product of measurable functions. Thus, by a standard result in the spirit of [20, p. 145], its inverse Fourier transform f_i exists and is smooth. We can now rewrite $I_{2j} = \langle \psi_j u, f_j \rangle = \langle u, \psi_j f_j \rangle$, which is well defined because $\psi_j f_j$ is smooth and compactly supported. Finally $|I_{2j}| \leq p_{\psi_j f_j}(u)$, where $p_{\psi_j f_j}(u) = |\langle u, \psi_j f_j \rangle|$, and we obtain

(3)
$$\begin{aligned} |\langle u, v \rangle| &\leq \sum_{j} \left(p_{\psi_j f_j}(u) + ||u||_{M, \operatorname{supp} \alpha'_j, \psi_j} C I_n^{M-m} \right. \\ &+ ||u||_{M, \operatorname{supp} \alpha'_j, \psi_j} ||v||_{N, \operatorname{supp} \beta'_j, \psi_j} I_n^{N+M} \end{aligned} \right). \end{aligned}$$

Thus, $\langle u, v \rangle$ is well defined because all the terms in the right hand side are finite and the sum is over a finite number of j. Note that $p_{\psi_j f_j}(u)$ and $||u||_{M, \text{supp } \alpha'_j, \psi_j}$ are seminorms of \mathcal{D}'_{Γ} because $\psi_j f_j \in \mathcal{D}(\Omega)$ and, by construction, $\operatorname{supp} \alpha'_j$ is a cone and $\operatorname{supp} \psi_j \times \operatorname{supp} \alpha'_i \cap \Gamma = \emptyset$.

Equation (4) obviously shows that $\lambda : u \mapsto \langle u, v \rangle$ is continuous.

The second case is $\Gamma = \dot{T}^*\Omega$ and $\Lambda = \emptyset$, so that $\mathcal{D}'_{\Gamma} = \mathcal{D}'(\Omega)$ and $\mathcal{E}'_{\Lambda} = \mathcal{D}(\Omega)$. The seminorm $|\langle u, v \rangle| = p_v(u)$ is then a seminorm of \mathcal{D}'_{Γ} since $v \in \mathcal{D}(\Omega)$. The last case is when $\Gamma = \emptyset$ and $\Lambda = \dot{T}^* \Omega$, so that $\mathcal{D}'_{\Gamma} = C^{\infty}(\Omega)$ and $\mathcal{E}'_{\Lambda} = \mathcal{E}'(\Omega)$. If we use the fact that the usual topology of $C^{\infty}(\Omega)$ is equivalent with the topology defined by $\|\cdot\|_{N,V,\chi}$ for all closed conical neighborhoods V and all $\chi \in \mathcal{D}(\Omega)$ [34], then we recover the fact that the elements of $\mathcal{E}'(\Omega)$ are continuous maps from $C^{\infty}(\Omega)$ to K [33, p. 89].

Finally, the pairing is separating because, if $\langle u, v \rangle = 0$ for all $v \in \mathcal{E}'_{\Lambda}$, then $\langle u, f \rangle = 0$ for all $f \in \mathcal{D}(\Omega)$ because $\mathcal{D}(\Omega) \subset \mathcal{E}'_{\Lambda}$ and a distribution u which is zero on $\mathcal{D}(\Omega)$ is the zero distribution. Similarly, v = 0 if $\langle u, v \rangle = 0$ for all $v \in \mathcal{E}'_{\Lambda}$ because $\mathcal{D}(\Omega) \subset \mathcal{E}'_{\Lambda}.$

To simplify the discussion, we used Eskin's α_j and β_j functions to build maps from $v \in \mathcal{E}'_{\Lambda}$ to $f_j \in C^{\infty}(\Omega)$. This can be improved by defining maps from \mathcal{E}'_{Λ} to the Schwartz space \mathcal{S} of rapidly decreasing functions. The idea is to use a smooth positive function h bounded by 1, equal to 1 outside the unit ball and to 0 in a neighborhood of the origin of \mathbb{R}^n . If, furthermore, the functions α_j and β_j

have polynomially bounded derivatives, then the functions $\hat{g}_j(k) = h(k)\alpha_j(-k)(1 - \beta_j(k))\widehat{\psi_j v}(k)$ are in \mathcal{S} as well as their Fourier transforms.

2.3. Normal space of distributions. The usual spaces of distribution theory (e.g. $\mathcal{D}, \mathcal{S}, C^{\infty}, \mathcal{D}', \mathcal{S}', \mathcal{E}'$), are normal spaces of distributions [35, p. 10], which enjoy useful properties with respect to duality. They are defined as follows:

Definition 4. A Hausdorff locally convex space E is said to be a normal space of distributions if there are continuous injective linear maps $i : \mathcal{D}(\Omega) \to E$ and $j : E \to \mathcal{D}'(\Omega)$, where $\mathcal{D}'(\Omega)$ is equipped with its strong topology, such that: (i) The image of i is dense in E, (ii) for any f and g in $\mathcal{D}(\Omega) \langle j \circ i(f), g \rangle = \int_{\Omega} f(x)g(x)dx$ [27, p. 319].

To transform \mathcal{D}'_{Γ} into a normal space of distributions we need to make its topology finer. In the case of \mathcal{D}'_{Γ} condition (ii) is obviously satisfied because the injections i and j are the identity. The fact that j is a continuous injection means that the topology of \mathcal{D}'_{Γ} must be finer than the topology induced on it by the strong topology of $\mathcal{D}'(\Omega)$ [36, p. 302]. Therefore, we now equip \mathcal{D}'_{Γ} with the topology defined by the seminorms $p_B(u) = \sup_{f \in B} |\langle u, f \rangle|$ of uniform convergence on the bounded sets B of $\mathcal{D}(\Omega)$ (instead of only the seminorms $p_f = |\langle u, f \rangle|$) and we keep the seminorms $||u||_{N,V,\chi}$ defined in section 2.1. Since p_B are the seminorms of $\mathcal{D}'(\Omega)$, \mathcal{D}'_{Γ} has more seminorms than $\mathcal{D}'(\Omega)$ [27, p. 98]. We call this topology the *normal topology* of \mathcal{D}'_{Γ} , while the usual topology will be called the *Hörmander topology* of \mathcal{D}'_{Γ} . Note that \mathcal{D}'_{Γ} is Hausdorff for the normal topology because it is Hausdorff for the coarser Hörmander topology. It remains to show that

Lemma 5. The injection of $\mathcal{D}(\Omega)$ in \mathcal{D}'_{Γ} is continuous.

Proof. We have to prove that the identity map $\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}'_{\Gamma}$ is continuous. Because of the inductive limit topology of $\mathcal{D}(\Omega)$, we must show that, for any compact subset K of Ω , the map $\mathcal{D}(K) \hookrightarrow \mathcal{D}'_{\Gamma}$ is continuous for the topology of $\mathcal{D}(K)$ [33, p. 66]. Recall that $\mathcal{D}(K)$ is the set of elements of $\mathcal{D}(\Omega)$ whose support is contained in K. Its topology is defined by the seminorms $\pi_{m,K}(f) = \sup_{|\alpha| \le m} \sup_{x \in K} |\partial^{\alpha} f(x)|$.

Continuity is proved by showing that all the seminorms of \mathcal{D}'_{Γ} are bounded by seminorms of $\mathcal{D}(K)$ [27, p. 98]. Let *B* be a bounded set of $\mathcal{D}(\Omega)$ and $p_B(f) = \sup_{g \in B} |\langle f, g \rangle|$ with $\langle f, g \rangle = \int_K f(x)g(x)dx$. The function f(x) is bounded by $\pi_{0,K}(f)$, all the g(x) in *B* are bounded by a common number M_0 because *B* is bounded [33, p. 69]. Thus, $p_B(f) \leq |K| M_0 \pi_{0,K}(f)$, where |K| is the volume of *K*.

We still must estimate the seminorms $||f||_{N,V,\chi} = \sup_{k \in V} (1+|k|)^N |\widehat{f\chi}(k)|$. By using $(1+|k|) \leq \beta(1+|k|^2)$, with $\beta = (1+\sqrt{2})/2$, we find

$$\begin{aligned} (1+|k|)^{N}|\widehat{f\chi}(k)| &\leq \beta^{N} \Big| (1+|k|^{2})^{N} \int e^{ik \cdot x} f(x)\chi(x)dx \\ &\leq \beta^{N} \Big| \int e^{ik \cdot x} (1-\Delta)^{N} (f\chi)(x)dx \Big| \end{aligned}$$

We expand $(1 - \Delta)^N = \sum_{i=0}^N {N \choose i} (-\Delta)^i$ and we estimate each $|\Delta^i(f\chi)(x)| \leq n^i \pi_{2N,K}(f\chi)$. This gives us $(1 + |k|)^N |\widehat{f\chi}(k)| \leq ((1 + n)\beta)^N |K| \pi_{2N,K}(f\chi)$. To calculate $\pi_{2N,K}(f\chi)$ we notice that, for any multi-index α such that $|\alpha| \leq m$, we have

$$\begin{aligned} |\partial^{\alpha}(f\chi)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^{\beta} f| |\partial^{\alpha-\beta} \chi| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \pi_{m,K}(f) \pi_{m,K}(\chi) \\ &\leq 2^{m} \pi_{m,K}(f) \pi_{m,K}(\chi). \end{aligned}$$

Thus,

(4)
$$(1+|k|)^N |\widehat{f\chi}(k)| \leq (4(n+1)\beta)^N |K| \pi_{2N,K}(\chi) \pi_{2N,K}(f),$$

with a bound independent of k and $||f||_{N,V,\chi} \leq C\pi_{2N,K}(f)$, where $C = (4(n + 1)\beta)^N |K|\pi_{2N,K}(\chi)$. The proof that the identity is continuous is complete.

It is now clear that \mathcal{D}'_{Γ} with its normal topology is a normal space of distribution because $\mathcal{D}(\Omega)$ is dense in \mathcal{D}'_{Γ} (since sequential convergence for the weak and strong topologies of $\mathcal{D}'(\Omega)$ are equivalent [33, p. 70] and from Hörmander's density result [16, p. 262]). From the general properties of normal spaces of distributions we obtain:

Proposition 6. If we (temporarily) denote by \mathcal{D}_{Γ} the dual of \mathcal{D}'_{Γ} , then

- (i) The restriction map induces an injection $\mathcal{D}_{\Gamma} \hookrightarrow \mathcal{D}'(\Omega)$ [27, p. 259]
- (ii) If \mathcal{D}_{Γ} is equipped with the strong topology $\beta(\mathcal{D}_{\Gamma}, \mathcal{D}'_{\Gamma})$, then the injection $\mathcal{D}_{\Gamma} \hookrightarrow \mathcal{D}'(\Omega)$ is continuous [27, p. 259]
- (iii) If \mathcal{D}_{Γ} is equipped with the Arens topology [37] $\kappa(\mathcal{D}_{\Gamma}, \mathcal{D}'_{\Gamma})$ of the uniform convergence on the convex compact sets for the normal topology of \mathcal{D}'_{Γ} , then \mathcal{D}_{Γ} is a normal space of distributions [27, p. 259] and the dual of \mathcal{D}_{Γ} is \mathcal{D}'_{Γ} [27, p. 235]
- (iv) A distribution $v \in \mathcal{D}'(\Omega)$ belongs to \mathcal{D}_{Γ} if and only if it is continuous on $\mathcal{D}(\Omega)$ for the topology induced by \mathcal{D}'_{Γ} [27, p. 319]
- (v) D(Ω) is dense in D_Γ equipped with any topology compatible with duality [35, p. 10]

We are now ready to prove

Proposition 7. The dual of \mathcal{D}'_{Γ} for its normal topology is \mathcal{E}'_{Λ} .

Proof. We already proved that $\mathcal{E}'_{\Lambda} \hookrightarrow \mathcal{D}_{\Gamma}$ because, by lemma 3, any $v \in \mathcal{E}'_{\Lambda}$ defines a continuous map $\mathcal{D}'_{\Gamma} \to \mathbb{K}$ (for the Hörmander and thus for the normal topology) and the injectivity is obvious by density of $\mathcal{D}(\Omega)$ in \mathcal{D}'_{Γ} . It remains to show that any continuous linear map $\lambda : \mathcal{D}'_{\Gamma} \to \mathbb{K}$ defines a distribution in \mathcal{E}'_{Λ} . By item (i) of proposition 6, we know that λ is a distribution. We first show that this distribution is compactly supported, then that its wavefront set is included in Λ .

Since the map λ is continuous for the normal topology of \mathcal{D}'_{Γ} , there exists a finite number of seminorms p_i and a constant M such that $|\lambda(u)| \leq M \sup_i p_i(u)$ for all uin \mathcal{D}'_{Γ} [27, p. 98]. In other words, there is a bounded set B in $\mathcal{D}(\Omega)$ (one is enough because $\sup_i p_{B_i} \leq p_B$ where $B = \bigcup_i B_i$), and there are r integers N_i , r functions χ_i in $\mathcal{D}(\Omega)$ and r closed cones V_i such that $\operatorname{supp} \chi_i \times V_i \cap \Gamma = \emptyset$ and

$$|\lambda(u)| \le M \sup(p_B(u), ||u||_{N_1, V_1, \chi_1}, \dots, ||u||_{N_r, V_r, \chi_r}).$$

We first show that λ is a compactly supported distribution. Indeed, B is a bounded set of $\mathcal{D}(\Omega)$ if and only if there is a compact subset K of Ω and constants M_m such that all $g \in B$ are supported on K and $\pi_{m,K}(g) \leq M_m$ [33, p. 68]. According to the definition of the support of a distribution [16, p. 42], $\langle u, g \rangle = 0$ if $\sup u \cap \sup g = \emptyset$. Thus $p_B(u) = \sup_{g \in B} |\langle u, g \rangle| = 0$ if $\sup u \cap K = \emptyset$. Similarly, $||u||_{N_i, V_i, \chi_i} = 0$ if $\sup u \cap \sup \chi_i = \emptyset$. Finally, for any $f \in \mathcal{D}(\Omega)$ whose support does not meet $K_{\lambda} = \bigcup_i \operatorname{supp} \chi_i \cup K$, we have $|\lambda(f)| = 0$. This implies that the support of λ is included in the compact set K_{λ} [16, p. 42].

Then we show that $WF(\lambda) \subset \Lambda_M = \bigcup_{i=1}^M \operatorname{supp} \chi_i \times (-V_i)$. We consider an integer N, a smooth function ψ and a closed cone W such that $\operatorname{supp} \chi \times W \cap \Lambda_M = \emptyset$. We now fix such W, χ and we define $f_k = (1 + |k|)^N \psi e_k$, where $e_k(x) = e^{ik \cdot x}$. Hence,

$$||\lambda||_{N,W,\psi} = \sup_{k \in W} (1+|k|)^N |\lambda \psi(k)| = \sup_{k \in W} |\lambda(f_k)|,$$

where we used the fact that the Fourier transform of the compactly supported distribution $\lambda \psi$ is $\lambda(\psi e_k)$ [16, p. 165]. Since, by continuity, $|\lambda(f_k)| \leq M \sup_i p_i(f_k)$, it suffices to bound $\sup_{k \in W} p_i(f_k)$.

We first estimate $p_B(f_k)$. Since B is a bounded set in $\mathcal{D}(\Omega)$, the support of all $g \in B$ is contained in a common compact set K [33, p. 88] and

$$\begin{aligned} |\langle f_k, g \rangle| &= (1+|k|)^N |\langle \psi e_k, g \rangle| = (1+|k|)^N |\widehat{\psi}g(k)| \\ &\leq (4(n+1)\beta)^N |K| \pi_{2N,K}(g) \pi_{2N,K}(\psi), \end{aligned}$$

where we used eq. (4). Moreover, all the seminorms of elements of B are bounded [33, p. 88]. Thus, there is a number M_{2N} such that $\pi_{2N,K}(g) \leq M_{2N}$ for all $g \in B$ and we obtain $|\langle f_k, g \rangle| \leq (8\beta)^N |K| \pi_{2N,K}(\psi) M_{2N}$. Since this bound is independent of k, we obtain our first bound $\sup_{k \in \mathbb{R}^n} p_B(f_k) < \infty$.

Consider now the second type of seminorms and calculate $p_i(f_k) = ||f_k||_{N_i, V_i, \chi_i}$. We have two cases:

(i) If $(\operatorname{supp} \psi \cap \operatorname{supp} \chi_i) = \emptyset$, then $\operatorname{sup}_{k \in \mathbb{R}^n} p_i(f_k) = 0$ and we are done.

(ii) If supp $\psi \cap$ supp $\chi_i \neq \emptyset$, we want to estimate

$$||f_k||_{N_i,V_i,\chi_i} = \sup_{q \in V_i} (1+|q|)^{N_i} |\widehat{f_k \chi_i}(q)| = \sup_{q \in V_i} (1+|q|)^{N_i} (1+|k|)^N |\widehat{e_k \psi \chi_i}(q)|.$$

We have $\widehat{e_k\psi\chi_i}(q) = \langle e_k e_q, \psi\chi_i \rangle = \widehat{\psi\chi_i}(k+q)$. Since we chose W such that $(-V_i) \cap W = \emptyset$, by compactness of the intersection of V_i and W with the unit sphere, there is a $1 \ge c > 0$ such that |k-q|/|k| > c, |q-k|/|q| > c for all $k \in W, q \in -V_i$. We thus deduce:

$$||f_k||_{N_i, V_i, \chi_i} \leq c^{-N-N_i} \sup_{q \in V_i} (1+|k+q|)^{N+N_i} \widehat{\psi\chi_i}(k+q)$$

The function $\psi \chi_i$ is smooth and compactly supported. We can use eq. (4) again to show that the right hand side of this inequality is bounded uniformly in k.

This concludes the proof of WF(λ) $\subset \Lambda_M$. Finally, supp $\chi_i \times V_i \cap \Gamma = \emptyset$ implies supp $\chi_i \times (-V_i) \subset \Lambda$ and $\Lambda_M \subset \Lambda$. Thus, WF(λ) $\subset \Lambda$ and since λ is compactly supported we have $\lambda \in \mathcal{E}'_{\Lambda}$.

In the following, we shall use \mathcal{E}'_{Λ} (instead of \mathcal{D}_{Γ}) to denote the dual of \mathcal{D}'_{Γ} . Note that a similar proof shows that \mathcal{E}'_{Λ} is the topological dual of \mathcal{D}'_{Γ} equipped with the Hörmander topology. Indeed, lemma 3 shows in fact that the pairing is continuous for the Hörmander topology because $p_{\psi_i f_i}$ in Eq. (4) is a seminorm of the weak topology of $\mathcal{D}'(\Omega)$, and the proof of the reverse inclusion just requires to replace p_B by p_f .

3. Topologies on \mathcal{E}'_{Λ}

Our purpose in this section is to show that, if $(\mathcal{E}'_{\Lambda}, \beta)$ denotes the space \mathcal{E}'_{Λ} equipped with the strong $\beta(\mathcal{E}'_{\Lambda}, \mathcal{D}'_{\Gamma})$ topology, then the topological dual of $(\mathcal{E}'_{\Lambda}, \beta)$ is \mathcal{D}'_{Γ} . This implies immediately that \mathcal{D}'_{Γ} is semi-reflexive and \mathcal{E}'_{Λ} is barrelled. However, we shall not work directly with the strong topology $\beta(\mathcal{E}'_{\Lambda}, \mathcal{D}'_{\Gamma})$. It will be convenient (especially to show that \mathcal{E}'_{Λ} is nuclear and \mathcal{D}'_{Γ} is complete) to define a topology on \mathcal{E}'_{Λ} as an inductive limit. Then, we prove that the inductive topology is compatible with duality and we conclude by showing that this inductive topology is equivalent to the strong topology.

3.1. Inductive limit topology on \mathcal{E}'_{Λ} . We want to define a topology on \mathcal{E}'_{Λ} as the topological inductive limit of some topological spaces E_{ℓ} . We shall first determine the vector spaces E_{ℓ} , then we equip them with a topology.

Let us express \mathcal{E}'_{Λ} as the union of increasing spaces E_{ℓ} . Inspired by the work of Brunetti and coll. [26], we take E_{ℓ} to be a set of distributions whose wavefront

set is contained in some closed cone, that we denote by Λ_{ℓ} . To determine Λ_{ℓ} we notice that Λ is an open set and the projection π_i of a product space into each of its coordinate spaces is open [38, p. 90]. Thus, $\pi_1(\Lambda)$ is an open subset of Ω . On the other hand, the singular support of $v \in \mathcal{E}'_{\Lambda}$ (i.e. $\Sigma(v) = \pi_1(WF(v))$ [16, p. 254]) is closed [20, p. 108]. It is even compact because it is a closed subset of the support of v, which is compact. Hence, if we exhaust $\pi_1(\Lambda)$ by an increasing sequence of compact sets K_{ℓ} we know that, for any $v \in \mathcal{E}'_{\Lambda}$, $\Sigma(v)$ will be contained in K_{ℓ} for ℓ large enough (because $\Sigma(v) \subset \pi_1(\Lambda)$ implies that the distance between the compact set $\Sigma(v)$ and the closed set $\pi_1(\Lambda)^c$ is strictly positive). Let us define K_{ℓ} to be the set of points that are at a distance smaller than ℓ from the origin and at a distance larger than $1/\ell$ from the boundary of Ω and from the boundary of $\pi_1(\Lambda)$: $K_\ell = \{x \in \Omega; |x| \le \ell, d(x, \Omega^c) \ge 1/\ell, d(x, \partial \pi_1(\Lambda)) \ge 1/\ell\}$, where $\partial \pi_1(\Lambda)$ is the boundary of $\pi_1(\Lambda)$ and $d(x, A) = \inf\{|x - y|, y \in A\}$ is the distance between a point x and a subset A of Ω . If A is empty, we consider that $d(x, A) = +\infty$. The sets K_{ℓ} are obviously compact (they are intersections of closed sets with a compact ball), $K_{\ell} \subset K_{\ell+1}$ and $\pi_1(\Lambda) = \bigcup_{\ell=1}^{\infty} K_{\ell}$. Indeed, Ω^c is closed because Ω is open and $\partial \pi_1(\Lambda)$ is a closed set disjoint from $\pi_1(\Lambda)$ because $\pi_1(\Lambda)$ is open [38, p. 46]. Thus, any point of $\pi_1(\Lambda)$ is at a finite distance ϵ_1 from Ω^c , ϵ_2 from $\partial \pi_1(\Lambda)$ and M from zero. Then $x \in K_{\ell}$ for all integers ℓ greater than $1/\epsilon_1$, $1/\epsilon_2$ and M.

We can now build the closed cones Λ_{ℓ} , that will be subsets of $\pi_1^{-1}(K_{\ell})$ at a finite distance from Γ' : $\Lambda_{\ell} = \{(x;k) \in \dot{T}^*\Omega; x \in K_{\ell}, d((x;k/|k|), \Gamma') \geq 1/\ell\}$. This set is clearly a cone because it is defined in terms of k/|k| and it is closed in $\dot{T}^*\Omega$ because it is the intersection of two close sets: $\pi_1^{-1}(K_{\ell})$ and $\{(x;k) \in \dot{T}^*\Omega; d((x;k/|k|), \Gamma') \geq 1/\ell\}$. The first set is closed because K_{ℓ} is compact and π_1 is continuous and the second set is closed because the function $(x;k) \mapsto d((x;k/|k|), \Gamma')$ is continuous on $\dot{T}^*\Omega$.

For some proofs, it will be useful for the support of the distributions to be contained in a fixed compact set. Therefore, we also consider an increasing sequence of compact sets $\{L_\ell\}_{\ell\in\mathbb{N}}$ exhausting Ω and such that L_ℓ is a compact neighborhood of $K_\ell \cup L_{\ell-1}$ ($L_0 = \emptyset$). Finally, we define $E_\ell = \mathcal{E}'_{\Lambda_\ell}(L_\ell)$ to be the set of distributions in $\mathcal{E}'(\Omega)$ whose support is contained in L_ℓ and whose wavefront set is contained in Λ_ℓ . Note that E_ℓ will be equipped with the topology induced by $\mathcal{D}'_{\Lambda_\ell}$ as a closed subset.

This is an increasing sequence of spaces exhausting \mathcal{E}'_{Λ} . It is increasing because $L_{\ell} \subset L_{\ell+1}$ and $\Lambda_{\ell} \subset \Lambda_{\ell+1}$ imply $\mathcal{E}'_{\Lambda_{\ell}}(L_{\ell}) \subset \mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$. To show that it is exhausting, consider any $v \in \mathcal{E}'_{\Lambda}$. Since the support of v is compact, it is contained in some L_{ℓ_0} and then in L_{ℓ} for all $\ell \geq \ell_0$. To show that $WF(v) \subset \Lambda_{\ell_1}$ for some ℓ_1 , consider the set $S_v = \{(x;k); |k| = 1 \text{ and } (x;k) \in WF(v)\}$. It is compact because it is closed and bounded (the support of v being compact). Since $WF(v) \subset \Lambda$ and $\Lambda \cap \Gamma' = \emptyset$, we have $S_v \cap \Gamma' = \emptyset$. There is a number $\delta > 0$ such that $d((x;k), \Gamma') > \delta$ for all $(x;k) \in S_v$ because S_v is compact and Γ' is closed. Thus, $S_v \subset \Lambda_{\ell}$ for $\ell > 1/\delta$. Since both S_v and Λ_{ℓ} are cones we have $WF(v) \subset \Lambda_{\ell}$. Finally, $v \in E_{\ell}$ for all ℓ larger than ℓ_0 and $1/\delta$.

We obtained the first part of

Lemma 8. If Λ is an open cone in $\dot{T}^*\Omega$, then

$$\mathcal{E}'_{\Lambda} = \bigcup_{\ell=1}^{\infty} E_{\ell},$$

where $E_{\ell} = \mathcal{E}'_{\Lambda_{\ell}}(L_{\ell})$ is the set of distributions in $\mathcal{E}'(\Omega)$ with a wavefront set contained in Λ_{ℓ} and a support contained in L_{ℓ} . If E_{ℓ} is equipped with the topology induced by $\mathcal{D}'_{\Lambda_{\ell}}$ (with its normal topology) we define on \mathcal{E}'_{Λ} the topological inductive limit

$$\mathcal{E}'_{\Lambda} = \lim E_{\ell}$$

This topology will be called the inductive topology on \mathcal{E}'_{Λ} .

Proof. The inductive limit of E_{ℓ} defines a topology on \mathcal{E}'_{Λ} iff the injections $E_{\ell} \hookrightarrow E_{\ell+1}$ are continuous [39, p. 221]. Since $E_{\ell} \subset \mathcal{D}'_{\Lambda_{\ell}}$, we can equip E_{ℓ} with the topology induced by $\mathcal{D}'_{\Lambda_{\ell}}$, which is defined by the seminorms $p_B(v)$ for all bounded sets B of $\mathcal{D}(\Omega)$ and $|| \cdot ||_{N,V\chi}$, where $\operatorname{supp} \chi \times V \cap \Lambda_{\ell} = \emptyset$. We prove that $E_{\ell} \hookrightarrow E_{\ell+1}$ is continuous by showing that E_{ℓ} has more seminorms than $E_{\ell+1}$: we have $\Lambda_{\ell} \subset \Lambda_{\ell+1}$. Thus, $\Lambda^c_{\ell} \supset \Lambda^c_{\ell+1}$, $\operatorname{supp} \chi \times V \cap \Lambda_{\ell} = \emptyset$ if $\operatorname{supp} \chi \times V \cap \Lambda_{\ell+1} = \emptyset$ and all the seminorms $||v||_{N,V\chi}$ on $\mathcal{E}'_{\Lambda_{\ell+1}}$ are also seminorms on $\mathcal{E}'_{\Lambda_{\ell}}$. The seminorms p_B are the same for $\mathcal{E}'_{\Lambda_{\ell+1}}$ and $\mathcal{E}'_{\Lambda_{\ell}}$ because the sets B are identical (i.e. the bounded sets of $\mathcal{D}(\Omega)$). \Box

This inductive limit is not strict if the open cone Λ is not closed. Indeed, if the inductive limit were strict, then the Dieudonné-Schwartz theorem [27, p. 161] would imply that each bounded set of \mathcal{E}'_{Λ} is included and bounded in an E_{ℓ} , which is wrong when Λ is not both open and closed, as we shall prove in section 4.3.

3.2. Duality of the inductive limit. In this section, we show that the inductive topology on \mathcal{E}'_{Λ} is compatible with the pairing:

Proposition 9. The topological dual of \mathcal{E}'_{Λ} equipped with its inductive topology is \mathcal{D}'_{Γ} .

Proof. We first show that $\mathcal{D}'_{\Gamma} \hookrightarrow (\mathcal{E}'_{\Lambda})'$. We already know that, for any $u \in \mathcal{D}'_{\Gamma}$, $\langle u, v \rangle$ is well defined for all $v \in E_{\ell}$ because $E_{\ell} \subset \mathcal{E}'_{\Lambda}$. Note that injectivity is obvious since smooth compactly supported functions, which form a separating set for distributions, are in \mathcal{E}'_{Λ} . A linear map from an inductive limit into a locally convex space is continuous if and only if its restriction to all E_{ℓ} is continuous [39, p. 217]. Therefore, we must show that, for any ℓ , the map $\lambda : v \mapsto \langle u, v \rangle$ is continuous from E_{ℓ} to \mathbb{K} . The proof is so close to the derivation of lemma 3 that it suffices to list the differences. We define a finite number of compactly supported smooth functions ψ_j such that $\sum_j \psi_j^2 = 1$ on a compact neighborhood of L_{ℓ} (here we use the fact that the support of all $v \in E_{\ell}$ is contained in a common compact set) and closed cones V_{uj} and V_{vj} satisfying the three conditions (i) $V_{uj} \cap (-V_{vj}) = \emptyset$, (ii) $\supp \psi_j \times V_{uj}^c \cap WF(u) = \emptyset$ and (iii) $\supp \psi_j \times V_{vj}^c \cap \Lambda_{\ell} = \emptyset$. The integral I_{2j} is calculated as I_{3j} in lemma 3 if we interchange u and v, α and β : $|I_{2j}| \leq ||v||_{N, \operatorname{supp} \beta'_j, \psi_j} C I_n^{N-m}$, where m is the order of v, and I_{3j} is bounded as I_{2j} in lemma 3: $|I_{3j}| \leq p_{\psi_j g_j}(v)$, where $\hat{g}_j(k) = \beta_j(k)(1 - \alpha_j(-k))\widehat{\psi_j u}(-k)$. We obtain

$$\begin{aligned} |\langle u, v \rangle| &\leq \sum_{j} \left(p_{\psi_{j}g_{j}}(v) + ||v||_{N, \operatorname{supp}\beta'_{j}, \psi_{j}} CI_{n}^{N-m} \right. \\ &+ ||u||_{M, \operatorname{supp}\alpha'_{j}, \psi_{j}} ||v||_{N, \operatorname{supp}\beta'_{j}, \psi_{j}} I_{n}^{N+M} \Big), \end{aligned}$$

for any N > m + n (the condition N + M > n being then satisfied for any nonnegative integer M). This shows the continuity of λ because the right hand side is a finite sum of terms involving seminorms of $\mathcal{D}'_{\Lambda_{\ell}}$, which induce the topology of E_{ℓ} .

Conversely, to prove that $(\mathcal{E}'_{\Lambda})' \hookrightarrow \mathcal{D}'_{\Gamma}$, we show that any element λ of $(\mathcal{E}'_{\Lambda})'$ defines by restriction to $\mathcal{D}(\Omega)$ a distribution and then that its wavefront set is contained in Γ . This will be enough since by density of $\mathcal{D}(\Omega)$ in \mathcal{E}'_{Λ} the restriction then extends uniquely to \mathcal{E}'_{Λ} and is thus the inverse of the reverse embedding. A linear map $\lambda : \mathcal{E}'_{\Lambda} \to \mathbb{K}$ is continuous if its restriction to all E_{ℓ} is continuous. In

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other words, for each E_{ℓ} there is a bounded set B in $\mathcal{D}(\Omega)$ and there are smooth functions χ_i and closed conic neighborhoods V_i such that $\operatorname{supp} \chi_i \times V_i \cap \Lambda_{\ell} = \emptyset$ and

(5)
$$|\lambda(v)| \leq M \sup(p_B(v), ||v||_{N_1, V_1, \chi_1}, \dots, ||v||_{N_r, V_r, \chi_r})$$

We first prove that λ is a distribution, i.e. a continuous linear map from $\mathcal{D}(\Omega)$ to \mathbb{K} . Recall that the space $\mathcal{D}(\Omega)$ is the inductive limit of $\mathcal{D}(L_{\ell})$ because L_{ℓ} is an increasing sequence of compact sets exhausting Ω [33, p. 66]. Thus, a map λ is a distribution if the restriction of λ to each $\mathcal{D}(K_{\ell})$ is continuous. For any $f \in \mathcal{D}(K_{\ell})$, we must show that all the seminorms on the right hand side of eq. (5) can be bounded by some $\pi_m(f)$. But this is a consequence of the fact that $\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}'_{\Lambda_{\ell}}$ is continuous, which was established in lemma 5.

Since λ is a distribution, it has a wavefront set. To prove that $WF(\lambda) \subset \Gamma$ consider a smooth compactly supported function ψ and a closed cone W such that $\operatorname{supp} \psi \times W \cap \Gamma = \emptyset$, i.e. $\operatorname{supp} \psi \times (-W) \subset \Lambda$. Since the restriction of $\operatorname{supp} \psi \times (-W)$ to the unit sphere is compact, there is an ℓ such that $\operatorname{supp} \psi \times (-W) \subset \Lambda_{\ell}$. Note also that $\operatorname{supp} \psi \subset \pi_1(\Lambda_{\ell}) \subset L_{\ell}$ so that $f_k = (1 + |k|)^N \psi e_k$ is in E_{ℓ} . We can now repeat the same reasoning as for the proof of proposition 7 to show that $||\lambda||_{N,W,\psi} = \operatorname{sup}_{k \in W} |\lambda(f_k)|$ is bounded. This shows that $WF(\lambda) \subset \Gamma$, which implies $\lambda \in \mathcal{D}'_{\Gamma}$ and $(\mathcal{E}'_{\Lambda})' \subset \mathcal{D}'_{\Gamma}$.

This completes the proof that
$$(\mathcal{E}'_{\Lambda})' = \mathcal{D}'_{\Gamma}$$
.

3.3. The strong topology on \mathcal{E}'_{Λ} . We showed that the coupling between \mathcal{E}'_{Λ} and \mathcal{D}'_{Γ} is compatible with duality. Thus, the inductive topology on \mathcal{E}'_{Λ} is coarser than the Mackey topology [32, p. IV.4]. The strong topology $\beta(\mathcal{E}'_{\Lambda}, \mathcal{D}'_{\Gamma})$ is always finer than the Mackey topology [32, p. IV.4]. Therefore, if we can show that the inductive topology is finer than the strong topology, we prove the identity of the inductive, Mackey and strong topologies.

Lemma 10. The inductive, Mackey and strong topologies on \mathcal{E}'_{Λ} are equivalent.

Proof. To show that the identity map from \mathcal{E}'_{Λ} with the inductive topology to \mathcal{E}'_{Λ} with the strong topology is continuous we must prove that the identity map is continuous from all E_{ℓ} to \mathcal{E}'_{Λ} with the strong topology. In other words, for any bounded set B' of \mathcal{D}'_{Γ} , we must show that $p_{B'}(v) = \sup_{u \in B'} |\langle u, v \rangle|$ is bounded on E_{ℓ} by some seminorms of E_{ℓ} .

We proceed as in the proof of lemma 3. From the fact that $\Gamma' \cap \Lambda_{\ell} = \emptyset$ and supp $v \subset L_{\ell}$ we can build a finite number of smooth compactly supported functions ψ_j such that $\sum_j \psi_j^2 = 1$ on a compact neighborhood K' of L_{ℓ} , and closed cones V_{uj} and V_{vj} satisfying the three conditions (i) $V_{uj} \cap (-V_{vj}) = \emptyset$, (ii) $\operatorname{supp} \psi_j \times V_{uj}^c \cap \Gamma = \emptyset$ and (iii) $\operatorname{supp} \psi_j \times V_{vj}^c \cap \Lambda_{\ell} = \emptyset$. The support of all ψ_j is assumed to be contained in a common compact neighborhood K of K'. Then, we define again homogeneous functions α_j and β_j of degree 0, measurable, smooth except at the origin, nonnegative and bounded by 1 on \mathbb{R}^n , such that the closed conical neighborhoods $\operatorname{supp} \alpha_j$ and $\operatorname{supp} \beta_j$ satisfy the three conditions (i), (ii) and (iii), with $\alpha_j = 1$ on V_{uj} and $\beta_j = 1$ on V_{vj} and, as in the proof of lemma 3, we write $\langle u, v \rangle =$ $\sum_j (I_{1j} + I_{2j} + I_{3j} + I_{4j})$. We have again $I_{1j} = 0$ because the supports of α_j and β_j are disjoint, and $|I_{4j}| \leq ||u||_{M,\operatorname{supp} \alpha'_j,\psi_j} ||v||_{N,\operatorname{supp} \beta'_j,\psi_j} I_n^{N+M}$ for any integers Nand M such that N + M > n. It is important to remark that ψ_j , α_j and β_j depend only on Γ , L_{ℓ} and Λ_{ℓ} and not on u and v.

To estimate I_{2j} and I_{3j} , we need to establish some properties of the bounded sets of \mathcal{D}'_{Γ} . The continuity of the injection $\mathcal{D}'_{\Gamma} \hookrightarrow \mathcal{D}'(\Omega)$ implies that a set B' which is bounded in \mathcal{D}'_{Γ} is also bounded in $\mathcal{D}'(\Omega)$ [27, p. 109]. According to Schwartz [33, p. 86], a subset B' is bounded in $\mathcal{D}'(\Omega)$ iff, for any relatively compact open set $U \subset \Omega$, there is an index m such that every $u \in B'$ can be expressed in U as $u = \partial^{\alpha} f_u$ for $|\alpha| \leq m$, where f_u a continuous function. Moreover, there is a number M such that $|f_u(x)| \leq M$ for all $x \in U$ and $u \in B'$. The elements of E_{ℓ} are supported on L_{ℓ} and we need only consider bounded sets of \mathcal{D}'_{Γ} that are defined on the compact neighborhood K of L_{ℓ} . Thus, we can take for U any relatively compact open set containing K.

To calculate I_{2j} , as in the proof of lemma 3, we define $\hat{g}_j(k) = \alpha_j(-k)(1 - \beta_j(k))\widehat{\psi_j v}(k)$ and we obtain $I_{2j} = (2\pi)^{-n} \int \widehat{\psi_j u}(-k)\widehat{g}_j(k)dk = \langle u, \psi_j g_j \rangle$. At this stage one might apply the Banach-Steinhaus theorem but we shall use an equivalent method using $u = \partial^{\alpha} f_u$:

$$\langle u, \psi_j g_j \rangle = \langle \partial^{\alpha} f_u, \psi_j g_j \rangle = (-1)^{|\alpha|} \langle f_u, \partial^{\alpha}(\psi_j g_j) \rangle = (-1)^{|\alpha|} \langle \varphi f_u, \partial^{\alpha}(\psi_j g_j) \rangle,$$

where φ is a smooth function, equal to 1 on K and supported on U. Thus

$$\begin{aligned} \langle u, \psi_j g_j \rangle &= (-1)^{|\alpha|} (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\varphi f_u}(-k) \partial^{\widehat{\alpha}}(\psi_j g_j)(k) dk \\ &= i^{|\alpha|} (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\varphi f_u}(-k) k^{\alpha} \widehat{\psi_j g_j}(k) dk. \end{aligned}$$

We must estimate $\widehat{\psi_j g_j}(k) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\psi_j}(k-q) \widehat{g_j}(q) dq$. The functions α_j and $(1-\beta_j)$ are bounded by 1 and $\widehat{\psi_j v}$ is rapidly decreasing on the support of $\beta'_j = 1-\beta_j$. Thus, $|\widehat{g_j}(q)| \leq ||v||_{N, \operatorname{supp} \beta'_j, \psi_j} (1+|q|)^{-N}$ for all integers N. In the proof of lemma 5, we estimated the Fourier transform of a smooth compactly supported function: $|\psi_j(k-q)| \leq C_j^{N'} (1+|k-q|)^{-N'}$ for all integers N', where $C_j^{N'} = ((1+n)\beta)^{N'} |K|\pi_{2N',K}(\psi_j)$. If we take N = n+m+1, where $m = |\alpha|$ is the degree of ∂^{α} , and N' = 2N we obtain

$$\begin{aligned} |\widehat{\psi_j g_j}(k)| &\leq (2\pi)^{-n} ||v||_{N, \operatorname{supp} \beta'_j, \psi_j} C_j^{2N} \int_{\mathbb{R}^n} (1+|k-q|)^{-2N} (1+|q|)^{-N} dq \\ &\leq ||v||_{N, \operatorname{supp} \beta'_j, \psi_j} C_j^{2N} I_n^N (1+|k|)^{-N}, \end{aligned}$$

where we used $(1 + |q|)^{-N} \le (1 + |k - q|)^N (1 + |k|)^{-N}$ [24, p. 50]. This estimate enables us to calculate

$$\begin{aligned} |I_{2j}| &= |\langle u, \psi_j g_j \rangle| \le (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{\varphi f_u}(-k)| |k|^m |\widehat{\psi_j g_j}(k)| dk \\ &\le (2\pi)^{-n} |U|M| |v||_{N, \text{supp } \beta'_j, \psi_j} C_j^{2N} I_n^N \int_{\mathbb{R}^n} \frac{|k|^m}{(1+|k|)^{n+m+1}} dk \\ &\le |U|M| |v||_{N, \text{supp } \beta'_j, \psi_j} C_j^{2N} I_n^N I_n^{n+1}, \end{aligned}$$

where N = n + m + 1, |U| is the volume of U and we used the obvious bound $|\widehat{\varphi f_u}(-k)| \leq |U|M$.

The estimate of I_{3j} is a little more subtle. We start from $I_{3j} = (2\pi)^{-n} \int \widehat{g_j^u}(-k) \widehat{\psi_j v}(k) dk$, where $\widehat{g_j^u}(-k) = (1 - \alpha_j(-k))\beta_j(k)\widehat{\psi_j u}(-k)$. Thus $|I_{3j}| = |\langle \psi_j g_j^u, v \rangle|$ can be bounded by $p_{B_j}(v) = \sup_{f \in B_j} |\langle f, v \rangle|$ if the set $B_j = \{\psi_j g_j^u; u \in B'\}$ is a bounded set in $\mathcal{D}(\Omega)$. It is clear that all $f \in B_j$ are supported on $K = \sup \psi_j$ and that all elements of $\psi_j g_j^u$ are smooth because ψ_j is smooth and the Fourier transform of g_j^u is rapidly decreasing. It remains to show that all the derivatives of $\psi_j g_j^u$ are bounded by a constant independent of u. For this we write

$$\partial^{\alpha}(\psi_{j}g_{j}^{u})(x) = (2\pi)^{-n}(-i)^{|\alpha|} \int_{\mathbb{R}^{n}} e^{-ik \cdot x} k^{\alpha} \widehat{\psi_{j}g_{j}^{u}}(k) dk.$$

If $|\alpha| \leq m$, we use the estimate of $\hat{\psi}_j g_j$ obtained in the previous section and we interchange u and v, α_j and β_j

$$\widehat{\psi_j g_j^u}(k)| \leq ||u||_{N, \operatorname{supp} \alpha'_j, \psi_j} C_j^{2N} I_n^N (1+|k|)^{-N},$$

where N = n + m + 1. A set B' is bounded in \mathcal{D}'_{Γ} iff it is bounded for all the seminorms of \mathcal{D}'_{Γ} [27, p. 109]. In particular, there is a constant $M_{N, \operatorname{supp} \alpha'_j, \psi_j}$ such that $||u||_{N, \operatorname{supp} \alpha'_j, \psi_j} \leq M_{N, \operatorname{supp} \alpha'_j, \psi_j}$ for all $u \in B'$. Thus, for all $f \in B_j$, $|\widehat{f}(k)| \leq M_{N, \operatorname{supp} \alpha'_j, \psi_j} C_j^{2N} I_n^N (1 + |k|)^{-N}$ and

$$|\partial^{\alpha} f(x)| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |k|^m |\widehat{f}(k)| dk \leq M_{N, \operatorname{supp} \alpha'_j, \psi_j} C_j^{2N} I_n^N I_n^{n+1},$$

where N = n + m + 1 as for the estimate of I_{3j} . In other words, for any α there is a constant $C_{|\alpha|}$ such that $|\partial^{\alpha} f| \leq C_{|\alpha|}$ for all $f \in B_j$. Thus, $\pi_m(f) \leq \sup_{0 \leq k \leq m} C_k$ is bounded independently of f, and we proved that B_j is a bounded set of $\mathcal{D}(\Omega)$. Hence, $|I_{3j}| \leq p_{B_j}(v)$ where p_{B_j} is a seminorm of \mathcal{D}'_{Γ} .

If we gather our results we obtain

$$p_{B'}(v) \leq \sum_{j} \left(M_{n, \text{supp } \alpha'_{j}, \psi_{j}} ||v||_{n, \text{supp } \beta'_{j}, \psi_{j}} I_{n}^{2n} + M ||v||_{N, \text{supp } \beta'_{j}, \psi_{j}} C_{j}^{2N} I_{n}^{N} I_{n}^{n+1} + p_{B_{j}}(v) \right),$$

where the sum over j is finite, N = n + m + 1 where m is the maximum order of the distributions of B'. The proof is complete.

4. Bornological properties

We study the bornological properties of \mathcal{D}'_{Γ} because they enable us to prove that \mathcal{D}'_{Γ} is complete and because they have a better behaviour than the topological properties with respect to the tensor product of sections. More precisely, if $\Gamma_c(E)$ is the space of compactly supported sections of a vector bundle E over M, then there is a bornological isomorphism but no topological isomorphism between $\Gamma_c(E \otimes F)$ and $\Gamma_c(E) \otimes_{C^{\infty}(M)}^{\beta} \Gamma(F)$, where F is another vector bundle over M [40]. As a consequence, there is a bornological (and topological) isomorphism between the distribution spaces $\Gamma_c(E \otimes F)'$ and $\Gamma(E^*) \otimes_{C^{\infty}(M)}^{\beta} \Gamma_c(F)'$ [40].

4.1. Bornological concepts. We start by recalling some elementary concepts of bornology theory [41].

Definition 11. A bornology on a set X is a family \mathcal{B} of subsets of X satisfying the following axioms:

- B.1: \mathcal{B} is a covering of X, i.e. $X = \bigcup_{B \in \mathcal{B}} B$.
- B.2: \mathcal{B} is hereditary under inclusion: if $A \in \mathcal{B}$ and $B \subset A$, then $B \in \mathcal{B}$.
- B.3: \mathcal{B} is stable under finite union.

A pair (X, \mathcal{B}) is called a *bornological set* and the elements of \mathcal{B} are called the *bounded subsets* (or the bounded sets) of X. Bornological concepts are often inspired by similar topological concepts. For example, a bornology \mathcal{B}_1 on a set X is *finer* than a bornology \mathcal{B}_2 if $\mathcal{B}_1 \subset \mathcal{B}_2$. If \mathcal{B} is a bornology on a set X, a subset \mathcal{B}_1 of \mathcal{B} is said to be a *basis* of \mathcal{B} iff every set of \mathcal{B} is contained in a set of \mathcal{B}_1 .

If A is a subset of a vector space E, then the convex balanced envelope of A, denoted by $\Gamma(A)$, is the smallest convex balanced set that contains A. It is also the convex envelope of $A \cup (-A)$ [32, p. II.10]. The envelope $\Gamma(A)$ is the set of finite linear combinations $\sum_i \lambda_i x_i$ where $x_i \in A$ and $\sum_i |\lambda_i| \leq 1$ [32, p. II.10]. If $f: E \to F$ is a linear map between vector spaces, then $f(\Gamma(A)) = \Gamma(f(A))$ [32, p. II.20].

p. II.10]. A convex balanced set is also called a *disk* [42, p. 1] or an *absolutely* convex set [42, p. 1].

To define a convex bornological space we need the concept of a disked hull [41, p. 6]. We recall that a subset A of a vector space is a disk if it is convex (i.e. for any x and y in A and any real λ such that $0 \leq \lambda \leq 1$, then $\lambda x + (1 - \lambda)$ is in A) and balanced (i.e. if $x \in A$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, then $\lambda x \in A$) [41, p. 4].

Lemma 12. A subset A is a disk if and only if $\lambda x + \mu y \in A$ whenever x and y are in A and λ and μ in \mathbb{K} satisfy $|\lambda| + |\mu| \leq 1$ [41, p. 4].

Definition 13. If E is a vector space, the disked hull of a subset A of E, denoted by $\Gamma(A)$, is the smallest disk containing A.

Lemma 14. Let A be a subset of the vector space E. The disked hull $\Gamma(A)$ of A is the set of finite linear combinations of the form $\sum_{i \in I} \lambda_i x_i$, with $x_i \in A$, $\lambda_i \in \mathbb{K}$ and $\sum_{i \in I} |\lambda_i| \leq 1$ [41, p. 6].

Definition 15. Let E be a vector space on \mathbb{K} . A bornology \mathcal{B} on E is said to be a a convex bornology if, for every A and B in \mathcal{B} and every $t \in \mathbb{K}$, the sets A + B, tA and $\Gamma(A)$ (i.e. the disked hull of A) belong to \mathcal{B} . Then E or (E, \mathcal{B}) is called a convex bornological space.

A convex bornological space is *separated* if the only vector space of \mathcal{B} is $\{0\}$.

4.2. Completeness of \mathcal{D}'_{Γ} . The set of bounded maps from a convex bornological space E to \mathbb{K} is called the *bornological dual* of E and is denoted by E^{\times} .

A powerful theorem of bornology states that, if a convex bornological space E is regular (i.e. if E^{\times} separates points in E [41, p. 66]), then its bornological dual E^{\times} , endowed with its natural topology, is a complete locally convex space [41, p. 77].

We are now going to build a bornological space E such that E^{\times} with its natural topology is equal to \mathcal{D}'_{Γ} with its normal topology. This implies the completeness of \mathcal{D}'_{Γ} .

Recall that E_{ℓ} is the space $\mathcal{E}'_{\Lambda_{\ell}}(L_{\ell})$ of the distributions compactly supported on L_{ℓ} whose wavefront set is included in Λ_{ℓ} , where the set of L_{ℓ} exhausts Ω and the set of Λ_{ℓ} exhausts Λ . To every locally convex space E_{ℓ} we associate the convex bornological space ${}^{b}E_{\ell}$ which is the vector space E_{ℓ} equipped with the von Neumann bornology (i.e. the bornology defined by the bounded sets of the locally convex space E_{ℓ}) [41, p. 48]. Let E be the bornological inductive limit of ${}^{b}E_{\ell}$, which is the vector space \mathcal{E}'_{Λ} equipped with the bornology defined by the bounded sets of E_{ℓ} for all integers ℓ [41, p. 33].

The bornological dual $(F)^{\times}$ of a convex bornological space F is a locally convex space for the natural topology defined by the bounded sets of F. In other words, the seminorms of $(F)^{\times}$ are of the form $p_{B'}(u) = \sup_{v \in B'} |\langle u, v \rangle|$, where B' runs over the bounded sets of F.

We start by two lemmas:

Lemma 16. $\mathcal{D}(\Omega)$ is Mackey-sequentially-dense in E.

Proof. Take $u \in {}^{b}E_{\ell} = \mathcal{E}'_{\Lambda_{\ell}}(L_{\ell})$. It suffices to find $u_n \in \mathcal{D}(\Omega)$ such that $u_n - u$ tends bornologically to 0 in $\mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$.

From the proof of Hörmander's density Theorem [16, Th 8.2.3 p. 262] we see that there exists a sequence $u_n \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(u_n) \subset L_{\ell+1}$ such that $u_n \to u$ in $\mathcal{D}'_{\Lambda_{\ell+1}}$ thus in $\mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$.

Moreover, in $\mathcal{D}'(\Omega)$, Mackey convergence of a sequence is equivalent to convergence in the strong topology (Mackey convergence for a von Neumann bornology always implies by definition convergence for the topology from which the bornology is derived). Moreover, strong convergence implies Mackey convergence when the von Neumann bornology is Schwartz, [41, p. 26], and this is the case for $\mathcal{D}'(\Omega)$ since this is a complete, thus bornologically complete [41, p. 46], co-nuclear locally convex space so that its von Neumann bornology is nuclear thus Schwartz [43, p. 69]). Thus, there exists $\nu_n \to 0$ such that $\{\frac{1}{\nu_n}(u_n - u)\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{D}'(\Omega)$.

Finally, the supplementary seminorms (beyond those of the strong topology) for convergence in $\mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$ can be chosen in a countable set $\{p_n\}_{n\in\mathbb{N}}$ [22, 34, p. 80]. Extracting a subsequence, one can assume that for all $k \leq n$, $p_k(u_n - u) \leq 1/n$. As a consequence, if $M_k = \max_{n < k} (np_k(u_n - u), 1)$, then $p_k(u_n - u) \leq M_k/n$ for all n, k.

Finally, let $\lambda_n = \max(\nu_n, \frac{1}{n}) \to 0$. We see that $\frac{1}{\lambda_n}(u_n - u) = \frac{1}{\nu_n} \frac{\nu_n}{\lambda_n}(u_n - u)$ is obviously still bounded in $\mathcal{D}'(\Omega)$ (since $|\frac{\nu_n}{\lambda_n}| \leq 1$). Moreover, $\frac{1}{\lambda_n} p_k(u_n - u) \leq np_k(u_n - u) \leq M_k$, so that $\frac{1}{\lambda_n}(u_n - u)$ is actually bounded in $\mathcal{E}'_{\Lambda_{\ell+1}}(L_{\ell+1})$, i.e. u_n tends to u bornologically.

Lemma 17. Let B be a bounded set in $\mathcal{D}'(\Omega)$, then for every $f \in \mathcal{D}(U)$ there exists M such that

$$\sup_{u\in B}\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{-M}|\widehat{fu}(\xi)|<\infty.$$

Proof. This lemma is an easy consequence of uniform boundedness principle/Banach-Steinhauss Theorem. Consider the space $\mathscr{S}'_{0,M}$ of C^0 functions g on \mathbb{R}^n such that:

$$\sup_{\xi \in \mathbb{R}^n} (1+|\xi|)^{-M} |g(\xi)| < \infty$$

This is obviously a Banach space continuously embedded in the space of tempered distribution \mathscr{S}' . One can consider $\mathscr{S}'_0 = \varinjlim_{M \to \infty} \mathscr{S}'_{0,M}$ with the inductive limit topology. Define $T_f: \mathcal{D}' \to \mathscr{S}'_0$ by $T_f(u) = \widehat{fu}$ (by polynomial boundedness and smoothness of fu it is well defined.) Now let us show this map is bounded (for the von Neumann bornologies). From [43, II.4 p10 Prop 1] or [41, 4:2.3 lemma (1) p52], it suffices to prove that the image of any sequence Mackey convergent to 0 is bounded. But a Mackey convergent sequence (for the von Neumann bornology) obviously converges topologically, and a weakly convergent sequence in \mathcal{D}' has polynomially bounded Fourier transform by uniform boudedness principle, i.e. is bounded in some $\mathscr{S}'_{0,M}$ and a fortiori in \mathscr{S}'_0 . This shows T_f is bounded. Now since \mathcal{D}' is complete, one may assume the bounded set B to be absolutely convex and complete, and since \mathcal{D}' is Montel in its strong topology B is compact. Since \mathcal{D}' is bornological with this topology, T_f is actually continuous, and thus $T_f(B)$ is compact, thus bounded and complete in $\mathcal{S}_0',$ and moreover absolutely convex. But finally, from [39, §19.5.(5)], as in any (LF) space, $T_f(B)$ has to be bounded in some $\mathscr{S}'_{0,M}$. This concludes. \square

Proposition 18. If E is the bornological inductive limit of the spaces ${}^{b}E_{\ell}$ as above, then $(E)^{\times} = \mathcal{D}'_{\Gamma}$ and its natural topology as a dual is equivalent to the normal topology we defined on \mathcal{D}'_{Γ} .

Proof. From lemma 3 and proposition 9, any $u \in \mathcal{D}'_{\Gamma}$ defines a continuous linear form on each E_l thus a bounded linear form of ${}^{b}E_{\ell}$, i.e. an element of $(E)^{\times}$. This gives an embedding $\mathcal{D}'_{\Gamma} \hookrightarrow (E)^{\times}$ since injectivity comes from the fact $\mathcal{D}(\Omega) \subset E$.

Conversely, we want to prove that each bounded linear form λ on E: (i) defines a distribution when restricted to $\mathcal{D}(\Omega) \subset E$; (ii) with wavefront set contained in Γ .

This will be enough to conclude the computation of the bornological dual since, from the previous lemma and the fact that a bounded linear functional is Mackeycontinuous, the restriction of a bounded linear functional to $\mathcal{D}(\Omega)$ has a unique extension to E, proving the second map above is injective. To prove that λ restricts to a distribution, we notice that the injection $\mathcal{D}(L_{\ell}) \hookrightarrow E_{\ell}$ is continuous because E_{ℓ} is a normal space of distributions. Any bounded set B of $\mathcal{D}(\Omega)$, which is actually in some E_{ℓ} , is bounded in E_{ℓ} thus in E because the image of a bounded set by a continuous linear map is a bounded set [27, p. 109]. Thus, λ is also a bounded map from $\mathcal{D}(\Omega)$ to \mathbb{K} . It is well-known that $\mathcal{D}(\Omega)$ is bounded map from $\mathcal{D}(\Omega)$ to \mathbb{K} because any bounded map from a bounded locally convex space to \mathbb{K} is continuous [27, p. 220]. In other words, λ is an element of $\mathcal{D}'(\Omega)$.

We still have to show that $\lambda \in \mathcal{D}'_{\Gamma}$, i.e. that for any $\chi \in \mathcal{D}(\Omega)$ and any closed convex neighborhood V such that $\operatorname{supp} \chi \times V \cap \Gamma = \emptyset$, the seminorm $||\lambda||_{N,V,\chi}$ is finite for all integers N. For this we use again the remark made in the proof of proposition 7 that $||\lambda||_{N,V,\chi} = \sup_{k \in V} |\lambda(f_k)|$, where $f_k = (1 + |k|)^N \chi e_k$. Thus, if $B' = \{f_k; k \in V\}$ is a bounded set in E, then we know that $p_{B'}(\lambda) = \sup_{k \in V} |\lambda(f_k)| < +\infty$ because the image of the bounded set B' by the bounded map λ is bounded. It remains to show that B' is a bounded set of some E_ℓ . We proceed as in the proof of lemma 10.

First, $\operatorname{supp} \chi$ is a compact subset of the open set $\pi_1(\Lambda)$. Therefore, there is an integer ℓ such that L_ℓ is a compact neighborhood of $\operatorname{supp} \chi$ and $U^*\Omega \cap \Lambda_\ell$ is a compact neighborhood of $U^*\Omega \cap (\operatorname{supp} \chi \times (-V))$ because L_ℓ exhausts Ω and Λ_ℓ exhausts Λ . This space E_ℓ contains B' because each f_k is smooth and compactly supported and we want to show B' is bounded in this E_ℓ .

Consider the seminorm $||f_k||_{N',W,\psi}$ where $\sup \psi \times W \cap \Lambda_{\ell} = \emptyset$. If $\sup \psi \cap \sup \chi = \emptyset$, then $||f_k||_{N',W,\psi} = 0$ is bounded. If $\sup \psi \cap \sup \chi \neq \emptyset$, then $W \cap (-V) = \emptyset$ and thus by compactness of the intersections of these cones with the unit sphere, there is a c > 0 such that |k + q|/|q| > c, |k + q|/|k| > c for all $k \in V, q \in W$. Therefore, we follow the proof of proposition 7 to show that

$$||f_k||_{N',W,\psi} \leq c^{-N-N'} \sup_{q \in W} (1+|k+q|)^{N+N'} |\widehat{\psi}\widehat{\chi}(k+q)|.$$

According to eq. (4), there is a constant $C_{N+N',\psi\chi}$ such that $||f_k||_{N',W,\psi} \leq c^{-N-N'}C_{N+N',\psi\chi}$. Therefore, $||f_k||_{N',W,\psi}$ is uniformly bounded for all values of $k \in V$.

To conclude the proof of the boundedness of B' in E_{ℓ} , we show that $p_B(f_k)$ is bounded for all bounded sets $B \subset \mathcal{D}(\Omega)$. We know that $\mathcal{D}(\Omega)$ is a Montel space [36, p. 357]. Thus, it is barrelled and it is enough to show that B' is weakly bounded: i.e. that, for any $g \in \mathcal{D}(\Omega)$, $\langle f_k, g \rangle$ is bounded. Indeed we have $|\langle f_k, g \rangle| =$ $(1 + |k|)^N |\langle e_k, \chi g \rangle| = (1 + |k|)^N |\widehat{\chi g}(k)|$, which is bounded uniformly in $k \in \mathbb{R}^n$, as seen from eq. (4).

Finally, we have shown that B' is bounded in E_{ℓ} , which implies that B' is bounded in E and that $||\lambda||_{N,V,\chi} = p_{B'}(\lambda) < +\infty$ for all integers N and all V, χ such that supp $\chi \times V \cap \Gamma = \emptyset$. In other words, this concludes our proof of WF(λ) $\subset \Gamma$.

Moreover, this also shows the natural topology of E^{\times} is finer than the normal topology of \mathcal{D}'_{Γ} (we have shown there are more seminorms defining the former than the later, since the seminorms p_B for $B \subset \mathcal{D}(\Omega)$ are clearly in both.). It remains to show the converse, i.e. continuity of the map $\mathcal{D}'_{\Gamma} \mapsto E^{\times}$. Since E^{\times} is defined as a projective limit, it suffices to show, continuity of the injection obtained by composition with $E^{\times} \to E_{\ell}^{\times}$ for all ℓ . Said otherwise, we have to show that the bound (3) we obtained in lemma 3 can be made uniform in $v \in B$ for some bounded set B in E_{ℓ} . First note that the choices of functions ψ, α, β can be made uniformly for $v \in B$, B a bounded set in E_{ℓ} .

Second, using lemma 17, one see that the m, C used in the proof of the bound (3) can be made uniform in $v \in B$ so that $\sup_{v \in B} |\widehat{v\psi_j}(\xi)| \leq C(1+|\xi|)^m$. Moreover, by definition of boundedness $\sup_{v \in B} ||v||_{N,\sup \beta'_j,\psi_j} \leq M_{N,\sup \beta'_j,\psi_j}$.

We thus obtain:

$$p_B(u) = \sup_{v \in B} |\langle u, v \rangle| \leq \sum_j \left(p_{B'_j}(u) + ||u||_{M, \operatorname{supp} \alpha'_j, \psi_j} CI_n^{M-m} + ||u||_{M, \operatorname{supp} \alpha'_j, \psi_j} M_{N, \operatorname{supp} \beta'_j, \psi_j} I_n^{N+M} \right),$$

where $B'_j := \{\psi_j f^v_j : v \in B\}$ with $\widehat{f^v_j}(k) = \alpha_j(-k)(1-\beta_j(k))\widehat{\psi_j v}(k)$. To prove the expected continuity, it thus only remains to show B' is bounded in $\mathcal{D}(\Omega)$ so that $p_{B'}$ will be a seminorm of \mathcal{D}'_{Γ} .

But, let $K_j = \operatorname{supp} \psi_j$, we deduce:

$$\begin{aligned} \pi_{N,K_j}(\psi_j f_j^v) &\leq 2^N \pi_{N,K_j}(\psi_j) \pi_{N,K_j}(f_j^v) \\ &\leq 2^N \pi_{N,K_j}(\psi_j) \sup_{|\gamma| \leq N} |\int_{\operatorname{supp} \beta'_j} dk(k^\gamma) \alpha_j(-k)(1-\beta_j(k)) \widehat{\psi_j v}(k)| \\ &\leq 2^N \pi_{N,K_j}(\psi_j) I_n^{n+1} ||v||_{N+n+1,\operatorname{supp} \beta'_j,\psi_j}, \end{aligned}$$

and the last seminorm is a seminorm in E_{ℓ} since $\operatorname{supp} \beta'_j$ has been chosen (in the process of choosing ψ, α, β independent of $v \in B$) so that $\operatorname{supp} (\psi_j) \times \operatorname{supp} \beta'_j \cap \Lambda_{\ell} = \emptyset$. The above estimate thus concludes.

Corollary 19. \mathcal{D}'_{Γ} with its normal topology is complete.

Proof. From the result we stated [41, p. 77], it remains to check that E, as a convex bornological space, is regular. From our computation of the dual, it was already proved in lemma 3 that E^{\times} separates points in E. Thus, E is a regular convex bornological space and its dual \mathcal{D}'_{Γ} is complete with its normal topology, because it is equivalent to the natural topology.

4.3. \mathcal{E}'_{Λ} is bornological. A locally convex space is *bornological* if its balanced, convex and bornivorous subsets are neighborhoods of zero [27, p. 220]. Bornological spaces have very convenient properties. For example, every linear map f from a bornological locally convex space E to a locally convex space F is continuous iff it is bounded (i.e. if f sends every bounded set of E to a bounded set of F) [27, p. 220]. Thus, it is worthwile to prove the following:

Proposition 20. \mathcal{E}'_{Λ} is a bornological locally convex space.

Proof. By a standard theorem [27, p. 221], a locally convex Hausdorff space E is bornological iff the topology of E is the Mackey topology and any bounded linear map from E to \mathbb{K} is continuous. We already know from lemma 10 that the inductive topology on \mathcal{E}'_{Λ} is equivalent to the Mackey topology. Thus, it remains to show that a linear map $\lambda : \mathcal{E}'_{\Lambda} \to \mathbb{K}$ is continuous if $\sup_{v \in B'} |\lambda(v)| < \infty$ for every bounded subset B' of \mathcal{E}'_{Λ} . Since λ is a fortiori bounded for the coarser bornology of E, we know from proposition 18 that it defines by restriction on $\mathcal{D}(\Omega)$ an element of \mathcal{D}'_{Γ} . Then this element extends to a continuous linear form on \mathcal{E}'_{Λ} , and since by lemma 16, $\mathcal{D}(\Omega)$ is Mackey dense in E thus a fortiori in \mathcal{E}'_{Λ} , the extension has to coincide with the original λ (which is bounded thus Mackey sequentially continuous). Finally, λ is thus continuous.

Note that the previous argument says \mathcal{E}'_{Λ} has the same bornological dual as E, but not necessarily with the same natural topology. Indeed, the natural topology of $(\mathcal{E}'_{\Lambda})^{\times}$ is the strong $\beta(\mathcal{D}'_{\Gamma}, \mathcal{E}'_{\Lambda})$ topology on \mathcal{D}'_{Λ} . If the normal topology of \mathcal{D}'_{Λ} were the strong topology, then \mathcal{E}'_{Λ} would be semi-reflexive because the dual of \mathcal{D}'_{Γ} for the normal topology is \mathcal{E}'_{Λ} . Thus, \mathcal{E}'_{Λ} would be quasi-complete and we shall prove in section 5.4 that this is not the case when the open cone Λ is not closed. This implies another consequence regarding the regularity of the inductive limit. Recall that an inductive limit of locally convex spaces is said to be *regular* if each bounded set of E is contained and bounded in some E_{ℓ} [44, 45]. If the inductive limit defining the topology of \mathcal{E}'_{Λ} were regular, then the bornology of \mathcal{E}'_{Λ} would be the bornology of E (because we already know that every bounded set of E is bounded in \mathcal{E}'_{Λ}). In that case, the natural topologies of their bornological dual \mathcal{D}'_{Γ} would be identical and the normal topology on \mathcal{D}'_{Γ} would be the strong topology. Thus, the inductive limit is not regular when Λ is not both open and closed.

5. Functional properties of \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ}

In this section, we put together the results derived up to now to determine the main functional properties of \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} .

5.1. General functional properties.

Proposition 21. The space \mathcal{D}'_{Γ} is a normal space of distributions. It is Hausdorff, nuclear and semi-reflexive. Its topological dual is \mathcal{E}'_{Λ} which is Hausdorff, nuclear, and barrelled.

Proof. We saw that \mathcal{D}'_{Γ} is Hausdorff. Its dual \mathcal{E}'_{Λ} is also Hausdorff because the pairing $\langle \cdot, \cdot \rangle$ is separating (see lemma 3) and the topology of \mathcal{E}'_{Λ} is finer than the weak topology $\sigma(\mathcal{E}'_{\Lambda}, \mathcal{D}'_{\Gamma})$ [27, p. 185]. We proved that \mathcal{D}'_{Γ} is the dual of \mathcal{E}'_{Λ} for the inductive topology and that the inductive topology of \mathcal{E}'_{Λ} is equivalent to the strong topology $\beta(\mathcal{E}'_{\Lambda}, \mathcal{D}'_{\Gamma})$. Therefore, \mathcal{D}'_{Γ} is the topological dual of \mathcal{E}'_{Λ} , which is the strong dual of \mathcal{D}'_{Γ} . This implies that \mathcal{D}'_{Γ} is semi-reflexive [27, p. 227].

The space \mathcal{E}'_{Λ} is barrelled because it is the strong dual of a semi-reflexive space [27, p. 228]. This can also be deduced from the fact that the inductive topology of \mathcal{E}'_{Λ} is equal to its strong topology [32, p. IV.5]. \mathcal{D}'_{Γ} is well-known to be nuclear with its Hörmander topology [12]. Since the normal topology is the locally convex kernel [39, p. 225] of this Hörmander topology and the strong topology of $\mathcal{D}'(\Omega)$ which is well known to be nuclear, it is again nuclear [36, p. 514]. Of course, one could give a direct proof in seeing the normal topology directly as a locally convex kernel of nuclear spaces of tempered distributions and $\mathcal{D}'(\Omega)$.

Since E_{ℓ} is a linear subspace of the nuclear space $\mathcal{D}'_{\Lambda_{\ell}}$, we immediately obtain that \mathcal{E}'_{Λ} is a nuclear space because a linear subspace of a nuclear space is nuclear and a countable inductive limit of nuclear spaces is nuclear [36, p. 514].

The duality pairing $\mathcal{D}'_{\Gamma} \times \mathcal{E}'_{\Lambda} \to \mathbb{K}$ defined by $\langle u, v \rangle$ is hypocontinuous but not continuous. Generally, the canonical pairing between a topological space E and its dual can only be continuous if E is normable [27, p. 359]. The spaces \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} are not normable. In fact, a nuclear space is normable if and only if it is finite dimensional [36, p. 520].

5.2. Completeness properties of \mathcal{D}'_{Γ} . We state the results concerning the completeness of \mathcal{D}'_{Γ} :

Proposition 22. In \mathcal{D}'_{Γ} :

- D'_Γ is complete for all topologies finer than the normal topology and coarser than the Mackey topology.
- \mathcal{D}'_{Γ} is quasi-complete for all topologies compatible with the duality between \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} ; all the bounded closed subsets are complete for these topologies. In particular, \mathcal{D}'_{Γ} is quasi-complete for the Hörmander topology.

Proof. We have proved that \mathcal{D}'_{Γ} is complete for the normal topology. Thus, it is complete for all topologies that are finer than the normal topology and that are compatible with duality [32, p. IV.5]. We have also showed that \mathcal{D}'_{Γ} is semi-reflexive.

As a consequence, it is quasi-complete for the weak topology $\sigma(\mathcal{D}'_{\Gamma}, \mathcal{E}'_{\Lambda})$ [27, p. 228]. This implies that \mathcal{D}'_{Γ} is quasi-complete for every topology compatible with the duality between \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} , in particular for the normal topology [32, p. IV.5]. Since Bourbaki's proof is rather sketchy, we give it in more detail. Assume that E is quasi-complete for the weak topology $\sigma(E, E')$ and consider a topology \mathcal{T} compatible with duality. The space E is quasi-complete for \mathcal{T} iff every \mathcal{T} -closed \mathcal{T} -bounded subset of E is complete. Consider a subset C of E which is closed and bounded for \mathcal{T} . By the theorem of the bipolars, the bipolar $C^{\circ\circ}$ of C is a balanced, convex, $\sigma(E, E')$ -closed set containing C. We also know that C is bounded for \mathcal{T} iff it is bounded for $\sigma(E, E')$ because \mathcal{T} is compatible with duality [27, p. 209]. Then, we use the fact that C is bounded for $\sigma(E, E')$ iff C° is absorbing [27, p. 191]. But $C^{\circ} = (C^{\circ\circ})^{\circ}$ so that $C^{\circ\circ}$ is weakly bounded if and only if C is weakly bounded. Therefore, $C^{\circ\circ}$ is bounded, convex and closed for $\sigma(E, E')$, and also for the other topologies compatible with duality by the first two items of the proposition. Consider now a Cauchy filter on $C^{\circ\circ}$ for the topology \mathcal{T} . It is also a Cauchy filter for the weak topology. Indeed a filter \mathfrak{F} is Cauchy iff, for any neighborhood V of zero, there is an $F \in \mathfrak{F}$ such that $F - F \subset V$. The topology \mathcal{T} being compatible with duality, it is finer than the weak topology. Thus, any weak neighborhood V is also a neighborhood of \mathcal{T} and \mathfrak{F} is a Cauchy filter for the weak topology. This Cauchy filter converges to a point x because E is quasi-complete for the weak topology. Moreover, x is in $C^{\circ\circ}$ because $C^{\circ\circ}$ is weakly closed. Therefore, the Cauchy filter converges in $C^{\circ\circ}$ and $C^{\circ\circ}$ is complete for \mathcal{T} . As a consequence, Citself is also complete because it is a closed subset of a complete set [27, p. 128]. \Box

This brings us to the following result

Proposition 23. The space \mathcal{D}'_{Γ} with its normal topology is semi-Montel. The space \mathcal{E}'_{Λ} is a normal space of distributions on which the strong, Mackey, inductive limit and Arens topologies are equivalent.

Proof. We saw that \mathcal{D}'_{Γ} is quasi-complete and nuclear for its normal topology. Thus, its bounded subsets are relatively compact [36, p. 520] and \mathcal{D}'_{Γ} is semi-Montel by definition of semi-Montel spaces [27, p. 231]. We already know that the strong, Mackey and inductive limit topologies are equivalent. It is known that on the dual of a semi-Montel space, the Arens topology is equivalent to the strong and Mackey ones [27, p. 235]. By item (iii) of proposition 6, we obtain that \mathcal{E}'_{Λ} is a normal space of distributions.

Semi-Montel spaces have interesting stability properties [27, § 3.9], [46, § 11.5] (for example, a closed subspace of a semi-Montel space is semi-Montel [27, p. 232], as well as a strict inductive limit of semi-Montel spaces [27, p. 240]). Moreover, if *B* is a bounded subset of \mathcal{D}'_{Γ} , then the topology induced on *B* by the normal topology is the same as that induced by the weak $\sigma(\mathcal{D}'_{\Gamma}, \mathcal{E}'_{\Lambda})$ topology [27, p. 231] and *B* is metrizable (because \mathcal{E}'_{Λ} , the strong dual of \mathcal{D}'_{Γ} , is nuclear [47, p. 217]).

The following properties of semi-Montel spaces are a characterization of convergence [27, p. 232] which is useful in renormalization theory:

Proposition 24. If u_i is a sequence of elements of \mathcal{D}'_{Γ} such that $\langle u_i, v \rangle$ converges to some number $\lambda(v)$ in \mathbb{K} for all $v \in \mathcal{E}'_{\Lambda}$, then u_i converges to λ in \mathcal{D}'_{Γ} .

Proposition 25. If $(u_{\epsilon})_{0 < \epsilon < \alpha}$ is a family of elements of \mathcal{D}'_{Γ} such that $\langle u_{\epsilon}, v \rangle$ converges to some number $\lambda(v)$ in \mathbb{K} as $\epsilon \to 0$ for all $v \in \mathcal{E}'_{\Lambda}$, then $u_{\epsilon} \to \lambda$ in \mathcal{D}'_{Γ} as $\epsilon \to 0$.

By proposition 22, we see that \mathcal{D}'_{Γ} is quasi-complete for the Hörmander topology. However, it is generally not complete because $\mathcal{D}'(\Omega)$ is not complete for the weak topology (otherwise, every linear map from $\mathcal{D}(\Omega)$ to \mathbb{K} would be continuous, whereas it is well known that the algebraic dual of $\mathcal{D}(\Omega)$ is larger than $\mathcal{D}'(\Omega)$ [48]).

5.3. Bounded sets. The bounded sets of \mathcal{D}'_{Γ} are important in renormalization theory because they are used to define the scaling degree [3] of a distribution and the weakly homogeneous distributions [49].

The bounded sets of \mathcal{D}'_{Γ} were characterized in the proof of lemma 10: a subset B' of \mathcal{D}'_{Γ} is bounded if B' is a bounded set of $\mathcal{D}'(\Omega)$ and for every integer N, every $\psi \in \mathcal{D}(\Omega)$ and every closed cone V such that $\sup \psi \times V \cap \Gamma = \emptyset$, there is a constant $M_{N,V,\chi}$ such that $||u||_{N,V\chi} \leq M_{N,V,\chi}$ for all $u \in B'$. The bounded sets of $\mathcal{D}'(\Omega)$ have several characterizations (see [33, pp. 86 and 195] and [50, pp. 330 and 493]).

We can now list the main properties of the bounded sets of \mathcal{D}'_{Γ} , which correspond to a Banach-Steinhaus theorem for \mathcal{D}'_{Γ} :

Theorem 26. In \mathcal{D}'_{Γ} :

- The bounded subsets are the same for all topologies finer than the weak topology $\sigma(\mathcal{D}'_{\Gamma}, \mathcal{E}'_{\Lambda})$ and coarser than the strong topology $\beta(\mathcal{D}'_{\Gamma}, \mathcal{E}'_{\Lambda})$. In particular, they are the same for the normal and the Hörmander topologies.
- The bounded sets are equicontinuous.
- The closed bounded sets are compact and identical for the weak, Hörmander and normal topologies.

Proof. In general, the bounded subsets of a topological vector space E are the same for all locally convex Hausdorff topologies on E compatible with the duality between E and E' [36, p. 371], i.e. for all topologies finer than the weak topology and coarser than the Mackey topology [36, p. 369]. The barrelledness of \mathcal{E}'_{Λ} implies that these bounded sets are also identical with the strongly bounded sets [27, p. 212]. In the dual \mathcal{D}'_{Γ} of a barrelled space \mathcal{E}'_{Λ} , a set is bounded if and only if it is equicontinuous [27, p. 212]. In a quasi-complete nuclear space, every closed bounded subset is compact [36, p. 520]. Especially, using propositions 21 and 22, this implies that bounded subsets closed for the Hörmander and normal topologies are compact for these topologies. In the dual of a barrelled space, the weakly closed bounded sets are weakly compact [27, p. 212]. After the proof of prop. 7, we showed that the Hörmander topology is compatible with the pairing [27, p. 198]. Thus, by the Mackey-Arens theorem [27, p. 205], it is finer than the weak topology and coarser than the Mackey one.

In the remarks following Proposition 23, we show that the weak and normal topologies are equivalent on the bounded sets. Therefore, the Hörmander topology is equivalent to those since it is finer than the weak topology and coarser than the normal one. As a consequence, the closed and bounded sets are the same for the three topologies. Indeed, it suffices to remember we noted above bounded sets closed for one of the topologies are compact for the corresponding induced topology, and compactness is an internal topological property so that they are compact for all the induced topologies since they coincide. Finally, compactness implies in a Hausdorff space they are closed for all the three topologies. \Box

In concrete terms, this means that a subset B' is bounded in \mathcal{D}'_{Γ} if and only if one (and then all) of the following conditions is satisfied:

- (i) For every $v \in \mathcal{E}'_{\Lambda}$, there is a constant M_v such that $|\langle u, v \rangle| \leq M_v$ for all $u \in B'$. This defines weakly bounded sets.
- (ii) For every bounded set B of \mathcal{E}'_{Λ} , there is a constant M_B such that $|\langle u, v \rangle| \leq M_B$ for all $u \in B'$ and all $v \in B$. This defines strongly bounded sets.
- (iii) There is a constant C and a finite set of seminorms p_i of \mathcal{E}'_{Λ} such that $|\langle u, v \rangle| \leq C \max_i p_i(v)$. This defines equicontinuous sets [27, p. 200].

With respect to item (ii) recall that, the inductive limit being not regular, there are bounded sets in \mathcal{E}'_{Λ} that are not contained and bounded in any E_{ℓ} . However, of course, as we already used, the bounded sets of every E_{ℓ} are bounded in \mathcal{E}'_{Λ} .

Note also that the closed *convex* subsets are the same for all topologies compatible with the duality between \mathcal{D}'_{Γ} and \mathcal{E}'_{Λ} [36, p. 370].

5.4. Completeness properties of \mathcal{E}'_{Λ} . By contrast with \mathcal{D}'_{Γ} , the completeness properties of \mathcal{E}'_{Λ} are very poor. More precisely, we have

Theorem 27. Assume Λ is an open cone which is not closed, then \mathcal{E}'_{Λ} with its strong topology (i.e. inductive limit topology) is not (even weakly) sequentially complete. In particular, if Ω is connected and the dimension of spacetime is n > 1, then \mathcal{E}'_{Λ} is not sequentially complete when Λ is any open conical proper subset of $\dot{T}^*\Omega$.

Proof. In fact, if Λ is an open cone which is not closed in $\overline{T}^*\Omega$, we exhibit an explicit counterexample showing that \mathcal{E}'_{Λ} is not sequentially complete. Since the construction of this counterexample is a bit elaborate, we first describe its main idea. Consider a point $(x;\eta)$ in the boundary of Λ . There is a sequence of points $(x_m;\eta_m) \in \Lambda$ such that $(x_m;\eta_m) \to (x;\eta)$. By using an example due to Hörmander, we construct a distribution v_m whose wavefront set is exactly the line $\{(x_m;\lambda\eta_m);\lambda>0\}$. Then we show that the sum $v = \sum_m v_m/m!$ is a well defined distribution which does not belong to \mathcal{E}'_{Λ} because the point $(x;\eta)$ belongs to its wavefront set. Since the series defining v is a Cauchy sequence, we have defined a Cauchy sequence in \mathcal{E}'_{Λ} whose limit is not in \mathcal{E}'_{Λ} .

The proof consists of several steps: (i) description of Hörmander's example, (ii) construction of the couter-example $v = \sum v_m/m!$, (iii) choice of the sequence $(x_m; \eta_m)$ and of the closed cones Γ_M , (iv) calculation of the seminorms of v_m in \mathcal{D}'_{Γ_M} , (v) determination of the wavefront set of v, (vi) proof that the series is Cauchy in \mathcal{E}'_{Λ} , (vii) discussion of the case where Λ is both open and closed.

Step 1: Hörmander's distribution

To build this counterexample we start from a family of distributions, defined by Hörmander [51, p. 188], whose wavefront sets are made of a single point x and a single direction λk and whose order is arbitrary: Let $\chi \in C^{\infty}(\mathbb{R}, [0, 1])$ be equal to 1 in $(-\infty, 1/2)$ and to 0 in $(1, +\infty)$ and fix $0 < \rho < 1$. Let $\eta \in \mathbb{R}^n$ a unit vector, and take an orthonormal basis $(e_1 = \eta, e_2, ..., e_n)$ and write coordinates in this coordinate system.

Define $u_{\eta,s} \in \mathcal{S}'(\mathbb{R}^n)$, for $s \in \mathbb{R}$, by

$$\widehat{\mu_{\eta,s}}(\xi) = (1 - \chi(\xi_1))\xi_1^{-s}\chi((\xi_2^2 + \dots + \xi_n^2)/\xi_1^{2\rho}).$$

Then WF $(u_{\eta,s}) = \{(0;\xi);\xi_2 = \cdots = \xi_n = 0, \xi_1 > 0\} = \{0\} \times \mathbb{R}^+_+ \eta$ and $u_{\eta,s}$ coincides with a function in $\mathcal{S}(\mathbb{R}^n)$ outside a neighborhood of the origin [51, p. 188]. It is clear that, if $\xi = \lambda \eta$ and $\lambda > 1$, then $|\hat{u}_{\eta,s}(\xi)| = \lambda^{-s}$ for any $\lambda > 1$, where s is an arbitrary real number. Thus, the degree of growth can be an arbitrary polynomial degree. Moreover, Hörmander actually proves that for any real number t and any integer m, there is a constant C(t,m), such that if α,β are multi-indexes, and $|\alpha| \geq C(t,m)$ then $\{x^{\alpha}\partial^{\beta}u_{\eta,s}, s \geq t, |\beta| \leq m, \eta \in S^n\}$ are bounded continuous functions on \mathbb{R}^n , uniformly bounded by a constant D(t,m).

One should also note that when the last factor in the definition does not vanish, we have $\xi_1^{2\rho} \ge (\xi_2^2 + \ldots + \xi_n^2)$ so that $|\xi_1|^2 \ge \frac{|\xi_1|^2 + (\xi_2^2 + \ldots + \xi_n^2)^{1/\rho}}{2} \ge |\xi|^2/2$ as soon as $\xi_2^2 + \ldots + \xi_n^2 \ge 1$, and otherwise $|\xi|^2 \le |\xi_1|^2 + 1 \le 5|\xi_1|^2$ when $(1 - \chi)(\xi_1) \ne 0$

(which implies $\xi_1 \ge 1/2$). Moreover, when the first factor does not vanish $|\xi| \ge 1/2$ so that $|\xi| \ge (1+2|\xi|)/4 \ge (1+|\xi|)/4$. As a consequence, we note for $s \ge 0$: (6) $|\widehat{u_{n,s}}(\xi)| \le (1-\chi(\xi_1))80^{s/2}(1+|\xi|)^{-s}\chi((\xi_2^2+\cdots+\xi_n^2)/\xi_1^{2\rho}) \le 10^s(1+|\xi|)^{-s}$.

Step 2: Construction of the counterexample

Since Λ is open and not closed, its boundary $\partial \Lambda = \overline{\Lambda} \setminus \Lambda$ is not empty and $\partial \Lambda \cap \Lambda = \emptyset$ [38, p. 46]. Moreover, any point $(x; \eta)$ of $\partial \Lambda$ is the limit of a sequence of points $(x_m; \eta_m)$ in Λ [52, p. 9].

By starting from Hörmander's example, we build a family of distributions v_m such that the wavefront set of v_m is $\{(x_m; \lambda \eta_m); \lambda > 0\}$ and $|\widehat{v_m}(\lambda \eta_m)| = (\lambda |\eta_m|)^{-m}$. For this we use the translation operator T_x acting on test functions by $(T_x f)(y) = f(y-x)$ and extend it to distributions by $\langle T_x u, f \rangle = \langle u, T_{-x} f \rangle$. Thus $T_{x_m} u_{\eta_m,m}$ has the desired properties. However, we want all distributions v_m to be compactly supported on Ω . Thus, we define the compact set $X = \bigcup_{m=1}^{\infty} \{x_m\} \cup \{x\} \subset \Omega$, so that $\delta = d(X, \Omega^c) > 0$, and χ a smooth function compactly supported on $B(0, \delta/2)$ and equal to 1 on a neighborhood of the origin. Then $v_m = T_{x_m}(\chi u_{\eta_m,m})$ is a distribution in $\mathcal{E}'(\Omega)$ with the desired properties.

It is easy to show that the series $v = \sum_{m=1}^{\infty} v_m/m!$ converges to a distribution in $\mathcal{E}'(\Omega)$. Indeed, it is enough to prove that, for any $f \in \mathcal{D}(\Omega)$, the numerical series $\sum_m \langle v_m, f \rangle/m!$ converges in \mathbb{K} [20, p. 13]. We have

$$\langle v_m, f \rangle = \langle T_{x_m} \chi u_{\eta_m, m}, f \rangle = \langle u_{\eta_m, m}, \chi T_{-x_m} f \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u_{\eta_m, m}}(k) \widehat{\chi f_{-x_m}}(-k).$$

where $f_{-x_m} = T_{-x_m} f$. For every integer N we have by Eq.(4)

$$|\widehat{\chi f_{-x_m}}(k)| \le (1+|k|)^{-N} (4(n+1)\beta)^N |K| \pi_{2N,K}(\chi) \pi_{2N,K}(f_{-x_m}),$$

where K is a compact neighborhood of supp χ and |K| its volume. Now, $\pi_{2N,K}(f_{-x_m}) \leq \pi_{2N,K'}(f)$, where K' is a compact neighborhood of supp f. Thus, there is a constant $C_N = (4(n+1)\beta)^N |K| \pi_{2N,K}(\chi) \pi_{2N,K'}(f)$, independent of m, such that $|\widehat{\chi f_{-x_m}}(k)| \leq C_N (1+|k|)^{-N}$. The estimate (6) gives us, for N = n,

$$\begin{aligned} |\langle v_m, f \rangle| &\leq C_n (2\pi)^{-n} 10^m \int_{\mathbb{R}^n} (1+|k|)^{-n-m} dk \\ &\leq C_n (2\pi)^{-n} 10^m \int_{\mathbb{R}^n} (1+|k|)^{-n-1} dk \leq C_n 10^m I_n^{n+1}, \end{aligned}$$

because $m \ge 1$, and the series is absolutely convergent with $|\langle v, f \rangle| \le C_n I_n^{n+1} e^{10}$.

We know that the distribution v is well defined but we have no control of its wavefront set. Indeed, the wavefront set of v can contain points that are not in any $WF(v_m)$ and there can be points that are in the wavefront set of some v_m but not in WF(v) (see refs. [53, 4] for concrete examples). Therefore, we must carefully choose the sequence $(x_m; \eta_m)$ so that $(x; \eta)$ is indeed in the wavefront set of v. This is done in the next step.

Step 3: Choice of the sequence and construction of the cones

We want to ensure that all points $(x_m; \eta_m)$ actually belong to WF(v). Thus, we choose the elements $(x_m; \eta_m)$ so that each direction η_m is at a finite distance from the other ones (except when n = 1, in which case we will choose x_m at a finite distance from one another), to avoid that their overlap concurs to remove $(x; \eta)$ from the wavefront set of v. Since Λ is a cone, we can choose $|\eta| = |\eta_m| = 1$ and, up to extraction and since Λ is open, it is possible to shift the points $(x_m; \eta_m)$ so that if $n = 1, x_m \neq x$ and $\eta_m = \eta, |x_{m+1} - x| < |x_m - x|/2, |x_m - x| < 1$, and if $n \neq 1$ $\eta_m \neq \eta, |\eta_{m+1} - \eta| < \min(|\eta_m - \eta|, d(\eta_m, -\Gamma_{x_m}))/2$, where $\Gamma_{x_m} = \{\xi; (x_m; \xi) \in \Gamma\}$, and $|\eta_m - \eta| < 1$ for all m. Let $\rho_m = \min(|\eta_m - \eta|, d(\eta_m, -\Gamma_{x_m})) < 1$ if $n \neq 1$ and set $\rho_m = 1/3^m$ if n = 1, and note that if $n \neq 1, \rho_{m+1} < \rho_m/2$ implies $|\eta_m - \eta_k| > \rho_m/2$ for all k > m, since $\rho_m/2 \ge \rho_m/2^{k-m} > \rho_k \ge |\eta_k - \eta|$ so that if $|\eta_m - \eta_k| \le \rho_m/2$ were true, we would deduce $\rho_m \le |\eta_m - \eta| \le |\eta_m - \eta_k| + |\eta_k - \eta| < \rho_m$, yielding a contradiction. Recall that $v_m = T_{x_m}(\chi u_{\eta_m,m})$ so that $v_m \in \mathcal{E}'_{\Lambda}$ and WF $(v_m) = \{x_m\} \times \mathbb{R}^*_+ \eta_m$.

To control the wavefront set, we define partial sums $S_m = \sum_{i=1}^m v_i/i!$, and we show that the cotangent directions of the wavefront set of $v - S_m$ do not meet $(x_i; \eta_i)$ for $i \leq m$. Thus, we have the finite sum $v = (v - S_m) + \sum_{i=1}^m v_i/i!$ and, since the cotangent directions of the wavefront set of the terms do not overlap, there can be no cancellation and all $(x_i; \eta_i)$ belong to the wavefront set for $i \leq m$. Then, we have indeed $(x_m; \eta_m) \subset WF(v)$ for all m because this procedure can be applied for all values of m.

It remains to show that the wavefront set of $v - S_m$ belongs to a closed conical set Γ_m which does not meet $(x_i; \eta_i)$ for $i \leq m$. We first build these Γ_m as follows: Let $X_m = \bigcup_{l>m}^{\infty} \{x_l\} \cup \{x\} \subset \Omega$ and $\gamma_{m,i} = X_m \times (\mathbb{R}^*_+ \overline{B(\eta_i, \rho_i/4)})$. It is clear that if $n \neq 1$, $\gamma_{m,i} \cap \gamma_{m,j} = \emptyset$ because, for j > i, we have $|\eta_i - \eta_j| > \rho_i/2$ and $\rho_j < \rho_i$. Thus, $|\eta_i - \eta_j| > (\rho_i + \rho_j)/4$ and since this expression is symmetric in iand j, it holds for all $i \neq j$. This shows that the balls $\overline{B(\eta_i, \rho_i/4)}$ and $\overline{B(\eta_j, \rho_j/4)}$ do not meet and the result follows. The closed cones $\gamma_{m,i}$ are then used to define $\Gamma_m = (\bigcup_{i>m} \gamma_{m,i}) \cup (X_m \times \mathbb{R}^*_+ \eta)$.

To show that the wavefront set of $v - S_m$ belongs to Γ_m , we prove that the series $\sum_{i=m+1}^{\infty} v_i/i!$ converges in \mathcal{D}'_{Γ_m} .

Step 4: Estimates on seminorms of v_m in \mathcal{D}'_{Γ_M} , m > M.

Fix $\psi \in \mathcal{D}(\Omega)$ and any closed cone W such that $\operatorname{supp} \psi \times W \cap \Gamma_M = \emptyset$. For convenience we define the distance $||x - y||_{\infty} = \sup_{i=1,\dots,n} |x^i - y^i|$, where x^i is the *i*th coordinate of x in a given orthonormal basis. Then, we define the distance between two sets to be $d_{\infty}(A, B) = \inf_{x \in A, y \in B} ||x - y||_{\infty}$.

We first consider the case when $X_M \cap \operatorname{supp} \psi = \emptyset$. Then, $v_m \psi$ is smooth, and we want to show that $\{v_m \psi, m \in \mathbb{N}\}$ is bounded in $\mathcal{D}(\Omega)$, since W above can be taken arbitrary. This is equivalent to prove that $\{\chi \psi_{-x_m} u_{\eta_m,m}, m \in \mathbb{N}\}$ is bounded, where $\psi_{-x_m} = T_{-x_m} \psi$. Let $\epsilon = d_{\infty}(X_M, \operatorname{supp} \psi) > 0$. Since ψ vanishes in a neighborhood of x_m on the ball $B_{\infty}(x_m, \epsilon)$ with $\epsilon > 0$, we deduce that $\chi \psi_{-x_m}(y)$ vanishes when $||y||_{\infty} \leq \epsilon$. Thus, we can consider that $||y||_{\infty}/\epsilon \geq 1$.

Then, using the properties of Hörmander's construction, we bound uniformly in m. Fix y and choose y^i such that $|y^i| = ||y||_{\infty}$. Then,

$$\begin{aligned} |\partial^{\alpha} \chi \psi_{-x_{m}} u_{\eta_{m},m}(y)| &\leq \frac{1}{\epsilon^{C(0,|\alpha|)}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^{\beta} \chi \psi_{-x_{m}}| |(y^{i})^{C(0,|\alpha|)} \partial^{\alpha-\beta} u_{\eta_{m},m}| \\ &\leq \frac{1}{\epsilon^{C(0,|\alpha|)}} 2^{|\alpha|} \pi_{|\alpha|, \operatorname{supp}(\psi_{-x_{m}})}(\chi \psi_{-x_{m}}) D(0,|\alpha|). \end{aligned}$$

To establish Eq. (4) we showed that

 $\pi_{|\alpha|,\operatorname{supp}(\psi_{-x_m})}(\chi\psi_{-x_m}) \leq 2^{|\alpha|}\pi_{|\alpha|,\operatorname{supp}(\psi_{-x_m})}(\chi)\pi_{|\alpha|,\operatorname{supp}(\psi_{-x_m})}(\psi_{-x_m}).$

But $\pi_{|\alpha|, \text{supp }(\psi_{-x_m})}(\chi) \leq \pi_{|\alpha|, \text{supp }\chi}(\chi)$ and $\pi_{|\alpha|, \text{supp }(\psi_{-x_m})}(\psi_{-x_m}) = \pi_{|\alpha|, \text{supp }\psi}(\psi)$. Thus,

$$\left|\partial^{\alpha}\chi\psi_{-x_{m}}u_{\eta_{m},m}(y)\right| \leq \frac{1}{\epsilon^{C(0,|\alpha|)}}2^{2|\alpha|}\pi_{|\alpha|,\operatorname{supp}\chi}(\chi)\pi_{|\alpha|,\operatorname{supp}\psi}(\psi)D(0,|\alpha|)$$

is bounded independently of m.

In the case $X_M \cap \operatorname{supp} \psi \neq \emptyset$, we have $y \in \operatorname{supp} \psi$ for some $y \in X_M$ and $\{y\} \times W \cap \gamma_{M,m} = \emptyset$ for all m > M by our assumption. Thus, $W \cap \mathbb{R}^*_+ \overline{B(\eta_m, \rho_m/4)} = \emptyset$ for all m > M. Arguing as usual by a compactness argument, one can prove there is a constant 1 > c > 0 (independent of m) such that for all $k \in \mathbb{R}^*_+ B(\eta_m, \rho_m/4)^c$, for all $(k - q) \in \mathbb{R}^*_+ B(\eta_m, \rho_m/8)$, $|q| \ge c\rho_m |k - q|$. We deduce from this and our previous estimates:

$$\begin{split} ||v_m||_{N,W,\psi} &\leq \sup_{k \in W} (1+|k|)^N \int_{\mathbb{R}^n} dq |\widehat{u_{\eta_m,m}}(k-q) \widehat{\chi\psi_{-x_m}}(q)| \\ &\leq 10^m \sup_{k \in W} \left(\int_{(k-q) \in \mathbb{R}^*_+ B(\eta_m,\rho_m/8)} + \int_{(k-q) \notin \mathbb{R}^*_+ B(\eta_m,\rho_m/8)} \right) dq \\ (1-\chi(\langle k-q,\eta_m \rangle)) \chi(\frac{|k-q|^2 - |\langle k-q,\eta_m \rangle|^2}{|\langle k-q,\eta_m \rangle|^{2\rho}})(1+|k-q|)^{N-m} |\widehat{\chi\psi_{-x_m}}(q)|(1+|q|)^N \\ &\leq 10^m \max(1, (c\rho_m)^{m-N}) \sup_{k \in W} \int_{q \in \mathbb{R}^n} dq \max(1, (1+|q|)^{N-m})(1+|q|)^{-n-1-N} ||\chi_{x_m}||_{n+1+2N,\mathbb{R}^n,\psi} \\ &+ 10^m \max(1, (1+r_m)^{N-m}) \int_{q \in \mathbb{R}^n} dq (1+|q|)^N (1+|q|)^{-n-1-N} ||\chi_{x_m}||_{n+1+N,\mathbb{R}^n,\psi} \\ &\leq 10^m I_n^{n+1} ||\chi_{x_m}||_{n+1+2N,\mathbb{R}^n,\psi} \max\left(2, (c\rho_m)^{m-N} + (1+r_m)^{N-m}\right) \\ &\sim_{m \to \infty} C_{n,N} 10^m, \end{split}$$

where $r_m = (\rho_m/8 - (\rho_m/16)^2)^{-1/2(1-\rho)}$. We used $(1+|k|)^N \leq (1+|q|)^N(1+|k-q|)^N$, (6) starting at the second inequality, the fact that $||\chi_{x_m}||_{n+1+2N,\mathbb{R}^n,\psi}$ can be bounded independently of m in the last line and where we also used that

$$\begin{aligned} \{q; (k-q)/|k-q| \notin B(\eta_m, \rho_m/8), \langle k-q, \eta_m \rangle > 0, \\ |\langle (k-q), \eta_m \rangle|^{2\rho} \ge |k-q|^2 - |\langle k-q, \eta_m \rangle|^2\} \subset B(k, r_m), \end{aligned}$$

since $|(k-q)/|k-q| - \eta_m|^2 = 2(|k-q| - \langle k-q, \eta_m \rangle)/|k-q| \ge \rho_m/8$ implies $|k-q|(1-\rho_m/16) \ge \langle k-q, \eta_m \rangle$ which implies with the two other inequalities: $|k-q|^2(1-(1-\rho_m/16)^2) \le |k-q|^{2\rho}(1-\rho_m/16)^{2\rho} \le |k-q|^{2\rho}$. Since by our choices $c\rho_m < 1$ the terms $\max(1, (c\rho_m)^{m-N}), \max(1, (1+|q|)^{N-m})$ and the max term of the next line correspond to the case when $N-m \le 0$ or $N-m \ge 0$.

Thus, we showed that, for any W and ψ such that $\operatorname{supp} \psi \times W \cap \Gamma_M = \emptyset$ and any integer N, the set $\{10^{-m} || v_m ||_{N,W,\psi}; m > M\}$ is bounded in \mathbb{R} . To show that the set $A = \{10^{-m} v_m; m > M\}$ is bounded in \mathcal{D}'_{Γ_M} , we still have to show that it is bounded for the seminorms p_B with B bounded in $\mathcal{D}(\Omega)$. In the course of step 2, we showed that, for any $f \in \mathcal{D}$, the set $p_f(A)$ is bounded in \mathbb{R} . This means that A is bounded in \mathcal{D}'_{Γ_M} equipped with the Hörmander topology. But we proved that this is equivalent to being bounded for the normal topology. Thus, A is bounded in \mathcal{D}'_{Γ_M} with its normal topology.

Step 5: Let $S_m := \sum_{k=1}^m \frac{1}{k!} v_k$ $(S_0 = 0)$. Then for any $M \ge 0$, the sequence $(S_m - S_M)_{m \ge M}$ is a Cauchy sequence in \mathcal{D}'_{Γ_M} . As a consequence, $S_m - S_M$ converges to $v - S_M$ in \mathcal{D}'_{Γ_M} and $WF(v) \supset \{(x_m; \eta_m), m \in \mathbb{N}^*\}$.

In the previous step we showed that the set $A = \{10^{-m}v_m; m > M\}$ is bounded in \mathcal{D}'_{Γ_M} . Thus, for every seminorm p_i of \mathcal{D}'_{Γ_M} and any $p \ge q > M$, we have $p_i(S_p - S_q) \le C_i \sum_{m=q}^p 10^m / m!$, and each $p_i(S_m - S_M)$ is a Cauchy sequence in \mathbb{R} . By the completeness of \mathcal{D}'_{Γ_M} , it implies that $S_m - S_M$ converges to $v - S_M$ in \mathcal{D}'_{Γ_M} .

 $\mathcal{D}'_{\Gamma_M}.$ Since the wavefront set is known for each v_m ($WF(v_m) = \{x_m\} \times \mathbb{R}^*_+ \eta_m$), v_M is the only one among the distributions $v - S_M, v_M, ..., v_1$ which is singular in direction $\mathbb{R}^*_+ \eta_M$ at x_M (because $(x_M; \eta_M) \notin \Gamma_M$ by construction either because $x_m \neq x_M$ if n = 1 or because $\eta_m \neq \eta_M$ if $n \neq 1$, for m > M), one deduces $\{x_M\} \times \mathbb{R}^*_+ \eta_M \subset WF(v)$. Indeed, by choosing a test function ψ such that $\psi(x_M) \neq 0$ and a closed cone $V \subset \mathbb{R}^*_+ B(\eta_M, \rho_M/4)$, we have $\operatorname{supp} \psi \times V \cap \operatorname{WF}(v_m) = \emptyset$ for m < M and $\operatorname{supp} \psi \times V \cap \operatorname{WF}(v - S_M) = \emptyset$. Therefore, $||v - v_M||_{N,V,\psi}$ is finite for all N and $\psi(v - v_M)(\lambda\eta_M)$ cannot compensate for the slow decrease of $\widehat{\psi}v_M(\lambda\eta_M)$, which is ensured by the fact that $\operatorname{WF}(\psi v_M) = \operatorname{WF}(v_M)$ when $\psi(x_M) \neq 0$ [15, p. 121]. Since this is valid for any M, this concludes about the wavefront set statement.

It remains to show that the sequence is also Cauchy in \mathcal{E}'_{Λ} .

Step 6: $S_m := \sum_{k=1}^m \frac{1}{k!} v_k$ is Cauchy in \mathcal{E}'_{Λ} for the strong topology coming from its duality with \mathcal{D}'_{Γ} . Especially, \mathcal{E}'_{Λ} is not sequentially complete.

By construction $WF(S_m) \subset \Lambda$. Assume proved the statement about its Cauchy nature, then the last step enables to show that if it were (even weakly) convergent in \mathcal{E}'_{Λ} , then the limit would be v (since it would be weakly convergent in \mathcal{D}'_{Λ} where the limit is v) as a distribution, but since the wavefront set is closed, $(x; \eta) \in WF(v)$ and since $(x; \eta) \notin \Lambda$ this gives a contradiction, implying S_m is a Cauchy sequence not (weakly) converging in \mathcal{E}'_{Λ} .

Thus it remains to show that S_m is Cauchy. Take $B \subset \mathcal{D}'_{\Gamma}$ bounded, we want to show that $p_B(S_m) = \sum_{k=1}^m p_B(v_k)/k!$ is a Cauchy sequence. First choose $\tilde{\chi} \in D(\Omega)$ which is identically one on the compact set $\bigcup_{y \in X} \operatorname{supp} \chi_y = X + \operatorname{supp} \chi$ (the sum of 2 compact sets is compact), where $\chi_y = T_y \chi$. Using lemma 17, since *B* is bounded in $\mathcal{D}'(\Omega)$, fix *M* such that $\sup_{u \in B} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-M} |\hat{\chi}u(\xi)| = D < \infty$. Then, for $y \in X$, we bound:

$$\sup_{u \in B} ||u||_{M,\mathbb{R}^{n},\chi_{y}} = \sup_{u \in B} \sup_{\xi \in \mathbb{R}^{n}} (1+|\xi|)^{-M} |\widehat{\chi_{y}u}(\xi)| = \sup_{u \in B} \sup_{\xi \in \mathbb{R}^{n}} (1+|\xi|)^{-M} |\chi_{y}\tilde{\chi}u(\xi)|$$

$$\leq \sup_{u \in B} \sup_{\xi \in \mathbb{R}^{n}} \int_{\mathbb{R}^{n}} dq (1+|\xi-q|)^{M} |\widehat{\chi_{y}}(\xi-q)\widehat{\chi}u(q)| (1+|q|)^{-M}$$

$$\leq DI_{n}^{n+1} ((1+n)\beta)^{(M+n+1)} \pi_{2(M+n+1), \operatorname{supp}(\chi)}(\chi) = C < \infty.$$

It now suffices to estimate $p_B(v_m)$ for $m \ge M + n + 1$. Thus, using this inequality and (6), we deduce for $m \ge M + n + 1$:

$$\begin{split} \sup_{u \in B} |\langle u, v_m \rangle| &= \sup_{u \in B} \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} dk \widehat{\chi}_{T_{-x_m}} u(k) \widehat{u_{\eta_m,m}}(-k) \right| \\ &\leq C 10^m \int_{\mathbb{R}^n} dk (1+|k|)^M (1+|k|)^{-m} \leq C 10^m I_n^{n+1}. \end{split}$$

Thus, for $p \ge q \ge M + n + 1$, $p_B(S_p - S_q) \le CI_n^{n+1} \sum_{k=q+1}^p \frac{10^m}{m!}$, and thus $p_B(S_m)$ is Cauchy as we wanted.

Step 7: Characterization of closed Λ .

To conclude the proof we give some information on the case when Λ is open and closed. A subset of a topological space X is called *clopen* if it is both open and closed in X [52, p. 10]. A topological space X is connected if and only if its only clopen subsets are X and \emptyset [52, p. 10]. Now, if Ω is connected, its cotangent bundle $T^*\Omega$ is connected. If the dimension of Ω is n > 1 the set $\dot{T}^*\Omega$, which is $T^*\Omega$ with the zero section removed, is also connected. In that case Λ is clopen if and only if it is either empty (so that $\mathcal{E}'_{\Lambda} = \mathcal{D}(\Omega)$) or $\dot{T}^*\Omega$ (so that $\mathcal{E}'_{\Lambda} = \mathcal{E}'(\Omega)$). Since both $\mathcal{D}(\Omega)$ and $\mathcal{E}'(\Omega)$ are complete, our theorem is optimal for connected $\dot{T}^*\Omega$.

Corollary 28. If Λ is an open cone which is not closed, \mathcal{E}'_{Λ} is not sequentially complete for any topology that is coarser than the normal topology and finer than the weak topology of distributions induced by $\mathcal{D}'(\Omega)$. In particular, the inductive limit of E_{ℓ} equipped with the Hörmander topology is also not sequentially complete.

Proof. This result is a consequence of the proof above rather than of the statement. A sequence which is Cauchy for the normal topology remains Cauchy for topologies that are coarser than it, thus our counterexample above is Cauchy for the topologies considered. Therefore, it thus converges weakly in $\mathcal{D}'(\Omega)$ and we showed the limit cannot be in \mathcal{E}'_{Δ} so that \mathcal{E}'_{Δ} is not sequentially complete. \Box

Corollary 29. If Λ is an open cone which is not closed, then \mathcal{E}'_{Λ} is not a regular inductive limit with the inductive limit defining its inductive limit topology and it is not semi-reflexive. If $(\Gamma')^c = \Lambda$, \mathcal{D}'_{Γ} is neither bornological nor barrelled in its normal topology

Proof. If \mathcal{E}'_{Λ} were semi-reflexive it would be weakly sequentially complete [27, p. 228]. If the inductive limit were regular, it would be semi-reflexive as explained at the end of section 4.3. Alternatively, one can see that the set of the Cauchy sequence $\{S_m, m \geq 1\}$ we built is bounded in \mathcal{E}'_{Λ} and not in any E_{ℓ} .

The space \mathcal{D}'_{Γ} is not bornological because the strong dual of a separated bornological space is complete [41, p. 77]. If \mathcal{D}'_{Γ} were barrelled in its normal topology so that, since it is semi-Montel, it would be a Montel space [27, p. 231], thus its strong dual \mathcal{E}'_{Λ} would also be a Montel space [27, p. 234] and thus again semi-reflexive. Note that Bourbaki states that a space that is semi-reflexive and semi-barrelled is complete [32, p. IV.60], but this is wrong [54].

6. CONCLUSION

This paper determined the main functional properties of Hörmander's space of distributions \mathcal{D}'_{Γ} and its dual. In view of applications to the causal approach of quantum field theory, we derived simple rules to determine whether a distribution belongs to \mathcal{D}'_{Γ} , whether a sequence converges in \mathcal{D}'_{Γ} and whether a subset of \mathcal{D}'_{Γ} is bounded.

By using the functional properties of \mathcal{D}'_{Γ} , the proof of renormalizability of scalar quantum field theory in curved space times can be considerably simplified and streamlined with respect to the original derivation given by Brunetti and Fredenhagen [3].

This paper is also the first step of a detailed investigation of the microcausal functionals discussed by Brunetti, Dütsch, Fredenhagen, Rejzner and Ribeiro [26, 12, 55, 14], which are the basis of a new and powerful formulation of quantum field theory. As noticed in ref. [55], the space of microcausal functionals is based on spaces of the type \mathcal{E}'_{Λ} which have very poor completeness properties. This problem can be solved by using the completion of \mathcal{E}'_{Λ} , which is, because of the nuclearity of \mathcal{E}'_{Λ} , also the bornological dual of \mathcal{D}'_{Γ} [47, p. 140]. The topological and bornological

properties of this completion will be discussed in a forthcoming publication by the first author [56].

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