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# Lie rank in groups of finite Morley rank with solvable local subgroups

Adrien Deloro\* and Éric Jaligot†

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## Abstract

We prove a general dichotomy theorem for groups of finite Morley rank with solvable local subgroups and of Prüfer  $p$ -rank at least 2, leading either to some  $p$ -strong embedding, or to the Prüfer  $p$ -rank being exactly 2.

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Éric Jaligot left us on July 10, 2013.

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## 1 Introduction

The “size” of a simple algebraic group over an algebraically closed field can be captured by several means. One can measure its Zariski dimension, but one can also consider its Lie rank, which is the Zariski dimension of its maximal algebraic tori. For instance, it is often straightforward to argue by induction on the Zariski dimension, as is typically the case with solvable groups. On the other hand the Lie rank is sometimes necessary in classification problems: it leads to the notions of thin/quasi-thin/generic groups which are essential in the Classification of the Finite Simple Groups, as well as in algebraic groups.

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Now the only quasi-simple algebraic groups of Zariski dimension 3 are of the form  $\mathrm{PSL}_2$  or  $\mathrm{SL}_2$ ; they also are the only quasi-simple algebraic groups of Lie rank 1. On the other hand  $\mathrm{PSL}_2$  and  $\mathrm{SL}_2$  are the only “small” quasi-simple algebraic groups from a *purely group-theoretic* point of view, namely a “local solvability” condition: the normalizer of each infinite solvable subgroup remains solvable. Hence an algebraic group with the latter property must have small Lie rank. In the present paper we give a precise meaning to, and prove, such a statement in a much more general context.

Our framework will be that of groups of finite Morley rank, for the global theory of which we refer to [BN94] or [ABC08]. Briefly put, groups of finite Morley rank are groups equipped with a rudimentary notion of dimension on their first-order definable subsets, called the Morley rank for historical reasons in model theory. Since the Morley rank satisfies basic axioms reminiscent of the Zariski dimension of algebraic varieties over algebraically closed fields, groups of finite Morley rank generalize algebraic groups over algebraically closed fields. Conversely, a major question which has stirred a huge body of work is the Cherlin-Zilber algebraicity conjecture, which postulates that infinite simple groups of finite Morley rank are in fact isomorphic to algebraic groups over algebraically closed fields. The algebraicity conjecture holds true at least of groups containing an infinite elementary abelian 2-group [ABC08]; so far, the proof is part of the Borovik program for groups with involutions, and based on ideas modelled on the Classification of the Finite Simple Groups. On the other hand there are potential configurations of simple groups of finite Morley without involutions for which the Borovik program is helpless. Here we may refer to the configurations of simple “bad” groups of Morley rank 3 discovered in [Che79], or more generally to the “full Frobenius” groups of finite Morley rank studied in [Jal01].

In the context of groups of finite Morley, the “local solvability” condition mentioned above is equivalent to the following.

**Definition.** *A group of finite Morley rank is  $\ast$ -locally $^\circ$  solvable if  $N^\circ(A)$  is solvable for each nontrivial abelian connected definable subgroup  $A$ .*

We now relate our group-theoretic notion of smallness to an abstract version of the Lie rank as follows. Since there is a priori no satisfactory first-order analogue of the notion of an algebraic torus, we shall deal with certain torsion subgroups throughout. Although basic matters such as the conjugacy and structure of Sylow  $p$ -subgroups of a group of finite Morley rank are not settled in general, enough is known about abelian divisible  $p$ -subgroups, which are called  *$p$ -tori*. The maximal ones are conjugate [Che05], and direct powers of a *finite* number of copies of the Prüfer  $p$ -group  $\mathbb{Z}_{p^\infty}$  [BP90]. The latter number is called the *Prüfer  $p$ -rank* of the ambient group. This will, quite naturally, be our analogue of the Lie rank. Indeed, it can easily be seen that in a quasi-simple algebraic group the Lie rank and the Prüfer  $p$ -rank agree: maximal tori of  $\mathrm{SL}_2$  or  $\mathrm{PSL}_2$  are of dimension 1 and of Prüfer  $p$ -rank 1 for any prime  $p$  different from the characteristic of the ground field. We note that in our more abstract

context we then have a notion of the Lie rank for each prime  $p$  such that the ambient group contains a non-trivial divisible abelian  $p$ -subgroup.

If  $S$  is an abelian  $p$ -group for some prime  $p$  and  $n$  is a natural number, then we denote by  $\Omega_n(S)$  the subgroup of  $S$  generated by all elements of order  $p^n$ . In technical terms that we will define shortly, our main theorem takes the following form.

**Main Theorem.** *Let  $G$  be a connected nonsolvable  $\ast$ -locally $^\circ$ -solvable group of finite Morley rank of Prüfer  $p$ -rank at least 2 for some prime  $p$ , and fix a maximal  $p$ -torus  $S$  of  $G$ . Assume that every proper definable connected subgroup containing  $S$  is solvable, that elements of  $S$  of order  $p$  are not exceptional, and let*

$$B = \langle C^\circ(s) \mid s \in \Omega_1(S) \setminus \{1\} \rangle.$$

*Then:*

- (1) *either  $B < G$ , in which case  $B$  is a Borel subgroup of  $G$ ; and if in addition  $S$  is a Sylow  $p$ -subgroup of  $N_{N(B)}(S)$ ,  $N(B)$  is  $p$ -strongly embedded in  $G$ ,*
- (2) *or  $B = G$ , in which case  $S$ , or equivalently  $G$ , has Prüfer  $p$ -rank 2.*

According to the algebraicity conjecture for simple groups of finite Morley rank, or rather a consequence of it, the Prüfer  $p$ -rank of a connected nonsolvable  $\ast$ -locally $^\circ$ -solvable group of finite Morley rank should be 1. Hence our Main Theorem deals with configurations which are not actually known to exist, but the dichotomy it gives severely limits possibilities in both cases. It is obvious in the second case, and in the first case it suffices to recall from [DJ11a] that a definable subgroup  $M$  of a group  $G$  is called  *$p$ -strongly embedded* if it contains non-trivial  $p$ -elements and  $M \cap M^g$  contains none for any  $g$  in  $G \setminus M$ . This mimics a similar notion in the theory of finite groups which had been crucial with  $p = 2$  in the Classification of the Finite Simple Groups. In any case we note that for groups of finite Morley rank, and typically for  $p \neq 2$ , the “bad” or “full Frobenius” groups mentioned above fit in case (1) of our Main Theorem.

The present paper is actually part of a series which aims at classifying configurations of nonsolvable  $\ast$ -locally $^\circ$  solvable groups of finite Morley rank. For much more details on such groups, including a few historical remarks, we refer to the preliminary article [DJ11a] of our series. We simply recall that this classification started in [CJ04] in the case of *minimal connected simple* groups: these are the infinite simple groups of finite Morley rank all of whose proper definable connected subgroups are solvable (as in  $\mathrm{PSL}_2$ ). After a series of generalizations of the original classification of [CJ04] it was realized that much of the theory of minimal connected simple groups transfers readily to  $\ast$ -locally $^\circ$ -solvable groups, which are defined as the  $\ast$ -locally $^\circ$ -solvable ones but where the main condition that  $N^\circ(A)$  is solvable is required for *any* nontrivial abelian subgroup  $A$ . This shift from minimal connected simple groups to  $\ast$ -locally $^\circ$ -solvable groups corresponds, in finite group theory, to a shift from the *minimal simple* groups studied for the Feit-Thompson (Odd Order) Theorem to the  *$N$ -groups* classified later by Thompson. Of the two main variations on the notion of “local solvability” from

[DJ11a] we shall work here with the most general  $\ast$ -local $^\circ$ -solvability, which allows  $\mathrm{SL}_2$  in addition to  $\mathrm{PSL}_2$ . Notice that if  $G$  is  $\ast$ -local $^\circ$ -solvable, then it can contain elements  $x$  (of order necessarily finite) with  $C_G^\circ(x)$  nonsolvable. Following [DJ11a] such elements  $x$  are called *exceptional*, referring somehow to the central involution of  $\mathrm{SL}_2$ .

Since [CJ04] it seems unrealistic to hope for a complete reduction of  $\ast$ -local $^\circ$  solvable groups to the algebraic ones, even assuming the presence of involutions in order to proceed to a much sharper analysis. However a reduction to a very small number of configurations will be obtained in [DJ11b], along the lines of [CJ04] and subsequent papers. This can be seen as part of the Borovik program for classifying simple groups with involutions, in the utterly critical case of minimal configurations. We simply note that the present part of the analysis does not depend on  $p$  and can be reached by very general means (and some of them, such as signalizer functors, could be borrowed from finite group theory).

We also note that our Main Theorem generalizes the dichotomy represented by Sections 6 and 7 of [CJ04], concerning  $p = 2$  in minimal connected simple groups also satisfying a simplifying “tameness” assumption. Sections 6 (resp. 7) there corresponds, in Prüfer 2-rank at least 2, to  $C_G^\circ(\Omega_1(S))$  not being (resp. being) a Borel subgroup, two cases corresponding respectively to our cases (2) and (1) here. Then came [BCJ07] where our corresponding case (1) was shown not to exist, still for minimal connected simple groups and for  $p = 2$ , but without the “tameness” assumption. In [DJ11b] we will reach essentially the same conclusions in the much more general context of  $\ast$ -local $^\circ$ -solvable groups, applying the general dichotomy of the present paper with  $p = 2$ . In particular we will bound by 2, in full generality, the Prüfer 2-rank of a nonsolvable  $\ast$ -local $^\circ$  solvable group. But in any case, case (2) of our Main Theorem still stands around, even with  $p = 2$  and in the context of tame minimal connected simple groups of [CJ04] where the configuration is described with high precision.

We collect some raw material in §2 and the proof of our Main Theorem takes place in §3. In §4 we will make more comments on the difficulty to deal with the configuration arising in case (2); we will also give a form of our Main Theorem more directly applicable in [DJ11b] (essentially explaining how to deal which the two extra assumptions that every proper definable connected subgroup containing  $S$  is solvable and that elements of  $S$  of order  $p$  are not exceptional).

## 2 Preliminaries

For general reference on groups of finite Morley rank we refer to [BN94] or [ABC08], and for more specific facts about  $\ast$ -local $^\circ$  solvable groups of finite Morley rank we refer to [DJ11a]. We simply recall that groups of finite Morley rank satisfy the descending chain condition on definable subgroups, and that any group  $G$  of finite Morley rank has a *connected component*, i.e., a smallest (normal) definable subgroup of finite index, denoted by  $G^\circ$ .

## 2.1 Unipotence theory

A delicate point in our proof is the use of the abstract unipotence theory for groups of finite Morley rank. We follow the general treatment of [FJ08] and [DJ11a, §2.1], and for the sake of self-containment we shall try and put unipotence theory in a nutshell.  $\mathcal{P}$  denotes the set of prime numbers.

**Definition.** A unipotence parameter is a pair  $\tilde{p} = (p, r) \in (\{\infty\} \cup \mathcal{P}) \times (\mathbb{N} \cup \{\infty\})$  with  $p < \infty$  if and only if  $r = \infty$ . In addition,  $p$  is called the characteristic of  $\tilde{p}$  and  $r$  is called the unipotence degree of  $\tilde{p}$ .

Such a notion enables a parallel treatment to two theories which had been considered distinct, namely  $p$ -unipotence theory ( $p$  a prime), and Burdges' more recent notion of null characteristic graded unipotence which originated in [Bur04]. For  $p$  a prime, a  $p$ -unipotent subgroup is by definition a nilpotent definable connected  $p$ -group of bounded exponent, and for  $p = \infty$  a definition is given in terms of generation by certain definable connected abelian subgroups with torsion-free quotients of rank  $r$  [FJ08, §2.3]. In the extreme case where  $(p, r) = (\infty, 0)$  one makes use of so-called decent tori [Che05]. All this gives rise, for any group  $G$  of finite Morley rank, to the group  $U_{\tilde{p}}(G)$  as the group generated by subgroups as above. This group is always definable and connected, and for  $p$  a prime  $U_{(p, \infty)}(G)$  is simply denoted by  $U_p(G)$ .

Imposing nilpotence, it leads to a general notion of  $\tilde{p}$ -subgroup for any unipotence parameter  $\tilde{p}$ , and now imposing maximality with respect to inclusion, it provides a notion of *Sylow  $\tilde{p}$ -subgroup* [FJ08, §2.4], with properties much similar to those of Sylow  $p$ -subgroups of finite groups.

In groups of finite Morley rank, the *Fitting* subgroup, i.e., the group generated by all normal nilpotent subgroups, is always definable and nilpotent, and in particular the *unique* maximal normal nilpotent subgroup. If one lets  $d_p(H)$  be the maximal  $r$  such that  $H$  contains a non-trivial  $(p, r)$ -unipotent subgroup [DJ11a, Definition 2.5], then the interest for the unipotence theory comes from the following in connected solvable groups.

**Fact 2.1.** [DJ11a, Fact 2.8] *Let  $H$  be a connected solvable group of finite Morley rank and  $\tilde{p} = (p, r)$  a unipotence parameter with  $r > 0$ . Assume  $d_p(H) \leq r$ . Then  $U_{\tilde{p}}(H) \leq F^\circ(H)$ , and in particular  $U_{\tilde{p}}(H)$  is nilpotent.*

## 2.2 Uniqueness Theorem in $\ast$ -locally $^\circ$ -solvable groups

The main result about  $\ast$ -locally $^\circ$ -solvable groups is a Uniqueness Theorem analogous to a similar result of Bender about minimal simple groups in finite group theory. We give here the most general statement as proved in [DJ11a]. We will not use it as such, but the proof of our Main Theorem will be based on an elaborate version of this Uniqueness Theorem.

**Fact 2.2.** [DJ11a, Uniqueness Theorem 4.1] *Let  $G$  be a  $\ast$ -locally $^\circ$ -solvable group of finite Morley rank,  $\tilde{p} = (p, r)$  a unipotence parameter with  $r > 0$ ,*

and  $U$  a Sylow  $\tilde{p}$ -subgroup of  $G$ . Assume that  $U_1$  is a nontrivial definable  $\tilde{p}$ -subgroup of  $U$  containing a nonempty (possibly trivial) subset  $X$  of  $G$  such that  $d_p(C^\circ(X)) \leq r$ . Then  $U$  is the unique Sylow  $\tilde{p}$ -subgroup of  $G$  containing  $U_1$ , and in particular  $N(U_1) \leq N(U)$ .

A Borel subgroup is a maximal definable connected solvable subgroup. Fact 2.2 has the following consequence on Borel subgroups of  $\ast$ -locally $^\circ$ -solvable groups of finite Morley rank.

**Fact 2.3.** [DJ11a, Corollary 4.4] *Let  $G$  be a  $\ast$ -locally $^\circ$ -solvable group of finite Morley rank,  $\tilde{p} = (p, r)$  a unipotence parameter with  $r > 0$  such that  $d_p(G) = r$ . Let  $B$  be a Borel subgroup of  $G$  such that  $d_p(B) = r$ . Then  $U_{\tilde{p}}(B)$  is a Sylow  $\tilde{p}$ -subgroup of  $G$ , and if  $U_1$  is a nontrivial definable  $\tilde{p}$ -subgroup of  $B$ , then  $U_{\tilde{p}}(B)$  is the unique Sylow  $\tilde{p}$ -subgroup of  $G$  containing  $U_1$ ,  $N(U_1) \leq N(U_{\tilde{p}}(B)) = N(B)$ , and  $B$  is the unique Borel subgroup of  $G$  containing  $U_1$ .*

## 2.3 Torsion

The results we shall use about torsion are rather elementary. We first give the general decomposition of nilpotent groups of finite Morley rank, in terms of both classical Sylow  $p$ -subgroups and Sylow  $\tilde{p}$ -subgroups.

**Fact 2.4.** [DJ11a, Fact 2.3] *Let  $G$  be a nilpotent group of finite Morley rank.*

- (1)  *$G$  is the central product of its Sylow  $p$ -subgroups and its Sylow  $(\infty, r)$ -subgroups, which are all divisible for  $\tilde{p}$  of the form  $(\infty, r)$  [FJ08, p. 16].*
- (2) *If  $G$  is connected, then  $G$  is the central product of its Sylow  $\tilde{p}$ -subgroups.*

Throughout the rest of this subsection,  $p$  denotes a prime number. For an arbitrary subgroup  $S$  of a group of finite Morley rank, one defines the (generalized) *connected component* of  $S$  by  $S^\circ = H^\circ(S) \cap S$ , where  $H(S)$  is the smallest definable subgroup of  $G$  containing  $S$ . (Of course,  $H(S)$  exists by descending chain condition on definable subgroups.) It is easily checked that  $S^\circ$  has finite index in  $S$ .

**Fact 2.5.** [BN94, Corollary 6.20] *Let  $p$  be a prime and  $S$  a  $p$ -subgroup of a solvable group of finite Morley rank, or more generally a locally finite  $p$ -subgroup of any group of finite Morley rank. Then:*

- (1)  *$S^\circ$  is a central product of a  $p$ -torus and a  $p$ -unipotent subgroup.*
- (2) *If  $S$  is infinite and has bounded exponent, then  $Z(S)$  contains infinitely many elements of order  $p$ .*

We also recall that Sylow  $p$ -subgroups are conjugate in solvable groups of finite Morley rank [BN94, Theorem 9.35]. Furthermore they are connected in connected solvable groups of finite Morley rank by [BN94, Theorem 9.29]; in this case a Sylow  $p$ -subgroup  $S$  satisfies  $S = S^\circ$  and has a decomposition as in Fact 2.5 (1), a point that will be frequently used below.

If  $G$  is a group of finite Morley rank, we denote by  $O_{p'}(H)$  the largest normal definable connected subgroup without  $p$ -torsion. It exists by ascending chain condition on definable connected subgroups and the following elementary property of lifting of torsion valid in groups of finite Morley rank (or more generally in groups with the descending chain condition on definable subgroups):

**Fact 2.6.** [ABC08, Lemma 2.18] *Let  $G$  be a group of finite Morley rank,  $N$  a normal definable subgroup of  $G$ , and  $x$  an element of  $G$  such that  $x$  has finite order  $n$  modulo  $N$ . Then the coset  $xN$  contains an element of finite order, involving the same prime divisors as  $n$ .*

The following will be useful when dealing with  $p$ -strongly embedded subgroups.

**Lemma 2.7.** (Compare with [CJ04, Lemma 3.2]) *Let  $H$  be a connected solvable group of finite Morley rank such that  $U_p(H) = 1$ . Then  $H/O_{p'}(H)$  is divisible abelian.*

*Proof.* Dividing by  $O_{p'}(H)$ , we may assume it is trivial and we want to show that  $H$  is divisible abelian.

Let  $F = F^\circ(H)$ . As  $O_{p'}(H) = 1$ ,  $O_{p'}(F) = 1$  as well, and  $U_q(H) = 1$  for any prime  $q$  different from  $p$ . By assumption  $U_p(H) = 1$  also, and  $F$  is divisible by Fact 2.4. As  $F'$  is torsion-free, by [BN94, Theorem 2.9] or Fact 2.4 and [DJ11a, Corollary 2.2], it must be trivial by assumption. (In connected groups of finite Morley rank, derived subgroups are definable and connected by a well-known corollary of Zilber's indecomposability theorem.) Hence  $F$  is divisible abelian.

To conclude it suffices to show that  $F$  is central in  $H$ , as then  $H$  is nilpotent, hence equal to  $F$ , and hence divisible abelian, as desired. Let  $h$  be any element of  $H$ ; we want to show that  $[h, F] = 1$ . Notice that that map  $f \mapsto [h, f]$ , from  $F$  to  $F$ , is a definable group homomorphism. As the torsion subgroup of  $F$  is central in  $H$  (by [DJ11a, Fact 2.7 (1)], or using [DJ11a, Fact 2.1]), it is contained in the kernel of the previous map and Fact 2.6 shows that the image of the previous map, i.e.,  $[h, F]$ , is torsion-free. Hence  $[h, F] \leq O_{p'}(F) = 1$ , as desired.  $\square$

## 2.4 Generation by centralizers

In the present subsection we prove miscellaneous lemmas concerning generation by centralizers.

**Lemma 2.8.** *Let  $\tilde{p}$  be a unipotence parameter and  $q$  a prime number. Let  $H$  be a  $\tilde{p}$ -group of finite Morley rank without elements of order  $q$ , and assume  $K$  is a definable solvable  $q$ -group of automorphisms of  $H$  of bounded exponent. Then  $C_H(K)$  is a definable  $\tilde{p}$ -subgroup of  $H$ .*

*Proof.* By descending chain condition on centralizers,  $C_H(K)$  is the centralizer of a finitely generated subgroup of  $K$ , and by local finiteness of the latter we may assume  $K$  finite. In particular  $C_H(K)$  is connected by [Bur04, Fact 3.4].



When  $\tilde{p} = (\infty, 0)$ ,  $H$  is a good torus in the sense of [Che05], and in particular  $(\infty, 0)$ -homogeneous in the sense of [FJ08, Lemma 2.17], and the connected subgroup  $C_H(K)$  is also a good torus. Otherwise,  $C_H(K)$  is also a  $\tilde{p}$ -group, by [Bur04, Lemma 3.6] when the unipotence parameter is finite, or by [FJ08, Lemma 2.17-c] when the characteristic is finite.  $\square$

**Fact 2.9.** [Bur04, Fact 3.7] *Let  $H$  be a solvable group of finite Morley rank without elements of order  $p$  for some prime  $p$ . Let  $E$  be a finite elementary abelian  $p$ -group acting definably on  $H$ . Then:*

$$H = \langle C_H(E_0) \mid E_0 \leq E, [E : E_0] = p \rangle.$$

We recall that a *Carter* subgroup of a group of finite Morley is, by definition, a definable connected nilpotent subgroup of finite index in its normalizer.

**Lemma 2.10.** *Let  $H$  be a connected solvable group of finite Morley rank such that  $U_p(H) = 1$  for some prime  $p$ . Suppose that  $H$  contains an elementary abelian  $p$ -group  $E$  of order  $p^2$ . Then:*

$$H = \langle C_H^\circ(E_0) \mid E_0 \text{ is a cyclic subgroup of order } p \text{ of } E \rangle.$$

*Proof.* By assumption, Fact 2.5, and [BN94, Theorem 9.29], Sylow  $p$ -subgroups of  $H$  are  $p$ -tori. Hence  $E$  is in a maximal  $p$ -torus of  $H$ , which is included in a Carter subgroup  $Q$  of  $H$  by [FJ08, Theorem 3.3]. By Lemma 2.7,  $H/O_{p'}(H)$  is abelian. As Carter subgroups cover all abelian quotients in connected solvable groups of finite Morley rank by [FJ08, Corollary 3.13],  $H = O_{p'}(H) \cdot Q$ . As  $E \leq Z(Q)$ , it suffices to show that:

$$O_{p'}(H) = \langle C_{O_{p'}(H)}^\circ(E_0) \mid E_0 \text{ is a cyclic subgroup of order } p \text{ of } E \rangle.$$

But the generation by the full centralizers is given by Fact 2.9, and these centralizers are connected by [Bur04, Fact 3.4].  $\square$

A subgroup is called *p-toral*, or just *toral*, if it is contained in a  $p$ -torus of the ambient group.

**Lemma 2.11.** *Let  $H$  be a connected solvable group of finite Morley rank with an elementary abelian  $p$ -toral subgroup  $E$  of order  $p^2$  for some prime  $p$ . Then:*

$$H = \langle C_H^\circ(E_0) \mid E_0 \text{ is a cyclic subgroup of order } p \text{ of } E \rangle.$$

*Proof.* For a connected nilpotent group of finite Morley rank  $L$ , we define the “complement”  $C_p(L)$  of  $U_p(L)$ , namely the product of all factors of  $L$  as in Fact 2.4 (2), except  $U_p(L)$ .

Now if  $H$  is any connected solvable group of finite Morley rank and  $Q$  a Carter subgroup of  $H$ , then  $H = QF^\circ(H)$  by [FJ08, Corollaire 3.13], and  $H$  is the product of the definable connected subgroup  $C_p(Q)C_p(F^\circ(H))$  with the normal definable connected subgroup  $U_p(H)$ , and the first factor has trivial  $p$ -unipotent subgroups.

In our particular case,  $E$  is by torality contained in a  $p$ -torus, and the latter is contained in a Carter subgroup  $Q$  of  $H$  by [FJ08, Théorème 3.3]. By [BN94, Theorem 9.29] and Fact 2.5,  $E$  centralizes the normal definable connected subgroup  $U_p(H)$ , so it suffices to show the generation by the connected components of centralizers in  $C_p(Q)C_p(F^\circ(H))$ . But this follows from Lemma 2.10.  $\square$

### 3 Proof of our theorem

We now turn to proving our Main Theorem which we restate.

**Main Theorem.** *Let  $G$  be a connected nonsolvable  $\ast$ -locally $^\circ$ -solvable group of finite Morley rank of Prüfer  $p$ -rank at least 2 for some prime  $p$ , and fix a maximal  $p$ -torus  $S$  of  $G$ . Assume that every proper definable connected subgroup containing  $S$  is solvable, that elements of  $S$  of order  $p$  are not exceptional, and let*

$$B = \langle C^\circ(s) \mid s \in \Omega_1(S) \setminus \{1\} \rangle.$$

*Then:*

- (1) *either  $B < G$ , in which case  $B$  is a Borel subgroup of  $G$ ; and if in addition  $S$  is a Sylow  $p$ -subgroup of  $N_{N(B)}(S)$ ,  $N(B)$  is  $p$ -strongly embedded in  $G$ ,*
- (2) *or  $B = G$ , in which case  $S$  has Prüfer  $p$ -rank 2.*

**Here begins the proof.** Let  $M = N(B)$ . As  $B$  is definable (and connected) by Zilber's generation lemma,  $M$  is definable as well. As  $B$  contains a generous Carter subgroup  $Q$  of  $G$  containing  $S$  by [DJ11a, Fact 3.32], the conjugacy of generous Carter subgroups of [Jal06] and a Frattini argument give  $M = N(B) \subseteq BN(Q)$ , and as  $Q$  is almost selfnormalizing  $B = M^\circ$ . (With the same notation, this holds of course for an arbitrary  $p$ -torus  $S$  in an arbitrary group  $G$  of finite Morley rank.)

#### 3.1 Getting rid of (1)

Assume first

$$(1) \quad B < G.$$

By assumption  $B \leq B_1$  for some Borel subgroup  $B_1$  of  $G$ . As  $S \leq B \leq B_1$ , Lemma 2.11 implies that  $B = B_1$ , and thus  $B$  is a Borel subgroup of  $G$ .

In particular, Sylow  $p$ -subgroups of  $B$  are conjugate, as in any solvable group of finite Morley rank.

We now make the new assumption that  $S$  is a Sylow  $p$ -subgroup of  $N_{N(B)}(S)$ .

**Lemma 3.1.**  *$U_p(C(s)) = 1$  for every element  $s$  of order  $p$  of  $S$ .*

*Proof.* It suffices to apply Fact 2.5 in the connected solvable group  $B$ , again with the fact that Sylow  $p$ -subgroups of connected solvable groups of finite Morley rank are connected.  $\square$

We claim that  $M = N(B)$  is  $p$ -strongly embedded in  $G$  in this case by using a “black hole” principle (a term going back to Harada) similar to the one used in [BCJ07, §2.2], and already contained in [CJ04, Lemma 7.3]. We note that Lemma 3.1 implies that  $S$  is a Sylow  $p$ -subgroup of  $B$  indeed, and of  $M$  as well, as  $M = BN_{N(B)}(S)$  by a Frattini Argument. In particular  $M/B$  has trivial Sylow  $p$ -subgroups by lifting of torsion, Fact 2.6.

Assume that  $M \cap M^g$  contains an element  $s$  of order  $p$  for some  $g$  in  $G$ . Notice that  $s$  is actually in  $B \cap B^g$ , and  $p$ -toral. By connectedness and conjugacy of Sylow  $p$ -subgroups in connected solvable groups, the definition of  $B$  implies that  $C^\circ(s') \leq B$  for any element  $s'$  of order  $p$  of  $B$ . Similarly,  $C^\circ(s') \leq B^g$  whenever  $s'$  has order  $p$  and is in  $B^g$ . By conjugacy in  $B$  we may assume  $s$  in  $S$ , and  $\Omega_1(S) \leq S \leq C^\circ(s) \leq B \cap B^g$ . By Lemma 2.10 or 2.11 applied in  $B$  and in  $B^g$  we get  $B^g = \langle C^\circ(s) \mid s \in \Omega_1(S) \setminus \{1\} \rangle = B$ . Thus  $g$  normalizes  $B$ , and is in  $M$ .

Hence  $M = N(B)$  is  $p$ -strongly embedded in  $G$  under the extra assumption adopted here, and this proves clause (1) of the Main Theorem.

### 3.2 Case (2); $p$ -elements

We now pass to the second case

$$(2) \quad B = G.$$

We will eventually show that clause (2) of our Main Theorem holds by reworking the beginning of Section 6 of [CJ04]. We first put aside  $p$ -unipotent subgroups.

**Lemma 3.2.** *Any Borel subgroup containing a toral element of order  $p$  has trivial  $p$ -unipotent subgroups.*

*Proof.* Assuming the contrary, we may assume after conjugacy of decent tori [Che05] that a Borel subgroup  $L$  with  $U_p(L)$  nontrivial contains an element  $s$  of  $S$  of order  $p$ . Then  $U_p(C(s))$  is nontrivial by Fact 2.5, contained in a unique Borel subgroup  $B_1$  of  $G$  by Fact 2.3. (Actually  $B_1 = L$ .) By Fact 2.3,  $B_1$  is the unique Borel subgroup containing any given nontrivial  $p$ -unipotent subgroup of  $U_p(C(s))$ . Now any element  $s'$  of order  $p$  of  $S$  normalizes  $U_p(C(s))$ , and thus  $U_p(C(s, s')) \neq 1$  by Fact 2.5, and as  $B_1$  is the unique Borel subgroup containing the latter group we get  $C^\circ(s') \leq B_1$ . This shows that  $B \leq B_1$ , a contradiction as  $B = G$  is nonsolvable under the current assumption.  $\square$

In other words, nontrivial  $p$ -toral elements commute with no nontrivial  $p$ -unipotent subgroups. This can be stated more carefully as follows.

**Corollary 3.3.** *Any connected solvable subgroup which is  $\langle s \rangle$ -invariant for some  $p$ -toral element  $s$  of order  $p$  has trivial  $p$ -unipotent subgroups.*

*Proof.* Otherwise  $s$  would normalize a nontrivial  $p$ -unipotent subgroup, and by Fact 2.5 it would centralize a nontrivial  $p$ -unipotent subgroup.  $\square$

Our assumption (2) on  $B$  yields similarly a property antisymmetric to the black hole principle implied by assumption (1). Let  $E$  denote the elementary abelian  $p$ -group  $\Omega_1(S)$ .

**Lemma 3.4.** *Let  $E_1$  be a subgroup of  $E$  of order at least  $p^2$ . Then for any proper definable connected subgroup  $L$  there exists an element  $s$  of order  $p$  of  $E_1$  such that  $C^\circ(s) \not\leq L$ . In particular  $G = \langle C^\circ(s) \mid s \in E_1 \setminus \{1\} \rangle$ .*

*Proof.* Assume on the contrary  $C^\circ(s) \leq L$  for any element  $s$  of order  $p$  of  $E_1$ .

We claim that  $C^\circ(t) \leq L$  for any element  $t$  of order  $p$  of  $E$ . In fact, as  $E_1 \leq S \leq C^\circ(t)$ ,  $C^\circ(t)$  is by Lemma 2.10 generated by its subgroups of the form  $C^\circ(t, s)$ , with  $s$  of order  $p$  in  $E_1$ . As these groups are all contained in  $L$  by assumption, our claim follows.

Hence we have  $B \leq L < G$ . But under our current assumption  $B = G$ , and this is a contradiction.

Our last claim follows immediately.  $\square$

**Corollary 3.5.** *There exists an element  $s$  of order  $p$  of  $E$  such that  $C^\circ(S) < C^\circ(s)$ .*

*Proof.*  $C^\circ(S)$  is  $S$ -local $^\circ$ , and thus solvable by  $\ast$ -local $^\circ$  solvability of  $G$  and [DJ11a, Lemma 3.4]. As  $C^\circ(S) \leq C^\circ(s)$  for any element  $s$  of order  $p$  of  $S$ , it suffices to apply Lemma 3.4.  $\square$

Recall from [DJ11a, Definition 2.5] that for any group  $G$  of finite Morley rank,  $d(G)$  denotes the maximum of  $d_\infty(G)$  and of  $\max_{p \in \mathcal{P}}(d_p(G))$ , i.e., the infinite symbol if  $G$  contains a nontrivial  $p$ -unipotent subgroup, and otherwise the maximal  $r$  in  $\mathbb{N}$  such that  $G$  contains a nontrivial  $(\infty, r)$ -subgroup. This number exists and belongs to  $\mathbb{N} \sqcup \{\infty\}$  when  $G$  is infinite [DJ11a, Lemma 2.6].

**Lemma 3.6.** *There exists an element  $s$  of order  $p$  of  $E$  such that*

$$d(O_{p'}(C^\circ(s))) \geq 1.$$

*Proof.* Assume the contrary, and let  $s$  be an arbitrary element of order  $p$  of  $E$ . By our assumption that  $d(O_{p'}(C^\circ(s))) \leq 0$ ,  $O_{p'}(C^\circ(s))$  is trivial or a good torus by [DJ11a, Lemma 2.6], and central in  $C^\circ(s)$  by [DJ11a, Fact 2.7 (1)]. Notice that  $U_p(C^\circ(s)) = 1$  by Lemma 3.2. As  $C^\circ(s)/O_{p'}(C^\circ(s))$  is abelian by Lemma 2.7,  $C^\circ(s)$  is nilpotent. Now  $S$  is central in  $C^\circ(s)$  by Fact 2.4 (2). In particular  $C^\circ(S) = C^\circ(s)$ , and this holds for any element  $s$  of order  $p$  of  $E$ . We get a contradiction to Corollary 3.5.  $\square$

It follows in particular from Lemma 3.6 that there exist definable connected subgroups  $L$  containing  $C^\circ(s)$  for some element  $s$  of order  $p$  of  $E$  and such that  $O_{p'}(L)$  is not a good torus. Choose then a unipotence parameter  $\tilde{q} = (q, r)$

different from  $(\infty, 0)$  such that  $r$  is maximal in the set of all  $d_q(O_{p'}(L))$ , where  $L$  varies in the set of all definable connected solvable subgroups with the above property.

Notice that there might exist several such maximal unipotence parameters  $\tilde{q}$ , maybe one for  $q = \infty$  and several ones for  $q$  prime, except for  $q = p$  by Corollary 3.3.

It will also be shortly and clearly visible below that the notion of maximality for  $\tilde{q}$  is the same when  $L$  varies in two smaller subsets of all definable connected solvable subgroups containing  $C^\circ(s)$  for some  $s$  of order  $p$  of  $E$ : the set of Borel subgroups with this property on the one hand, and exactly the finite set of subgroups of the form  $C^\circ(s)$  on the other.

**Lemma 3.7.** *Let  $L$  be any definable connected solvable subgroup containing  $C^\circ(s)$  for some element  $s$  of order  $p$  of  $E$ . Then  $U_{\tilde{q}}(O_{p'}(L))$  is a normal definable connected nilpotent subgroup of  $L$ .*

*Proof.* As  $O_{p'}(L)$  is normal in  $L$ , it suffices to show that its definably characteristic subgroup  $U_{\tilde{q}}(O_{p'}(L))$  is nilpotent. But the latter is in  $F(O_{p'}(L))$  by Fact 2.1 and the maximality of  $r$ .  $\square$

**Corollary 3.8.** *Let  $L$  be any definable connected solvable subgroup containing  $C^\circ(s)$  for some element  $s$  of order  $p$  of  $E$ . Then any definable  $\tilde{q}$ -subgroup of  $L$  without elements of order  $p$  is in  $U_{\tilde{q}}(F(O_{p'}(L)))$ .*

*Proof.* Let  $U$  be such a subgroup. As  $U_p(L) = 1$  by Lemma 3.2,  $L/O_{p'}(L)$  is (divisible) abelian by Lemma 2.7, and thus  $U \leq O_{p'}(L)$ , and  $U \leq U_{\tilde{q}}(O_{p'}(L))$ . Now it suffices to apply the normality and the nilpotence of the latter.  $\square$

### 3.3 A Uniqueness Theorem via elementary $p$ -groups

We now prove a version of the Uniqueness Theorem (Fact 2.2) with a combined action, more precisely where the assumption on unipotence degrees of centralizers is replaced by an assumption of invariance by a sufficiently “large”  $p$ -toral subgroup. For this purpose we first note the following.

**Lemma 3.9.** *Let  $E_1$  be a subgroup of order at least  $p^2$  of  $E$ , and  $H$  a definable connected solvable  $E_1$ -invariant subgroup. Then  $d_q(O_{p'}(H)) \leq r$ .*

*Proof.* Assume toward a contradiction  $r' > r$ , where  $r'$  denotes  $d_q(O_{p'}(H))$ . In this case  $r$  is necessarily finite, and  $q = \infty$ . By Fact 2.1,  $U_{(\infty, r')}(O_{p'}(H)) \leq F^\circ(O_{p'}(H))$ , and this nontrivial definable  $(\infty, r')$ -subgroup is  $E_1$ -invariant. Fact 2.9 gives an element  $s$  of order  $p$  in  $E_1$  such that

$$C_{U_{(\infty, r')}(O_{p'}(H))}(s) \neq 1.$$

But the latter is an  $(\infty, r')$ -group by Lemma 2.8. Now considering the definable connected solvable subgroup  $C^\circ(s)$  gives a contradiction to the maximality of  $r$ , as  $C^\circ(s)/O_{p'}(C^\circ(s))$  is (divisible) abelian as usual and the centralizer above is connected without elements of order  $p$ , and thus contained in  $O_{p'}(C^\circ(s))$ .  $\square$

As already mentioned around the definition of maximal parameters  $\tilde{q}$  (after Lemma 3.6), the same argument shows that  $r$  is also exactly the maximum of the  $d_q(O_{p'}(L))$  different from 0, with  $L$  varying in the set of *Borel* subgroups containing  $C^\circ(s)$  for some element  $s$  of order  $p$  of  $E$  (instead of all definable connected solvable subgroups  $L$  with the same property), and similarly with  $L$  varying in the set of subgroups  $C^\circ(s)$  for some element  $s$  of order  $p$  of  $E$ .

We now prove our version of the Uniqueness Theorem, Fact 2.2, specific to the configuration considered here.

**Theorem 3.10.** *Let  $E_1$  be a subgroup of order at least  $p^2$  of  $E$ . Then any  $E_1$ -invariant nontrivial definable  $\tilde{q}$ -subgroup without elements of order  $p$  is contained in a unique maximal such.*

*Proof.* Let  $U_1$  be the  $\tilde{q}$ -subgroup under consideration. Let  $U$  be a maximal  $E_1$ -invariant definable  $\tilde{q}$ -subgroup without elements of order  $p$  containing  $U_1$ .

Assume  $V$  is another such subgroup, distinct from  $U$ , and chosen so as to maximize the rank of  $U_2 = U_{\tilde{q}}(U \cap V)$ . As  $1 < U_1 \leq U_2$ , the subgroup  $U_2$  is nontrivial. As  $U_2$  is nilpotent,  $N := N^\circ(U_2)$  is solvable by  $\ast$ -local $^\circ$  solvability of  $G$ . Note that  $U_2 < U$ , as otherwise  $U = U_2 \leq V$  and  $U = V$  by maximality of  $U$ . Similarly  $U_2 < V$ , as otherwise  $V = U_2 \leq U$  and  $V = U$  by maximality of  $V$ . In particular, by normalizer condition [FJ08, Proposition 2.8],  $U_2 < U_{\tilde{q}}(N_U(U_2))$  and  $U_2 < U_{\tilde{q}}(N_V(U_2))$ .

We claim that  $d_q(O_{p'}(N)) = r$ . Actually  $d_q(O_{p'}(N)) \leq r$  by Lemma 3.9, and as  $O_{p'}(N)$  contains  $U_2$  which is nontrivial and of unipotence degree  $r$  in characteristic  $q$  we get  $d_q(O_{p'}(N)) = r$ .

By Fact 2.1 and the fact that  $r \geq 1$  we get  $U_{\tilde{q}}(O_{p'}(N)) \leq F^\circ(O_{p'}(N))$ . In particular  $U_{\tilde{q}}(O_{p'}(N))$  is nilpotent, and contained in a maximal definable  $E_1$ -invariant  $\tilde{q}$ -subgroup without elements of order  $p$ , say  $\Gamma$ . Notice that  $N$ , being  $E_1$ -invariant, satisfies  $U_p(N) = 1$ , and  $N/O_{p'}(N)$  is abelian as usual. Now  $U_1 \leq U_2 < U_{\tilde{q}}(N_U(U_2)) \leq \Gamma$ , so our maximality assumption implies that  $\Gamma = U$ . In particular  $U_{\tilde{q}}(N_V(U_2)) \leq \Gamma = U$ . But then  $U_2 < U_{\tilde{q}}(N_V(U_2)) \leq U_{\tilde{q}}(U \cap V) = U_2$ , a contradiction.  $\square$

**Corollary 3.11.** *Let  $E_1$  be a subgroup of order at least  $p^2$  of  $E$ .*

- (1) *If  $U_1$  is a nontrivial  $E_1$ -invariant definable  $\tilde{q}$ -subgroup without elements of order  $p$ , then  $U_1$  is contained in a unique maximal  $E_1$ -invariant definable connected solvable subgroup  $B$ . Furthermore  $U_{\tilde{q}}(O_{p'}(B))$  is the unique maximal  $E_1$ -invariant definable  $\tilde{q}$ -subgroup without elements of order  $p$  containing  $U_1$ , and, for any element  $s$  of order  $p$  of  $E_1$  with a nontrivial centralizer in  $U_1$ ,  $C^\circ(s) \leq B$  and  $B$  is a Borel subgroup of  $G$ .*
- (2)  *$U_{\tilde{q}}(O_{p'}(C^\circ(E_1)))$  is trivial.*

*Proof.* (1). Assume  $B_1$  and  $B_2$  are two maximal  $E_1$ -invariant definable connected solvable subgroups containing  $U_1$ . We have  $U_p(B_1) = U_p(B_2) = 1$ . Hence  $B_1$  and  $B_2$  are both abelian modulo their  $O_{p'}$  subgroups.

Let  $U = U_{\tilde{q}}(O_{p'}(B_1 \cap B_2))$ . This group contains  $U_1$  and is in particular nontrivial, and is  $E_1$ -invariant, as well as  $U_{\tilde{q}}(O_{p'}(B_1))$  and  $U_{\tilde{q}}(O_{p'}(B_2))$ . Now all these three subgroups are contained in a (unique) common maximal  $E_1$ -invariant definable  $\tilde{q}$ -subgroup without elements of order  $p$  by the Uniqueness Theorem 3.10, say  $\tilde{U}$ . Notice that  $B_1 = N^\circ(U_{\tilde{q}}(O_{p'}(B_1)))$  and  $B_2 = N^\circ(U_{\tilde{q}}(O_{p'}(B_2)))$  by maximality of  $B_1$  and  $B_2$ . Now applying the normalizer condition [FJ08, Proposition 2.8] in the subgroup  $\tilde{U}$  without elements of order  $p$  yields easily  $U_{\tilde{q}}(O_{p'}(B_1)) = \tilde{U} = U_{\tilde{q}}(O_{p'}(B_2))$ . Taking their common connected normalizers,  $B_1 = B_2$ .

Our next claim follows from the same argument.

For the last claim, we note that there exists an element  $s$  in  $E_1$  of order  $p$  such that  $C_{U_1}(s)$  is nontrivial. By Lemma 2.8 the latter is a  $\tilde{q}$ -group, and of course it is  $E_1$ -invariant. So the preceding uniqueness applies to  $C_{U_1}(s)$ , and as  $C_{U_1}(s) \leq U_1 \leq B$  we get that  $B$  is the unique maximal  $E_1$ -invariant definable connected solvable subgroup containing  $C_{U_1}(s)$ . But  $C_{U_1}(s) \leq C^\circ(s) \leq B_s$  for some Borel subgroup  $B_s$  and  $E_1 \leq B_s$ , so  $B_s$  satisfies the same conditions as  $B$ , so  $B_s \leq B$  and  $B = B_s$  is a Borel subgroup of  $G$ .

(2). Suppose toward a contradiction  $U := U_{\tilde{q}}(O_{p'}(C^\circ(E_1)))$  nontrivial. It is of course  $E_1$ -invariant. Recall that  $Q$  is a fixed Carter subgroup of  $G$  containing the maximal  $p$ -torus  $S$ . As  $Q \leq C^\circ(E_1)$ ,  $Q$  normalizes the subgroup  $U$ . Now for any element  $s$  of order  $p$  in  $E_1$  we have  $UQ \leq C^\circ(s)$ .

As  $E_1 \leq Q$ , any Borel subgroup containing  $UQ$  is  $E_1$ -invariant, and by the first point there is a *unique* Borel subgroup containing  $UQ$ . Now  $C^\circ(s)$  is necessarily contained in this unique Borel subgroup containing  $UQ$ , and this holds for any element  $s$  of order  $p$  of  $E_1$ . We get a contradiction to Lemma 3.4.  $\square$

We note that the proof of the second point in Corollary 3.11 actually shows that any definable connected subgroup containing  $E_1$  and  $U_1$  for some nontrivial  $E_1$ -invariant definable  $\tilde{q}$ -subgroup  $U_1$  without elements of order  $p$  is contained in a *unique* Borel subgroup of  $G$ . Furthermore with the notation of Corollary 3.11 (1) we have in any case  $N(U_1) \cap N(E_1) \leq N(U_{\tilde{q}}(O_{p'}(B))) = N(B)$ .

### 3.4 Bounding the Prüfer rank via “signalizer functors”

There are two possible ways to prove that the Prüfer  $p$ -rank is 2 at this stage. One may use the Uniqueness Theorem 3.10 provided by the  $\ast$ -local $^\circ$  solvability of the ambient group, or use the general signalizer functor theory, which gives similar consequences in more general contexts. We now explain how to use the signalizer functor theory to get the bound on the Prüfer  $p$ -rank, but we will rather continue the analysis with the Uniqueness Theorem 3.10 which is closer in spirit to [CJ04, Lemma 6.1] and our original proof. It also gives much more information in the specific context under consideration, including when the Prüfer  $p$ -rank is 2, while the general signalizer functor theory just provides the bound.

For  $s$  a nontrivial element of  $E$  we let

$$\theta(s) = U_{\tilde{q}}(O_{p'}(C(s))).$$

If  $t$  is another nontrivial element of  $E$ , then it normalizes the connected nilpotent  $\tilde{q}$ -group without  $p$ -elements of order  $\theta(s)$ , and by Lemmas 2.8 and 2.7,  $C_{\theta(s)}(t) \leq U_{\tilde{q}}(O_{p'}(C(t))) = \theta(t)$ . Hence one has the two following properties:

- (1)  $\theta(s)^g = \theta(s^g)$  for any  $s$  in  $E \setminus \{1\}$  and any  $g$  in  $G$ .
- (2)  $\theta(s) \cap C_G(t) \leq \theta(t)$  for any  $s$  and  $t$  in  $E \setminus \{1\}$ .

In the parlance of finite group theory one says that  $\theta$  is an *E-signalizer functor* on  $G$ . In groups of finite Morley rank one says that  $\theta$  is a *connected nilpotent E-signalizer functor*, as any  $\theta(s)$  is connected (by definition) and nilpotent, which follows from Corollary 3.8. When  $E_1$  is a subgroup of  $E$  one defines

$$\theta(E_1) = \langle \theta(s) \mid s \in E_1 \setminus \{1\} \rangle.$$

In groups of finite Morley rank there is no “Solvable Signalizer Functor Theorem” available as in the finite case [Asc93, Chapter 15] (see [Gol72a, Gol72b, Gla76, Ben75] for the history of the finite case). However Borovik imported from finite group theory a “Nilpotent Signalizer Functor Theorem” for groups of finite Morley rank [Bor95] [BN94, Theorem B.30], stated as follows in [Bur04, Theorem A.2] (and which suffices by the unipotence theory of [Bur04] for which it has been designed originally).

**Fact 3.12. (Nilpotent Signalizer Functor Theorem)** *Let  $G$  be a group of finite Morley rank,  $p$  a prime, and  $E \leq G$  a finite elementary abelian  $p$ -group of order at least  $p^3$ . Let  $\theta$  be a connected nilpotent  $E$ -signalizer functor. Then  $\theta(E)$  is nilpotent. Furthermore  $\theta(E) = O_{p'}(\theta(E))$  and  $\theta(s) = C_{\theta(E)}(s)$  for any  $s$  in  $E \setminus \{1\}$ .*

(In the finite group theory terminology one says that  $\theta$  is *complete* when it satisfies the two properties of the last statement.)

In our situation one thus has, assuming toward a contradiction the Prüfer  $p$ -rank to be at least 3, that  $\theta(E)$  is nilpotent. Notice that the definable connected subgroup  $\theta(E)$  is nontrivial, as  $\theta(s)$  is nontrivial at least for some  $s$  by Lemma 2.8 and Fact 2.9. In particular  $N^\circ(\theta(E))$  is solvable by  $*$ -local $^\circ$  solvability of  $G$ .

From this point on one can use arguments formally identical to those of [Bor95, §6.2-6.3] used there for dealing with “proper 2-generated cores”.

If  $E_1$  and  $E_2$  are two subgroups of  $E$  of order at least  $p^2$ , then for any  $s$  in  $E_1 \setminus \{1\}$  one has  $\theta(s) \leq \langle C_{\theta(s)}(t) \mid t \in E_2 \setminus \{1\} \rangle \leq \theta(E_2)$  and thus  $\theta(E_1) = \theta(E_2)$ .

In particular  $\theta(E) = \theta(E_1)$  for any subgroup  $E_1$  of  $E$  of order at least  $p^2$ .

Now if  $g$  in  $G$  normalizes such a subgroup  $E_1$ , then  $\theta(E)^g = \theta(E_1)^g = \theta(E_1^g) = \theta(E_1) = \theta(E)$  and thus  $g \in N(\theta(E))$ .

Take now as in Lemma 3.4 an element  $s$  of order  $p$  in  $E$  such that  $C^\circ(s) \not\leq N^\circ(\theta(E))$ .



Then, still assuming  $E$  of order at least  $p^3$ , there exists a subgroup  $E_2$  of  $E$  of order at least  $p^2$  and disjoint from  $\langle s \rangle$ . By Lemma 2.10,

$$C^\circ(s) = \langle C_{C^\circ(s)}(t) \mid t \in E_2 \setminus \{1\} \rangle.$$

But now if  $t$  is in  $E_2$  as in the above equality, then  $E_1 := \langle s, t \rangle$  has order  $p^2$  as  $E_2$  is disjoint from  $\langle s \rangle$ , hence  $C_{C^\circ(s)}(t) \leq C(s, t) \leq N(\langle s, t \rangle) = N(E_1) \leq N(\theta(E))$ , and this shows that  $C^\circ(s) \leq N^\circ(\theta(E))$ . This is a contradiction, and as our only extra assumption was that the Prüfer  $p$ -rank was at least 3, it must be 2.

### 3.5 Bounding the Prüfer rank: Uniqueness Methods

Anyway, we can get the bound similarly, by using more directly the Uniqueness Theorem 3.10 here instead of the axiomatized signalizer functor machinery. Actually the proof below is the core of the proof of the Nilpotent Signalizer Functor Theorem, and the Uniqueness Theorem here shortcuts the passage to a quotient for the induction in the general case (see [Bur04]).

**Theorem 3.13.**  *$S$  has Prüfer  $p$ -rank 2.*

*Proof.* Assume towards a contradiction  $E$  has order at least  $p^3$ .

We then claim that there exists a *unique* maximal nontrivial  $E$ -invariant definable  $\tilde{q}$ -subgroup without elements of order  $p$ . Let  $U_1$  and  $U_2$  be two such subgroups. Then by Lemma 2.8 and Fact 2.9  $C_{U_1}(E_1)$  and  $C_{U_2}(E_2)$  are nontrivial  $\tilde{q}$ -subgroups for some subgroups  $E_1$  and  $E_2$  of  $E$ , each of index  $p$  in  $E$ . Assuming  $|E| \geq p^3$  then gives an element  $s$  of order  $p$  in  $E_1 \cap E_2$ . Now  $C_{U_1}(s)$  and  $C_{U_2}(s)$  are nontrivial  $\tilde{q}$ -subgroups by Lemma 2.8. Clearly both are  $E$ -invariant, as  $E$  centralizes  $s$ , and included in  $U_{\tilde{q}}(O_{p'}(C^\circ(s)))$  as usual, which is also  $E$ -invariant. Now the Uniqueness Theorem 3.10 gives  $U_1 = U_2$ , as desired.

Hence there is a unique maximal  $E$ -invariant definable  $\tilde{q}$ -subgroup without elements of order  $p$ , say “ $\theta(E)$ ” in the notation of the signalizer functor theory. For the same reasons as mentioned above, Lemma 2.8 and Fact 2.9, it is nontrivial.

Now by Lemma 2.8 and Fact 2.9 again,  $C_{\theta(E)}(E_1)$  is a nontrivial definable  $\tilde{q}$ -subgroup of  $\theta(E)$  for some subgroup  $E_1$  of  $E$  of index  $p$ . As  $U_p(C^\circ(E_1)) = 1$ , the quotient  $C^\circ(E_1)/O_{p'}(C^\circ(E_1))$  is abelian as usual, and the definable connected subgroup  $C_{\theta(E)}(E_1)$  is in  $O_{p'}(C^\circ(E_1))$ , and in  $U_{\tilde{q}}(O_{p'}(C^\circ(E_1)))$ .

But as  $|E| \geq p^3$ ,  $|E_1| \geq p^2$ , and we get a contradiction to Corollary 3.11 (2).  $\square$

This proves clause (2) of the Main Theorem and completes its proof.  $\square$

## 4 Afterword

We can also record informally some information gained along the proof of case (2) of our Main Theorem, which can be compared to [CJ04, 6.1-6.6]. We let  $G$  and  $S$  be as in case (2) of the Main Theorem and  $Q$  be a Carter subgroup of

$G$  containing  $S$ . Then  $Q$  is contained in at least two distinct Borel subgroups of  $G$  by Lemma 3.4, and in particular  $Q$  is divisible abelian by Fact 2.3 and [DJ11a, Proposition 4.46]. Now there are unipotence parameters  $\tilde{q} \neq (\infty, 0)$  as in the proof of case (2) of the Main Theorem (maybe one for  $q = \infty$ , several for  $q$  prime, but none for  $q = p$  by Lemma 3.2). All the results of the above analysis apply, now with  $|\Omega_1(S)| = p^2$  necessarily.

By Corollary 3.11,

$$U_{\tilde{q}}(O_{p'}(C^\circ(\Omega_1(S)))) = 1.$$

As  $\Omega_1(S)$  has order  $p^2$ , it contains in particular

$$\frac{p^2 - 1}{p - 1} = p + 1$$

pairwise noncollinear elements. It follows that there are at most  $p + 1$  nontrivial subgroups of the form  $U_{\tilde{q}}(O_{p'}(C^\circ(s)))$  for some nontrivial element  $s$  of order  $p$  of  $S$ , and at most  $p + 1$  Borel subgroups  $B$  containing  $Q$  (actually  $\Omega_1(S)$ -invariant suffices as noticed after Corollary 3.11) and such that  $U_{\tilde{q}}(O_{p'}(B)) \neq 1$ . By Corollary 3.11, any such Borel subgroup would contain  $C^\circ(s)$  for any element  $s$  of order  $p$  of  $S$  having a nontrivial centralizer in  $U_{\tilde{q}}(O_{p'}(B))$ , and  $\Omega_1(S)$  has a trivial centralizer in  $U_{\tilde{q}}(O_{p'}(B))$ .

The following corollary to the Main Theorem will be of crucial use in [DJ11b] to get a bound on Prüfer 2-ranks.

**Corollary 4.1.** *Let  $G$  be a connected nonsolvable  $\ast$ -locally $\mathcal{F}_\infty$ -solvable group of finite Morley rank and of Prüfer  $p$ -rank at least 2 for some prime  $p$ , and fix a maximal  $p$ -torus  $S$  of  $G$ . Let  $X$  be a maximal exceptional (finite) subgroup of  $S$  (as in [DJ11a, Lemma 3.29]),  $\bar{H} = C^\circ(X)/X$ ,  $\bar{K}$  a minimal definable connected nonsolvable subgroup of  $\bar{H}$  containing  $\bar{S}$ , and let*

$$\bar{B} = \langle C_{\bar{K}}^\circ(\bar{s}) \mid \bar{s} \in \Omega_1(\bar{S}) \setminus \{\bar{1}\} \rangle.$$

*Then:*

- (1) *either  $\bar{B} < \bar{K}$ , in which case  $\bar{B}$  is a Borel subgroup of  $\bar{K}$ ; and if in addition  $\bar{S}$  is a Sylow  $p$ -subgroup of  $N_{N_{\bar{K}}(\bar{B})}(\bar{S})$ , then  $N_{\bar{K}}(\bar{B})$  is  $p$ -strongly embedded in  $\bar{K}$ ,*
- (2) *or  $\bar{B} = \bar{K}$ , in which case  $\bar{S}$ , as well as  $S$ , has Prüfer  $p$ -rank 2.*

*Proof.* It suffices to apply our Main Theorem in  $\bar{K}$ . We note that  $\bar{S}$  and  $S$  have the same Prüfer  $p$ -rank, as  $X$  is finite by [DJ11a, Lemma 3.18].  $\square$

Cases (1) and (2) of the Main Theorem and Corollary 4.1 correspond respectively to Sections 7 and 6 of [CJ04] in presence of divisible torsion.

For  $p = 2$  case (1) will entirely disappear in [DJ11b] by an argument similar to the one used in [BCJ07, Case I] for minimal connected simple groups.

## References

- [ABC08] T. Altinel, A. V. Borovik, and G. Cherlin. *Simple groups of finite Morley rank*, volume 145 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.
- [Asc93] M. Aschbacher. *Finite group theory*, volume 10 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993. Corrected reprint of the 1986 original.
- [BCJ07] J. Burdges, G. Cherlin, and E. Jaligot. Minimal connected simple groups of finite Morley rank with strongly embedded subgroups. *J. Algebra*, 314(2):581–612, 2007.
- [Ben75] H. Bender. Goldschmidt’s 2-signalizer functor theorem. *Israel J. Math.*, 22(3-4):208–213, 1975.
- [BN94] A. Borovik and A. Nesin. *Groups of finite Morley rank*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
- [Bor95] A. V. Borovik. Simple locally finite groups of finite Morley rank and odd type. In *Finite and locally finite groups (Istanbul, 1994)*. Kluwer Acad. Publ., Dordrecht, 1995.
- [BP90] A. V. Borovik and B. P. Poizat. Tores et  $p$ -groupes. *J. Symbolic Logic*, 55(2):478–491, 1990.
- [Bur04] J. Burdges. A signalizer functor theorem for groups of finite Morley rank. *J. Algebra*, 274(1):215–229, 2004.
- [Che79] G. Cherlin. Groups of small Morley rank. *Ann. Math. Logic*, 17(1-2):1–28, 1979.
- [Che05] G. Cherlin. Good tori in groups of finite Morley rank. *J. Group Theory*, 8(5):613–622, 2005.
- [CJ04] G. Cherlin and E. Jaligot. Tame minimal simple groups of finite Morley rank. *J. Algebra*, 276(1):13–79, 2004.
- [DJ11a] A. Deloro and E. Jaligot. Groups of finite Morley rank with solvable local subgroups. To appear in *Communications in Algebra*, 2011.
- [DJ11b] A. Deloro and E. Jaligot. Groups of finite Morley rank with solvable local subgroups and of odd type. In preparation, 2011.
- [FJ08] O. Frécon and E. Jaligot. Conjugacy in groups of finite Morley rank. In *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Math. Soc. Lecture Note Ser.*, pages 1–58. Cambridge Univ. Press, Cambridge, 2008.

- [Gla76] G. Glauberman. On solvable signalizer functors in finite groups. *Proc. London Math. Soc. (3)*, 33(1):1–27, 1976.
- [Gol72a] D. M. Goldschmidt. 2-signalizer functors on finite groups. *J. Algebra*, 21:321–340, 1972.
- [Gol72b] D. M. Goldschmidt. Solvable signalizer functors on finite groups. *J. Algebra*, 21:137–148, 1972.
- [Jal01] E. Jaligot. Full Frobenius groups of finite Morley rank and the Feit-Thompson theorem. *Bull. Symbolic Logic*, 7(3):315–328, 2001.
- [Jal06] E. Jaligot. Generix never gives up. *J. Symbolic Logic*, 71(2):599–610, 2006.