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# GENERALIZATION OF ISOMORPHISM THEOREMS GROUPS TO PARTIAL GROUPS

#### YAHYA N'DAO AND ADLENE AYADI

ABSTRACT. In this paper, we define a new structure analogous to group, called partial group. This structure concerns the partial stability by the composition inner law. We generalize the three isomorphism theorems for groups to partial groups.

## 1. Introduction

A semigroup is an algebraic structure consisting of a set together with an associative binary operation. A semigroup generalizes a monoid in that a semigroup need not have an identity element. It also (originally) generalized a group (a monoid with all inverses) to a type where every element did not have to have an inverse, thus the name semigroup(See [1], [2]). Several other algebraic structure generalize the notion of group as quasigroup, hypergroup. In abstract algebra, a quasigroup is an algebraic structure resembling a group in the sense that "division" is always possible. Quasigroups differ from groups mainly in that they need not be associative. A quasigroup with an identity element is called a loop(see [3], [4]). In [5], Kamran extended the notion of AG-groupoid to AG-group. Later on the significant results on the topic were published in [6]. An AG-group is a generalization of abelian group and a special case of quasigroup. The structure of AG-group is a very interesting structure in which one has to play with brackets. There is no commutativity or associativity in general.

We are interested in this paper to introduce a lower structure than those of groups structure, called *partial group*. Most of algebraic structures like groups, semigroups, quasigroup and monoid all contain an inner composition law which is total (i.e. the law acts on any element of a set G and remains there). Order our new structure is based on the partial action of this law. There is a subset E of G which is stable by the law and it admits the group structure. Firstly, the set G is chosen as a part of a group  $\Gamma$  and it is defined as a product (by the law of  $\Gamma$ ) of a subgroup E of  $\Gamma$  with a part D which is free with E, that is to say D is contained in a group  $\widetilde{D}$  satisfying  $\Gamma = E.\widetilde{D} := \{xd : x \in E, d \in \widetilde{D}\}$  and  $\widetilde{D} \cap E = \{e\}$ , where e is the identity element of  $\Gamma$ . We say that  $\widetilde{D}$  is supplement of E in  $\Gamma$ . We define a partial inner law '.' in G as follows: Let  $a, b \in G$  and write a = xd and

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b = yd' with  $x, y \in E$  and  $d, d' \in D$ , one has: a.b = xy. This is means that

$$': G \times G \longrightarrow E$$

 $(a,b) \mapsto a.b = xy$ For any  $a = xd \in G$  with  $x \in E$  and  $d \in D$ , denote by

$$Inv(a) = \{x^{-1}d: d \in D\}$$

the set of all inverses element of a. An element of Inv(a) is denoted by  $a^*$ .

In a partial group, the definition of the identity element e differs from group as follow:

$$a.e = x$$
 for every  $a = xd$ ,  $x \in E$  and  $d \in D$ 

Moreover, we know that in a group every element has an only inverse element (symmetric element) but for a partial group we have "for every element a of G, we have  $Inv(a) \neq \emptyset$ ".

Secondly, in the definition of subgroup, we replace the phrase "every element of H has an inverse" by "every element  $a \in H$ , Inv(a) meets H", and we give by analogy, the definition of partial subgroup H of G as follow: let  $H \subset G$ . Then H is a *partial subgroup* of G if:

(i) 
$$e \in H$$

(ii)  $a.b \in H$  for every  $a, b \in H$ 

(iii)  $Inv(a) \cap H \neq \emptyset$  for every  $a \in H$ .

Define the right (resp. left) cosets of an element  $a \in G$  by

 $aH = \{a.h: h \in H\},$  (resp.  $Ha = \{h.a: h \in H\}$ ).

We define the normal partial subgroup N as follow: N is a normal partial subgroup, that is, Na = aN for all  $a \in G$ . The space X, quotient of G by the normal partial subgroup N is defined by the following equivalence relation :

 $a \sim_N b \iff$  there exists  $h \in N$  such that  $a \cdot e = b \cdot h$ 

A homomorphism of partial group is a map  $f: G_1 \longrightarrow G_2$  between two partial groups  $G_1$  and  $G_2$  and satisfying f(a.b).e = f(a).f(b). In particular, if  $G_1 = E_1D_1$  and  $G_2 = E_2D_2$  then  $f(E_1) \subset E_2$  and  $f_{/E_1}: E_1 \longrightarrow E_2$  is a homomorphism of groups. (see Lemma 9.3).

By analogy, we will prove that the kernal  $Ker(f) = \{a \in G_1 : f(a) = e\}$  of f, is a normal partial subgroup of  $G_1$ .

Finally, we generalize the isomorphism's theorem for groups to partial groups.

Our principal results are the following:

**Theorem 1.1.** (An image is a natural quotient). Let  $G_1 = E_1D_1$ ,  $G_2 = E_2D_2$  be two partial groups and  $f : G_1 \longrightarrow G_2$  be a partial group homomorphism. Let its default kernel and the image be  $K = f^{-1}(D_2)$ ;  $H = \{f(a) : a \in G_1\}$ ; respectively a normal partial subgroup of  $G_1$  and a partial subgroup of  $G_2$ . Then there is a natural isomorphism  $\tilde{f} : G_{/K} \longrightarrow H$ ;  $aK \longmapsto f(a)$ . **Theorem 1.2.** Let G be a partial group. Let H be a partial subgroup of G and let K be a normal partial subgroup of G. Then there is a natural isomorphism of partial group

$$\begin{array}{c} HK_{/K} \longrightarrow H_{/(H \cap K)} \\ aK \longmapsto a(H \cap K) \end{array}$$

**Theorem 1.3.** (Absorption property of quotients). Let G be a partial group. Let K be a normal partial subgroup of G, and let N be a subgroup of K that is also a normal partial subgroup of G. Then  $K_{/N}$  is a normal partial subgroup of  $G_{/N}$ ; and there is a natural isomorphism

$$(G_{/N})_{/(K_{/N})} \longrightarrow G_{/K}: aN.(K_{/N}) \longmapsto aK$$

# 2. Supplement subgroups and free subsets

Let  $\Gamma$  be a group. The composition law of  $\Gamma$  is denoted as simple multiplication. Let E and D be two subgroup of  $\Gamma$ . We say that E and D are supplement in  $\Gamma$  and denoted by  $E_{\top}D = \Gamma$ , if  $E \cap D = \{e\}$  and  $E.D = \Gamma$ , where

$$E.D = \{xd, x \in E, d \in D\}$$

and e is the neutral element of  $\Gamma$ . We say that  $\Gamma$  is the internal direct product of E and D. A subset  $D \subset \Gamma$  is called *free* with E in  $\Gamma$  if  $e \in D$  and there exists a supplement D of E in  $\Gamma$  containing D.

For example:

- if  $\Gamma = \mathbb{Z}^2$ ,  $E = \{0\} \times \mathbb{Z}$  and  $\widetilde{D} = \mathbb{Z} \times \{0\}$  then  $E_{\top}\widetilde{D} = \Gamma$ . - if  $\Gamma = \mathbb{Z}$ ,  $E = 2\mathbb{Z}$  and  $\widetilde{D} = 3\mathbb{Z}$  then  $E_{\top}\widetilde{D} = \Gamma$ .

**Proposition 2.1.** Let E be a subgroup of  $\Gamma$  and D be a free subset of  $\Gamma$  with E. Denote by  $E.D := \{xd, x \in E, d \in D\}$ . Then:

(i) for every  $a \in E.D$  there exist only one  $x \in E$  and only one  $d \in D$  such that a = xd.

(ii) there exists an equivalence relation in E.D defined by:

 $a \sim b \iff$  there exists  $x \in E$  such that  $x^{-1}a, x^{-1}b \in D$ 

*Proof.* (i) Let  $a \in E.\widetilde{D}$ . If a = xd = x'd' with  $x, x' \in E$  and  $d, d' \in \widetilde{D}$ . Then  $x^{-1}x = dd'^{-1} \in E \cap D$ . It follows that x = x' and d = d' since  $E \cap D = \{e\}$ .

(ii) - Reflexivity:  $a \sim a$  since if a = xd with  $x \in E$  and  $d \in \widetilde{D}$ . Then  $x^{-1}a =$  $x^{-1}a = d \in D.$ 

- Symmetry:  $a \sim b$  then  $b \sim a$  (it is obvious).

- Transitivity: If  $a \sim b$  and  $b \sim c$  then there exist  $x, y \in E$  such that  $x^{-1}a, x^{-1}b \in \widetilde{D}$ and  $y^{-1}b, y^{-1}c \in \widetilde{D}$ . It follows that b = xd = yd' for some  $d, d' \in \widetilde{D}$ . By (i) we have x = y. Hence,  $x^{-1}a$ ,  $x^{-1}c \in \widetilde{D}$  and so  $a \sim c$ . We conclude that  $\sim$  is an equivalence relation. 

#### 3. Partial inner law and partial group

Let  $\Gamma$  be a group, E be a subgroup of  $\Gamma$  and  $D \subset \Gamma$  be a subset free with E in  $\Gamma$ and containing the neutral element e. Denote by  $G := E.D = \{xd, x \in E, d \in D\}$ . We define a partial inner law '.' in G as follows: Let  $a, b \in G$  and write a = xd and b = yd' with  $x, y \in E$  and  $d, d' \in D$ :

a.b = xy

This means that

$$a.b = \begin{cases} xy, & \text{if } a, b \notin E \cup D \\ ab, & \text{if } a, b \in E \\ a, & \text{if } a \in E \text{ and } b \in D \\ b, & \text{if } b \in E \text{ and } a \in D \\ e, & \text{if } a, b \in D \end{cases}$$

We obtain also

$$a.e = \begin{cases} x, & \text{if } a \notin D \\ e, & \text{if } a \in D \end{cases}$$

We can write also if a = xd with  $x \in E$  and  $d \in D$  then

$$a = (a.e)d$$

**Definition 3.1.** (G, .) is a partial group with support the group E and defect the set  $D \subset (G \setminus E) \cup \{e\}$  (where e is the neutral element of E), if: (i)  $G = E.D = \{xd, x \in E, d \in D\}$ (ii)  $e \in D$ (iii) For every  $a, b \in G$ , one has a.b = xy, where a = xd and b = yd' with  $x, y \in E$  and  $d, d' \in D$ .

In all the following, we denote by G = E.D any partial group with support E and defect D.

**Proposition 3.2.** Let (G, .) be a partial group with support the group E and defect the set D. Then: (i)  $a.b \in E$  for every  $a, b \in G$ (ii) a.e = x for every  $a = xd \in G$ (iii) d.d' = e for every  $d, d' \in D$ (iii) (a.b).c = a.(b.c) = a.b.c for every  $a, b, c \in G$  (i.e. the partial law (.) is associative) (iv) x.d.d' = d.x.d' = d.d'.x = x for every  $x \in E$ ,  $d, d' \in D$ (v) a.e = a if and only if  $a \in E$ 

*Proof.* Let  $a, b, c \in G$  and write a = xd, b = yd' and c = zd" with  $x, y, z \in E$  and d, d', d"  $\in D$ .

Assertion (iii): We have (a.b).c = (xy).zd" = xyz, a.(b.c) = xd.(yz) = (xd).(yze) = xyz and a.b.c = (xd).(yd').(zd") = xyz.

Assertion (i) follows directly from the definition.

Assertion (ii): We have a.e = (xd).(ee) = xe = x.

Assertion (iv): By (iii), we have x.d.d' = (xe).(ed).(ed') = xee = x, d.x.d' = (ed).(xe).(ed') = exe = x and d.d'.x = (ed).(ed').(xe) = eex = x.

Assertion (v): If a.e = a then  $a \in E$  by definition. Conversely, if  $a \in E$  then a = ae then a.e = (ae).(ee) = ae = a.

**Proposition 3.3.** Let (G, .) be a partial group with support the group E and defect the set D. Then: (i) a.b = e if and only if  $y = x^{-1}$ , for every  $a = xd, b = yd' \in G$   $(x, y \in E$  and

 $d, d' \in D$ ) (ii) for every  $a = xd \in G$  the set of all reverse points of a is  $Inv(a) := \{x^{-1}d', d' \in D\}$ .

(iii) 
$$\underbrace{a...a}_{n-times} = x^n$$
 for every  $a = xd \in G$ ,  $(x \in E \text{ and } d \in D)$ 

*Proof.* Let  $a, b, c \in G$  and write a = xd, b = yd' and c = zd" with  $x, y, z \in E$  and d, d', d"  $\in D$ .

(i) We have a.b = (xd).(yd') = xy = e so  $y = x^{-1}$ . Conversely, if  $y = x^{-1}$  then  $a.b = (xd).(x^{-1}d') = xx^{-1} = e$ .

(ii) Let  $b \in G$ . The proof results directly from (i) because one has a.b = e if and only if  $b = x^{-1}d'$  with  $d' \in D$ .

(iii) By Proposition 3.2.(iii), we have  $\underbrace{a...a}_{n-times} = \underbrace{(xd)....(xd)}_{n-times} = \underbrace{x...x}_{n-times} = x^n$ .  $\Box$ 

**Example 3.4.** We give two trivial examples:

(i) Every group E is a partial group with support E and defect  $D = \{e\}$ (ii) Every subset D of a group  $\Gamma$ , containing the identity element e of  $\Gamma$ , is a partial group with support  $E = \{e\}$  and defect D.

**Proposition 3.5.** Let H be a group, K and L are two subgroups of H. Suppose that L is free with K in H. Then:

(i) The set  $G := K.L = \{xd, x \in K, d \in L\}$  is a partial group with support K and defect L. We say that G is melted in K.

(ii) The set  $G' := \{e\}.K = \{ed, d \in K\}$  is a partial group with support  $\{e\}$  and defect K. We say that K is totally melted in  $\{e\}$ .

*Proof.* The proof follows directly from the definition.

**Proposition 3.6.** The quotient space X of E.D by the above equivalence relation is a group isomorphic to E.

*Proof.* Denote by  $\overline{a} = \{b \in E.D : a \sim b\}$  the equivalence classes of a point  $a \in E.D$ . Write a = x.d with  $x \in E$  and  $d \in D$ . See that  $\overline{a} = \overline{x}$ . This means that  $X = (E.D)_{/\sim} = E_{/\sim}$ . Define the inner law '\*' on X by:

$$\overline{a} * \overline{b} = \overline{a.b}$$

We have (X, \*) is a group; Indeed:

- '\*' is an inner law in X.

- '\*' has  $\overline{e}$  as the neutrally element, since  $\overline{e} * \overline{a} = \overline{a} * \overline{e} = \overline{a.e} = \overline{x} = \overline{a}$  for every  $a = xd \in E.D$ .

- The inverse of any element  $\overline{a} = \overline{x}$  is  $\overline{x^{-1}}$ , since  $\overline{x} * \overline{x^{-1}} = \overline{x^{-1}} * \overline{x} = \overline{x \cdot x^{-1}} = \overline{e}$ .

- '\*' is associative since is the law '.' of E.

Denote by  $\pi : E \longrightarrow X$  be the canonical projection given by  $\pi(x) = \overline{x}$ . The projection  $\pi$  is a homomorphism of groups by construction. Now,  $\pi$  is injective:  $\pi(x) = \overline{e}$  implies that  $\overline{x} = \overline{e}$  so  $x \in D \cap E$ . As E and D are

free, x = e.

 $\pi$  is surjective by construction. It follows that  $\pi$  is an isomorphism of group.  $\Box$ 

# 4. Partial subgroup of a partial group

In words, this tells us that for a subset to be a partial subgroup, it must be nonempty and contain the products and inverses of all its elements.

**Definition 4.1.** Let G be a partial group with partial neutral element e and  $H \subset G$ . Then H is a *partial subgroup* of G if: (i)  $e \in H$ (ii)  $a.b \in H$  for every  $a, b \in H$ (iii)  $Inv(h) \cap H \neq \emptyset$  for every  $h \in H$ 

Remark 4.2. If G is a group then it is a partial group with defect  $\{e\}$ . Moreover, any subgroup H of G is also a partial subgroup of G with defect  $\{e\}$ .

**Proposition 4.3.** Let H be a partial subgroup of the partial group G with support E and defect D. Then H = F.D' with F is a subgroup of E and defect  $D' \subset D$ .

*Proof.* By definition we have  $Inv(h) \cap H \neq \emptyset$  for every  $h \in H$ , in particular  $Inv(e) \cap H \neq \emptyset$ . Denote by  $D' = Inv(e) \cap H$ , so  $e \in D'$ . Since  $Inv(e) = \{e^{-1}d, d \in D\} = D$  then  $D' = D \cap H \subset D$ . Now, denote by

 $F = \{x \in E, \text{ for which there is } d \in D \text{ such that } xd \in H\}.$ 

We can verify that  $F = E \cap H$  and that F is a subgroup of E.

**Corollary 4.4.** Let G be a partial group with support E and defect D and H be a partial subgroup of G with support F and defect D'. Then: (i)  $F = E \cap H$ (ii)  $D' = D \cap H$  *Proof.* We have G = E.D and H = F.D'.

(i) By Proposition 4.3,  $F \subset E$  so  $F \subset E \cap H$ . For the converse, let  $a \in E \cap H$ . Then a = xd with  $x \in F$  and  $d \in D'$ . By Proposition 4.3,  $D' \subset D$ . Since D is free with E so is D'. Since  $a \in E$  then d = e and so  $a = x \in F$ .

(ii) By Proposition 4.3,  $D' \subset D$  so  $D' \subset D \cap H$ . For the converse, let  $a \in D' \cap H$ . Then a = xd with  $x \in F$  and  $d \in D'$ . By Proposition 4.3,  $F \subset E$ . Since D is free with E so is D'. Since  $a \in D'$  then x = e and so  $a = d \in D'$ .

**Corollary 4.5.** Any partial subgroup of a partial group G is a partial group with the restriction of the partial inner law of G to H.

Proof.

**Definition 4.6.** Let G be a partial group with support E and defect D and H be a partial subgroup of G with support F and defect D'.

(i) We say that H has a total support if E = F

(ii) We say that H has a total defect if D' = D

**Proposition 4.7.** Let G with support E and defect D and H be a partial subgroup of G with support F and defect D'. Then:

(i) H has a total defect if and only if  $Inv(h) \subset H$  for every  $h \in H$ (ii) H has a total current if and only if for every  $n \in F$  there is  $d \in D'$ 

(ii) H has a total support if and only if for every  $x \in E$  there is  $d \in D'$  such that  $xd \in H$ .

*Proof.* (i) If H has a total defect then D' = D, so for every  $h = xd \in H$  with  $x \in F$  and  $d \in D$ , we have  $Inv(h) = \{x^{-1}d', d' \in D\} \subset H$ . Conversely, since  $Inv(e) = \{e^{-1}d, d \in D\} = D$  then  $D \subset H$ , so by Corollary 4.4,  $D' = D \cap H = D$ . (ii) The directly sense is obvious by definition. For the converse, let  $x \in E$  then there exists  $d \in D'$  such that  $a := xd \in H$ , so  $a.e = x \in F$ .

#### 5. Partial Cosets

Given any partial subgroup H of a partial group G, we can construct a partition of G into partial cosets of H, just as we did for rings. But for partial groups, things are a bit more complicated. Because the partial group operation may not be commutative, we have to define two different sorts of partial cosets. Write H = F.D'and define a relation  $\sim_r$  on G by the rule  $a \sim_r b$  if and only if  $b.a^* \in F$  for some  $a^* \in Inv(a)$ . We claim that  $\sim_r$  is an equivalence relation:

- reflexive: For any  $a \in G$ , we have  $a \cdot a^* = e \in F$ , so  $a \sim_r a$ .

- symmetric: Suppose that  $a \sim_r b$ , with a = xd and b = yd', so that  $h = b.a^* = yx^{-1} \in F$ . Then  $h^{-1} = xy^{-1} = a.b^* \in F$ , so  $b \sim_r a$ .

- transitive: Suppose that  $a \sim_r b$  and  $b \sim_r c$ , with a = xd, b = yd' and c = zd'', so that  $h = b.a^* = yx^{-1} \in F$  and  $k = c.b^* = zy^{-1} \in F$ . Then  $kh = (zy^{-1})(yx^{-1}) = zx^{-1} = c.a^* \in F$ , so  $a \sim_r c$ .

The equivalence classes of this equivalence relation are called the *right partial* cosets of H in G. A right partial coset is a set of elements of the form  $Ha = \{h.a : h \in H\}$ , for some fixed element  $a \in G$  called the partial coset representative. For

 $b \in Ha$ ,  $\iff$   $b.e = h.a \text{ for some } h \in H \iff b.a^* \in F \iff a \sim_r b.$ 

We summaries all this as follows:

**Proposition 5.1.** If H is a partial subgroup of the partial group G = E.D, then E is partitioned into right partial cosets of H in G, sets of the form  $Ha = \{h.a : h \in H\}$ .

In a similar way, the relation  $\sim_l$  defined on G by the rule  $a \sim_l b$  if and only if  $a^*.b \in F$  is an equivalence relation on G, and its equivalence classes are the left partial cosets of H = F.D' in G, the sets of the form

$$aH = \{a.h: h \in H\}.$$

If G is an abelian partial group, the left and right partial cosets of any partial subgroup coincide, since  $Ha = \{h.a: h \in H\} = \{a.h: h \in H\} = aH$ .

We say that G is abelian if a.b = b.a for every  $a, b \in G$ .

# 6. Normal partial subgroups

A normal partial subgroup is a special kind of partial subgroup of a partial group. Recall from the last chapter that any partial subgroup H = F.D' has right and left cosets, which may not be the same. We say that H is a normal subgroup of G if the right and left cosets of H in G are the same; that is, if Ha = aH for any  $a \in G$ . There are several equivalent ways of saying the same thing. We define

$$a^{*}Ha = \{a^{*}.h.x: h \in H\}$$

for any element  $a \in G$  and  $a^* \in Inv(a)$ . We can show that  $a^*Ha = x^{-1}Fx$  and a = xd.

**Proposition 6.1.** Let H = F.D' be a partial subgroup of the partial group G = E.D. Then the following are equivalent:

(a) H is a normal partial subgroup, that is, Ha = aH for all  $a \in G$ . (b)  $a^*Ha = F$  for all  $a \in G$ .

(c)  $a^*.h.a \in F$ , for all  $a \in G$  and  $h \in H$ .

(d)  $x^{-1}Fx = F$  for all  $x \in E$ .

*Proof.* If Ha = aH, then  $a^*Ha = a^*.aH = eH = F$ , and conversely. So (a) and (b) are equivalent. If (b) holds then every element  $a^*.h.a$  belongs to  $a^*Ha$ , and so to H, so (c) holds. Conversely, suppose that (c) holds. Then every element of  $a^*Ha$  belongs to H, and we have to prove the reverse inclusion. So take  $h \in H$ . Putting  $b = a^*$ , we have  $c = b^*.h.b = a.h.a^* \in F$ , so  $h.e \in a^*Ha$ , finishing the proof. Now the important thing about normal patial subgroups is that, like ideals, they are kernels of homomorphisms.

**Proposition 6.2.** If G is Abelian, then every partial subgroup H of G is normal.

*Proof.* The proof is obvious because, if G is Abelian, then aH = Ha for all  $a \in$ G. 

For the record, here is a normal subgroup test:

**Proposition 6.3.** (Normal partial subgroup test) A non-empty subset H = F.D'of a partial group G = E.D is a normal partial subgroup of G if the following hold: (a) for any  $h, k \in H$ , we have  $h.k^* \in F$ . (b) for any  $h \in H$  and  $a \in G$ , we have  $a^*.h.a \in F$ .

Proof. (a) is the condition of the second partial subgroup Test, and we saw that (b) is a condition for a partial subgroup to be normal.

### 7. Quotient of partial group

Let H = F.D' be a normal partial subgroup of a partial group G = E.D. We define the quotient group  $G_{/H}$  as follows:

 $\diamond$  The elements of  $G_{/H}$  are the cosets of H in G (left or right doesn't matter, since H is normal).

 $\diamond$  The group operation is defined by (Ha)(Hb) = Hab for all  $a, b \in G$ ; in other words, to multiply cosets, we multiply their representatives.

**Definition 7.1.** If G is a partial group and  $H \subset G$  is a partial subgroup, define a relation on G by

 $a \sim_H b \iff$  there exists  $h \in H$  such that  $a \cdot e = b \cdot h$ 

We say that a is congruent to  $b \mod H$ .

Now, write H = F.D' then saying that a.e = b.h for some  $h \in H$  is the same as saying that there exists  $a^* \in Inv(a)$  such that  $a^*.b \in F$ . Thus:

Lemma 7.2. Congruence mod H is an equivalence relation.

What do the equivalence classes look like?

**Definition 7.3.** If H is apartial subgroup of the partial group G and  $g \in G$ , then define the left coset of H containing g as  $gH = \{g.h \mid h \in H\}$ .

**Proposition 7.4.** (Congruence mod H and Cosets). Let H = F.D' be a partial subgroup of the partial group G = E.D. Then (a) The following are equivalent:

- (i)  $a \sim_H b$
- (ii)  $a^*b \in F$  for every  $a^* \in Inv(a)$
- (iii) b.e = a.h for some  $h \in H$
- (iv)  $b.e \in aH$
- (v)  $bH \subset aH$
- (vi) bH = aH

(b) The left cosets aH are the equivalence classes of equivalence mod H. Thus, two left cosets are either equal or disjoint (this being true of equivalence classes in general).

*Proof.* By definition, we prove (a). For part (b), denote the equivalence class of  $a \in G$  by [a]. One has

 $b \in [a] \iff a \sim_H b$  (Definition of equivalence classes)  $\iff b.e \in aH$ 

By part (a), (i) $\implies$  (iv) whence [a] = aH. That is, the equivalence classes are just the left cosets, as required.

**Proposition 7.5.** If N is a normal partial subgroup of G, then the quotient group  $G_{/N}$  as defined above is a group. Moreover, the map  $\pi$  from G to  $G_{/N}$  defined by  $\pi(a) = Na$  is a homomorphism whose kernel is N and whose image is  $G_{/N}$ .

*Proof.* First we have to show that the definition of the group operation is a good one. In other words, suppose that we chose different coset representatives a' and b' for the cosets Ha and Nb; is it true that Na.b = Ha'.b'? We have x' = h.x and y' = k.y, for some  $h, k \in N$ . Now a.k belongs to the left coset aN. Since N is normal, this is equal to the right coset Ha, so that a.k = l.a for some  $l \in N$ . Then  $a'b' = h.a.k.b = (h.l).(a.b) \in Na.b$ , since  $h.l \in N$ . Thus the operation is indeed well defined.

**Lemma 7.6.** (The Quotient Group). If N is a normal partial subgroup of the partial group G, then the multiplication of left cosets turns  $G_{/N}$  into a group, called the quotient partial group.

*Proof.* We just check the axioms.

### 8. Homomorphisms of partial groups

An isomorphism between partial groups has two properties: it is a bijection; and it preserves the partial group operation. If we relax the first property but keep the second, we obtain a homomorphism. Let  $G_1 = E_1 D_1$  and  $G_2 = E_2 D_2$  two partial groups. Just as for rings, let  $f: G_1 \longrightarrow G_2$  such that  $f(E_1) \subset E_2$  and  $f(D_1) \subset D_2$ . We say that a function f is :

 $\diamond$  a *homomorphism* of partial group if it satisfies

$$f(g.h).e = f(g).f(h) \tag{1}$$

 $\diamond$  a *monomorphism* of partial group if it satisfies (1) and is one-to-one;

 $\diamond$  an *epimorphism* of partial group if it satisfies (1) and is onto.

 $\diamond$  an *isomorphism* of partial group if it satisfies (1) and is one-to-one and onto.

We have the following lemma, proved in much the same way as for rings:

# 9. Proof of Theorems 1.1, 1.2 and 1.3

**Lemma 9.1.** (Inverse of a Homomorphism). Let  $G_1 = E_1D_1$  and  $G_2 = E_2D_2$  be two partial group. If  $f: G_1 \longrightarrow G_2$  is an homomorphism of partial group, bijective with  $f^{-1}(E_2) \subset E_2$  and  $f^{-1}(D_2) \subset D_1$ , then  $f^{-1}: G_2 \longrightarrow G_1$  is also a partial group homomorphism.

Proof. All we need to show is that  $f^{-1}(a.b) = f^{-1}(a).f^{-1}(b)$  for every  $a, b \in G_2$ . But, since f is a homomorphism, we have  $f(f^{-1}(a).f^{-1}(b)).e_2 = f(f^{-1}(a)).f(f^{-1}(b)) = a.b$ . Then there exists  $d \in D_2$  such that  $f(f^{-1}(a).f^{-1}(b)) = (a.b)d$ . As  $f^{-1}(a).f^{-1}(b) \in E_1$  then  $f(f^{-1}(a).f^{-1}(b)) \in f(E_1) \subset E_2$ , so  $(a.b)d \in E_2$ , hence  $d = e_1$ . It follows that  $f(f^{-1}(a).f^{-1}(b)) = a.b$ . Since f is bijective then  $f^{-1}(a).f^{-1}(b) = f(a.b)$ .  $\Box$ 

**Lemma 9.2.** Let  $f : G_1 \longrightarrow G_2$  be a homomorphism of partial group. Then f(e) = e;  $f(a^*) \in Inv(f(a))$ ; and  $f(a.b^*).e = (f(a)).(f(b))^*$ , for all  $a, b \in G_1$ .

Now, if  $f: G_1 \longrightarrow G_2$  is a homomorphism, we define the image of f to be the subset

 $Im(f):=\{b\in G_2:\ b=f(a) \text{ for some } a\in G_1\} \ of \ G_2,$  and the kernel of f to be the subset

$$Ker(f) := \{a \in G_1 : f(a) = e\}$$
 of  $G_1$ .

**Lemma 9.3.** Let  $G_1 = E_1.D_1$  and  $G_2 = E_2.D_2$  be two partial groups and  $f : G_1 \longrightarrow G_2$  be an homomorphism of partial group. Then: (i)  $f_{/E_1} : E_1 \longrightarrow E_2$  is a homomorphism of group. (ii)  $f(e_1) = e_2$ . (iii)  $f(a^*) \in Inv(f(a))$  for every  $a \in G_1$ . (iv) for every  $a = xd \in G_1$ , with  $x \in E_1$  and  $d \in D_1$ , one has f(a) = f(x)d'', for some  $d'' \in D_2$ .

Proof. (i)  $f(xy) = f(x.y).e_2 = f(x).f(y) = f(x)f(y)$  for every  $x, y \in E_1$ . (ii)  $f(e_1) = f(x.x^{-1}).e_2 = f(x).f(x^{-1}) = f(x)(f(x))^{-1} = e_2$ . (iii) Let  $a \in G_1$ . We have  $e_2 = f(e_1) = f(a.a^*) = f(a.a^*).e_2 = f(a).f(a^*)$ , so  $f(a^*) \in Inv(f(a^*))$ . (iv) We have  $f(a.e_1) = f(x) \in E_2$ , so  $f(a.e_1) = f(a.e_1).e_2 = f(a).f(e_1) = f(a).e_2$ . Hence f(a) = f(x).d'' for some  $d'' \in D_2$ .

Define the default kernel of  $f \ \widetilde{Ker}(f) = f^{-1}(D_2)$ . We have  $Ker(f) \subset \widetilde{Ker}(f)$  since  $e_2 \in D_2$ , so  $f^{-1}(e_2) \subset f^{-1}(D_2)$ .

**Proposition 9.4.** Let  $f : G_1 \longrightarrow G_2$  be an isomorphism of partial group. Then the default kernel  $\widetilde{Ker}(f)$  and the image Im(f) are respectively normal partial subgroup of  $G_1$  and partial subgroup of  $G_2$ .

*Proof.*  $\diamond Ker(f)$  is a partial subgroup of  $G_1$ :

- By Lemma 9.3,(i), we have  $e_1 \in Ker(f)$ .

- Take  $a, b \in Ker(f)$ . Then  $f(a), f(b) \in D_2$ . Write f(a) = d and f(b) = d', so  $f(a.b).e_2 = f(a).f(b) = d.d' = e_2$ . Then  $f(a.b) \in D_2$ .

- Let  $a \in Ker(f)$  and  $a^* \in Inv(a)$ , so  $f(a^*) \in Inv(f(a))$  (By Lemma 9.3,(ii)). Write  $f(a) = d \in D_2$  then by definition  $Inv(f(a)) = Inv(d) = \{d': d' \in D_2\} = D_2$ , so  $a^* \in f^{-1}(D_2)$ . It follows that  $Inv(a) \subset Ker(f)$ .

We conclude that Ker(f) is a partial subgroup of  $G_1$ .

Suppose that  $b \in Ker(f)$  and  $a \in G$ . Write  $f(b) = d \in D_2$ . Then  $f(a^*.b.a) = f(a^*)f(b).f(a) = (f(a))^*.d.f(a) = (f(a))^*.f(a) = e_2$ , so  $a^*.b.a \in ker(f) \subset Ker(f)$ . Hence, Ker(f) is a normal subgroup of  $G_1$ .

 $\diamond Im(f)$  is a normal partial subgroup of  $G_2$ :

- By Lemma 9.3,(i), we have  $f(e_1) = e_2 \in Im(f)$ .

- Take  $a, b \in Im(f)$ . Write a = f(x).d and b = f(y).d' for some  $x, y \in E_1$  and  $d, d' \in D_2$ . Since  $f(E_1) \subset E_2$  then  $f(x), f(y) \in E_2$ , so  $a.b = f(x)f(y) = f(xy) \in Im(f)$ .

- Let  $a \in Im(f)$  and  $a^* \in Inv(a)$ . Write a = f(x)d with  $x \in E_1$  and  $d \in D_2$ . By definition,  $Inv(a) = \{f(x)^{-1}d': d' \in D_2\}$ , so  $a^* = f(x)^{-1}d'$  for some  $d' \in D_2$ . By Lemma 9.3.(i),  $f(x)^{-1} = f(x^{-1}) \in Im(f)$  then  $a^* \in Im(f)$ , so  $Inv(a) \subset Im(f)$ . We conclude that Im(f) is a partial subgroup of  $G_2$ .

**Proposition 9.5.** Let H = F.D' and K = F'.D'' two partial subgroups of G = E.D. Then:

(i) H.K = F.F'. (ii) aH = xF and Ha = Fx for every  $a = xd \in G$  and  $x \in E$ . (iii) H is normal in G if and only if F is normal in E. (iv) if H is normal then  $G_{/H} = E_{/F}$ . (v)  $H \cap K = (F \cap F').(D' \cap D'').$ (vi) For every  $a \in H$ , aH = e.H = Ha, where e is the identity element of H.

*Proof.* (i) We have  $H.K = \{a.b: a \in H, b \in K\}$ . If  $a = xd \in H$  and  $b = yd' \in K$  with  $x \in F$  and  $y \in F'$ , then a.b = xy, so  $H.K = \{x.y: x \in F, y \in F'\}$ . (ii) Write a = xd with  $x \in F$  and  $d \in D$  then

$$aH = \{a.b: b \in K\} = \{xy: y \in F'\} = xF.$$

It follows then Ha = Fx with the same proof.

(iii) Suppose that H is normal in G then aH = Ha for every  $a = xd \in G$ . By (i) we have aH = xF and Ha = Fx so xF = Fx for every  $x \in E$ . Hence, F is normal subgroup in E. We use the same proof for the converse.

(iv) If H is normal partial subgroup in G then by (iii), F is normal subgroup in E. Let  $[a] \in G_{/H}$  with  $a = xd \in G$  and [a] = aH. By (ii), aH = xF, so  $[a] \in E_{/F}$ . The same proof is used for the converse.

(v) It is clear that  $(F \cap F').(D' \cap D'') \subset H \cap K$ . For the converse, let  $a \in H \cap K$ . Then there exists  $x \in F$ ,  $y \in F'$ ,  $d \in D'$  and  $d' \in D''$  such that a = xd = yd'. By Proposition 4.3, we have  $D' \cup D'' \subset D$  and  $F \cup F' \subset E$ . By proposition 2.1, x = y and d = d'. It follows that  $x \in F \cap F'$  and  $y \in D' \cap D''$ . (vi) Let  $a \in K$  and write a = xd with  $x \in F$  and  $d \in D'$ , then  $aH = \{a.b : b \in H\} = \{xy : y \in F\} = xF = F$ . In the same way we have F = Fx = Ha.

Proof of Theorem 1.1. By Proposition 9.4, the default kernel and default image of f be  $K = \widetilde{ker(f)}$ ; H = Im(f); respectively a normal partial subgroup of  $G_1$  and a patial subgroup of  $G_2$ . Write  $G_1 = E_1D_1$ ,  $G = E_2D_2$ ,  $K = F_1D'_1$  and  $H = F_2D'_2$ . Then there is a natural isomorphism  $\tilde{f} : G_{/K} \longrightarrow H$ ; given by  $aK \longmapsto f(a)$ . Denote by  $\tilde{e} = eK$ . The map  $\tilde{f}$  is well defined because if a'K = aK then a'.e = a.k for some  $k \in K$  and so

$$f(a').e = f(a.k) = f(a).f(k) = f(a).\widetilde{e} = f(a)$$

The map  $\tilde{f}$  is a homomorphism of partial group because f is a homomorphism of partial group,

$$\widetilde{f}(aK.a'K).e_2 = \widetilde{f}((a.a')K).e_2$$
 by definition of coset multiplication  
 $= f(a.a').e_2$  by definition of  $\widetilde{f}$   
 $= f(a).f(a')$  because  $f$  is a homomorphism of partial group  
 $= \widetilde{f}(aK).\widetilde{f}(a'K)$  by definition of  $\widetilde{f}$ 

To show that  $\tilde{f}$  injects, it suffices to show that  $ker(\tilde{f})$  is only the trivial element  $\tilde{e}$  of  $G_{/K}$ . Compute that if  $\tilde{f}(aK) = \tilde{e}$  then  $f(a) = \tilde{e}$ , and so  $a \in K$ . It follows by Proposition 9.5,(vi) that  $aK = eK = \tilde{e}$ . The map  $\tilde{f}$  surjects because H = Im(f).

Proof of Theorem 1.2. Write  $H = F.D'_1$  and  $K = F'.D'_2$ . By proposition 9.5.(i), we have HK = FF' and by By proposition 9.5.(iv), we have  $HK_{/K} = FF'_{/F'}$  which is a group. It can be considered as a partial group with support  $E_2 := FF'_{/F'}$  and defect  $D_2 := \{e\}$ . Routine verifications show that HK is a partial group having Kas a normal partial subgroup and that  $H \cap K$  is a normal partial subgroup of H. The map  $f : H \longrightarrow HK_{/K}$  given by  $h \longmapsto hK$  is a surjective homomorphism having kernel  $H \cap K$ . In this case the default kernel of f is equal its kernel since  $D_2 = \{e_2\}$ . Therefore, the first theorem gives an isomorphism  $H_{/(H \cap K)} \longrightarrow HK_{/K}$  given by  $h(H \cap K) \longmapsto hK$ . The desired isomorphism is the inverse of the isomorphism in the display.

Proof of Theorem 1.3. Write G = E.D, K = F.D' and N = F'.D''. By proposition 9.5.(iv), we have  $G_{/N} = E_{/F'}$ ,  $K_{/N} = F_{/F'}$  and  $G_{/K} = E_{/F}$  which are groups. By applying the third theorem of group isomorphism we have the map

$$f: (E_{/F'})_{/(F_{/F'})} \longrightarrow E_{/F}: xF'.(E_{/F'}) \longmapsto xF$$

is an isomorphism. By Proposition 9.5.(ii), the map f is also defined as follow:

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$$f: (G_{/N})_{/(K_{/N})} \longrightarrow G_{/K}: aN.(K_{/N}) \longmapsto aK.$$

The proof is completed

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