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# GENERALIZATION OF ISOMORPHISM THEOREMS GROUPS TO PARTIAL GROUPS 

YAHYA N'DAO AND ADLENE AYADI


#### Abstract

In this paper, we define a new structure analogous to group, called partial group. This structure concerns the partial stability by the composition inner law. We generalize the three isomorphism theorems for groups to partial groups.


## 1. Introduction

A semigroup is an algebraic structure consisting of a set together with an associative binary operation. A semigroup generalizes a monoid in that a semigroup need not have an identity element. It also (originally) generalized a group (a monoid with all inverses) to a type where every element did not have to have an inverse, thus the name semigroup(See [1], [2]). Several other algebraic structure generalize the notion of group as quasigroup, hypergroup. In abstract algebra, a quasigroup is an algebraic structure resembling a group in the sense that "division" is always possible. Quasigroups differ from groups mainly in that they need not be associative. A quasigroup with an identity element is called a loop(see [3], [4]). In [5], Kamran extended the notion of AG-groupoid to AG-group. Later on the significant results on the topic were published in [6]. An AG-group is a generalization of abelian group and a special case of quasigroup. The structure of AG-group is a very interesting structure in which one has to play with brackets. There is no commutativity or associativity in general.

We are interested in this paper to introduce a lower structure than those of groups structure, called partial group. Most of algebraic structures like groups, semigroups, quasigroup and monoid all contain an inner composition law which is total (i.e. the law acts on any element of a set $G$ and remains there). Order our new structure is based on the partial action of this law. There is a subset $E$ of $G$ which is stable by the law and it admits the group structure. Firstly, the set G is chosen as a part of a group $\Gamma$ and it is defined as a product (by the law of $\Gamma$ ) of a subgroup $E$ of $\Gamma$ with a part $D$ which is free with $E$, that is to say $D$ is contained in a group $\widetilde{D}$ satisfying $\Gamma=E . \widetilde{D}:=\{x d: \quad x \in E, \quad d \in \widetilde{D}\}$ and $\widetilde{D} \cap E=\{e\}$, where $e$ is the identity element of $\Gamma$. We say that $\widetilde{D}$ is supplement of $E$ in $\Gamma$. We define a partial inner law ${ }^{\prime} .!$ in $G$ as follows: Let $a, b \in G$ and write $a=x d$ and

[^0]$b=y d^{\prime}$ with $x, y \in E$ and $d, d^{\prime} \in D$, one has: $a . b=x y$. This is means that
\[

$$
\begin{aligned}
\prime \prime \\
\prime
\end{aligned}
$$ \quad $$
\begin{aligned}
& \\
& \quad(a, b) \longmapsto a . b=x y
\end{aligned}
$$
\]

For any $a=x d \in G$ with $x \in E$ and $d \in D$, denote by

$$
\operatorname{Inv}(a)=\left\{x^{-1} d: \quad d \in D\right\}
$$

the set of all inverses element of $a$. An element of $\operatorname{Inv}(a)$ is denoted by $a^{*}$.
In a partial group, the definition of the identity element $e$ differs from group as follow:

$$
\text { a.e }=x \text { for every } a=x d, x \in E \text { and } d \in D
$$

Moreover, we know that in a group every element has an only inverse element (symmetric element) but for a partial group we have "for every element $a$ of $G$, we have $\operatorname{Inv}(a) \neq \emptyset$ ".

Secondly, in the definition of subgroup, we replace the phrase "every element of $H$ has an inverse" by "every element $a \in H, \operatorname{Inv}(a)$ meets $H$ ", and we give by analogy, the definition of partial subgroup $H$ of $G$ as follow: let $H \subset G$. Then $H$ is a partial subgroup of $G$ if:
(i) $e \in H$
(ii) $a . b \in H$ for every $a, b \in H$
(iii) $\operatorname{Inv}(a) \cap H \neq \emptyset$ for every $a \in H$.

Define the right (resp. left) cosets of an element $a \in G$ by

$$
a H=\{a . h: \quad h \in H\}, \quad(\text { resp. } H a=\{h . a: \quad h \in H\})
$$

We define the normal partial subgroup $N$ as follow: $N$ is a normal partial subgroup, that is, $N a=a N$ for all $a \in G$. The space $X$, quotient of $G$ by the normal partial subgroup $N$ is defined by the following equivalence relation :

$$
a \sim_{N} b \Longleftrightarrow \text { there exists } h \in N \text { such that } a . e=b . h
$$

A homomorphism of partial group is a map $f: G_{1} \longrightarrow G_{2}$ between two partial groups $G_{1}$ and $G_{2}$ and satisfying $f(a . b) . e=f(a) \cdot f(b)$. In particular, if $G_{1}=E_{1} D_{1}$ and $G_{2}=E_{2} D_{2}$ then $f\left(E_{1}\right) \subset E_{2}$ and $f_{/ E_{1}}: E_{1} \longrightarrow E_{2}$ is a homomorphism of groups. (see Lemma 9.3).

By analogy, we will prove that the kernal $\operatorname{Ker}(f)=\left\{a \in G_{1}: \quad f(a)=e\right\}$ of $f$, is a normal partial subgroup of $G_{1}$.
Finally, we generalize the isomorphism's theorem for groups to partial groups.
Our principal results are the following:
Theorem 1.1. (An image is a natural quotient). Let $G_{1}=E_{1} D_{1}, G_{2}=E_{2} D_{2}$ be two partial groups and $f: G_{1} \longrightarrow G_{2}$ be a partial group homomorphism. Let its default kernel and the image be $K=f^{-1}\left(D_{2}\right) ; H=\left\{f(a): a \in G_{1}\right\}$; respectively a normal partial subgroup of $G_{1}$ and a partial subgroup of $G_{2}$. Then there is a natural isomorphism $\widetilde{f}: G_{/ K} \longrightarrow H ; a K \longmapsto f(a)$.

Theorem 1.2. Let $G$ be a partial group. Let $H$ be a partial subgroup of $G$ and let $K$ be a normal partial subgroup of $G$. Then there is a natural isomorphism of partial group

$$
\begin{gathered}
H K_{/ K} \longrightarrow H_{/(H \cap K)} \\
a K \longmapsto a(H \cap K)
\end{gathered}
$$

Theorem 1.3. (Absorption property of quotients). Let $G$ be a partial group. Let $K$ be a normal partial subgroup of $G$, and let $N$ be a subgroup of $K$ that is also a normal partial subgroup of $G$. Then $K_{/ N}$ is a normal partial subgroup of $G_{/ N}$; and there is a natural isomorphism

$$
\left(G_{/ N}\right)_{/\left(K_{/ N}\right)} \longrightarrow G_{/ K}: \quad a N .\left(K_{/ N}\right) \longmapsto a K
$$

## 2. Supplement subgroups and free subsets

Let $\Gamma$ be a group. The composition law of $\Gamma$ is denoted as simple multiplication. Let $E$ and $\widetilde{D}$ be two subgroup of $\Gamma$. We say that $E$ and $\widetilde{D}$ are supplement in $\Gamma$ and denoted by $E_{\top} \widetilde{D}=\Gamma$, if $E \cap \widetilde{D}=\{e\}$ and $E \cdot \widetilde{D}=\Gamma$, where

$$
E . \widetilde{D}=\{x d, \quad x \in E, d \in \widetilde{D}\}
$$

and $e$ is the neutral element of $\Gamma$. We say that $\Gamma$ is the internal direct product of $E$ and $\widetilde{D}$. A subset $D \subset \Gamma$ is called free with $E$ in $\Gamma$ if $e \in D$ and there exists a supplement $\widetilde{D}$ of $E$ in $\Gamma$ containing $D$.

For example:

- if $\Gamma=\mathbb{Z}^{2}, E=\{0\} \times \mathbb{Z}$ and $\widetilde{D}=\mathbb{Z} \times\{0\}$ then $E_{\top} \widetilde{D}=\Gamma$.
- if $\Gamma=\mathbb{Z}, E=2 \mathbb{Z}$ and $\widetilde{D}=3 \mathbb{Z}$ then $E_{\top} \widetilde{D}=\Gamma$.

Proposition 2.1. Let $E$ be a subgroup of $\Gamma$ and $D$ be a free subset of $\Gamma$ with $E$. Denote by $E . D:=\{x d, \quad x \in E, d \in D\}$. Then:
(i) for every $a \in E . D$ there exist only one $x \in E$ and only one $d \in D$ such that $a=x d$.
(ii) there exists an equivalence relation in E.D defined by:

$$
a \sim b \Longleftrightarrow \text { there exists } x \in E \quad \text { such that } \quad x^{-1} a, x^{-1} b \in D
$$

Proof. (i) Let $a \in E . \widetilde{D}$. If $a=x d=x^{\prime} d^{\prime}$ with $x, x^{\prime} \in E$ and $d, d^{\prime} \in \widetilde{D}$. Then $x^{-1} x=d d^{\prime-1} \in E \cap \widetilde{D}$. It follows that $x=x^{\prime}$ and $d=d^{\prime}$ since $E \cap \widetilde{D}=\{e\}$.
(ii) - Reflexivity: $a \sim a$ since if $a=x d$ with $x \in E$ and $d \in \widetilde{D}$. Then $x^{-1} a=$ $x^{-1} a=d \in \widetilde{D}$.

- Symmetry: $a \sim b$ then $b \sim a$ (it is obvious).
- Transitivity: If $a \sim b$ and $b \sim c$ then there exist $x, y \in E$ such that $x^{-1} a, x^{-1} b \in \widetilde{D}$ and $y^{-1} b, y^{-1} c \in \widetilde{D}$. It follows that $b=x d=y d^{\prime}$ for some $d, d^{\prime} \in \widetilde{D}$. By (i) we have $x=y$. Hence, $x^{-1} a, x^{-1} c \in \widetilde{D}$ and so $a \sim c$.
We conclude that $\sim$ is an equivalence relation.


## 3. Partial inner law and partial group

Let $\Gamma$ be a group, $E$ be a subgroup of $\Gamma$ and $D \subset \Gamma$ be a subset free with $E$ in $\Gamma$ and containing the neutral element $e$. Denote by $G:=E . D=\{x d, x \in E, d \in D\}$. We define a partial inner law ${ }^{\prime} .{ }^{\prime}$ in $G$ as follows: Let $a, b \in G$ and write $a=x d$ and $b=y d^{\prime}$ with $x, y \in E$ and $d, d^{\prime} \in D:$

$$
a . b=x y
$$

This means that

$$
a . b= \begin{cases}x y, & \text { if } a, b \notin E \cup D \\ a b, & \text { if } a, b \in E \\ a, & \text { if } a \in E \text { and } b \in D \\ b, & \text { if } b \in E \text { and } a \in D \\ e, & \text { if } a, b \in D\end{cases}
$$

We obtain also

$$
a . e=\left\{\begin{array}{lll}
x, & \text { if } & a \notin D \\
e, & \text { if } & a \in D
\end{array}\right.
$$

We can write also if $a=x d$ with $x \in E$ and $d \in D$ then

$$
a=(a . e) d
$$

Definition 3.1. ( $G$, .) is a partial group with support the group $E$ and defect the set $D \subset(G \backslash E) \cup\{e\}$ (where $e$ is the neutral element of $E$ ), if:
(i) $G=E \cdot D=\{x d, \quad x \in E, d \in D\}$
(ii) $e \in D$
(iii) For every $a, b \in G$, one has $a . b=x y$, where $a=x d$ and $b=y d^{\prime}$ with $x, y \in E$ and $d, d^{\prime} \in D$.

In all the following, we denote by $G=E . D$ any partial group with support $E$ and defect $D$.

Proposition 3.2. Let $(G,$.$) be a partial group with support the group E$ and defect the set $D$. Then:
(i) $a . b \in E$ for every $a, b \in G$
(ii) a.e $=x$ for every $a=x d \in G$
(iii) $d . d^{\prime}=e$ for every $d, d^{\prime} \in D$
(iii) (a.b).c a.(b.c) =a.b.c for every $a, b, c \in G$ (i.e. the partial law (.) is associative)
(iv) $x . d . d^{\prime}=d . x \cdot d^{\prime}=d . d^{\prime} \cdot x=x$ for every $x \in E, d, d^{\prime} \in D$
(v) $a . e=a$ if and only if $a \in E$

Proof. Let $a, b, c \in G$ and write $a=x d, b=y d^{\prime}$ and $c=z d^{\prime \prime}$ with $x, y, z \in E$ and $d, d^{\prime}, d^{\prime \prime} \in D$.

Assertion (iii): We have $(a \cdot b) \cdot c=(x y) \cdot z d "=x y z, a \cdot(b \cdot c)=x d \cdot(y z)=(x d) \cdot(y z e)=$ $x y z$ and a.b.c $=(x d) \cdot\left(y d^{\prime}\right) \cdot\left(z d^{\prime \prime}\right)=x y z$.

Assertion (i) follows directly from the definition.
Assertion (ii): We have a.e $=(x d) .(e e)=x e=x$.
Assertion (iv): By (iii), we have x.d.d $d^{\prime}=(x e) \cdot(e d) \cdot\left(e d^{\prime}\right)=x e e=x$, d.x. $\cdot d^{\prime}=$ $(e d) \cdot(x e) \cdot\left(e d^{\prime}\right)=e x e=x$ and $d \cdot d^{\prime} \cdot x=(e d) \cdot\left(e d^{\prime}\right) \cdot(x e)=e e x=x$.

Assertion (v): If a.e $=a$ then $a \in E$ by definition. Conversely, if $a \in E$ then $a=a e$ then $a \cdot e=(a e) \cdot(e e)=a e=a$.

Proposition 3.3. Let $(G$, .) be a partial group with support the group $E$ and defect the set $D$. Then:
(i) $a . b=e$ if and only if $y=x^{-1}$, for every $a=x d, b=y d^{\prime} \in G(x, y \in E$ and $\left.d, d^{\prime} \in D\right)$
(ii) for every $a=x d \in G$ the set of all reverse points of $a$ is $\operatorname{Inv}(a):=\left\{x^{-1} d^{\prime}, d^{\prime} \in\right.$ D\}.
(iii) $\underbrace{a \ldots \ldots a}_{n-\text { times }}=x^{n}$ for every $a=x d \in G,(x \in E$ and $d \in D)$

Proof. Let $a, b, c \in G$ and write $a=x d, b=y d^{\prime}$ and $c=z d^{\prime \prime}$ with $x, y, z \in E$ and $d, d^{\prime}, d^{\prime \prime} \in D$.
(i) We have $a \cdot b=(x d) \cdot\left(y d^{\prime}\right)=x y=e$ so $y=x^{-1}$. Conversely, if $y=x^{-1}$ then $a . b=(x d) .\left(x^{-1} d^{\prime}\right)=x x^{-1}=e$.
(ii) Let $b \in G$. The proof results directly from (i) because one has $a . b=e$ if and only if $b=x^{-1} d^{\prime}$ with $d^{\prime} \in D$.
(iii) By Proposition 3.2.(iii), we have $\underbrace{a \ldots \ldots a}_{n-\text { times }}=\underbrace{(x d) \ldots(x d)}_{n-\text { times }}=\underbrace{x \ldots x}_{n-\text { times }}=x^{n}$.

Example 3.4. We give two trivial examples:
(i) Every group $E$ is a partial group with support $E$ and defect $D=\{e\}$
(ii) Every subset $D$ of a group $\Gamma$, containing the identity element $e$ of $\Gamma$, is a partial group with support $E=\{e\}$ and defect $D$.

Proposition 3.5. Let $H$ be a group, $K$ and $L$ are two subgroups of $H$. Suppose that $L$ is free with $K$ in $H$. Then:
(i) The set $G:=K . L=\{x d, x \in K, d \in L\}$ is a partial group with support $K$ and defect $L$. We say that $G$ is melted in $K$.
(ii) The set $G^{\prime}:=\{e\} . K=\{e d, d \in K\}$ is a partial group with support $\{e\}$ and defect $K$. We say that $K$ is totally melted in $\{e\}$.

Proof. The proof follows directly from the definition.

Proposition 3.6. The quotient space $X$ of E.D by the above equivalence relation is a group isomorphic to $E$.

Proof. Denote by $\bar{a}=\{b \in E . D: a \sim b\}$ the equivalence classes of a point $a \in E . D$. Write $a=x . d$ with $x \in E$ and $d \in D$. See that $\bar{a}=\bar{x}$. This means that $X=(E . D)_{/ \sim}=E_{/ \sim}$. Define the inner law ' $*$ ' on $X$ by:

$$
\bar{a} * \bar{b}=\overline{a . b}
$$

We have $(X, *)$ is a group; Indeed:

- ' $*$ ' is an inner law in $X$.
- ' $^{\prime}$ ' has $\bar{e}$ as the neutrally element, since $\bar{e} * \bar{a}=\bar{a} * \bar{e}=\overline{a . e}=\bar{x}=\bar{a}$ for every $a=x d \in E . D$.
- The inverse of any element $\bar{a}=\bar{x}$ is $\overline{x^{-1}}$, since $\bar{x} * \overline{x^{-1}}=\overline{x^{-1}} * \bar{x}=\overline{x \cdot x^{-1}}=\bar{e}$.
$-^{\prime} *$ ' is associative since is the law '.' of $E$.
Denote by $\pi: E \longrightarrow X$ be the canonical projection given by $\pi(x)=\bar{x}$. The projection $\pi$ is a homomorphism of groups by construction.
Now, $\pi$ is injective: $\pi(x)=\bar{e}$ implies that $\bar{x}=\bar{e}$ so $x \in D \cap E$. As $E$ and $D$ are free, $x=e$.
$\pi$ is surjective by construction. It follows that $\pi$ is an isomorphism of group.


## 4. Partial subgroup of a partial group

In words, this tells us that for a subset to be a partial subgroup, it must be nonempty and contain the products and inverses of all its elements.
Definition 4.1. Let $G$ be a partial group with partial neurtral element $e$ and $H \subset G$. Then $H$ is a partial subgroup of $G$ if:
(i) $e \in H$
(ii) $a . b \in H$ for every $a, b \in H$
(iii) $\operatorname{Inv}(h) \cap H \neq \emptyset$ for every $h \in H$

Remark 4.2. If $G$ is a group then it is a partial group with defect $\{e\}$. Moreover, any subgroup $H$ of $G$ is also a partial subgroup of $G$ with defect $\{e\}$.

Proposition 4.3. Let $H$ be a partial subgroup of the partial group $G$ with support $E$ and defect $D$. Then $H=F . D^{\prime}$ with $F$ is a subgroup of $E$ and defect $D^{\prime} \subset D$.

Proof. By definition we have $\operatorname{Inv}(h) \cap H \neq \emptyset$ for every $h \in H$, in particular $\operatorname{Inv}(e) \cap$ $H \neq \emptyset$. Denote by $D^{\prime}=\operatorname{Inv}(e) \cap H$, so $e \in D^{\prime}$. Since $\operatorname{Inv}(e)=\left\{e^{-1} d, d \in D\right\}=D$ then $D^{\prime}=D \cap H \subset D$. Now, denote by

$$
F=\{x \in E, \text { for which there is } d \in D \text { such that } x d \in H\}
$$

We can verify that $F=E \cap H$ and that $F$ is a subgroup of $E$.

Corollary 4.4. Let $G$ be a partial group with support $E$ and defect $D$ and $H$ be a partial subgroup of $G$ with support $F$ and defect $D^{\prime}$. Then:
(i) $F=E \cap H$
(ii) $D^{\prime}=D \cap H$

Proof. We have $G=E . D$ and $H=F . D^{\prime}$.
(i) By Proposition 4.3, $F \subset E$ so $F \subset E \cap H$. For the converse, let $a \in E \cap H$. Then $a=x d$ with $x \in F$ and $d \in D^{\prime}$. By Proposition $4.3, D^{\prime} \subset D$. Since $D$ is free with $E$ so is $D^{\prime}$. Since $a \in E$ then $d=e$ and so $a=x \in F$.
(ii) By Proposition $4.3, D^{\prime} \subset D$ so $D^{\prime} \subset D \cap H$. For the converse, let $a \in D^{\prime} \cap H$. Then $a=x d$ with $x \in F$ and $d \in D^{\prime}$. By Proposition $4.3, F \subset E$. Since $D$ is free with $E$ so is $D^{\prime}$. Since $a \in D^{\prime}$ then $x=e$ and so $a=d \in D^{\prime}$.

Corollary 4.5. Any partial subgroup of a partial group $G$ is a partial group with the restriction of the partial inner law of $G$ to $H$.

Proof.

Definition 4.6. Let $G$ be a partial group with support $E$ and defect $D$ and $H$ be a partial subgroup of $G$ with support $F$ and defect $D^{\prime}$.
(i) We say that $H$ has a total support if $E=F$
(ii) We say that $H$ has a total defect if $D^{\prime}=D$

Proposition 4.7. Let $G$ with support $E$ and defect $D$ and $H$ be a partial subgroup of $G$ with support $F$ and defect $D^{\prime}$. Then:
(i) $H$ has a total defect if and only if $\operatorname{Inv}(h) \subset H$ for every $h \in H$
(ii) $H$ has a total support if and only if for every $x \in E$ there is $d \in D^{\prime}$ such that $x d \in H$.

Proof. (i) If $H$ has a total defect then $D^{\prime}=D$, so for every $h=x d \in H$ with $x \in F$ and $d \in D$, we have $\operatorname{Inv}(h)=\left\{x^{-1} d^{\prime}, \quad d^{\prime} \in D\right\} \subset H$. Conversely, since $\operatorname{Inv}(e)=\left\{e^{-1} d, \quad d \in D\right\}=D$ then $D \subset H$, so by Corollary 4.4, $D^{\prime}=D \cap H=D$.
(ii) The directly sense is obvious by definition. For the converse, let $x \in E$ then there exists $d \in D^{\prime}$ such that $a:=x d \in H$, so a.e $=x \in F$.

## 5. Partial Cosets

Given any partial subgroup $H$ of a partial group $G$, we can construct a partition of $G$ into partial cosets of $H$, just as we did for rings. But for partial groups, things are a bit more complicated. Because the partial group operation may not be commutative, we have to define two different sorts of partial cosets. Write $H=F . D^{\prime}$ and define a relation $\sim_{r}$ on $G$ by the rule $a \sim_{r} b$ if and only if $b . a^{*} \in F$ for some $a^{*} \in \operatorname{Inv}(a)$. We claim that $\sim_{r}$ is an equivalence relation:

- reflexive: For any $a \in G$, we have $a \cdot a^{*}=e \in F$, so $a \sim_{r} a$.
- symmetric: Suppose that $a \sim_{r} b$, with $a=x d$ and $b=y d^{\prime}$, so that $h=b . a^{*}=$ $y x^{-1} \in F$. Then $h^{-1}=x y^{-1}=a . b^{*} \in F$, so $b \sim_{r} a$.
- transitive: Suppose that $a \sim_{r} b$ and $b \sim_{r} c$, with $a=x d, b=y d^{\prime}$ and $c=z d^{\prime \prime}$, so that $h=b . a^{*}=y x^{-1} \in F$ and $k=c . b^{*}=z y^{-1} \in F$. Then $k h=\left(z y^{-1}\right)\left(y x^{-1}\right)=$ $z x^{-1}=c . a^{*} \in F$, so $a \sim_{r} c$.

The equivalence classes of this equivalence relation are called the right partial cosets of $H$ in $G$. A right partial coset is a set of elements of the form $H a=\{h . a$ : $h \in H\}$, for some fixed element $a \in G$ called the partial coset representative. For
$b \in H a, \Longleftrightarrow b . e=h . a$ for some $h \in H \quad \Longleftrightarrow \quad b . a^{*} \in F \quad \Longleftrightarrow \quad a \sim_{r} b$.

We summaries all this as follows:
Proposition 5.1. If $H$ is a partial subgroup of the partial group $G=E . D$, then $E$ is partitioned into right partial cosets of $H$ in $G$, sets of the form $H a=\{$ h.a : $h \in H\}$.

In a similar way, the relation $\sim_{l}$ defined on $G$ by the rule $a \sim_{l} b$ if and only if $a^{*} . b \in F$ is an equivalence relation on $G$, and its equivalence classes are the left partial cosets of $H=F . D^{\prime}$ in $G$, the sets of the form

$$
a H=\{a . h: \quad h \in H\} .
$$

If $G$ is an abelian partial group, the left and right partial cosets of any partial subgroup coincide, since $H a=\{h . a: \quad h \in H\}=\{a . h: \quad h \in H\}=a H$.

We say that $G$ is abelian if $a . b=b$.a for every $a, b \in G$.

## 6. Normal partial subgroups

A normal partial subgroup is a special kind of partial subgroup of a partial group. Recall from the last chapter that any partial subgroup $H=F . D^{\prime}$ has right and left cosets, which may not be the same. We say that $H$ is a normal subgroup of $G$ if the right and left cosets of $H$ in $G$ are the same; that is, if $H a=a H$ for any $a \in G$. There are several equivalent ways of saying the same thing. We define

$$
a^{*} H a=\left\{a^{*} . h \cdot x: \quad h \in H\right\}
$$

for any element $a \in G$ and $a^{*} \in \operatorname{Inv}(a)$. We can show that $a^{*} H a=x^{-1} F x$ and $a=x d$.

Proposition 6.1. Let $H=F . D^{\prime}$ be a partial subgroup of the partial group $G=$ E.D. Then the following are equivalent:
(a) $H$ is a normal partial subgroup, that is, $H a=a H$ for all $a \in G$.
(b) $a^{*} H a=F$ for all $a \in G$.
(c) $a^{*} . h . a \in F$, for all $a \in G$ and $h \in H$.
(d) $x^{-1} F x=F$ for all $x \in E$.

Proof. If $H a=a H$, then $a^{*} H a=a^{*} . a H=e H=F$, and conversely. So (a) and (b) are equivalent. If (b) holds then every element $a^{*} . h . a$ belongs to $a^{*} H a$, and so to $H$, so (c) holds. Conversely, suppose that (c) holds. Then every element of $a^{*} H a$ belongs to $H$, and we have to prove the reverse inclusion. So take $h \in H$. Putting $b=a^{*}$, we have $c=b^{*} . h . b=a . h . a^{*} \in F$, so h.e $\in a^{*} H a$, finishing the proof. Now the important thing about normal patial subgroups is that, like ideals, they are kernels of homomorphisms.

Proposition 6.2. If $G$ is Abelian, then every partial subgroup $H$ of $G$ is normal.

Proof. The proof is obvious because, if $G$ is Abelian, then $a H=H a$ for all $a \in$ $G$.

For the record, here is a normal subgroup test:
Proposition 6.3. (Normal partial subgroup test) A non-empty subset $H=F . D^{\prime}$ of a partial group $G=E . D$ is a normal partial subgroup of $G$ if the following hold:
(a) for any $h, k \in H$, we have $h . k^{*} \in F$.
(b) for any $h \in H$ and $a \in G$, we have $a^{*} . h . a \in F$.

Proof. (a) is the condition of the second partial subgroup Test, and we saw that (b) is a condition for a partial subgroup to be normal.

## 7. Quotient of partial group

Let $H=F . D^{\prime}$ be a normal partial subgroup of a partial group $G=E . D$. We define the quotient group $G_{/ H}$ as follows:
$\diamond$ The elements of $G_{/ H}$ are the cosets of $H$ in $G$ (left or right doesn't matter, since $H$ is normal).
$\diamond$ The group operation is defined by $(H a)(H b)=H a b$ for all $a, b \in G$; in other words, to multiply cosets, we multiply their representatives.

Definition 7.1. If $G$ is a partial group and $H \subset G$ is a partial subgroup, define a relation on $G$ by

$$
a \sim_{H} b \Longleftrightarrow \text { there exists } h \in H \text { such that } a . e=b . h
$$

We say that $a$ is congruent to $b \bmod H$.
Now, write $H=F . D^{\prime}$ then saying that $a . e=b . h$ for some $h \in H$ is the same as saying that there exists $a^{*} \in \operatorname{Inv}(a)$ such that $a^{*} . b \in F$. Thus:

Lemma 7.2. Congruence mod $H$ is an equivalence relation.
What do the equivalence classes look like?
Definition 7.3. If $H$ is apartial subgroup of the partial group $G$ and $g \in G$, then define the left coset of $H$ containing $g$ as $g H=\{g . h \mid h \in H\}$.

Proposition 7.4. (Congruence mod $H$ and Cosets). Let $H=F . D^{\prime}$ be a partial subgroup of the partial group $G=E . D$. Then
(a) The following are equivalent:
(i) $a \sim_{H} b$
(ii) $a^{*} b \in F$ for every $a^{*} \in \operatorname{Inv}(a)$
(iii) $b . e=a . h$ for some $h \in H$
(iv) $b . e \in a H$
(v) $b H \subset a H$
(vi) $b H=a H$
(b) The left cosets $a H$ are the equivalence classes of equivalence mod $H$. Thus, two left cosets are either equal or disjoint (this being true of equivalence classes in general).

Proof. By definition, we prove (a). For part (b), denote the equivalence class of $a \in G$ by $[a]$. One has

$$
\begin{aligned}
b \in[a] & \Longleftrightarrow a \sim_{H} \quad b \quad \text { (Definition of equivalence classes) } \\
& \Longleftrightarrow b . e \in a H
\end{aligned}
$$

By part $(\mathrm{a}),(\mathrm{i}) \Longrightarrow(i v)$ whence $[a]=a H$. That is, the equivalence classes are just the left cosets, as required.

Proposition 7.5. If $N$ is a normal partial subgroup of $G$, then the quotient group $G_{/ N}$ as defined above is a group. Moreover, the map $\pi$ from $G$ to $G_{/ N}$ defined by $\pi(a)=N a$ is a homomorphism whose kernel is $N$ and whose image is $G_{/ N}$.

Proof. First we have to show that the definition of the group operation is a good one. In other words, suppose that we chose different coset representatives $a^{\prime}$ and $b^{\prime}$ for the cosets $H a$ and $N b$; is it true that $N a . b=H a^{\prime} . b^{\prime}$ ? We have $x^{\prime}=h . x$ and $y^{\prime}=k . y$, for some $h, k \in N$. Now $a . k$ belongs to the left coset $a N$. Since $N$ is normal, this is equal to the right coset $H a$, so that $a . k=l . a$ for some $l \in N$. Then $a^{\prime} b^{\prime}=h . a . k . b=(h . l) .(a . b) \in N a . b$, since $h . l \in N$. Thus the operation is indeed well defined.

Lemma 7.6. (The Quotient Group). If $N$ is a normal partial subgroup of the partial group $G$, then the multiplication of left cosets turns $G_{/ N}$ into a group, called the quotient partial group.

Proof. We just check the axioms.

## 8. Homomorphisms of partial groups

An isomorphism between partial groups has two properties: it is a bijection; and it preserves the partial group operation. If we relax the first property but keep the second, we obtain a homomorphism. Let $G_{1}=E_{1} \cdot D_{1}$ and $G_{2}=E_{2} . D_{2}$ two partial groups. Just as for rings, let $f: G_{1} \longrightarrow G_{2}$ such that $f\left(E_{1}\right) \subset E_{2}$ and $f\left(D_{1}\right) \subset D_{2}$. We say that a function $f$ is :
$\diamond$ a homomorphism of partial group if it satisfies

$$
\begin{equation*}
f(g \cdot h) \cdot e=f(g) \cdot f(h) \tag{1}
\end{equation*}
$$

$\diamond$ a monomorphism of partial group if it satisfies (1) and is one-to-one;
$\diamond$ an epimorphism of partial group if it satisfies (1) and is onto.
$\diamond$ an isomorphism of partial group if it satisfies (1) and is one-to-one and onto.

We have the following lemma, proved in much the same way as for rings:

## 9. Proof of Theorems 1.1, 1.2 and 1.3

Lemma 9.1. (Inverse of a Homomorphism). Let $G_{1}=E_{1} D_{1}$ and $G_{2}=E_{2} D_{2}$ be two partial group. If $f: G_{1} \longrightarrow G_{2}$ is an homomorphism of partial group, bijective with $f^{-1}\left(E_{2}\right) \subset E_{2}$ and $f^{-1}\left(D_{2}\right) \subset D_{1}$, then $f^{-1}: G_{2} \longrightarrow G_{1}$ is also a partial group homomorphism.

Proof. All we need to show is that $f^{-1}(a . b)=f^{-1}(a) \cdot f^{-1}(b)$ for every $a, b \in G_{2}$. But, since $f$ is a homomorphism, we have $f\left(f^{-1}(a) \cdot f^{-1}(b)\right) \cdot e_{2}=f\left(f^{-1}(a)\right) \cdot f\left(f^{-1}(b)\right)=$ a.b. Then there exists $d \in D_{2}$ such that $f\left(f^{-1}(a) . f^{-1}(b)\right)=(a . b) d$. As $f^{-1}(a) \cdot f^{-1}(b) \in$ $E_{1}$ then $f\left(f^{-1}(a) . f^{-1}(b)\right) \in f\left(E_{1}\right) \subset E_{2}$, so $(a . b) d \in E_{2}$, hence $d=e_{1}$. It follows that $f\left(f^{-1}(a) \cdot f^{-1}(b)\right)=a . b$. Since $f$ is bijective then $f^{-1}(a) \cdot f^{-1}(b)=f(a . b)$.

Lemma 9.2. Let $f: G_{1} \longrightarrow G_{2}$ be a homomorphism of partial group. Then $f(e)=e ; f\left(a^{*}\right) \in \operatorname{Inv}(f(a)) ;$ and $f\left(a . b^{*}\right) \cdot e=(f(a)) \cdot(f(b))^{*}$, for all $a, b \in G_{1}$.

Now, if $f: G_{1} \longrightarrow G_{2}$ is a homomorphism, we define the image of $f$ to be the subset

$$
\operatorname{Im}(f):=\left\{b \in G_{2}: \quad b=f(a) \text { for some } a \in G_{1}\right\} \text { of } G_{2},
$$

and the kernel of $f$ to be the subset

$$
\operatorname{Ker}(f):=\left\{a \in G_{1}: \quad f(a)=e\right\} \quad \text { of } \quad G_{1} .
$$

Lemma 9.3. Let $G_{1}=E_{1} . D_{1}$ and $G_{2}=E_{2} . D_{2}$ be two partial groups and $f$ : $G_{1} \longrightarrow G_{2}$ be an homomorphism of partial group. Then:
(i) $f_{/ E_{1}}: E_{1} \longrightarrow E_{2}$ is a homomorphism of group.
(ii) $f\left(e_{1}\right)=e_{2}$.
(iii) $f\left(a^{*}\right) \in \operatorname{Inv}(f(a))$ for every $a \in G_{1}$.
(iv) for every $a=x d \in G_{1}$, with $x \in E_{1}$ and $d \in D_{1}$, one has $f(a)=f(x) d^{\prime \prime}$, for some $d^{\prime \prime} \in D_{2}$.

Proof. (i) $f(x y)=f(x . y) \cdot e_{2}=f(x) . f(y)=f(x) f(y)$ for every $x, y \in E_{1}$.
(ii) $f\left(e_{1}\right)=f\left(x \cdot x^{-1}\right) \cdot e_{2}=f(x) \cdot f\left(x^{-1}\right)=f(x)(f(x))^{-1}=e_{2}$.
(iii) Let $a \in G_{1}$. We have $e_{2}=f\left(e_{1}\right)=f\left(a \cdot a^{*}\right)=f\left(a \cdot a^{*}\right) \cdot e_{2}=f(a) \cdot f\left(a^{*}\right)$, so $f\left(a^{*}\right) \in \operatorname{Inv}\left(f\left(a^{*}\right)\right)$.
(iv) We have $f\left(a \cdot e_{1}\right)=f(x) \in E_{2}$, so $f\left(a \cdot e_{1}\right)=f\left(a \cdot e_{1}\right) \cdot e_{2}=f(a) \cdot f\left(e_{1}\right)=f(a) \cdot e_{2}$. Hence $f(a)=f(x) \cdot d^{\prime \prime}$ for some $d^{\prime \prime} \in D_{2}$.

Define the default kernel of $f \widetilde{\operatorname{Ker}(f)}=f^{-1}\left(D_{2}\right)$.
We have $\operatorname{Ker}(f) \subset \widetilde{\operatorname{Ker}(f)}$ since $e_{2} \in D_{2}$, so $f^{-1}\left(e_{2}\right) \subset f^{-1}\left(D_{2}\right)$.
Proposition 9.4. Let $f: G_{1} \longrightarrow G_{2}$ be an isomorphism of partial group. Then the default kernel $\widetilde{K e r(f)}$ and the image $\operatorname{Im}(f)$ are respectively normal partial subgroup of $G_{1}$ and partial subgroup of $G_{2}$.

Proof. $\diamond \widetilde{\operatorname{Ker}(f)}$ is a partial subgroup of $G_{1}$ :

- By Lemma 9.3,(i), we have $e_{1} \in \widetilde{\operatorname{Ker}(f)}$.
- Take $a, b \in \widehat{\operatorname{Ker}(f)}$. Then $f(a), f(b) \in D_{2}$. Write $f(a)=d$ and $f(b)=d^{\prime}$, so $f(a . b) \cdot e_{2}=f(a) \cdot f(b)=d \cdot d^{\prime}=e_{2}$. Then $f(a . b) \in D_{2}$.
- Let $a \in \widehat{\operatorname{Ker}(f)}$ and $a^{*} \in \operatorname{Inv(a)}$, so $f\left(a^{*}\right) \in \operatorname{Inv}(f(a))$ (By Lemma 9.3,(ii)). Write $f(a)=d \in D_{2}$ then by definition $\operatorname{Inv}(f(a))=\operatorname{Inv}(d)=\left\{d^{\prime}: \quad d^{\prime} \in D_{2}\right\}=D_{2}$, so $a^{*} \in f^{-1}\left(D_{2}\right)$. It follows that $\operatorname{Inv}(a) \subset \widehat{\operatorname{Ker}(f)}$.

We conclude that $\widetilde{\operatorname{Ker}(f)}$ is a partial subgroup of $G_{1}$.
Suppose that $b \in \widetilde{\operatorname{Ker}(f)}$ and $a \in G$. Write $f(b)=d \in D_{2}$. Then $f\left(a^{*} . b . a\right)=$ $f\left(a^{*}\right) f(b) \cdot f(a)=(f(a))^{*} . d \cdot f(a)=(f(a))^{*} \cdot f(a)=e_{2}$, so $\left.a^{*} . b \cdot a \in \operatorname{ker}(f) \subset \widetilde{\operatorname{Ker}(f}\right)$. Hence, $\operatorname{Ker}(f)$ is a normal subgroup of $G_{1}$.
$\diamond \operatorname{Im}(f)$ is a normal partial subgroup of $G_{2}$ :

- By Lemma 9.3,(i), we have $f\left(e_{1}\right)=e_{2} \in \operatorname{Im}(f)$.
- Take $a, b \in \operatorname{Im}(f)$. Write $a=f(x) \cdot d$ and $b=f(y) \cdot d^{\prime}$ for some $x, y \in E_{1}$ and $d, d^{\prime} \in D_{2}$. Since $f\left(E_{1}\right) \subset E_{2}$ then $f(x), f(y) \in E_{2}$, so $a . b=f(x) f(y)=f(x y) \in$ $\operatorname{Im}(f)$.
- Let $a \in \operatorname{Im}(f)$ and $a^{*} \in \operatorname{Inv}(a)$. Write $a=f(x) d$ with $x \in E_{1}$ and $d \in D_{2}$. By definition, $\operatorname{Inv}(a)=\left\{f(x)^{-1} d^{\prime}: \quad d^{\prime} \in D_{2}\right\}$, so $a^{*}=f(x)^{-1} d^{\prime}$ for some $d^{\prime} \in D_{2}$. By Lemma 9.3.(i), $f(x)^{-1}=f\left(x^{-1}\right) \in \operatorname{Im}(f)$ then $a^{*} \in \operatorname{Im}(f)$, so $\operatorname{Inv}(a) \subset \operatorname{Im}(f)$.

We conclude that $\operatorname{Im}(f)$ is a partial subgroup of $G_{2}$.

Proposition 9.5. Let $H=F . D^{\prime}$ and $K=F^{\prime} . D^{\prime \prime}$ two partial subgroups of $G=$ E.D. Then:
(i) $H \cdot K=F \cdot F^{\prime}$.
(ii) $a H=x F$ and $H a=F x$ for every $a=x d \in G$ and $x \in E$.
(iii) $H$ is normal in $G$ if and only if $F$ is normal in $E$.
(iv) if $H$ is normal then $G_{/ H}=E_{/ F}$.
(v) $H \cap K=\left(F \cap F^{\prime}\right) .\left(D^{\prime} \cap D^{\prime \prime}\right)$.
(vi) For every $a \in H, a H=e . H=H a$, where $e$ is the identity element of $H$.

Proof. (i) We have $H . K=\{a . b: \quad a \in H, b \in K\}$. If $a=x d \in H$ and $b=y d^{\prime} \in K$ with $x \in F$ and $y \in F^{\prime}$, then $a . b=x y$, so $H . K=\left\{x . y: \quad x \in F, y \in F^{\prime}\right\}$.
(ii) Write $a=x d$ with $x \in F$ and $d \in D$ then

$$
a H=\{a . b: \quad b \in K\}=\left\{x y: \quad y \in F^{\prime}\right\}=x F .
$$

It follows then $H a=F x$ with the same proof.
(iii) Suppose that $H$ is normal in $G$ then $a H=H a$ for every $a=x d \in \in G$. By (i) we have $a H=x F$ and $H a=F x$ so $x F=F x$ foe every $x \in E$. Hence, $F$ is normal subgroup in $E$. We use the same proof for the converse.
(iv) If $H$ is normal partial subgroup in $G$ then by (iii), $F$ is normal subgroup in $E$. Let $[a] \in G_{/ H}$ with $a=x d \in G$ and $[a]=a H$. By (ii), $a H=x F$, so $[a] \in E_{/ F}$. The same proof is used for the converse.
(v) It is clear that $\left(F \cap F^{\prime}\right) .\left(D^{\prime} \cap D^{\prime \prime}\right) \subset H \cap K$. For the converse, let $a \in H \cap K$. Then there exists $x \in F, y \in F^{\prime}, d \in D^{\prime}$ and $d^{\prime} \in D^{\prime \prime}$ such that $a=x d=y d^{\prime}$. By

Proposition 4.3, we have $D^{\prime} \cup D^{\prime \prime} \subset D$ and $F \cup F^{\prime} \subset E$. By proposition 2.1, $x=y$ and $d=d^{\prime}$. It follows that $x \in F \cap F^{\prime}$ and $y \in D^{\prime} \cap D^{\prime \prime}$.
(vi) Let $a \in K$ and write $a=x d$ with $x \in F$ and $d \in D^{\prime}$, then $a H=\{a . b: \quad b \in$ $H\}=\{x y: y \in F\}=x F=F$. In the same way we have $F=F x=H a$.

Proof of Theorem 1.1. By Proposition 9.4, the default kernel and default image of $f$ be $K=\widetilde{\operatorname{ker}(f)} ; H=\operatorname{Im}(f)$; respectively a normal partial subgroup of $G_{1}$ and a patial subgroup of $G_{2}$. Write $G_{1}=E_{1} D_{1}, G=E_{2} D_{2}, K=F_{1} D_{1}^{\prime}$ and $H=F_{2} D_{2}^{\prime}$. Then there is a natural isomorphism $\tilde{f}: G_{/ K} \longrightarrow H$; given by $a K \longmapsto f(a)$. Denote by $\widetilde{e}=e K$. The map $\widetilde{f}$ is well defined because if $a^{\prime} K=a K$ then $a^{\prime} \cdot e=a . k$ for some $k \in K$ and so

$$
f\left(a^{\prime}\right) \cdot e=f(a \cdot k)=f(a) \cdot f(k)=f(a) \cdot \widetilde{e}=f(a)
$$

The map $\tilde{f}$ is a homomorphism of partial group because $f$ is a homomorphism of partial group,

$$
\begin{aligned}
\tilde{f}\left(a K \cdot a^{\prime} K\right) \cdot e_{2} & =\widetilde{f}\left(\left(a \cdot a^{\prime}\right) K\right) \cdot e_{2} \quad \text { by definition of coset multiplication } \\
& =f\left(a \cdot a^{\prime}\right) \cdot e_{2} \quad \text { by definition of } \tilde{f} \\
& =f(a) \cdot f\left(a^{\prime}\right) \quad \text { because } f \text { is a homomorphism of partial group } \\
& =\widetilde{f}(a K) \cdot \widetilde{f}\left(a^{\prime} K\right) \quad \text { by definition of } \widetilde{f}
\end{aligned}
$$

To show that $\tilde{f}$ injects, it suffces to show that $\operatorname{ker}(\tilde{f})$ is only the trivial element $\widetilde{e}$ of $G_{/ K}$. Compute that if $\widetilde{f}(a K)=\widetilde{e}$ then $f(a)=\widetilde{e}$, and so $a \in K$. It follows by Proposition $9.5,(\mathrm{vi})$ that $a K=e K=\widetilde{e}$. The map $\widetilde{f}$ surjects because $H=$ $\operatorname{Im}(f)$.

Proof of Theorem 1.2. Write $H=F . D_{1}^{\prime}$ and $K=F^{\prime} . D_{2}^{\prime}$. By proposition 9.5.(i), we have $H K=F F^{\prime}$ and by By proposition 9.5.(iv), we have $H K_{/ K}=F F_{/ F^{\prime}}^{\prime}$ which is a group. It can be considered as a partial group with support $E_{2}:=F F_{/ F^{\prime}}^{\prime}$ and defect $D_{2}:=\{e\}$. Routine verifications show that $H K$ is a partial group having $K$ as a normal partial subgroup and that $H \cap K$ is a normal partial subgroup of $H$. The map $f: H \longrightarrow H K_{/ K}$ given by $h \longmapsto h K$ is a surjective homomorphism having kernel $H \cap K$. In this case the default kernel of $f$ is equal its kernel since $D_{2}=\left\{e_{2}\right\}$. Therefore, the first theorem gives an isomorphism $H_{/(H \cap K)} \longrightarrow H K_{/ K}$ given by $h(H \cap K) \longmapsto h K$. The desired isomorphism is the inverse of the isomorphism in the display.

Proof of Theorem 1.3. Write $G=E . D, K=F . D^{\prime}$ and $N=F^{\prime} . D^{\prime \prime}$. By proposition 9.5.(iv), we have $G_{/ N}=E_{/ F^{\prime}}, K_{/ N}=F_{/ F^{\prime}}$ and $G_{/ K}=E_{/ F}$ which are groups. By applying the third theorem of group isomorphism we have the map

$$
f:\left(E_{/ F^{\prime}}\right)_{/\left(F_{/ F^{\prime}}\right)} \longrightarrow E_{/ F}: \quad x F^{\prime} .\left(E_{/ F^{\prime}}\right) \longmapsto x F
$$

is an isomorphism. By Proposition 9.5.(ii), the map $f$ is also defined as follow:

$$
f:\left(G_{/ N}\right)_{/\left(K_{/ N}\right)} \longrightarrow G_{/ K}: \quad a N .\left(K_{/ N}\right) \longmapsto a K .
$$

The proof is completed

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