



Reciprocal processes. A measure-theoretical point of view

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ON THE TIME SYMMETRY OF SOME STOCHASTIC PROCESSES

CHRISTIAN LÉONARD, SYLVIE RÖLLY, AND JEAN-CLAUDE ZAMBRINI

ABSTRACT. The bridges of a Markov process are also Markov. But an arbitrary mixture of these bridges fails to be Markov in general. However, it still enjoys the interesting properties of a reciprocal process.

This article analyses the structure of Markov and reciprocal processes, emphasizing their time symmetric properties. A unifying measure-theoretical approach to reciprocal processes is presented, regardless of the peculiar settings (diffusion or jump processes) that were the frameworks of this theory considered until now. Along the way, new results are obtained.

CONTENTS

Introduction	1
1. Time symmetry of Markov probability measures	3
2. Reciprocal probability measures and time symmetry	13
3. Reciprocal probability measures are solutions of entropy minimizing problems	23
References	27

INTRODUCTION

The Markov property is a standard probabilistic notion since its formalization at the beginning of the 20th century. It was presented by Doob [11] for the first time in a time-symmetric way in 1953. This remarkable point of view, which is often replaced by its intuitive time-asymmetric counterpart, was developed further by numerous authors, see the references in the monograph by Chung and Walsh [7] and also Wentzell [39].

Two decades after Markov, Bernstein [2] introduced the notion of *reciprocal process*¹ which, by its very definition, is obviously² invariant with respect to time reversal. Bernstein's motivation for introducing these random processes with a time symmetric structure is rooted into an outstandingly insightful article by Schrödinger entitled “*Über die Umkehrung der Naturgesetze*” [33] that was published in 1931.

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¹This terminology is due to Bernstein.

²Unlike the standard time-arrow-respectful statement of Markov property whose time symmetry is hidden.

Reciprocal diffusion processes with continuous paths were studied by Jamison [18] and others [36, 37, 10, 35]. Lévy processes [30] and pure jump processes [27] also appear in the literature.

In the present paper, a structural approach to the laws of reciprocal processes is proposed. They are called *reciprocal probability measures*. The peculiar settings that were the previous frameworks of this theory, are disregarded. We present a unifying outlook of reciprocal notions and compare different types of time symmetries that are satisfied either by Markov or by reciprocal processes. We see Markov and reciprocal processes as path measures and treat them with a measure-theoretic viewpoint (rather than a semigroup or a semimartingale viewpoint). As an instance of this way of looking at things, Theorems 2.14 and 2.15 identify large classes of reciprocal measures, dominated by some reference Markov measure, as time-symmetric versions of Doob's h -transforms.

New results are also obtained: Theorems 1.5 and 2.15 characterize the structure of the Radon-Nykodim derivative of a Markov measure with respect to another one. We also illustrate our abstract results with several examples and counter-examples.

A possible extension. We focus onto *probability* path measures. Our results could easily be extended to σ -finite path measures, e.g. a diffusion process admitting an unbounded measure as its initial law. For technical detail about this generalized framework, see [23].

Outline of the paper. Section 1 is devoted to the time-symmetric treatment of the Markov property. Reciprocal measures are introduced at Section 2 and their relationship with Markov measures is investigated. At Section 3, one sketches the tight connection between reciprocal families and some specific entropy minimization problems. So doing, we step back to Schrödinger's way of looking at some statistical physics problems with a time-symmetric viewpoint.

Some notation. We consider the set $\Omega = D([0, 1], \mathcal{X}) \subset \mathcal{X}^{[0, 1]}$ of càdlàg paths defined on the finite time interval $[0, 1]$ with state space \mathcal{X} , which is assumed to be Polish and equipped with its Borel σ -algebra. As usual Ω is endowed with the canonical filtration \mathcal{A} generated by the *canonical process* $X = (X_t)_{t \in [0, 1]}$:

$$X_t(\omega) := \omega_t, \quad \omega = (\omega_s)_{s \in [0, 1]} \in \Omega, \quad t \in [0, 1].$$

For any subset $\mathcal{S} \subset [0, 1]$ and for any probability measure P on Ω one denotes

- $X_{\mathcal{S}} = (X_s)_{s \in \mathcal{S}}$ the canonical process restricted to \mathcal{S} ,
- $\mathcal{A}_{\mathcal{S}} = \sigma(X_s; s \in \mathcal{S})$ the σ -algebra of the events observed during \mathcal{S} ,
- $P_{\mathcal{S}} = (X_{\mathcal{S}})_{\#} P$ the restriction of P to $\Omega_{\mathcal{S}} := X_{\mathcal{S}}(\Omega)$.

For $\mathcal{S} = [s, u] \subset [0, 1]$ we use the peculiar notations:

- $X_{[s, u]} := (X_t; s \leq t \leq u)$
- $\mathcal{A}_{[s, u]} := \sigma(X_{[s, u]})$, the σ -algebra generated by the events that occurred between time s and time u
- $\mathcal{A}_s := \sigma(X_s)$, the σ -algebra generated by the events that occur at time s
- $P_s := (X_s)_{\#} P$ is the projection of P at time s
- $P_{su} := (X_s, X_u)_{\#} P$ is the marginal law of P at times s and u simultaneously (P_{01} is therefore the endpoint marginal law of the process)
- $P_{[s, u]} := (X_{[s, u]})_{\#} P$ is the projection of P on the time interval $[s, u]$

The probability measure

$$P^* = (X^*)_{\#}P$$

is the law under P of the time reversed canonical process $X^* := (X_{1-t})_{0 \leq t \leq 1}$.

1. TIME SYMMETRY OF MARKOV PROBABILITY MEASURES

We present structural properties of both Markov probability measures and their bridges. Emphasis is put on their time symmetry which has already been studied in specific frameworks, see for instance [7].

1.1. Definition and basic properties. Let us begin with the symmetric definition of the Markov property.

Definition 1.1 (Markov probability measure). *A probability measure P on Ω is said to be Markov (or the law of a Markov process) if for any $t \in [0, 1]$ and for any events $A \in \mathcal{A}_{[0,t]}$, $B \in \mathcal{A}_{[t,1]}$*

$$P(A \cap B \mid X_t) = P(A \mid X_t)P(B \mid X_t), \quad P\text{-a.e.} \quad (1.1)$$

This means that, knowing the present state X_t , the future and past informations $\mathcal{A}_{[t,1]}$ and $\mathcal{A}_{[0,t]}$, are P -independent. This notion is obviously invariant with respect to time-reversal.

In Theorem 1.2 below, we recall equivalent descriptions of the Markov property, especially the standard identity (2) which states that a Markov process forgets its past history.

Theorem 1.2. *Let P be a probability measure on Ω . Then the following are equivalent:*

- (1) *The probability measure P is Markov.*
- (1*) *The time-reversed probability measure P^* is Markov.*
- (2) *For all $0 \leq t \leq 1$ and all sets $B \in \mathcal{A}_{[t,1]}$,*

$$P(B \mid X_{[0,t]}) = P(B \mid X_t), \quad P\text{-a.e.}$$

- (2*) *For all $0 \leq t \leq 1$ and all sets $A \in \mathcal{A}_{[0,t]}$,*

$$P(A \mid X_{[t,1]}) = P(A \mid X_t), \quad P\text{-a.e.}$$

- (3) *For all $0 \leq s \leq u \leq 1$ and all sets $A \in \mathcal{A}_{[0,s]}$, $C \in \mathcal{A}_{[u,1]}$*

$$P(A \cap C \mid X_{[s,u]}) = P(A \mid X_s)P(C \mid X_u), \quad P\text{-a.e.}$$

Proof. We shall prove $(3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$.

- Proof of $(3) \Rightarrow (1)$. It is clear by taking $s = u$.
- Proof of $(2) \Rightarrow (3)$. For all sets $A \in \mathcal{A}_{[0,s]}$ and $C \in \mathcal{A}_{[u,1]}$ and all sets $B \in \mathcal{A}_{[s,u]}$, the equality

$$P(A \cap B \cap C) = E[\mathbf{1}_B P(A \cap C \mid X_{[s,u]})]$$

holds, just as

$$\begin{aligned}
P(A \cap B \cap C) &= E[P(A \cap B \cap C \mid X_{[0,u]})] \\
&= E[\mathbf{1}_A \mathbf{1}_B P(C \mid X_{[0,u]})] \\
&= E[\mathbf{1}_A \mathbf{1}_B P(C \mid X_{[s,u]})] \\
&= E[\mathbf{1}_B P(A \mid X_{[s,u]}) P(C \mid X_{[s,u]})]
\end{aligned}$$

where property (2) is used in the third equality. Therefore

$$P(A \cap C \mid X_{[s,u]}) = P(A \mid X_{[s,u]}) P(C \mid X_{[s,u]}).$$

• Proof of (1) \Rightarrow (2). Let us show that if (1.1) is satisfied under P then P forgets its past history. Let $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$ be some events. Let us compute $P(B \mid X_{[0,t]})$ by using (1.1):

$$E[\mathbf{1}_A P(B \mid X_{[0,t]})] = P(A \cap B) = E(P(A \cap B \mid X_t)) = E[P(A \mid X_t) P(B \mid X_t)].$$

On the other hand,

$$E[\mathbf{1}_A P(B \mid X_t)] = E[P(A \mid X_t) P(B \mid X_t)].$$

One obtains for any set $A \in \mathcal{A}_{[0,t]}$, $E[\mathbf{1}_A P(B \mid X_{[0,t]})] = E[\mathbf{1}_A P(B \mid X_t)]$, which implies that $P(B \mid X_{[0,t]}) = P(B \mid X_t)$. This completes the proof of (1) \Rightarrow (2), together with the proof of (1) \Leftrightarrow (2) \Leftrightarrow (3).

Eventually the symmetry of the formulation of (3) leads to the equivalence between (2) and (1*). Assertion (2*) corresponds to (2) applied to P^* . \square

The first proof of (1) \Leftrightarrow (2) appears in the monograph by Doob [11, Eq. (6.8) & (6.8')]. Then, Dynkin [13] and Chung [6, Thm. 9.2.4] took it over. Meyer already remarked in [26] that *the Markov property is invariant under time reversal*, unlike other interesting properties of processes.

Identity (2) is often used as the definition of the Markov property. It is usually called *one-sided property* and may create the inaccurate illusion (frequent in the context of statistical physics) that the Markov property is time asymmetric.

Since each Markov process can be defined via its forward and backward transition probability kernels, we recall how to construct them in a *symmetric* way.

Definitions 1.3. Let P be a Markov probability measure on Ω .

- (1) The forward transition probability kernel associated with P is the family of conditional probability measures $(p(s, x; t, \cdot); 0 \leq s \leq t \leq 1, x \in \mathcal{X})$ defined for any $0 \leq s \leq t \leq 1$, and P_s -almost all x , by

$$p(s, x; t, dy) = P(X_t \in dy \mid X_s = x).$$

- (2) The backward transition probability kernel associated with P is the family of conditional probability measures $(p^*(s, \cdot; t, y); 0 \leq s \leq t \leq 1, y \in \mathcal{X})$ defined for any $0 \leq s \leq t \leq 1$, and P_t -almost all y , by

$$p^*(s, dx; t, y) := P(X_s \in dx \mid X_t = y).$$

Since these kernels satisfy the celebrated Chapman-Kolmogorov relations

$$\forall 0 \leq s \leq t \leq u \leq 1,$$

$$p(s, x; u, \cdot) = \int_{\mathcal{X}} p(s, x; t, dy) p(t, y; u, \cdot) \quad \text{for } P_s\text{-a.a. } x \quad (1.2)$$

$$p^*(s, \cdot; u, z) = \int_{\mathcal{X}} p^*(s, \cdot; t, y) p^*(t, dy; u, z) \quad \text{for } P_u\text{-a.a. } z, \quad (1.3)$$

one can construct the probability measure P in the following way.

Proposition 1.4. *The Markov probability measure P is uniquely determined by one time marginal P_u at some time $u \in [0, 1]$, its forward transition probability kernels starting from time u , $(p(s, x; t, \cdot); u \leq s \leq t \leq 1, x \in \mathcal{X})$ and the backward transition probability kernels since time u , $(p^*(s, \cdot; t, y); 0 \leq s \leq t \leq u, y \in \mathcal{X})$. Indeed, for any $0 \leq s_1 \leq \dots \leq s_k \leq u \leq t_1 \leq \dots \leq t_l \leq 1$ and $k, l \geq 1$, the finite dimensional projection of P are given by*

$$P_{s_1, \dots, s_k, u, t_1, \dots, t_l} = p_{s_1; s_2}^* \otimes \dots \otimes p_{s_k; u}^* \otimes P_u \otimes p_{u; t_1} \otimes \dots \otimes p_{t_{l-1}; t_l}.$$

where we used the following intuitive notation

$$P_u \otimes p_{u; t}(dx, dy) := P_u(dx) p(u, x; t, dy).$$

1.2. Probability measure dominated by a Markov probability measure. We consider the problem of knowing whether a probability measure on Ω , which is dominated by a reference Markov probability measure, inherits its Markov property. The following result states a criterion in terms of the multiplicative structure of the Radon-Nikodym derivative.

Theorem 1.5. *Let R be a reference probability measure on Ω and let $P \ll R$, a probability measure dominated by R . Then the following statements are equivalent:*

- (1) *The probability measure P is Markov.*
- (2) *For any time $t \in [0, 1]$, the Radon-Nikodym derivative of P with respect to R factorizes in the following way:*

$$\frac{dP}{dR} = \alpha_t \beta_t, \quad R\text{-a.e.} \quad (1.4)$$

where α_t and β_t are respectively nonnegative $\mathcal{A}_{[0, t]}$ -measurable and $\mathcal{A}_{[t, 1]}$ -measurable functions.

Proof. • Proof of (2) \Rightarrow (1). Take two events, $A \in \mathcal{A}_{[0, t]}$ and $B \in \mathcal{A}_{[t, 1]}$. In terms of Definition 1.1, we have to show that

$$P(A \cap B \mid X_t) = P(A \mid X_t) P(B \mid X_t), \quad P\text{-a.e.} \quad (1.5)$$

To this aim, note that although the product $\alpha_t \beta_t$ is R -integrable, it is not clear why α_t or β_t should be separately integrable. To prove this required integrability, one may use the following lemma of integration theory which assures the $R(\cdot \mid X_t)$ -integrability of the functions α_t and β_t , P -a.e.

Lemma 1.6. *Assume that statement (2) of the above theorem holds true. Then, the functions α_t and β_t are $R(\cdot | X_t)$ -integrable P -a.e. and*

$$\begin{cases} 0 < E_R(\alpha_t \beta_t | X_t) = E_R(\alpha_t | X_t) E_R(\beta_t | X_t), & P\text{-a.e.} \\ 0 \leq E_R(\alpha_t \beta_t | X_t) = \mathbf{1}_{\{E_R(\alpha_t | X_t) E_R(\beta_t | X_t) < +\infty\}} E_R(\alpha_t | X_t) E_R(\beta_t | X_t), & R\text{-a.e.} \end{cases}$$

Proof. See [23, §3]. □

Lemma 1.6 leads to

$$P(A \cap B | X_t) = \frac{E_R(\alpha_t \beta_t \mathbf{1}_A \mathbf{1}_B | X_t)}{E_R(\alpha_t \beta_t | X_t)} = \frac{E_R(\alpha_t \mathbf{1}_A | X_t)}{E_R(\alpha_t | X_t)} \frac{E_R(\beta_t \mathbf{1}_B | X_t)}{E_R(\beta_t | X_t)}, \quad P\text{-a.e.}$$

Choosing $A = \Omega$ or $B = \Omega$ in this formula, we obtain

$$\begin{aligned} P(B | X_t) &= E_R(\beta_t \mathbf{1}_B | X_t) / E_R(\beta_t | X_t), \\ P(A | X_t) &= E_R(\alpha_t \mathbf{1}_A | X_t) / E_R(\alpha_t | X_t). \end{aligned}$$

This completes the proof of (1.5).

• Proof of (1) \Rightarrow (2). Take a Markov probability measure P with derivative Z with respect to R : $dP = Z dR$. We denote by

$$Z_t := E_R(Z | X_{[0,t]}), \quad Z_t^* := E_R(Z | X_{[t,1]}) \quad \text{and} \quad \zeta_t(z) := E_R(Z | X_t = z) = \frac{dP_t}{dR_t}(z).$$

Remark that the last equality implies that $\zeta_t(X_t) > 0$, P -a.e.,

$$\zeta_t(X_t) = E_R(Z_t | X_t) = E_R(Z_t^* | X_t), \quad R\text{-a.e.} \quad (1.6)$$

and that $\zeta_t(X_t)$ is R -integrable.

Fix three bounded nonnegative functions f, g, h that are respectively $\mathcal{A}_{[0,t]}$, \mathcal{A}_t and $\mathcal{A}_{[t,1]}$ measurable. One obtains

$$\begin{aligned} E_P(fgh) &\stackrel{(i)}{=} E_P[E_P(f | X_t) g E_P(h | X_t)] \\ &\stackrel{(ii)}{=} E_P\left[\frac{E_R(f Z_t | X_t)}{E_R(Z_t | X_t)} g \frac{E_R(h Z_t^* | X_t)}{E_R(Z_t^* | X_t)}\right] \\ &\stackrel{(iii)}{=} E_P\left[g \frac{E_R(fh Z_t Z_t^* | X_t)}{\zeta_t(X_t)^2}\right] \\ &\stackrel{(iv)}{=} E_P[g E_{\tilde{P}}(fh | X_t)] \end{aligned}$$

where we successively used in (i): the Markov property of P , in (iii): identity (1.6) and the Markov property of R and in (iv), we introduce the probability measure

$$\tilde{P} := \mathbf{1}_{\{\zeta_t(X_t) > 0\}} \frac{Z_t Z_t^*}{\zeta_t(X_t)} R. \quad (1.7)$$

From all these identities one deduces that

$$P(\cdot | X_t) = \tilde{P}(\cdot | X_t), \quad P\text{-a.e.} \quad (1.8)$$

Define

$$\begin{cases} \alpha_t &= \mathbf{1}_{\{\zeta_t(X_t) > 0\}} Z_t / \zeta_t(X_t) \\ \beta_t &= Z_t^*. \end{cases}$$

Therefore (1.7) becomes

$$\tilde{P} = \alpha_t \beta_t R \quad (1.9)$$

and

$$E_R(\alpha_t \mid X_t) = \mathbf{1}_{\{\zeta_t(X_t) > 0\}} \text{ and } E_R(\beta_t \mid X_t) = \zeta_t(X_t).$$

In order to identify P with \tilde{P} , since (1.8) is satisfied, it is enough to show that their marginals at time t are the same. Let us prove it.

$$\begin{aligned} \tilde{P}_t(dz) &= E_R(\alpha_t \beta_t \mid X_t = z) R_t(dz) \\ &\stackrel{(i)}{=} E_R(\alpha_t \mid X_t = z) E_R(\beta_t \mid X_t = z) R_t(dz) \\ &= \zeta_t(z) R_t(dz) = P_t(dz) \end{aligned}$$

where the Markov property of R is used at (i). This fact, together with (1.8), implies the equality $P = \tilde{P}$. Eventually, since Z_t is $\mathcal{A}_{[0,t]}$ -measurable and Z_t^* is $\mathcal{A}_{[t,1]}$ -measurable, α_t and β_t are respectively $\mathcal{A}_{[0,t]}$ and $\mathcal{A}_{[t,1]}$ -measurable functions. \square

Example 1.7. In the extreme case where α_t is \mathcal{A}_0 -measurable (that is $\alpha_t = f_0(X_0)$) and β_t is \mathcal{A}_1 -measurable (that is $\beta_t = g_1(X_1)$), one obtains from the above theorem that any probability measure P of the form

$$P = f_0(X_0)g_1(X_1) R \quad (1.10)$$

is Markov.

In Theorem 2.15 we will see that, under some restrictions on R , the probability measures of the form (1.10) are the only ones which are Markov in the class of all probability measures of the form $P = h(X_0, X_1) R$.

1.3. A fundamental example: bridges of a Markov measure. Since our aim is to carefully analyze the time symmetry of probability measures on path spaces, it is reasonable to disintegrate them along their endpoint (initial and final) values. One then describes a probability measure P on Ω as a mixture of probability measures pinned at both times $t = 0$ and $t = 1$, i.e. a mixture of its own bridges:

$$P = \int_{\mathcal{X}^2} P(\cdot \mid X_0 = x, X_1 = y) P_{01}(dxdy). \quad (1.11)$$

Since \mathcal{X} is Polish, the product space \mathcal{X}^2 is Polish too and this disintegration is meaningful. Note however that the bridges of P are a priori only defined P -a.s.

To simplify the presentation of our results, from now on we will only consider probability measures P whose bridges can be constructed for *all* $x, y \in \mathcal{X}$ as a regular version of the family of conditional laws $(P(\cdot \mid X_0 = x, X_1 = y), x, y \in \mathcal{X})$. We denote them by $(P^{xy})_{x,y \in \mathcal{X}}$. It is uneasy to construct such an everywhere-defined version in a general non-Markov setting. But this is done in several relevant situations: When P is a Lévy process – see [20] and [30, Prop.3.1], a right process [14] or a Feller process – see the recent paper [5].

Proposition 1.8. *Let P be a Markov probability measure on Ω whose bridges are defined everywhere. Then, for any $x, y \in \mathcal{X}$, the bridge P^{xy} is also Markov.*

Proof. Let P be a Markov probability measure, t be a time in $[0, 1]$, $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$ be two events. We first prove the following equality:

$$P(A \cap B \mid X_0, X_t, X_1) = P(A \mid X_0, X_t)P(B \mid X_0, X_t, X_1), P\text{-a.e.} \quad (1.12)$$

Indeed,

$$\begin{aligned} P(A \cap B \mid X_0, X_t, X_1) &= E[P(A \cap B \mid X_0, X_{[t,1]}) \mid X_0, X_t, X_1] \\ &= E[\mathbf{1}_B P(A \mid X_0, X_{[t,1]}) \mid X_0, X_t, X_1] \\ &= E[\mathbf{1}_B P(A \mid X_0, X_t) \mid X_0, X_t, X_1] \\ &= P(A \mid X_0, X_t)P(B \mid X_0, X_t, X_1). \end{aligned}$$

Moreover, by Theorem 1.2-(2*), $P(A \mid X_0, X_t) = P(A \mid X_0, X_t, X_1)$. Therefore

$$P^{X_0, X_1}(A \cap B \mid X_t) = P^{X_0, X_1}(A \mid X_t)P^{X_0, X_1}(B \mid X_t) P\text{-a.e.},$$

which characterizes the Markov property of every bridge P^{xy} via (1.1). \square

In the sequel, we treat in particular the case where the Markov transition probability kernels are sufficiently regular to admit densities. We describe this situation as follows.

Hypothesis (H). There exists a σ -finite positive measure m on \mathcal{X} such that the transition probability kernels of the Markov probability measure P satisfy

$$\forall 0 \leq s < t \leq 1, p(s, x; t, \cdot) \ll m \text{ for } P_s\text{-a.e. } x \text{ and } p^*(s, \cdot; t, y) \ll m \text{ for } P_t\text{-a.e. } y.$$

For simplicity, one also writes p, p^* for the density functions with respect to m : for all $0 \leq s < t \leq 1$,

$$\begin{aligned} p(s, x; t, y) &:= \frac{dp(s, x; t, \cdot)}{dm}(y) \text{ for } R_s \otimes m\text{-a.e. } (x, y) \\ \text{and } p^*(s, x; t, y) &:= \frac{dp^*(s, \cdot; t, y)}{dm}(x) \text{ for } m \otimes R_t\text{-a.e. } (x, y). \end{aligned}$$

Remark that Hypothesis (H) is not always satisfied. If P is a Poisson process with a random initial law that admits a density on \mathbb{R} , at any time s , P_s admits a density too. But the support of the measure $p(s, x; t, dy)$ is discrete and equal to $x + \mathbb{N}$. Therefore there does not exist any measure m such that for a.e. x , $p(s, x; t, dy) \ll m(dy)$. We will see at Example 1.12 (ii) how to circumvent this difficulty.

In the rest of this section, we assume that the reference Markov probability measure R admits a family of bridges which can be defined everywhere and satisfies Hypothesis (H) with transition probability density denoted by r . Therefore

$$\begin{aligned} R_0(dx) &= \int r^*(0, x; 1, y)m(dy)R_1(dy) = \int r^*(0, x; 1, y)R_1(dy)m(dx) \\ &=: r_0(x)m(dx) \end{aligned}$$

and symmetrically,

$$R_1(dy) = \int r(0, x; 1, y)m(dx)R_0(dx) =: r_1(y)m(dy).$$

This leads us to

$$R_{01}(dxdy) = r_0(x)m(dx)r(0, x; 1, y)m(dy) = r_1(y)m(dy)r^*(0, x; 1, y)m(dx),$$

in such a way that the function defined by

$$c(x, y) := r_0(x)r(0, x; 1, y) = r_1(y)r^*(0, x; 1, y),$$

is the density of the joint marginal $R_{01}(dxdy)$ with respect to $m \otimes m$.

We now recall the general structural relation between the probability measure R and its bridges. These bridges are not globally absolutely continuous with respect to R (as the probability measures considered in the last section), but they are locally absolutely continuous with respect to R , on each time interval $[s, t]$ that is included in the open interval $(0, 1)$. The density is time symmetric and we do exhibit it in a simple way.

Theorem 1.9. *Consider a Markov probability measure R on Ω . Denote $r_0 := dR_0/dm$, $r_1 := dR_1/dm$, and*

$$c(x, y) := r_0(x)r(0, x; 1, y) = r^*(0, x; 1, y)r_1(y) \quad m \otimes m\text{-a.e.}$$

Then for all $0 < s \leq t < 1$ and all $x, y \in \mathcal{X}$, the bridge R^{xy} of R restricted to $\mathcal{A}_{[s, t]}$ is dominated by $R_{[s, t]}$ with its density given by

$$(R^{xy})_{[s, t]} = \frac{r^*(0, x; s, X_s)r(t, X_t; 1, y)}{c(x, y)} R_{[s, t]}. \quad (1.13)$$

Proof. We first show the following property:

$$c(x, y) = 0 \Rightarrow r^*(0, x; s, z)r(t, z'; 1, y) = 0, \quad \forall (z, z'), \quad R_{st}\text{-a.e.} \quad (1.14)$$

On the one hand,

$$R_{01}(dxdy) = c(x, y)m(dx)m(dy)$$

and on the other hand, following Proposition 1.4,

$$\begin{aligned} R_{01}(dxdy) &= \int_{\mathcal{X}^2} R_{0, s, t, 1}(dx, dz, dz', dy) \\ &= \int_{\mathcal{X}^2} r^*(0, dx; s, z)R_s(dz)r(s, z; t, dz')r(t, z'; 1, dy) \\ &= \int_{\mathcal{X}^2} r^*(0, x; s, z)r(s, z; t, z')r(t, z'; 1, y)R_s(dz)m(dz')m(dx)m(dy). \end{aligned}$$

Then

$$c(x, y) = \int_{\mathcal{X}^2} r^*(0, x; s, z)r(s, z; t, z')r(t, z'; 1, y)R_s(dz)m(dz')$$

and (1.14) holds. Moreover, for R_{st} -a.e. (z, z') , the probability measure $r^*(0, dx; s, z)r(t, z'; 1, dy)$ is dominated by $R_{01}(dxdy)$ and satisfies

$$r^*(0, dx; s, z)r(t, z'; 1, dy) = \frac{r^*(0, x; s, z)r(t, z'; 1, y)}{c(x, y)} R_{01}(dxdy). \quad (1.15)$$

Take two bounded measurable functions f, g and an event $B \in \mathcal{A}_{[s,t]}$. Thus,

$$\begin{aligned}
& E_R[f(X_0) \mathbf{1}_B g(X_1)] \\
&= E_R[\mathbf{1}_B E_R(f(X_0) \mid X_{[s,t]}) E_R(g(X_1) \mid X_{[s,t]})] \\
&= E_R[\mathbf{1}_B E_R(f(X_0) \mid X_s) E_R(g(X_1) \mid X_t)] \\
&= E_R\left[\mathbf{1}_B \int_{\mathcal{X}} f(x) r^*(0, dx; s, X_s) \int_{\mathcal{X}} g(y) r(t, X_t; 1, dy)\right] \\
&= E_R\left[\mathbf{1}_B \int_{\mathcal{X}^2} f(x) g(y) r^*(0, dx; s, X_s) r(t, X_t; 1, dy)\right] \\
&\stackrel{\vee}{=} E_R\left[\mathbf{1}_B \int_{\mathcal{X}^2} f(x) \frac{r^*(0, x; s, X_s) r(t, X_t; 1, y)}{c(x, y)} g(y) R_{01}(dxdy)\right] \\
&= \int_{\mathcal{X}^2} f(x) E_R\left[\mathbf{1}_B \frac{r^*(0, x; s, X_s) r(t, X_t; 1, y)}{c(x, y)}\right] g(y) R_{01}(dxdy),
\end{aligned}$$

where we used (1.15) at the marked equality. This proves (1.13). \square

Corollary 1.10 (Decomposition of a bridge). *Introducing $f_s(z) := r^*(0, x; s, z)$ and $g_t(z') := c(x, y)^{-1} r(t, z'; 1, y)$, (1.13) becomes*

$$(R^{xy})_{[s,t]} = f_s(X_s) g_t(X_t) R_{[s,t]}. \quad (1.16)$$

In particular, at each time $t \in (0, 1)$, the one dimensional marginal of the bridge R^{xy} is dominated by the marginal R_t of the Markov probability measure R . Its satisfies

$$R_t^{xy} = f_t(X_t) g_t(X_t) R_t.$$

One interprets (1.16) as a generalization of (1.10) on the time interval $[s, t]$: the density of the bridge decomposes into a product of functions of the process at boundary times s and t . This assures its Markov property.

Naturally, both forward and backward dynamics of the bridge are directly related to the dynamics of the reference process with free boundary conditions.

Proposition 1.11. *Let R be a Markov probability measure on Ω .*

- (1) *For any time $0 < t < 1$ and for any (x, y) , the bridge R^{xy} of R , restricted to $\mathcal{A}_{[0,t]}$ is given by*

$$(R^{xy})_{[0,t]} = \frac{r(t, X_t; 1, y)}{r(0, x; 1, y)} R_{[0,t]}(\cdot \mid X_0 = x). \quad (1.17)$$

- (2) *Analogously, for any time $0 < s < 1$ and for any (x, y) , the bridge R^{xy} of R restricted to $\mathcal{A}_{[s,1]}$ is given by*

$$(R^{xy})_{[s,1]} = \frac{r^*(0, x; s, X_s)}{r^*(0, x; 1, y)} R_{[s,1]}(\cdot \mid X_1 = y). \quad (1.18)$$

(3) The forward and backward transition probability kernels of R^{xy} satisfy for all $0 \leq s < t \leq 1$ and R_{st} -a.e. (z, z') ,

$$\begin{aligned} r^{xy}(s, z; t, dz') &= \mathbf{1}_{\{r(s, z; 1, y) > 0\}} \frac{r(s, z; t, z') r(t, z'; 1, y)}{r(s, z; 1, y)} m(dz') \\ r_*^{xy}(s, dz; t, z') &= \mathbf{1}_{\{r^*(0, x; t, z') > 0\}} \frac{r^*(0, x; s, z) r^*(s, z; t, z')}{r^*(0, x; t, z')} m(dz) \end{aligned}$$

with the conventions $r(1, z; 1, y) = \mathbf{1}_{\{z=y\}}$ and $r^*(0, x; 0, z) = \mathbf{1}_{\{z=x\}}$.

Proof. • Proof of (1). Define $P^{\widetilde{xy}} := \frac{r(t, X_t; 1, y)}{r(0, x; 1, y)} R_{[0, t]}(\cdot \mid X_0 = x)$ and take a bounded nonnegative map f and an event $B \in \mathcal{A}_{[0, t]}$. Then,

$$\begin{aligned} E_R \left(P^{\widetilde{xy}}(B) f(X_1) \mid X_0 = x \right) &= \int_{\mathcal{X}} r(0, x; 1, y) P^{\widetilde{xy}}(B) f(y) m(dy) \\ &= \int_{\mathcal{X}} E_R[\mathbf{1}_B r(t, X_t; 1, y) f(y) \mid X_0 = x] m(dy) \\ &= E_R[\mathbf{1}_B \int_{\mathcal{X}} r(t, X_t; 1, dy) f(y) m(dy) \mid X_0 = x] \\ &= E_R[\mathbf{1}_B E_R(f(X_1) \mid X_t) \mid X_0 = x] \\ &= E_R[\mathbf{1}_B E_R(f(X_1) \mid X_{[0, t]}) \mid X_0 = x] \\ &= E_R[\mathbf{1}_B f(X_1) \mid X_0 = x] \\ &= E_R[R^{xX_1}(B) f(X_1) \mid X_0 = x] \end{aligned}$$

which proves (1.17).

- Proof of (2). It is analogous to (1).
- Proof of (3). It is a direct corollary of (1) and (2). □

Examples 1.12. Let us provide examples of several kind of bridges.

- (i) The first example is standard. Let $R = \mathbf{W}$ be a *Wiener probability measure* on the set of real-valued continuous paths on $[0, 1]$, with initial marginal law \mathbf{W}_0 , admitting a density function r_0 with respect to Lebesgue measure $m(dx) = dx$. Brownian bridges can be constructed for any $x, y \in \mathbb{R}$ (as Paul Lévy already proposed). Hypothesis (H) is satisfied and the forward and backward transition probability densities are given, for any $s \leq t, x, y \in \mathbb{R}$, by:

$$\begin{aligned} r(s, x; t, y) &= \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}}, \\ r^*(s, x; t, y) &= \frac{\int r_0(z) r(0, z; s, x) r(s, x; t, y) dz}{\int r_0(z) r(0, z; t, y) dz}. \end{aligned}$$

Therefore, due to (1.17), the Brownian bridge restricted to $\mathcal{A}_{[0, t]}$ satisfies

$$(\mathbf{W}^{x, y})_{[0, t]} = \frac{1}{\sqrt{1-t}} e^{-\left(\frac{(y-X_t)^2}{2(1-t)} - \frac{(y-x)^2}{2}\right)} \mathbf{W}_{[0, t]}(\cdot \mid X_0 = x).$$

Similarly

$$(\mathbf{W}^{x,y})_{[s,1]} = F^{xy}(s, X_s) \mathbf{W}_{[s,1]}(\cdot \mid X_1 = y)$$

where

$$\begin{aligned} F^{xy}(s, z) &= \frac{r^*(0, x; s, z)}{r^*(0, x; 1, y)} = \frac{\int r_0(x') r(0, x'; 1, y) dx'}{\int r_0(x') r(0, x'; s, z) dx'} \frac{r(0, x; s, z)}{r(0, x; 1, y)} \\ &= \frac{\int r_0(x') e^{-\frac{(y-x')^2}{2}} dx'}{\int r_0(x') e^{-\frac{(z-x')^2}{2s}} dx'} e^{-\left(\frac{(z-x)^2}{2s} - \frac{(y-x)^2}{2}\right)}. \end{aligned}$$

Moreover, the density of the marginal at time $t > 0$ of the Brownian bridge pinned in x and y with respect to the marginal of the Brownian motion with initial density r_0 and free final condition is given by

$$\frac{1}{\sqrt{1-t}} \frac{r_0(x)}{\int r_0(x') e^{-\frac{(X_t-x')^2}{2t}} dx'} e^{-\left(\frac{(X_t-x)^2}{2t} + \frac{(y-X_t)^2}{2(1-t)}\right)}.$$

- (ii) Let \mathbf{P} be the law of a *Poisson process* with values in the set of càdlàg step functions with positive unit jumps. Poisson bridges can be constructed for all $x, y \in \mathbb{R}$ such that $y - x \in \mathbb{N}$.

Now suppose that X_0 under \mathbf{P} is random, real-valued and admits a density: $\mathbf{P}_0(dx) = r_0(x)dx$ on \mathbb{R} . As already remarked, such a process does not satisfy Hypothesis (H). However its dynamics is space- (and time-) homogeneous:

$$r(s, x; t, dy) = \delta_x * r(0, 0; t - s, dy)$$

and the transition kernel $r(0, 0; u, dy)$ admits a Poissonian density r with respect to the counting measure m on \mathbb{N} :

$$r(0, 0; u, dy) = r(u, y) m(dy) \quad \text{where} \quad r(u, n) = e^{-u} \frac{u^n}{n!}.$$

Therefore the proof of (1.17) can be generalized to this case, since one exhibits the density of the bridge, on the time interval $[0, t]$, of the Poisson process between 0 and n with respect to the standard Poisson process starting in 0. Then, the density on the time interval $[0, t]$ of the Poisson process pinned at x and y with respect to the Poisson process starting from x satisfies for \mathbf{P}_0 -a.e. x and $y \in x + \mathbb{N}$,

$$\begin{aligned} (\mathbf{P}^{xy})_{[0,t]} &= \frac{r(1-t, y - X_t)}{r(1, y - x)} \mathbf{P}_{[0,t]}(\cdot \mid X_0 = x) \\ &= e^t (1-t)^{y-X_t} \frac{(y-x)!}{(y-X_t)!} \mathbf{P}_{[0,t]}(\cdot \mid X_0 = x). \end{aligned}$$

- (iii) Let \mathbf{C} be the law of a *Cauchy process* on Ω . A regular version of Cauchy bridges can be constructed for all $x, y \in \mathbb{R}$, see [5]. The forward transition density $r(s, x; t, y)$ is given, for each $x, y \in \mathbb{R}$, by the Cauchy law with parameter $t - s$:

$$r(s, x; t, y) = \frac{t - s}{\pi((t - s)^2 + (y - x)^2)}$$

and for \mathbf{C}_0 -almost all x ,

$$(\mathbf{C}^{xy})_{[0,t]} = (1-t) \frac{1 + (y-x)^2}{(1-t)^2 + (y-X_t)^2} \quad \mathbf{C}_{[0,t]}(\cdot \mid X_0 = x).$$

The computation of the density of the bridge on the time interval $[s, 1]$ follows the same schema, using the backward transition density and the initial value \mathbf{C}_0 . One could also consider the reversible situation, corresponding to $\mathbf{C}_0(dx) = dx$. This reversible measure cannot be normalised but the techniques presented here remain valid for σ -finite measures, see [23].

(iv) Several alternate examples of Lévy bridges can be found in [30].

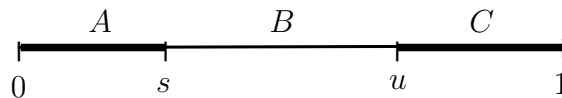
2. RECIPROCAL PROBABILITY MEASURES AND TIME SYMMETRY

We now enlarge our framework to the class of reciprocal probability measures. They are not necessarily Markov but exhibit a kind of intrinsic time symmetry. Bernstein introduced them in the particular framework of diffusion processes in his talk [2] at the International Congress in Zürich in 1932³, commenting on their associated transitions as being *stochastiquement parfaites*⁴. Their symmetry property justifies their relevance in the study of quantum mechanical systems, see [29, 28, 8]. In fact, this is in the context of such a motivation, inspired by Feynman's approach to quantum theory, that the dynamical properties of these diffusion processes were elaborated many years after Bernstein, cf. [17, 40]. See Section 3.1 for further detail.

2.1. Definition and basic properties. Let us begin with the definition.

Definition 2.1 (Reciprocal probability measure). *A probability measure P on Ω is called reciprocal (or the law of a reciprocal process) if for any times $s \leq u$ in $[0, 1]$ and for any events $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$, $C \in \mathcal{A}_{[u,1]}$*

$$P(A \cap B \cap C \mid X_s, X_u) = P(A \cap C \mid X_s, X_u)P(B \mid X_s, X_u) \quad P\text{-a.e.} \quad (2.1)$$



The above property, which was formalized by Jamison in [18], states that under P , given the knowledge of the process at both times s and u , the future of the time u and the past of the time s are conditionally independent. It is clearly a time symmetric notion, in contrast with the standard one-sided Markov property.

Paralleling Theorem 1.2, we present several characterizations of the reciprocal property.

Theorem 2.2. *Let P be a probability measure on Ω . Then the following assertions are equivalent:*

- (1) *The probability measure P is reciprocal.*
- (1*) *The time-reversed probability P^* is reciprocal.*

³For this historical reason, they are sometimes called Bernstein processes.

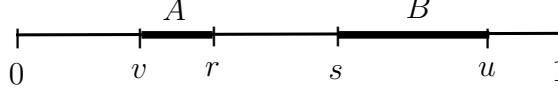
⁴stochastically perfect.

(2) For all $0 \leq s \leq u \leq 1$ and all sets $B \in \mathcal{A}_{[s,u]}$,

$$P(B \mid X_{[0,s]}, X_{[u,1]}) = P(B \mid X_s, X_u) \quad P\text{-a.e.} \quad (2.2)$$

(3) For all $0 \leq v \leq r \leq s \leq u \leq 1$ and all sets $A \in \mathcal{A}_{[v,r]}$, $B \in \mathcal{A}_{[s,u]}$,

$$P(A \cap B \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}) = P(A \mid X_v, X_r)P(B \mid X_s, X_u) \quad P\text{-a.e.}$$



Proof. • Proof of (1) \Leftrightarrow (1*). Straightforward.

• Proof of (1) \Rightarrow (2). Let us take $B \in \mathcal{A}_{[s,u]}$. $P(B \mid X_{[0,s]}, X_{[u,1]})$ is the unique random variable $\mathcal{A}_{[0,s]} \vee \mathcal{A}_{[u,1]}$ -measurable such that, for all $A \in \mathcal{A}_{[0,s]}$ and $C \in \mathcal{A}_{[u,1]}$,

$$P(A \cap B \cap C) = E[\mathbf{1}_A \mathbf{1}_C P(B \mid X_{[0,s]}, X_{[u,1]})].$$

But, due to (2.1), one has

$$\begin{aligned} P(A \cap B \cap C) &= E(P(A \cap B \cap C \mid X_s, X_u)) \\ &= E[P(A \cap C \mid X_s, X_u)P(B \mid X_s, X_u)] \\ &= E[E(\mathbf{1}_A \mathbf{1}_C P(B \mid X_s, X_u) \mid X_s, X_u)] \\ &= E[\mathbf{1}_A \mathbf{1}_C P(B \mid X_s, X_u)]. \end{aligned}$$

This implies (2).

• Proof of (2) \Rightarrow (1). Let us take $0 \leq s \leq u \leq 1$, $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$, $C \in \mathcal{A}_{[u,1]}$ and f, g some measurable nonnegative functions. By definition,

$$E[\mathbf{1}_A \mathbf{1}_B \mathbf{1}_C f(X_s)g(X_u)] = E[P(A \cap B \cap C \mid X_s, X_u)f(X_s)g(X_u)]$$

holds. But,

$$\begin{aligned} E[\mathbf{1}_A \mathbf{1}_B \mathbf{1}_C f(X_s)g(X_u)] &= E[E(\mathbf{1}_A \mathbf{1}_B \mathbf{1}_C f(X_s)g(X_u) \mid X_{[0,s]}, X_{[u,1]})] \\ &= E[\mathbf{1}_A \mathbf{1}_C P(B \mid X_s, X_u)f(X_s)g(X_u)] \\ &= E[P(A \cap C \mid X_s, X_u)P(B \mid X_s, X_u)f(X_s)g(X_u)]. \end{aligned}$$

Therefore

$$P(A \cap B \cap C \mid X_s, X_u) = P(A \cap C \mid X_s, X_u)P(B \mid X_s, X_u).$$

• Proof of (2) \Rightarrow (3). Take $A \in \mathcal{A}_{[v,r]}$ and $B \in \mathcal{A}_{[s,u]}$. Then

$$\begin{aligned} &P(A \cap B \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}) \\ &= E[P(A \cap B \mid X_{[0,v]}, X_{[r,1]}) \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}] \\ &\stackrel{\checkmark}{=} E[P(A \mid X_v, X_r) \mathbf{1}_B \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}] \\ &= E[E(P(A \mid X_v, X_r) \mathbf{1}_B \mid X_{[0,s]}, X_{[u,1]}) \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}] \\ &\stackrel{\checkmark}{=} E[P(A \mid X_v, X_r)P(B \mid X_s, X_u) \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}] \\ &= P(A \mid X_v, X_r)P(B \mid X_s, X_u) \end{aligned}$$

where we used assumption (2) at the \checkmark -marked equalities.

- Proof of (3) \Rightarrow (2). It is enough to take $A = \Omega$ and $v = t = s$. \square

Identity (2.2) states that a reciprocal probability measure is indeed a Markov field indexed by time, seen as a one-dimensional continuous parameter process: if one conditions the probability measure evolving during the time interval $[s, u]$ by the knowledge of the past of s and of the future of u , this is equivalent to conditioning it only by the knowledge at both boundary times s and u . This property is sometimes called *two-sided Markov property*, which is inadequate, because one could get mixed up with (1.1).

For a probability measure, being Markov is stronger than being reciprocal.

Proposition 2.3. *Any Markov probability measure on Ω is reciprocal, but the converse is false.*

Proof. Take P a Markov probability measure, $0 \leq s \leq u \leq 1$ and $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$ and $C \in \mathcal{A}_{[u,1]}$. The following holds:

$$\begin{aligned}
 P(A \cap B \cap C) &= E[P(A \cap B \cap C \mid X_{[s,u]})] \\
 &\stackrel{(i)}{=} E[P(A \mid X_s) \mathbf{1}_B P(C \mid X_u)] \\
 &= E[P(A \mid X_s) P(B \mid X_s, X_u) P(C \mid X_u)] \\
 &\stackrel{(ii)}{=} E[P(A \mid X_s) P(B \mid X_s, X_u) P(C \mid X_{[0,u]})] \\
 &= E[P(A \mid X_s) P(B \mid X_s, X_u) \mathbf{1}_C] \\
 &\stackrel{(iii)}{=} E[P(A \mid X_{[s,1]}) P(B \mid X_s, X_u) \mathbf{1}_C] \\
 &= E[\mathbf{1}_A P(B \mid X_s, X_u) \mathbf{1}_C]
 \end{aligned}$$

Equality (i) is due to Theorem 1.2 (3). To prove (ii) et (iii) we use the Markov property. Therefore (2.2) holds.

Examples 2.4 and 2.6-(ii) below provide counter-examples showing that the set of Markov probability measures is strictly included in the set of reciprocal probability measures. \square

In Section 2.6, we also identify reciprocal processes which are not Markov since their endpoint marginal laws do not have the required structure. The first proof of Proposition 2.3 was done in [17] in a Gaussian setting.

Example 2.4 (Reciprocal measures on a loop space). Let us mention the following class of reciprocal – but not Markov – probability measures. Take a Markov probability measure R on Ω whose bridges are defined everywhere and m any probability measure on \mathcal{X} . Then

$$P_{\text{loop}} := \int_{\mathcal{X}} R^{xx} m(dx)$$

is a probability measure that is concentrated on loop paths, i.e. such that $X_0 = X_1$ almost surely, with both initial and final marginal laws equal to m . One can see this path measure as describing a periodic random process.

Due to Proposition 2.7, P_{loop} is reciprocal with the mixture probability measure $\pi(dxdy) = m(dx)\delta_x(dy)$.

To see that the Markov property breaks down in general, take R^{xx} to be the usual Brownian bridge on \mathbb{R} between x and x , choose $m = (\delta_{-1} + \delta_{+1})/2$ and pick any intermediate time $0 < t < 1$. We have $P_{\text{loop}}(X_1 \geq 0 \mid X_{[0,t]}) = \mathbf{1}_{\{X_0=+1\}}$, while

$$P_{\text{loop}}(X_1 \geq 0 \mid X_t) = P_{\text{loop}}(X_0 = +1 \mid X_t).$$

In particular when $t = 1/2$, symmetry considerations lead us to $P_{\text{loop}}(X_0 = +1 \mid X_{1/2}) = 1/2$, implying that $P_{\text{loop}}(X_1 \geq 0 \mid X_{1/2}) = 1/2 \neq \mathbf{1}_{\{X_0=+1\}} = P_{\text{loop}}(X_1 \geq 0 \mid X_{[0,t]})$.

We will discuss in a short while the typical structure of reciprocal probability measures.

2.2. Pinning leads back to the Markov property. Proposition 1.8 states the stability of the Markov property by pinning. Similarly, it is remarkable that pinning a reciprocal probability measure not only respects its reciprocal property but also transforms it into a Markov one.

Theorem 2.5. *Let P be a reciprocal probability measure on Ω . Then, the conditional law $P(\cdot \mid X_0, X_1)$, defined P_{01} -almost surely, is a Markov probability measure.*

Proof. Take two events $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$. Equation (1.12) holds under P : it is enough to do similar computations as in the proof of Theorem 1.8 and to apply property (2.2) to A with $s = 0$ and $t = u$. By symmetry one then obtains

$$\begin{aligned} P(A \cap B \mid X_0, X_t, X_1) &= P(A \mid X_0, X_t)P(B \mid X_0, X_t, X_1) \\ &= P(A \mid X_0, X_t, X_1)P(B \mid X_0, X_t, X_1), \end{aligned}$$

which characterises the Markov property of $P(\cdot \mid X_0, X_1)$ via (1.1). \square

In the next subsection we study how to mix pinned probability measures without perturbing their nice properties.

2.3. Mixing properly preserves the reciprocal property. To complete the previous subsection we analyse in which way mixing probability measures perturbs their reciprocal and/or Markov properties.

Mixing Markov probability measures sometimes preserves the Markov property, but this is far from being the rule. Similarly, mixing reciprocal probability measures sometimes results in a reciprocal measure, but not always. The following examples illustrate these assertions. Moreover, we construct in (ii) an example of a reciprocal probability measure which is not Markov.

Examples 2.6 (Various mixtures of deterministic paths). Let $\mathcal{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be a state space with three elements. We denote by δ_w , $w \in \Omega$, the Dirac measure at the path w . Any δ_w is Markov since the path w is deterministic.

(i) One denotes by $\mathbf{acb} \in \Omega$ the following path w :

$$\mathbf{acb}(t) := \mathbf{1}_{[0,1/3)}(t) \mathbf{a} + \mathbf{1}_{[1/3,2/3)}(t) \mathbf{c} + \mathbf{1}_{[2/3,1]}(t) \mathbf{b}.$$

Similar notations are used for paths that only jump at times 0, 1/3 or 2/3.

The probability measure P on Ω defined by

$$P = \frac{1}{4}(\delta_{\mathbf{abc}} + \delta_{\mathbf{aba}} + \delta_{\mathbf{cba}} + \delta_{\mathbf{cbc}})$$

is the uniform mixture of deterministic Markov paths and is Markov too. Indeed $P_0 = \frac{1}{2}(\delta_a + \delta_c)$ and the nontrivial transition probability measures which are given by

$$P(X_{1/3} = \mathbf{b} \mid X_0 = \mathbf{a}) = P(X_{1/3} = \mathbf{b} \mid X_0 = \mathbf{c}) = 1$$

and

$$P(X_{2/3} = \mathbf{a} \mid X_{1/3} = \mathbf{b}) = P(X_{1/3} = \mathbf{c} \mid X_{1/3} = \mathbf{b}) = 1/2,$$

entirely specify the dynamics of P .

(ii) The probability measure on Ω

$$P = \frac{1}{2}(\delta_{\mathbf{abc}} + \delta_{\mathbf{cba}}),$$

is reciprocal but not Markov. It is reciprocal since each boundary condition determines the path. Nevertheless we observe that P is not Markov since

$$P(X_1 = \mathbf{a} \mid X_0 = \mathbf{a}, X_{1/3} = \mathbf{b}) = 0$$

while

$$P(X_1 = \mathbf{a} \mid X_{1/3} = \mathbf{b}) = 1/2.$$

(iii) Let us now define paths with four states and three jumps at fixed times $1/4, 1/2$ et $3/4$, such as

$$\mathbf{abab}(t) := \mathbf{1}_{[0,1/4)}(t) \mathbf{a} + \mathbf{1}_{[1/4,1/2)}(t) \mathbf{b} + \mathbf{1}_{[1/2,3/4)}(t) \mathbf{a} + \mathbf{1}_{[3/4,1]}(t) \mathbf{b}.$$

The probability measure $P := \frac{1}{2}(\delta_{\mathbf{abab}} + \delta_{\mathbf{cbcb}})$ on Ω , which is a mixture of reciprocal deterministic paths (they are their own bridges) is not reciprocal anymore. Indeed

$$P(X_{2/3} = \mathbf{a} \mid X_{[0,1/3]}, X_{[4/5,1]}) = \mathbf{1}_{\{X_0=\mathbf{a}\}}$$

while

$$P(X_{2/3} = \mathbf{a} \mid X_{1/3}, X_{4/5}) = P(X_{2/3} = \mathbf{a}) = 1/2.$$

Let R be a reciprocal probability measure on Ω . We would like to test the reciprocal character of mixtures of its bridges. Therefore let us assume – as in Section 1.3 – that bridges of R , denoted by R^{xy} , can be constructed for all $x, y \in \mathcal{X}$ as a regular version of the family of its conditional laws $(R(\cdot \mid X_0 = x, X_1 = y), x, y \in \mathcal{X})$.

Proposition 2.7. *Let R be a reciprocal probability measure on Ω , such that the mapping $(x, y) \in \mathcal{X}^2 \mapsto R^{xy}$ is defined everywhere and measurable. Then, for any probability measure π on \mathcal{X}^2 , the probability measure*

$$P(\cdot) = \int_{\mathcal{X}^2} R^{xy}(\cdot) \pi(dxdy)$$

is reciprocal. Moreover, the bridges of P coincide with those of R , P -a.e.

Proof. Let us show (2.2) under P . Let $0 \leq s \leq t \leq 1$, $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$ and $C \in \mathcal{A}_{[u,1]}$. Then,

$$\begin{aligned} E_P[\mathbf{1}_A P(B \mid X_{[0,s]}, X_{[u,1]}) \mathbf{1}_C] &= P(A \cap B \cap C) \\ &= \int_{\mathcal{X}^2} R^{xy}(A \cap B \cap C) \pi(dxdy) \\ &\stackrel{\vee}{=} \int_{\mathcal{X}^2} E_{R^{xy}}[\mathbf{1}_A R(B \mid X_s, X_t) \mathbf{1}_C] \pi(dxdy) \\ &= E_P[\mathbf{1}_A R(B \mid X_s, X_t) \mathbf{1}_C] \end{aligned}$$

where the reciprocal property is used at the marked equality. Thus $P(B \mid X_{[0,s]}, X_{[t,1]})$ only depends on (X_s, X_t) and

$$P(B \mid X_{[0,s]}, X_{[t,1]}) = R(B \mid X_s, X_t), \quad P\text{-a.e.},$$

which completes the proof. \square

Let us observe that this result does not contradict Example 2.6-(iii). Indeed, P was expressed as a mixture of its own bridges, but not as a mixture of bridges of a given reciprocal probability measure. It happens that there does not exist any reciprocal probability measure R such that $\delta_{\text{abab}} = R^{\text{ab}}$ and $\delta_{\text{cbcb}} = R^{\text{cb}}$.

Previous proposition allows to construct classes of reciprocal probability measures based on some reference reciprocal probability measure by varying the way of mixing bridges. Therefore, we now recall the definition of an important concept.

Definition 2.8 (Reciprocal class associated with R). *Suppose that R is a reciprocal probability measure on Ω such that $(x, y) \in \mathcal{X}^2 \mapsto R^{xy}$ is defined everywhere and measurable. The set of probability measures on Ω defined by*

$$\mathfrak{R}_c(R) := \left\{ P = \int_{\mathcal{X}^2} R^{xy}(\cdot) \pi(dxdy); \pi \text{ probability measure on } \mathcal{X}^2 \right\} \quad (2.3)$$

is called the reciprocal class associated with R .

The index c in $\mathfrak{R}_c(R)$ recalls the first letter of the word class. Later we will introduce another set of probability measures called reciprocal family, which will be denoted by $\mathfrak{R}_f(R)$.

In the case of a *discrete* state space \mathcal{X} , the hypothesis on R becomes useless. One should only make sure that the support of the mixing measure π is included in the support of R_{01} , in such a way that (2.3) makes sense.

Remarks 2.9 (about this definition). The concept of reciprocal class is due to Jamison, [18, §3], for a Markov reference probability measure whose transition kernels satisfy Hypothesis (H) of Section 1.3.

In the particular case where R is a Brownian diffusion defined on the space of continuous paths, the class $\mathfrak{R}_c(R)$ can be characterized by two functions of the drift of R , called reciprocal invariants. This was conjectured by Krener in [22] and proved by Clark in [9, theorem 1]. Thereafter, Thieullen and the second author derived an integration by parts formula on the path space that is expressed in terms of the reciprocal invariants of the Brownian diffusion R and that fully characterises the associated reciprocal class. See [31]

for one-dimensional diffusion processes and [32] for the multidimensional case.

When R is a counting process (i.e. $\mathcal{X} = \mathbb{N}$), Murr12 provides a description of a reciprocal invariant associated with $\mathfrak{R}_c(R)$, as well as a characterisation of the reciprocal class through a duality formula, see [27, 25]. An extension of this work for more general jump processes is in preparation.

2.4. Time reversal and reciprocal property. We already saw in Theorem 2.2 that a probability measure is reciprocal if and only if its time-reversed probability measure on Ω is reciprocal too. We can now make this assertion more precise.

Proposition 2.10. *Let R be a reciprocal probability measure on Ω as in Definition 2.8. Then*

$$P \in \mathfrak{R}_c(R) \iff P^* \in \mathfrak{R}_c(R^*).$$

We first prove the following auxiliary lemma.

Lemma 2.11.

(a) *Consider the diagram $\Omega \xrightarrow{\Phi} \Phi(\Omega) \xrightarrow{\theta} \mathcal{Y}$ where the mentioned sets and mappings are measurable. Then, for any bounded measurable function $f : \Phi(\Omega) \rightarrow \mathbb{R}$, we have*

$$E_{\Phi_{\#}P}(f|\theta) = \alpha(\theta)$$

with $\alpha(y) := E_P(f(\Phi)|\theta(\Phi) = y)$.

(b) *Consider the diagram $\mathcal{Y} \xleftarrow{\theta} \Omega \xrightarrow{\Phi} \Omega$ where the mentioned sets and mappings are measurable. Suppose that Φ is one-to-one with measurable inverse Φ^{-1} . Then,*

$$\Phi_{\#}[P(\cdot | \theta = y)] = [\Phi_{\#}P](\cdot | \theta \circ \Phi^{-1} = y), \quad y \in \mathcal{Y}.$$

Proof. • Proof of (a). For any bounded measurable function $u : \mathcal{Y} \rightarrow \mathbb{R}$,

$$\begin{aligned} E_{\Phi_{\#}P}[E_{\Phi_{\#}P}(f|\theta)u(\theta)] &= E_{\Phi_{\#}P}(fu(\theta)) = E_P[f(\Phi)u(\theta(\Phi))] \\ &= E_P[E_P(f(\Phi)|\theta(\Phi))u(\theta(\Phi))] = E_{\Phi_{\#}P}(\alpha(\theta)u(\theta)) \end{aligned}$$

• Proof of (b). We add a bounded measurable function u to the diagram:

$\mathcal{Y} \xleftarrow{\theta} \Omega \xrightarrow{\Phi} \Omega \xrightarrow{u} \mathbb{R}$ and compute, for $y \in \mathcal{Y}$,

$$\begin{aligned} E_{\Phi_{\#}P(\cdot|\theta=y)}(u) &= E_P[u(\Phi)|\theta = y] \\ &= E_P[u(\Phi)|\theta \circ \Phi^{-1} \circ \Phi = y] \stackrel{(i)}{=} E_{\Phi_{\#}P}(u|\theta \circ \Phi^{-1} = y) \end{aligned}$$

where equality (i) is a consequence of the above result (a). □

Proof of Proposition 2.10. In particular Lemma 2.11-(b) implies that

$$(R^{xy})^* = (R^*)^{yx}, \quad \text{pour } R\text{-a.e. } x, y \in \mathcal{X}. \quad (2.4)$$

Let $P \in \mathfrak{R}_c(R)$, then $P(\cdot) = \int_{\mathcal{X}^2} R^{xy}(\cdot) P_{01}(dxdy)$. We now compute the integral of a function u under P^* :

$$\begin{aligned} E_{P^*}[u(X)] &= E_P[u(X^*)] = \int_{\mathcal{X}^2} E_{(R^{xy})^*}(u) P_{01}(dxdy) \\ &\stackrel{(2.4)}{=} \int_{\mathcal{X}^2} E_{(R^*)^{yx}}(u) P_{01}(dxdy) = \int_{\mathcal{X}^2} E_{(R^*)^{xy}}(u) (P^*)_{01}(dxdy). \end{aligned}$$

This means that $P^*(\cdot) = \int_{\mathcal{X}^2} (R^*)^{xy}(\cdot) (P^*)_{01}(dxdy)$, completing the proof of Proposition 2.10. \square

2.5. Reciprocal families. To make precise our structural analysis of reciprocal probability measures, we prefer to introduce a slightly more restrictive concept than the one of reciprocal class, which we call *reciprocal family* associated with R . This set only contains probability measures which are dominated by the reference probability measure R .

Definition 2.12 (Reciprocal family associated with R). *Suppose that R is a reciprocal probability measure on Ω as in Definition 2.8. The set of probability measures on Ω defined by*

$$\mathfrak{R}_f(R) := \left\{ P : P = \int_{\mathcal{X}^2} R^{xy} \pi(dxdy), \text{ where } \pi \ll R_{01} \right\} \subset \mathfrak{R}_c(R) \quad (2.5)$$

is called the reciprocal family associated with R .

Remarks 2.13 (about this definition).

- (a) Due to Proposition 2.7, we notice that any probability measure belonging to a reciprocal family is reciprocal.
- (b) We write $P \prec R$ when P disintegrates as in (2.5). Note that the relation \prec is transitive. But it is not symmetric; this lack of symmetry arises when the marginal laws at time 0 and 1 are not equivalent in the sense of measure theory. Therefore a reciprocal family is not an equivalence class. If one wants to define a genuine equivalence relation \sim between probability measures on Ω one should assume that marginal laws at time 0 and 1 are equivalent. Then $P \sim R$ if and only if $P \prec R$ and $R \prec P$.

Elements of a reciprocal family have a remarkably simple structure.

Theorem 2.14. *Each probability measure P in the reciprocal family $\mathfrak{R}_f(R)$ defined by (2.5), is dominated by R and satisfies*

$$P = \frac{d\pi}{dR_{01}}(X_0, X_1) R.$$

Conversely, if P is defined by

$$P = h(X_0, X_1) R \quad (2.6)$$

for some nonnegative measurable function h on \mathcal{X}^2 , then $P \in \mathfrak{R}_f(R)$ and more precisely, P is a π -mixture of bridges of R with

$$\pi(dxdy) := h(x, y) R_{01}(dxdy).$$

Proof. Let $P \in \mathfrak{R}_f(R)$ and f any nonnegative bounded function. Due to Definition (2.5), since $\pi \ll R_{01}$,

$$\begin{aligned} E_P(f) &= \int_{\mathcal{X}^2} E_R(f \mid X_0 = x, X_1 = y) \pi(dxdy) \\ &= \int_{\mathcal{X}^2} E_R(f \mid X_0 = x, X_1 = y) \frac{d\pi}{dR_{01}}(x, y) R_{01}(dxdy) \\ &= E_R \left(E_R(f \mid X_0, X_1) \frac{d\pi}{dR_{01}}(X_0, X_1) \right) \\ &= E_R \left(\frac{d\pi}{dR_{01}}(X_0, X_1) f \right), \end{aligned}$$

which proves the first assertion.

For the second assertion, note that

$$P(\cdot) = \int_{\mathcal{X}^2} P^{xy}(\cdot) \pi(dxdy) = \int_{\mathcal{X}^2} h(x, y) R^{xy}(\cdot) R_{01}(dxdy).$$

This completes the proof of the theorem. \square

The specific structure of P which appears in (2.6) can be regarded as a time symmetric version of the h -transform introduced by Doob in [12].

2.6. Markov probability measures of a reciprocal family. Since the Markov property is more restrictive than the reciprocal property, it is interesting to describe the subset of a reciprocal family of all Markov measures. In other words, one is looking for the specific mixtures of probability measures which preserve Markov property.

In the remainder of the subsection, R is a reference Markov probability measure on Ω . If a probability measure in $\mathfrak{R}_f(R)$ admits a density with respect to R which is decomposable into a product, as in (1.10), we already know that it is Markov. This property is (almost) characteristic. Let us prove it.

Theorem 2.15. *Let R and P be two probability measures on Ω and suppose that R is Markov. Consider the following assertions:*

- (1) *The probability measure P belongs to $\mathfrak{R}_f(R)$ (see Definition 2.12) and is Markov.*
- (2) *There exist two measurable nonnegative functions f_0 and g_1 such that*

$$\frac{dP}{dR} = f_0(X_0)g_1(X_1), \quad R\text{-a.e.} \quad (2.7)$$

Then, (2) implies assertion (1).

If we suppose moreover that there exists $0 < t_0 < 1$ and a measurable subset $\mathcal{X}_o \subset \mathcal{X}$ such that $R_{t_0}(\mathcal{X}_o) > 0$ and for all $z \in \mathcal{X}_o$,

$$R_{01}(\cdot) \ll R_{01}^{t_0 z}(\cdot) := R((X_0, X_1) \in \cdot \mid X_{t_0} = z), \quad (2.8)$$

then (1) and (2) are equivalent.

Proof. • Proof of (2) \Rightarrow (1). It is contained in Example 1.7. Note that Hypothesis (2.8) is not necessary.

- Proof of (1) \Rightarrow (2). Since P is Markov, Theorem 1.5 applied with $t = t_0$ leads to

$$\frac{dP}{dR} = \alpha(X_{[0,t_0]})\beta(X_{[t_0,1]}) \quad R\text{-a.e.} \quad (2.9)$$

with α and β two measurable nonnegative functions. But, since P belongs to the reciprocal family of R , following Theorem 2.14, its Radon-Nikodym derivative is

$$\frac{dP}{dR} = h(X_0, X_1)$$

for some measurable nonnegative function h on \mathcal{X}^2 . This implies that

$$\alpha(X_{[0,t_0]})\beta(X_{[t_0,1]}) = h(X_0, X_1), \quad R\text{-a.e.}$$

which in turns implies that the functions α and β have the form

$$\alpha(X_{[0,t_0]}) = a(X_0, X_{t_0}) \text{ and } \beta(X_{[t_0,1]}) = b(X_{t_0}, X_1), \quad R\text{-a.e.}$$

with a and b two measurable nonnegative functions on \mathcal{X}^2 . It follows that

$$a(x, z)b(z, y) = h(x, y) \quad \forall (x, z, y) \in \mathcal{N}^c \subset \mathcal{X}^3,$$

where the set $\mathcal{N} \subset \mathcal{X}^3$ is $R_{0,t_0,1}$ -negligible. Now, with the notation

$$\mathcal{N}_z := \{(x, y); (x, z, y) \in \mathcal{N}\} \subset \mathcal{X}^2,$$

we obtain

$$0 = R_{0,t_0,1}(\mathcal{N}) = \int_{\mathcal{X}} R_{01}^{t_0 z}(\mathcal{N}_z) R_{t_0}(dz)$$

which implies that $R_{01}^{t_0 z}(\mathcal{N}_z) = 0$ for R_{t_0} -a.e. $z \in \mathcal{X}_0$. Due to condition (2.8), one deduces that there exists $z_o \in \mathcal{X}_o$ such that $R_{01}(\mathcal{N}_{z_o}) = 0$. Taking $f_0 = a(\cdot, z_o)$ and $g_1 = b(z_o, \cdot)$, we see that

$$h(x, y) = f_0(x)g_1(y), \quad R_{01}(dxdy)\text{-a.e.},$$

which proves that dP/dR has the form expressed in (2.7). \square

Remarks 2.16.

- (a) This result belongs to folk knowledge about reciprocal processes. It is often used without detailed proof. Our statement emphasizes the role of condition (2.8) which, up to our knowledge, appears for the first time. A partial version of Theorem 2.15 can be found in [18, Thm. 3.1]. Jamison proved, under the assumption that the Markov probability measure R admits a smooth transition density, that $P \in \mathfrak{R}_c(R)$ if and only if there exist two probability measures on \mathcal{X} , ν_0 and ν_1 such that

$$P_{01}(dxdy) = r(0, x; 1, y)\nu_0(dx)\nu_1(dy).$$

- (b) Since R is Markov, condition (2.8) is equivalent to

$$\forall z \in \mathcal{X}_o, \quad R_{01}(\cdot) \ll R(X_0 \in \cdot | X_{t_0} = z) \otimes R(X_1 \in \cdot | X_{t_0} = z).$$

- (c) Without any additional condition on R , both assertions of the above theorem fail to be equivalent. We provide a counter-example by constructing a probability measure R which does not satisfy condition (2.8) and a Markov probability measure P whose density with respect to R does not have the mentioned structure.

Let R be the Markov probability measure on Ω with state space $\mathcal{X} = \{\mathbf{a}, \mathbf{b}\}$, with

initial law $R_0 = (\delta_a + \delta_b)/2$ and infinitesimal generator $\begin{pmatrix} 0 & 0 \\ \lambda & -\lambda \end{pmatrix}$ for some $\lambda > 0$. The support of R is concentrated on two types of paths: those identically equal to **a** or **b**, and those starting from **b** with one jump onto **a** after an exponential waiting time in $(0, 1)$ with law $\mathcal{E}(\lambda)$. We see that R does not satisfy (2.8). Indeed, for all $t \in (0, 1)$,

(a) $R_{01}^{ta}(\mathbf{b}, \mathbf{b}) = 0$, but $R_{01}(\mathbf{b}, \mathbf{b}) = \frac{e^{-\lambda}}{2} > 0$. Thus, $R_{01} \not\ll R_{01}^{ta}$.

(b) $R_{01}^{tb}(\mathbf{a}, \mathbf{a}) = 0$, but $R_{01}(\mathbf{a}, \mathbf{a}) = \frac{1}{2} > 0$. Thus, $R_{01} \not\ll R_{01}^{tb}$.

Now consider the Markov probability measure P which gives half mass to the deterministic constant paths equal to **a** or **b**. It is dominated by R with density:

$$\frac{dP}{dR} = \begin{cases} 1, & \text{if } X \equiv \mathbf{a} \\ e^\lambda, & \text{if } X \equiv \mathbf{b} \\ 0, & \text{if } X_0 \neq X_1 \end{cases}. \quad \text{This density } dP/dR \text{ does not have the product form}$$

$$(2.7), \text{ since the system } \begin{cases} f(\mathbf{a})g(\mathbf{a}) &= 1 \\ f(\mathbf{b})g(\mathbf{b}) &= e^\lambda \\ f(\mathbf{b})g(\mathbf{a}) &= 0 \end{cases} \text{ admits no solution. Remark that the}$$

functions α and β defined in (2.9) could be chosen as follows: $\alpha(X) = \beta(X) = 1$ if $X \equiv \mathbf{a}$, $\alpha(X) = 1$ if $X \equiv \mathbf{b}$, $\beta(X) = e^\lambda$ if $X \equiv \mathbf{b}$ and $\alpha(X) = \beta(X) = 0$ otherwise.

3. RECIPROCAL PROBABILITY MEASURES ARE SOLUTIONS OF ENTROPY MINIMIZING PROBLEMS

We conclude this survey paper going back to the problem that was originally addressed by Schrödinger in [33] and developed in [34]. It was the starting point of the theory of time-reversed Markov [21] and reciprocal diffusion processes. A modern formulation of Schrödinger's problem is stated below at (S_{dyn}) .

Motivated by a probabilistic solution of this problem, Bernstein [2] introduced the notion of reciprocal process. It is likely that Bernstein wasn't aware of the fact that (S_{dyn}) 's solution is not only reciprocal, but also Markov as was clearly demonstrated four decades later by Jamison in [19].

The new ingredient of this section is the relative entropy. The relative entropy of a probability measure p with respect to another probability measure r on a measurable space \mathcal{Y} is given by

$$H(p|r) := \int_{\mathcal{Y}} \log \left(\frac{dp}{dr} \right) dp \in [0, +\infty]$$

when p is dominated by r , and $+\infty$ otherwise.

3.1. Schrödinger's problem. This problem is of a statistical physics nature.

Dynamical and static formulations of Schrödinger's problem. We now sketch some results which are presented with further detail in the review paper [24] (see the references therein too). The modern dynamical formulation of Schrödinger's problem is as follows. Take a reference probability measure R on $\Omega = D([0, 1], \mathcal{X})$ and fix two probability measures μ_0, μ_1 on \mathcal{X} (the marginal constraints). The aim is to minimize $P \mapsto H(P|R)$ where P varies in the set of all probability measures on Ω such that $P_0 = \mu_0$ and $P_1 = \mu_1$. A concise

statement of Schrödinger's dynamical problem is

$$H(P|R) \rightarrow \min; \quad P \in \text{Proba}(\Omega) : P_0 = \mu_0, P_1 = \mu_1 \quad (\text{S}_{\text{dyn}})$$

Projecting via (X_0, X_1) this variational problem onto the set \mathcal{X}^2 of endpoint configurations, one obtains the following associated static formulation which consists of minimizing $\pi \mapsto H(\pi|R_{01})$ where π is subject to vary in the set of all probability measures on \mathcal{X}^2 such that its marginals $\pi_0(dx) := \pi(dx \times \mathcal{X})$ and $\pi(dy) := \pi(\mathcal{X} \times dy)$ are prescribed to satisfy $\pi_0 = \mu_0$ and $\pi_1 = \mu_1$. A concise statement of Schrödinger's static problem is

$$H(\pi|R_{01}) \rightarrow \min; \quad \pi \in \text{Proba}(\mathcal{X}^2) : \pi_0 = \mu_0, \pi_1 = \mu_1 \quad (\text{S})$$

Let us recall the uniqueness result obtained by Föllmer [15] for a Brownian diffusion with drift. In the following proposition, we consider path measures on $\Omega = D([0, 1], \mathcal{X})$ since Föllmer's results extend without trouble to this general setting.

Proposition 3.1 (Föllmer). *The dynamical and static Schrödinger problems admit respectively at most one solution \hat{P} and $\hat{\pi}$. If \hat{P} denotes the solution of (S_{dyn}) , then $\hat{\pi} = \hat{P}_{01}$ is the solution of (S). Conversely, if $\hat{\pi}$ solves (S), then the solution of (S_{dyn}) is*

$$\hat{P}(\cdot) = \int_{\mathcal{X}^2} R^{xy}(\cdot) \hat{\pi}(dxdy) \in \mathfrak{R}_f(R). \quad (3.1)$$

Sketch of the proof. As strictly convex minimization problems, (S_{dyn}) and (S) admit at most one solution. Using the disintegration formula

$$H(P|R) = H(P_{01}|R_{01}) + \int_{\mathcal{X}^2} H(P^{xy}|R^{xy}) P_{01}(dxdy),$$

one obtains $H(P_{01}|R_{01}) \leq H(P|R)$ with equality (when $H(P|R) < +\infty$) if and only if $P^{xy} = R^{xy}$ for P_{01} -almost all $(x, y) \in \mathcal{X}^2$, which corresponds to $P \in \mathfrak{R}_f(R)$. Thus, \hat{P} is the solution of (S_{dyn}) if and only if it disintegrates as (3.1). \square

Since R is Markov, so is \hat{P} . We present an existence and uniqueness result for (S_{dyn}) and (S), obtained by the first author in [24].

Theorem 3.2. *Let R be a reference Markov probability measure on Ω , with identical⁵ marginal laws at time 0 and 1, denoted by m . Suppose that R satisfies the following assumptions:*

(i) *there exists $0 < t_0 < 1$ and a measurable set $\mathcal{X}_o \subset \mathcal{X}$ such that $R_{t_0}(\mathcal{X}_o) > 0$ and*

$$R_{01} \ll R((X_0, X_1) \in \cdot | X_{t_0} = z), \quad \forall z \in \mathcal{X}_o.$$

(ii) *there exists a nonnegative measurable function A on \mathcal{X} such that*

$$R_{01}(dxdy) \geq e^{-A(x)-A(y)} m(dx)m(dy).$$

⁵This restriction is done for simplifying the statements. We mostly have in mind a stationary reference measure R .

Suppose also that the constraints μ_0 and μ_1 satisfy

$$H(\mu_0|m) + H(\mu_1|m) < +\infty \text{ and } \int_{\mathcal{X}} A d\mu_0 + \int_{\mathcal{X}} A d\mu_1 < +\infty.$$

Then (S) admits a unique solution $\hat{\pi}$. It satisfies

$$\hat{\pi}(dxdy) = f_0(x)g_1(y) R_{01}(dxdy)$$

for some m -measurable nonnegative functions $f_0, g_1 : \mathcal{X} \rightarrow [0, \infty)$ which solve the so-called Schrödinger system:

$$\begin{cases} f_0(x) E_R[g_1(X_1) | X_0 = x] = d\mu_0/dm(x), & \text{for } m\text{-a.e. } x \\ g_1(y) E_R[f_0(X_0) | X_1 = y] = d\mu_1/dm(y), & \text{for } m\text{-a.e. } y. \end{cases} \quad (3.2)$$

Moreover, (S_{dyn}) admits the unique solution

$$\hat{P} = f_0(X_0)g_1(X_1) R. \quad (3.3)$$

It inherits the Markov property from R .

Remark 3.3. In the Schrödinger system, $E_R[f_0(X_0) | X_1]$ and $E_R[g_1(X_1) | X_0]$ are well defined even if $f_0(X_0)$ and $g_1(X_1)$ are not R -integrable. In fact, f_0 and g_1 are measurable and nonnegative; therefore, only positive integration is needed, see [23].

Generalizing Proposition 3.1, we obtain without additional effort the following result.

Corollary 3.4. *Let R be any reciprocal probability measure. The solution \hat{P} of the variational problem (S_{dyn}) , if it exists, belongs to the reciprocal family $\mathfrak{R}_f(R)$.*

A connection between Schrödinger's problem and PDEs. We give a PDE interpretation of the time-marginal flow $(\hat{P}_t)_{0 \leq t \leq 1}$ of the solution \hat{P} of (S_{dyn}) , with the aim of clarifying its dynamical content. Let us come back to Example 1.12-(i), where $\mathcal{X} = \mathbb{R}$, $m(dx) = dx$, R denotes the (unbounded⁶) reversible Wiener measure and $r(s, x; t, y)$ is its Gaussian kernel. Let us call $\rho_0(x) = \frac{d\mu_0}{dx}(x)$ and $\rho_1(y) = \frac{d\mu_1}{dy}(y)$. Then the system (3.2) reduces to

$$\begin{cases} f_0(x) \int r(0, x; 1, y) g_1(y) dy = \rho_0(x) \\ g_1(y) \int f_0(x) r(0, x; 1, y) dx = \rho_1(y). \end{cases} \quad (3.4)$$

Schrödinger raised the question of the existence and uniqueness of solutions (f_0, g_1) of this nonlinear system, given r and the probabilistic boundary data ρ_0 and ρ_1 . For f_0 and g_1 strictly positive, and r considerably more general than the Gaussian kernel, introducing an entropy minimizing problem close to (S), Beurling [3] answered positively to this question. This was extended later by several authors, see [16, 24] for instance, taking advantage of the tight connection between (3.4) and (S).

Let us denote by $f(t, z)$ the solution of the parabolic initial value problem

$$\begin{cases} (-\partial_t + \partial_{zz}^2/2)f = 0, & 0 < t \leq 1 \\ f(0, \cdot) = f_0, & t = 0 \end{cases} \quad (3.5)$$

⁶See [23] for the technical modifications that are necessary to handle the case of an unbounded reference measure.

and by $g(t, z)$ the solution of the adjoint final value problem

$$\begin{cases} (\partial_t + \partial_{zz}^2/2)g = 0, & 0 \leq t < 1 \\ g(1, \cdot) = g_1, & t = 1 \end{cases} \quad (3.6)$$

Remark that $f(t, z) = E_R(f_0(X_0) \mid X_t = z)$ and $g(t, z) = E_R(g_1(X_1) \mid X_t = z)$. Thanks to the Markov property of R , Theorem 1.2-(3) entails that for all $0 \leq t \leq 1$,

$$\hat{P}_t(dz) = f(t, z)g(t, z) dz.$$

This relation is analogous to Born's formula:

$$\rho_t(dz) = \psi_t(z)\overline{\psi_t(z)} dz$$

where ψ is the wave function. Indeed, as remarked in 1928 by the astrophysicist Edington (this is quoted in [34]), taking the complex conjugate of $e^{i\omega t}$ amounts to reverse time. Therefore, ψ_t and $\overline{\psi_t}$ can be interpreted as two wave functions carrying respectively information from past and future. Indeed, they solve the standard quantum Schrödinger equation with respect to both directions of time. Switching to the classical statistical physics problem (S_{dyn}) , one sees that the functions f_t and g_t share similar properties, replacing the complex-valued Schrödinger equations in both directions of time by the heat equations (3.5) and (3.6). This striking analogy was Schrödinger's main motivation for introducing (S_{dyn}) . See [34] and also [8], [24, § 6,7] for further detail.

Regarded as an element of $L^2(\mathbb{R}, dz)$ the solutions of (3.5) and (3.6) are analytic in the domain $\text{Re}(t) > 0$, continuous for $\text{Re}(t) \geq 0$ and their values on the imaginary axis respectively solve the (quantum mechanical) Schrödinger equation and its complex conjugate. It is in this way that the Markov measure \hat{P} is a quantum-like probability measure. The multiplicative structure of the density $d\hat{P}_t/dz$ appears as a stochastic analytic version of the complex conjugation of quantum functionals. When the Markov generator associated with R is not self-adjoint, the same idea holds. For (much) more on this PDE connection, see [38].

3.2. A modification of Schrödinger's problem. Having in mind these considerations about Schrödinger's problem, it appears that the notion of reciprocal measure was a technical intermediate step on the way to the solution of (S_{dyn}) . Indeed, Theorem 3.2 insures that \hat{P} is Markov, which is more specific than being reciprocal, and its proof doesn't rely on the reciprocal property. Nevertheless, there exist instances of non-Markov reciprocal measures that are interesting in their own right. Let us give a short presentation of two problems relating entropy minimization and genuine reciprocal measures.

Reciprocal measures and entropy minimization. Consider the following modification of Schrödinger's problem

$$H(P|R) \rightarrow \min; \quad P \in \text{Proba}(\Omega) : P_{01} = \pi \quad (S^\pi)$$

where R is Markov and $\pi \in \text{Proba}(\mathcal{X}^2)$ is given. Mimicking the sketch of proof of Proposition 3.1, it is easy to show that (S^π) admits a solution if and only if $H(\pi|R_{01}) < \infty$ and that, when this occurs, this solution is unique and is equal to

$$R^\pi(\cdot) := \int_{\mathcal{X}^2} R^{xy}(\cdot) \pi(dxdy).$$

When $\pi(dxdy) = f_0(x)g_1(y)R_{01}(dxdy)$ with (f_0, g_1) solution of the Schrödinger system (3.2), then $(S^\pi) = (S_{\text{dyn}})$. By (2.5), we see that R^π stands in the reciprocal family $\mathfrak{R}_f(R)$ of R . More precisely, when π describes $\text{Proba}(\mathcal{X}^2)$, defining

$$\mathfrak{R}_H(R) := \{P : P \text{ solution of } (S^\pi) \text{ with } \pi \in \text{Proba}(\mathcal{X}^2)\},$$

we see that

$$\mathfrak{R}_H(R) = \{R^\pi; \quad \pi \text{ such that } H(\pi|R_{01}) < \infty\}$$

which is a little smaller than $\mathfrak{R}_f(R)$ for which π is only required to satisfy $\pi \ll R_{01}$. Notice that

$$\mathfrak{R}_H(R) \subset \mathfrak{R}_f(R) \subset \mathfrak{R}_c(R)$$

where these three classes are convex subsets of $\text{Proba}(\Omega)$.

Loop measures. Example 2.4 exhibits a reciprocal loop measure $P_{\text{loop}} = \int_{\mathcal{X}} R^{xx} m(dx)$ which is not Markov in general. Denoting $\pi_m(dxdy) = m(dx)\delta_x(dy)$, we see that $P_{\text{loop}} = R^{\pi_m}$.

Remark that in the important case where R is the reversible Brownian motion⁷, then $\pi_m \not\ll R_{01}$ because $R_{01}(X_0 = X_1) = 0$ and $\pi_m(X_0 = X_1) = 1$. Consequently, (S^{π_m}) has no solution. The endpoint constraint $\pi = \pi_m$ of (S^π) is degenerate in the same way as $(\mu_0, \mu_1) = (\delta_x, \delta_y)$ is a degenerate constraint of (S_{dyn}) . Indeed, both π_m and $\delta_{(x,y)}$ verify $H(\pi_m|R_{01}), H(\delta_{(x,y)}|R_{01}) < \infty$ and can be approximated by finite entropy constraints.

Stochastic representation of incompressible hydrodynamical flows. Consider the following entropy minimization problem

$$H(P|R) \rightarrow \min; \quad P \in \text{Proba}(\Omega) : P_t = m, \forall 0 \leq t \leq 1, P_{01} = \pi \quad (3.7)$$

which consists of minimizing the relative entropy $H(P|R)$ of the path measure P with respect to the Markov measure R subject to the constraints that the time marginal flow $(P_t)_{0 \leq t \leq 1}$ is constantly equal to a given $m \in \text{Proba}(\mathcal{X})$ and that the endpoint marginal P_{01} is equal to a given $\pi \in \text{Proba}(\mathcal{X}^2)$. This problem is a natural stochastization of Arnold's approach to Euler equation for incompressible fluids [1] which is connected to the Navier-Stokes equation. The justification of this assertion is part of a work in progress by two of the authors. The incompressibility constraints is $P_t = m, \forall 0 \leq t \leq 1$ when m is the volume measure on the manifold \mathcal{X} . The constraint $P_{01} = \pi$ is Brenier's relaxation [4] of Arnold's final diffeomorphism. It can be proved using the results of the present paper that for a generic endpoint constraint π , the minimizer of (3.7) (whenever it exists) is reciprocal but not Markov.

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⁷See [23] for the technical modifications that are necessary to handle the case of an unbounded reference measure.

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