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Finsler Steepest Descent with Applications to Piecewise-regular Curve Evolution

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Abstract

This paper introduces a novel steepest descent flow in Banach spaces. This extends previous works on generalized gradient descent, notably the work of Charpiat et al. [12], to the setting of Finsler metrics. Such a generalized gradient allows one to take into account a prior on deformations (e.g., piecewise rigid) in order to favor some specific evolutions. We define a Finsler gradient descent method to minimize a functional defined on a Banach space and we prove a convergence theorem for such a method. In particular, we show that the use of non-Hilbertian norms on Banach spaces is useful to study non-convex optimization problems where the geometry of the space might play a crucial role to avoid poor local minima. We show some applications to the curve matching problem. In particular, we characterize piecewise rigid deformations on the space of curves and we study several models to perform piecewise rigid evolution of curves.

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Keywords: Curve evolution ; Finsler space ; gradient flow ; shape registration

1 Introduction

This paper introduces a new descent method to minimize energies defined over Banach spaces. This descent makes use of a generalized gradient which corresponds to a descent direction for a Finsler geometry. We show applications of this method to the optimization over the space of curves, where this Finsler gradient allows one to construct piecewise regular curve evolutions.

1.1 Previous Works

Energy minimization for curve evolution. The main motivation for this work is the design of novel shape optimization methods, with an emphasis toward curves evolutions. Shape optimization

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is a central topic in computer vision, and has been introduced to solve various problems such as image segmentation or shape matching. These problems are often solved by introducing an energy which is minimized over the space of curves. The first variational method proposed to perform image segmentation through curve evolution is the snake model [21]. This initial proposal has been formalized using intrinsic energies depending only on the geometry of the curves. A first class of energies corresponds to a weighted length of the curve, where the weight acts as an edge detector [8, 23]. A second class of segmentation energies, pioneered by the Mumford-Shah model [26], integrates a penalization both inside and outside the curve, see for instance [9]. Shape registration requires to compute a matching between curves, which in turn can be solved by minimizing energies between pairs of curves. An elegant framework to design such energies uses distances over a space of measures or currents, see [17] for a detailed description and applications in medical imaging.

Shape spaces as Riemannian spaces. Minimizing these energies requires to define a suitable space of shapes and a notion of gradient with respect to the geometry of this space. The mathematical study of spaces of curves has been largely investigated in the last years, see for instance [42, 24]. The set of curves is naturally modeled over a Riemannian manifold [25]. This corresponds to using a Hilbertian metric on each tangent plane of the space of curves, i.e. the set of vector fields which deform infinitesimally a given curve. This Riemannian framework allows one to define geodesics which are shortest paths between two shapes [43, 18]. Computing minimizing geodesics is useful to perform shape registration [31, 37, 35], tracking [31] and shape deformation [22]. The theoretical study of the existence of these geodesics depends on the Riemannian metric. For instance, a striking result [25, 40, 41] is that the natural L^2-metric on the space of curves, that has been largely used in several applications in computer vision, is a degenerate Riemannian metric: any two curves have distance equal to zero with respect to such a metric.

Beside the computation of minimizing geodesics, Riemannian metrics are also useful to define descent directions for shape optimization. Several recent works [25, 11, 41, 40] point out that the choice of the metric, which the gradient depends on, notably affects the results of a gradient descent algorithm. Carefully designing the metric is thus crucial to reach better local minima of the energy. Modifying the descent flow can also be important for shape matching applications. A typical example of such a Riemannian metrics are Sobolev-type metrics [32, 30, 34, 33] which lead to smooth curve evolutions.

Shape spaces as Finslerian spaces. It is possible to extend the Riemannian framework by considering more general metrics on the tangent planes of the space of curves. Finsler spaces make use of Banach norm instead of Hilbertian norms [6]. A few recent works [24, 41] have studied the theoretical properties of Finslerian spaces of curves. To the best of our knowledge, with the notable exception of [12], which is discussed in detailed in Section 1.4, no previous work has used Finslerian metrics for curve evolution.

Generalized gradient flow. Beyond shape optimization, the use of non-Euclidean geometries is linked to the study of generalized gradient flows. Optimization on manifolds requires the use of Riemannian gradients and Riemannian Hessians, see for instance [1]. Second order schemes on manifolds can be used to accelerate shape optimization over Riemannian spaces of curves, see [39]. Optimization over Banach spaces requires the use of convex duality to define the associated gradient

flow [19, 2, 20]. It is possible to generalize these flows for metric spaces using implicit descent steps, we refer to [4] for an overview of the theoretical properties of the resulting flows.

1.2 Motivation

The metrics defined over the tangent planes of the space of curves (e.g. an Hilbertian norm in the Riemannian case and a Banach norm in the Finsler case) have a major impact on the trajectory of the associated gradient descent. This choice thus allows one to favor specific evolutions. A first reason for introducing a problem-dependent metric is to enhance the performances of the optimization method. Energies minimized for shape optimization are non-convex, so a careful choice of the metric is helpful to avoid being trapped in a poor local minimum. A typical example is the curve registration problem, where reaching a non-global minimum makes the matching fail. A second reason is that, in some applications, one is actually interested in the whole descent trajectory, and not only in the local minimum computed by the algorithm. For the curve registration problem, the matching between the curves is obtained by tracking the bijection between the curves during the evolution. Taking into account desirable physical properties of the shapes, such as global or piecewise rigidity, is crucial to achieve state of the art results, see for instance [13, 10, 29]. In this article, we explore the use of Finsler gradient flows to encode piecewise regular deformations of the curves.

1.3 Contributions

Our first contribution is the definition of a novel generalized gradient flow, that we call Finsler descent, and the study of the convergence properties of this flow. This Finsler gradient is obtained from the L^2-gradient through the resolution of a constrained convex optimization problem. Our second contribution is the instantiation of this general framework to define piecewise regular curve evolutions, without knowing in advance the location of the articulations. This contribution includes the definition of novel Finsler penalties to encode piecewise rigid and piecewise similarity evolutions. It also includes the theoretical analysis of the convergence of the flow for BV^2-regular curves. Our last contribution is the application of these piecewise regular evolutions to the problem of curve registration. This includes the definition of a discretized flow using finite elements, and a comparison of the performances of Riemannian and Finsler flows for articulated shapes registration. The Matlab code to reproduce the numerical results of this article is available online[1].

1.4 Relationship with [12]

Our work is partly inspired by the generalized gradient flow originally defined in [12]. We use a different formulation for our Finsler gradient, and in particular consider a convex constrained formulation, which allows us to prove convergence results. An application to piecewise rigid evolutions is also proposed in [12], but it differs significantly from our method. In [12], piecewise rigid flows are obtained using a non-convex penalty, which poses both theoretical difficulties (definition of a suitable functional space to make the descent method well-defined) and numerical difficulties (computation of descent direction as the global minimizer of a non-convex energy). In our work we prove

[1] `https://www.ceremade.dauphine.fr/~peyre/codes/`

a characterization of piecewise rigid deformations that enables the definition of a penalty depending on the deformation (instead of instantaneous parameters as done in [12]). Then, we generalize this penalty to the BV^2-framework obtaining a convex penalty for BV^2-piecewise rigid deformations.

1.5 Paper Organization

Section 2 defines the Finsler gradient and the associated steepest descent in Banach spaces, for which we prove a convergence theorem. Section 3 introduces the space of BV^2-curves and study its main properties, in particular its stability to reparametrization. Section 4 characterizes C^2-piecewise rigid motions and defines a penalty in the case of BV^2-regular motions. We apply this method in Section 5 to the curve registration problem. We minimize a matching energy using the Finsler descent method for BV^2-piecewise rigid motions. Section 6 details the finite elements discretization of the method. Section 7 gives numerical illustrations of the Finsler descent method for curve matching. Section 8 refines the model introduced in Section 4 to improve the matching results by replacing piecewise rigid transforms with piecewise similarities.

2 Finsler Descent Method in Banach Spaces

We consider a Banach space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ and a differentiable energy on $\mathcal{B}$

$$E : \Gamma \in \mathcal{B} \mapsto E(\Gamma) \in \mathbb{R} \,.$$

We aim at solving the following minimization problem

$$\min_{\Gamma \in \mathcal{B}} E(\Gamma) \tag{2.1}$$

using a steepest descent method. We treat $\mathcal{B}$ as a manifold modeled on itself and denote by $T_\Gamma\mathcal{B}$ the tangent space at $\Gamma \in \mathcal{B}$. In the following we suppose that at every point $\Gamma \in \mathcal{B}$, the space $T_\Gamma\mathcal{B}$ coincides with $\mathcal{B}$, although our descent method can be adapted to more general settings.

We also suppose that $\mathcal{B}$ is a subset of a Hilbert space $(H, \langle\cdot,\,\cdot\rangle_H)$, so we can define an inner product $\langle\cdot,\,\cdot\rangle_{H(\Gamma)}$ on the tangent space $T_\Gamma\mathcal{B}$. Note that this inner product may be different from the inner product induced by $\langle\cdot,\,\cdot\rangle_H$ on $T_\Gamma\mathcal{B}$, and in particular it might depend on Γ. For instance in the case of Sobolev metrics for the space of curves we usually consider $H = W^{1,2}([0,1],\mathbb{R}^2)$ and set $T_\Gamma\mathcal{B} = W^{1,2}([0,1],\mathbb{R}^2)$ equipped with the measure defined by the arclength of Γ (see Remark 2.3).

In order to minimize the energy E we remind that the Gâteaux derivative of E at Γ in the direction $\Phi \in T_\Gamma\mathcal{B}$ is defined as

$$DE(\Gamma)(\Phi) = \lim_{t\to 0} \frac{E(\Gamma + t\Phi) - E(\Gamma)}{t} \,.$$

Moreover, by the Riesz representation theorem, there exists a unique vector $v \in T_\Gamma\mathcal{B}$ such that

$$DE(\Gamma)(\Phi) = \langle v,\, \Phi\rangle_{H(\Gamma)} \quad \forall\, \Phi \in T_\Gamma\mathcal{B} \,.$$

The vector v represents the gradient of E at Γ with respect to this inner product $\langle\cdot,\,\cdot\rangle_{H(\Gamma)}$ and it is denoted is denoted by $v = \nabla_H E(\Gamma)$.

2.1 Finsler Gradient

The Finsler gradient determines a descent direction by modifying $\nabla_H E(\Gamma)$ with respect to a penalty R_Γ that depends on Γ. It is defined by minimizing R_Γ under a constraint $\mathcal{L}_\Gamma$.

Definition 2.1 (Finsler gradient). For every $\Gamma \in \mathcal{B}$, let $R_\Gamma : T_\Gamma\mathcal{B} \to \mathbb{R}^+ \cup \{+\infty\}$ be a function such that $R_\Gamma \neq +\infty$ and $\mathcal{L}_\Gamma \subset T_\Gamma\mathcal{B}$ a set satisfying

$$\mathcal{L}_\Gamma \subset \left\{ \Phi \in T_\Gamma\mathcal{B} \; ; \; \left\langle \frac{\nabla_H E(\Gamma)}{\|\nabla_H E(\Gamma)\|_{H(\Gamma)}}, \frac{\Phi}{\|\Phi\|_{H(\Gamma)}} \right\rangle_{H(\Gamma)} \geqslant (1-\rho) \right\} \tag{2.2}$$

where $\rho \in (0,1)$. If R_Γ admits a minimum on $\mathcal{L}_\Gamma$ then a Finsler gradient for E at Γ with respect to R_Γ is defined as:

$$\nabla_{R_\Gamma} E(\Gamma) \in \operatorname{argmin} \; \{R_\Gamma(\Phi) \; ; \; \Phi \in \mathcal{L}_\Gamma\} \; . \tag{2.3}$$

Next theorem gives an existence result for the Finsler gradient.

Theorem 2.2. Let $T_\Gamma\mathcal{B}$ be a Banach space equipped with a topology $\mathcal{T}(T_\Gamma\mathcal{B})$ such that every bounded sequence in $T_\Gamma\mathcal{B}$ converges (up to a subsequence) with respect to the topology $\mathcal{T}(T_\Gamma\mathcal{B})$. Let R_Γ be coercive (i.e., $R_\Gamma(\Phi) \to +\infty$ as $\|\Phi\|_{T_\Gamma\mathcal{B}} \to +\infty$) and lower semi-continuous with respect to the topology $\mathcal{T}(T_\Gamma\mathcal{B})$ and we suppose that $\mathcal{L}_\Gamma$ is closed in $T_\Gamma\mathcal{B}$ with respect to the topology $\mathcal{T}(T_\Gamma\mathcal{B})$. Then Problem (2.3) admits at least a solution.

Proof. As R_Γ is coercive, every minimizing sequence is bounded in $T_\Gamma\mathcal{B}$ so it converges (up to a subsequence) with respect to the topology $\mathcal{T}(T_\Gamma\mathcal{B})$ toward an element of $\mathcal{L}_\Gamma$. Now, because of the lower semi-continuity of R_Γ, the theorem ensues. □

Such a result is the generalization of the usual existence theorem of calculus of variations on a reflexive Banach space. In fact, if $T_\Gamma\mathcal{B}$ is reflexive, the existence of the Finsler gradient is guaranteed whenever $\mathcal{L}_\Gamma$ is convex and closed with respect to the strong topology of $T_\Gamma\mathcal{B}$, and R_Γ is coercive, $R_\Gamma \neq +\infty$, convex, and lower semi-continuous with respect to the strong topology of $T_\Gamma\mathcal{B}$. These hypothesis guarantee in particular an existence result if $T_\Gamma\mathcal{B}$ is a Hilbert space.

We point out that the applications studied in this work concern a minimization problem on $T_\Gamma\mathcal{B} = BV^2(\mathbb{S}^1, \mathbb{R}^2)$. Such a space is not reflexive but the weak* topology of $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ verifies the hypothesis of the previous theorem (see Appendix). Then, for some suitable set $\mathcal{L}_\Gamma$ and penalty R_Γ, the existence of the Finsler gradient is guaranteed.

The set $\mathcal{L}_\Gamma$ imposes a constraint on the direction of the Finsler gradient. It is crucial to guarantee the convergence of the descent method (see Theorem 2.5). The parameter ρ controls the deviation of the Finsler gradient with respect to $\nabla_H E(\Gamma)$. This parameter can be tuned by the user to modify the geometry of the trajectory of the flows defined in Section 2.2. This impact is studied in the numerical simulations, see Section 7.1.

If the hypothesis of Theorem 2.2 are verified then the minimum in (2.3) exists, but in general it is not unique. A Finsler gradient is any minimum of the functional minimized in (2.3).

Condition (2.2) implies

$$\left\langle \frac{\nabla_H E(\Gamma)}{\|\nabla_H E(\Gamma)\|_{H(\Gamma)}}, \frac{\nabla_{R_\Gamma} E(\Gamma)}{\|\nabla_{R_\Gamma} E(\Gamma)\|_{H(\Gamma)}} \right\rangle_{H(\Gamma)} \geqslant (1-\rho) > 0 \qquad \forall\, \Gamma \in \mathcal{B}. \tag{2.4}$$

This shows that the Finsler gradient is a valid descent direction, in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\Gamma - t\nabla_{R_\Gamma}E(\Gamma))\Big|_{t=0} = -\langle \nabla_H E(\Gamma),\, \nabla_{R_\Gamma}E(\Gamma)\rangle_{H(\Gamma)} < 0\,.$$

Remark 2.3 (**Relationship with Sobolev gradient**). We consider $\mathcal{B} = W^{1,2}([0,1],\mathbb{R}^2)$, $H = L^2([0,1],\mathbb{R}^2)$. More precisely, for every $\Gamma \in \mathcal{B}$, we set $T_\Gamma\mathcal{B} = W^{1,2}([0,1],\mathbb{R}^2)$, $L^2(\Gamma) = L^2([0,1],\mathbb{R}^2)$ equipped with the norm

$$\|\Psi\|^2_{L^2(\Gamma)} = \int_0^1 \|\Psi(s)\|^2\|\Gamma'(s)\|\mathrm{d}s.$$

which depends on Γ. This setting models smooth parametric planar curves and their deformations Ψ. Note that the space of curves is further detailed in Section 3. To define smooth evolutions of these curves, we introduce

$$R_\Gamma(\Phi) = \|D\Phi\|^2_{L^2(\Gamma)}, \qquad \forall\, \Phi \in T_\Gamma\mathcal{B},$$

$$\mathcal{L}_\Gamma = \left\{\Phi \in T_\Gamma\mathcal{B}\ ;\ \|\nabla_{L^2}E(\Gamma) - \Phi\|_{L^2(\Gamma)} \leqslant \rho\|\nabla_{L^2}E(\Gamma)\|_{L^2(\Gamma)}\right\} \tag{2.5}$$

where we denote by $D\Phi$ the weak derivative of Φ. Note that $\mathcal{L}_\Gamma$ satisfies condition (2.2). For a given differentiable energy E, (2.3) becomes

$$\nabla_{R_\Gamma}E(\Gamma) \in \operatorname*{argmin}_{\Phi\in\mathcal{L}_\Gamma} \|D\Phi\|^2_{L^2(\Gamma)}\,. \tag{2.6}$$

Remark that, comparing with Proposition 4 p. 17 in [12], the Finsler gradient (2.6) represents a constrained version of the Sobolev gradient. Note also that in Definition 2.1, the penalty R_Γ needs not be quadratic so that the Finsler gradient can be understood as a generalization of the Sobolev gradient.

Remark 2.4 (**Relationship with [12]**). Our definition of Finsler gradient is partly inspired by the generalized gradient introduced in Section 6.1 of [12]. An important difference is that we introduce a constraint $\mathcal{L}_\Gamma$ whereas [12] defines the gradient as a minimum of $DE(\Gamma)(\Phi) + R_\Gamma(\Phi)$ on $T_\Gamma\mathcal{B}$. This is a crucial point because, as shown in the next section, this constraint guarantees the convergence of the descent method associated with the Finsler gradient toward a stationary point of E.

2.2 Finsler Descent Method

In this section we consider the minimization problem (2.1) of an energy E on $\mathcal{B}$. Given some initialization $\Gamma_0 \in \mathcal{B}$, the Finsler gradient descent is defined as

$$\Gamma_{k+1} = \Gamma_k - \tau_k\nabla_{R_{\Gamma_k}}E(\Gamma_k) \tag{2.7}$$

where $\nabla_{R_{\Gamma_k}}E(\Gamma_k)$ is any minimizer of (2.3) and the step size $\tau = \tau_k > 0$ is chosen in order to satisfy the Wolfe constraints

$$\left\{\begin{array}{lcl} E(\Gamma + \tau v) & \leqslant & E(\Gamma) + \alpha\tau\langle\nabla_H E(\Gamma),\, v\rangle \\ \langle\nabla_H E(\Gamma + \tau v),\, v\rangle & \geqslant & \beta\langle\nabla_H E(\Gamma),\, v\rangle \end{array}\right. \tag{2.8}$$

for some fixed $0 < \alpha < \beta < 1$ and with $v = -\nabla_{R_{\Gamma_k}}E(\Gamma_k)$, see for instance [27], p.37.

We have the following result.

Theorem 2.5. Let E be a non negative energy, $E \in C^1(\mathcal{B})$ with uniformly-Lipschitz continuous gradient $\nabla_{H(Gamma)}E$ with respect to the topology of $\mathcal{B}$. Then every accumulation point of the sequence $\{\Gamma_k\}_k$, defined in (2.7), is a critical point of E.

Proof. The energy E is a C^1 non negative function and

$$\langle \frac{\nabla_H E(\Gamma)}{\|\nabla_H E(\Gamma)\|}, \frac{\nabla_{R_\Gamma} E(\Gamma)}{\|\nabla_{R_\Gamma} E(\Gamma)\|} \rangle_{H(\Gamma)} \geqslant 1 - \rho > 0 \,. \tag{2.9}$$

Now, as τ_k is chosen by the Wolfe rule, from (2.9) and the first relationship of (2.8), it follows that the sequence $E(\Gamma_k)$ is decreasing. Moreover, $\nabla_H E$ is uniformly-Lipschitz continuous, so the Zoutendijk theorem (see [27]: Theorem 3.2 p.43) implies that

$$\sum_{k=0}^{\infty} \langle \frac{\nabla_H E(\Gamma)}{\|\nabla_H E(\Gamma)\|}, \frac{\nabla_{R_\Gamma} E(\Gamma)}{\|\nabla_{R_\Gamma} E(\Gamma)\|} \rangle_{H(\Gamma)} \, \|\nabla_H E(\Gamma)\|_{H(\Gamma)} \, < \, \infty \,.$$

Now, because of (2.9), this implies that

$$\nabla_H E(\Gamma_k) \to 0 \quad \text{in} \quad H \,, \tag{2.10}$$

and, as $\nabla_H E$ is continuous, the theorem ensues. □

Note that the sequence defined by the Finsler descent method could diverge (for instance if $E(\Gamma) \to 0$ as $\|\Gamma\|_{\mathcal{B}} \to +\infty$). However, if E is coercive and its level sets are compact with respect to the strong topology of $\mathcal{B}$, then previous theorem guarantees the convergence of the Finsler descent method toward a stationary point of the energy. In fact, as E is coercive, we have that $\{\Gamma_k\}$ is uniformly bounded in $\mathcal{B}$. Then, as the level sets of E are compact, $\{\Gamma_k\}$ converges (up to a subsequence) to an element of $\mathcal{B}$ which is, by (2.10), a stationary point of E.

3 Finsler Gradient in the Spaces of Curves

This section specializes our method to a space of piecewise-regular curves. We target applications to piecewise rigid evolutions to solve a curve matching problem (see Section 5). Note that, in order to perform piecewise rigid evolutions, we are led to deal with curves whose first and second derivatives are not continuous. This leads us to consider the setting of BV^2-regular functions. We refer the reader to Appendix for the definition and the main properties of BV and BV^2 functions.

In the following, for every BV^2-function Γ, we denote by $\mathrm{d}\Gamma(s)$ the measure $\|\Gamma'(s)\|\mathrm{d}s$ where Γ' denotes the approximate derivative of Γ (see for instance [3]).

3.1 BV^2-curves

In this section we define the space of BV^2-curves and introduce its main properties. This models closed, connected curves admitting a counter-clockwise BV^2 parameterization.

Definition 3.1 (BV^2**-curves).** We define $\mathcal{B} = BV^2(\mathbb{S}^1, \mathbb{R}^2)$ equipped with the BV^2-norm, where $\mathbb{S}^1$ is the unit circle. For any $\Gamma \in \mathcal{B}$, we set $T_\Gamma \mathcal{B} = BV^2(\Gamma)$ the space $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ equipped with the measure $\mathrm{d}\Gamma(s)$. In a similar way we define $W^{2,1}(\Gamma)$. In $BV^2(\Gamma)$ and $W^{2,1}(\Gamma)$, differentiation and

integration are done with respect to the measure $\mathrm{d}\Gamma(s)$. We point out that, because of the definition of the measure $\mathrm{d}\Gamma(s)$, $BV^2(\mathbb{S}^1,\mathbb{R}^2)$ and $BV^2(\Gamma)$ represent the same set of functions and the respective norms are equivalent.

Every $\Phi \in T_\Gamma\mathcal{B}$ operates on a curve $\Gamma \in \mathcal{B}$ as

$$(\Gamma+\Phi)(s) = \Gamma(s) + \Phi(s)\,, \quad \forall\, s \in \mathbb{S}^1.$$

Definition 3.2 (**Tangent, normal, and curvature).** For every $\Gamma \in \mathcal{B}$ we define the following vector

$$\nu_\Gamma(s) = \lim_{r\to 0} \frac{D\Gamma((s-r,s+r))}{|D\Gamma|((s-r,s+r))}$$

where $|D\Gamma|$ denotes the total variation of Γ and $D\Gamma$ denotes the vector-valued measure associated with the total variation. Note that $|D\Gamma|$ coincides with the measure $\mathrm{d}\Gamma(s)$. As Γ is a BV^2-function of one variable we can represent the previous measure by its good representatives (see [3], Theorem 3.28) which implies that the previous limit exists at each point and $\|\nu_\Gamma\| = 1$ for $\mathrm{d}\Gamma(s)$-a.e. $s \in \mathbb{S}^1$. The tangent and normal vectors to the curve at every point $\Gamma(s)$ are defined as

$$\mathbf{t}_\Gamma(s) = \frac{\nu_\Gamma(s)}{\|\nu_\Gamma(s)\|} \qquad \mathbf{n}_\Gamma(s) = \mathbf{t}_\Gamma(s)^\perp \tag{3.1}$$

$$\text{where} \quad (x,y)^\perp = (-y,x)\,, \quad \forall\,(x,y) \in \mathbb{R}^2.$$

By the previous definition we have $\mathbf{t}_\Gamma \cdot \mathbf{t}_\Gamma = 1$ at each point and, by differentiating with respect to s, we get that the measure $\mathbf{t}_\Gamma \cdot D\mathbf{t}_\Gamma$ is null. Then, there exists a real measure curv_Γ such that

$$D\mathbf{t}_\Gamma = \mathbf{n}_\Gamma\,\mathrm{curv}_\Gamma\,. \tag{3.2}$$

By the definition of $\mathbf{n}_\Gamma$ we also have

$$D\mathbf{n}_\Gamma = -\mathbf{t}_\Gamma\,\mathrm{curv}_\Gamma\,. \tag{3.3}$$

The measure curv_Γ is called generalized curvature of Γ and in the case of a smooth curve it coincides with the measure $\kappa_\Gamma\,\mathrm{d}s$ where κ_Γ denotes the standard curvature of Γ.

From the properties of the total variation (see instance [3] and Proposition 9.3 in the Appendix) it follows that

$$|\mathrm{curv}_\Gamma|(\mathbb{S}^1) \leqslant 2|\mathrm{D}^2\Gamma|(\mathbb{S}^1) \tag{3.4}$$

where $|\mathrm{curv}_\Gamma|(\mathbb{S}^1)$ denotes the total variation of the generalized curvature on the circle.

Example 3.3. Let Γ the square with vertices $\{(0,0),(1,0),(1,1),(0,1)\}$ parameterized as

$$\Gamma(s) = \begin{cases} (4s,0) & \text{if } s \in (0,1/4) \\ (1,4(s-1/4)) & \text{if } s \in (1/4,1/2) \\ (1-4(s-1/2),1) & \text{if } s \in (1/2,3/4) \\ (0,1-4(s-3/4)) & \text{if } s \in (3/4,1) \end{cases}$$

where we have identified $\mathbb{S}^1$ with the interval $[0,1]$. By the previous definition, for every $s \in [0,1]$,we have

$$\nu_\Gamma(s) = \begin{cases} (1,0) & \text{if } s \in (0,1/4) \\ (0,1) & \text{if } s \in (1/4,1/2) \\ (-1,0) & \text{if } s \in (1/2,3/4) \\ (0,-1) & \text{if } s \in (3/4,1) \end{cases}$$

and at the vertices we have

$$\nu_\Gamma(0) = \frac{1}{2}(1,-1),\ \nu_\Gamma(1/4) = \frac{1}{2}(1,1),\ \nu_\Gamma(1/2) = \frac{1}{2}(-1,1),\ \nu_\Gamma(3/4) = \frac{1}{2}(-1,-1)\,.$$

In particular we have that the tangent and normal vectors are constant on each edge of the square and at the vertices they coincide with a linear combination of the left and right limits. This implies that the generalized curvature is a Dirac measure concentrated at the vertices:

$$\mathrm{curv}_\Gamma = \delta_0 + \delta_{1/4} + \delta_{1/2} + \delta_{3/4}\,.$$

Definition 3.4 (Projectors). We denote by Π_Γ the projection on the normal vector field $\mathbf{n}_\Gamma$

$$\Pi_\Gamma(\Phi)(s) \;=\; \Big(\Phi(s)\cdot \mathbf{n}_\Gamma(s)\Big)\,\mathbf{n}_\Gamma(s), \tag{3.5}$$

where $\cdot$ is the inner product in $\mathbb{R}^2$. We denote by Σ_Γ the projection on the tangent vector field $\mathbf{t}_\Gamma$

$$\Sigma_\Gamma(\Phi)(s) \;=\; \Big(\Phi(s)\cdot \mathbf{t}_\Gamma(s)\Big)\,\mathbf{t}_\Gamma(s)\,. \tag{3.6}$$

Definition 3.5 (Hilbertian structure). The Banach space $\mathcal{B} = BV^2(\mathbb{S}^1,\mathbb{R}^2)$ is a subset of the Hilbert space $H = L^2(\mathbb{S}^1,\mathbb{R}^2)$. For every $\Gamma \in \mathcal{B}$, we define $L^2(\Gamma) = L^2(\mathbb{S}^1,\mathbb{R}^2)$, where integration is done with respect to the measure $\|\Gamma'(s)\|\mathrm{d}s$. This defines the following inner product on $T_\Gamma\mathcal{B}$

$$\langle \Phi,\ \Psi\rangle_{L^2(\Gamma)} = \int_{\mathbb{S}^1} \Phi(s)\cdot\Psi(s)\ \mathrm{d}\Gamma(s) \tag{3.7}$$

for every $\Phi, \Psi \in T_\Gamma\mathcal{B}$, where $\mathrm{d}\Gamma(s) = \|\Gamma'(s)\|\,\mathrm{d}s$ is the infinitesimal length.

The quantity $\langle \Phi,\ \Psi\rangle_{L^2(\Gamma)}$ is intrinsic in the sense that it is invariant to re-parameterization of Γ. Section 3.2 gives more details about parameterization of BV^2 curves.

Finally, recall that for a differentiable energy E, the L^2 gradient of E at Γ, is defined as the unique deformation $\nabla_{L^2}E(\Gamma)$ satisfying :

$$DE(\Gamma)(\Phi) \;=\; \langle \nabla_{L^2}E(\Gamma),\ \Phi\rangle_{L^2(\Gamma)}\,, \quad \forall\,\Phi \in T_\Gamma\mathcal{B}$$

where $DE(\Gamma)(\Phi)$ is the directional derivative.

3.2 Geometric Curves and Parameterizations

For applications in computer vision, it is important that the developed method (e.g. a gradient descent flow to minimize an energy) only depends on the actual geometry of the planar curve, and not on its particular parametrization. We denote $[\Gamma] = \Gamma(\mathbb{S}^1)$ the geometric realization of the curve, i.e. the image of the parameterization in the plane.

If for two curves $\Gamma_1, \Gamma_2 \in \mathcal{B}$ there exists a smooth invertible map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Gamma_2 = \Gamma_1 \circ \varphi$, then Γ_2 is a reparameterization of Γ_1, and these parameterizations share the same image, i.e. $[\Gamma_1] = [\Gamma_2]$. This section, shows in some sense the converse implication in the BV^2 framework, namely the existence of a reparameterization map between two curves sharing the same geometric

realization. This result is important since it shows the geometric invariance of the developed Finsler gradient flow.

Note however that this is clearly not possible without any constraint on the considered curve. For instance, there is no canonical parameterization of an eight-shaped curve in the plane. We thus only consider injective curves $\Gamma \in \mathcal{B}$ satisfying the following additional property

$$0 \notin \text{Conv}(\Gamma'(s^+), \Gamma'(s^-)) \quad \forall\, s \in \mathbb{S}^1 \,. \tag{3.8}$$

Here Conv denotes the convex envelope (a line segment) of the right and left limits $(\Gamma'(s^-), \Gamma'(s^+))$ of the derivative of Γ at s. Such a property gives a generalized definition of immersion for BV^2-curves and implies that the support of the curve has no cusp points.

We define the set of curves

$$\mathcal{B}_0 = \left\{\Gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2)\ ;\ \Gamma \text{ is injective and satisfies (3.8)}\right\}$$

equipped with the BV^2-norm.

Proposition 3.6 below shows that $\mathcal{B}_0$ is an open set of $\mathcal{B}$. Note however that it is not a linear space, so that, to guarantee that the evolution (2.7) stays inside $\mathcal{B}_0$, the step size τ_k should be small enough. This however contradicts the constraints (2.8) that are needed to ensure the convergence of the flow. This reflects the well known fact that during an evolution, a parametric curve can cross itself and become non-injective.

Proposition 3.6. $\mathcal{B}_0$ is an open set of $\mathcal{B} = BV^2(\mathbb{S}^1, \mathbb{R}^2)$.

Proof. As $\mathbb{S}^1$ is a compact set, by property (3.8), if $\Lambda \in \mathcal{B}_0$ then

$$m = \min_{s \in \mathbb{S}^1} \|\Lambda'(s)\| > 0 \,.$$

Now, as BV is embedded in L^∞ (i.e. there exists a constant $C > 0$ such that $\|\Gamma\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)} \leqslant C\|\Gamma\|_{BV(\mathbb{S}^1,\mathbb{R}^2)}$), every curve $\Gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$ such that

$$\|\Gamma' - \Lambda'\|_{BV(\mathbb{S}^1,\mathbb{R}^2)} < \frac{\eta}{2C}\ , \quad \eta < m$$

satisfies (3.8). Moreover, (3.8) implies that Λ is locally injective and, as $\mathbb{S}^1$ is compact, there exists $\varepsilon, \alpha > 0$ such that

$$\|\Lambda(s) - \Lambda(s')\| \geqslant \alpha|s - s'|, \quad \forall\, s, s' \in \mathbb{S}^1 \ \text{ such that } \ |s - s'| \leqslant \varepsilon \,. \tag{3.9}$$

Every Γ such that $\|\Gamma' - \Lambda'\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)} < \alpha/2$ satisfies (3.9) so it is locally injective. Then, as Λ is injective, if we take $\|\Gamma - \Lambda\|_{BV^2(\mathbb{S}^1,\mathbb{R}^2)} < \beta(\Lambda)$ where

$$\beta(\Lambda) = \frac{1}{2}\min\left\{\frac{\alpha}{C}, \inf_{s \in \mathbb{S}^1} \inf_{|s-s'|>\varepsilon} \|\Lambda(s) - \Lambda(s')\|\right\}$$

then Γ is also globally injective.

Then

$$\left\{\Gamma \in \mathcal{B}_0 \,\middle|\, \|\Gamma - \Lambda\|_{BV^2(\mathbb{S}^1,\mathbb{R}^2)} < \min\left\{\frac{\eta}{2C}, \beta(\Lambda)\right\}\right\} \subset \mathcal{B}_0$$

which proves that $\mathcal{B}_0$ is an open set of $BV^2(\mathbb{S}^1, \mathbb{R}^2)$. □

The next proposition extends two classical properties of smooth curves to $\mathcal{B}_0$.

Proposition 3.7. 1) (**Reparameterization**) For every $\Gamma_1, \Gamma_2 \in \mathcal{B}_0$ such that $[\Gamma_1] = [\Gamma_2]$, there exists a homeomorphism $\varphi \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$ such that

$$\Gamma_1 = \Gamma_2 \circ \varphi \,.$$

2) (**Local coordinates**) For every $\Gamma \in \mathcal{B}_0$ the set $\Gamma(\mathbb{S}^1)$ can be locally represented as the graph of a BV^2-function.

Proof. 1) For every $\Gamma \in \mathcal{B}_0$ we consider the arc-length parameterization defined by

$$\varphi_\Gamma : \mathbb{S}^1 \to [0, \text{Length}(\Gamma)]$$
$$\varphi_\Gamma(s) = \int_{s_0}^{s} \|\Gamma'(t)\| \, dt \,, \quad s_0 \in \mathbb{S}^1 \,.$$

As $\mathbb{S}^1$ si a compact set, by property (3.8), it follows that

$$\inf_{s \in \mathbb{S}^1} \|\Gamma'(s)\| > 0$$

which implies that $\varphi'_\Gamma > 0$ on $\mathbb{S}^1$. Then, $\varphi_\Gamma \in BV^2(\mathbb{S}^1, \mathbb{S}^1) \subset W^{1,\infty}(\mathbb{S}^1, \mathbb{S}^1)$ and its inverse $\varphi_\Gamma^{-1} \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$ can be defined (see Theorem 1 in [14]). It follows that

$$(\Gamma_1 \circ \varphi_{\Gamma_1} \circ \varphi_{\Gamma_2}^{-1})(s) = \Gamma_2(s) \quad \forall\, s \in \mathbb{S}^1.$$

2) It suffices to adapt the classical proof in the case of smooth curves using Theorem 1 in [14]. □

3.3 Geometric Invariance

For BV^2 curves, the geometric invariance of the developed methods should be understood as an invariance with respect to BV^2 reparameterizations.

Definition 3.8. Let G_{BV^2} denotes the set of homeomorphisms $\varphi \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$ such that $\varphi^{-1} \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$.

To ensure this invariance, we consider energies E and penalties R_Γ such that

$$E(\Gamma \circ \varphi) = E(\Gamma) \;\;, \quad R_{\Gamma\circ\varphi}(\Phi \circ \varphi) = R_\Gamma(\Phi) \qquad \forall\, \Gamma \in \mathcal{B}_0,\, \forall\, \varphi \in G_{BV^2},\, \forall\, \Phi \in T_\Gamma \mathcal{B} \,.$$

This implies that

$$\nabla_{R_{\Gamma\circ\varphi}} E(\Gamma \circ \varphi)(\Phi \circ \varphi) = \nabla_{R_\Gamma} E(\Gamma)(\Phi) \circ \varphi$$

so that the descent scheme (2.7) does not depend on the parameterization of Γ. Finally, as

$$(\Gamma - \tau \nabla_{R_\Gamma} E(\Gamma)) \circ \varphi = \Gamma \circ \varphi - \tau \nabla_{R_{\Gamma\circ\varphi}} E(\Gamma \circ \varphi),$$

for $\tau = \tau_k$, the descent step in (2.7) is also invariant under reparameterization.

This shows that the Finsler gradient flow can actually be defined over the quotient space $\mathcal{B}/G_{BV^2}$. To avoid unnecessarily technicalities, we decided not to use this framework and develop our analysis in the setting of the vector space $\mathcal{B}$.

Another consequence of this invariance is that, as long as the evolution (2.7) is in $\mathcal{B}_0$, the flow does not depend on the choice of curve parameterization. However, as already noted in Section 3.2, it might happen that the sequence leaves $\mathcal{B}_0$, in which case different choices of parameterizations of an initial geometric realization can lead to different evolutions.

4 Piecewise Rigidity

This section defines a penalty R_Γ that favors BV^2-piecewise rigid deformations of BV^2-curves. For the sake of clarity we present the construction of this penalty in two steps. We first characterize in Section 4.1 C^2-global rigid deformations for smooth curves. Then, in Section 4.2, we introduce a penalty that favors BV^2-piecewise rigid deformations for curves belonging to $\mathcal{B}$.

4.1 Rigid Curve Deformations

A continuous curve evolution $t \in \mathbb{R} \mapsto \Gamma_t \in \mathcal{B}$ reads

$$\forall\, t \in \mathbb{R}, \quad \frac{\partial \Gamma_t(s)}{\partial t} = \Phi_t(s) \quad \text{where} \quad \Phi_t \in T_{\Gamma_t}\mathcal{B}\,. \tag{4.1}$$

We further assume in this section that Γ_t is a C^2 curve. This evolution is said to be globally rigid if it preserves the pairwise distances between points along the curves, i.e.

$$\forall\, t \in \mathbb{R}, \quad \forall\,(s,s') \in \mathbb{S}^1 \times \mathbb{S}^1, \quad \|\Gamma_t(s) - \Gamma_t(s')\| = \|\Gamma_0(s) - \Gamma_0(s')\|. \tag{4.2}$$

The following proposition shows that the set of instantaneous motion Φ_t giving rise to a rigid evolution is, at each time, a linear sub-space of dimension 3 of $T_{\Gamma_t}\mathcal{B}$.

Proposition 4.1. The evolution (4.1) satisfies (4.2) if and only if $\Phi_t \in \mathcal{R}_{\Gamma_t}$ for all $t \in \mathbb{R}$, where

$$\mathcal{R}_\Gamma = \left\{ \Phi \in T_\Gamma\mathcal{B}\ ;\ \exists a \in \mathbb{R}, \exists b \in \mathbb{R}^2,\ \ \forall\, s \in \mathbb{S}^1,\ \Phi(s) = a\,\Gamma(s)^\perp + b \right\}\,. \tag{4.3}$$

Proof. By the definition of evolution, we have

$$\Gamma_t(s) = \Gamma_0(s) + \int_0^t \Phi_\tau(s)\mathrm{d}\tau \quad \forall s \in \mathbb{S}^1\,. \tag{4.4}$$

If the evolution satisfies (4.2), then the application $m(t) : \Gamma_0(\mathbb{S}^1) \to \Gamma_t(\mathbb{S}^1)$ defined as

$$m(t)(\Gamma_0(s)) = \Gamma_0(s) + \int_0^t \Phi_\tau(s)\mathrm{d}\tau \quad \forall s \in \mathbb{S}^1$$

is an isometry between the supports of the curves Γ_0 and Γ_t. Now, by considering three non-collinear points of $\Gamma_0(\mathbb{S}^1)$ denoted by $\Gamma_0(s_i)$ $(i = 1, 2, 3)$, we can extend such an isometry $m(t)$ to an isometry of the plane. We define $\tilde{m}(t) : \mathbb{R}^2 \to \mathbb{R}^2$ as it follows: for every $x \in \mathbb{R}^2$, $\tilde{m}(t)(x)$ is such that

$$\|\tilde{m}(t)(x) - \Gamma_t(s_i)\| = \|x - \Gamma_0(s_i)\| \quad i = 1, 2, 3\,. \tag{4.5}$$

Previous condition determines a unique isometry of the plane $\tilde{m}(t)$ such that

$$\tilde{m}(t)(\Gamma_0(s)) = m(t)(\Gamma_0(s)) \quad \forall s \in \mathbb{S}^1\,.$$

Now, by the characterization of the isometries, $\tilde{m}(t)$ can be written as the composition of a rotation and a translation, so that

$$\tilde{m}(t)(x) = A(t)x + c(t) \quad \forall x \in \mathbb{R}^2 \tag{4.6}$$

for some vector $c(t)$ and matrix $A(t)$ defined by

$$\begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}.$$

Note that, $A(t)$ and $c(t)$ are defined by condition (4.5) as two differentiable functions with respect to t. Moreover, as $\tilde{m}(t)$ is continuous with respect to t, it is a direct isometry, that justifies the previous definition of $A(t)$. In particular, as $\tilde{m}(t)$ coincides with $m(t)$ on $\Gamma_0(\mathbb{S}^1)$, we have

$$\Gamma_t(s) = A(t)\Gamma_0(s) + c(t) \quad \forall s \in \mathbb{S}^1 . \tag{4.7}$$

By differentiating (4.4) and (4.7) with respect to t, we get

$$\Phi_t(s) = A'(t)\Gamma_0(s) + c'(t) \quad \forall s \in \mathbb{S}^1, \tag{4.8}$$

where $(\cdot)'$ denotes here the derivative with respect to t. Note that, by the definition of rotation matrix in dimension two, we have

$$A'(t)A(t)^{-1}w = \theta'(t)w^{\perp} \tag{4.9}$$

for every $\mathrm{w} \in \mathbb{R}^2$. Then by (4.7), (4.8), and (4.9) we get, for all $s \in \mathbb{S}^1$,

$$\Phi_t(s) = A'(t)A(t)^{-1}\Gamma_t(s) + c'(t) - A'(t)A(t)^{-1}c(t) = \theta'(t)\Gamma_t(s)^{\perp} + c'(t) - A'(t)A(t)^{-1}c(t),$$

which implies that $\Phi_t \in \mathcal{R}_{\Gamma_t}$ with $a = \theta'(t)$ and $b = c'(t) - A'(t)A(t)^{-1}c(t)$.

On the other hand if $\Phi_t \in \mathcal{R}_{\Gamma_t}$ then

$$\Phi_t(s) - \Phi_t(s') = a(\Gamma_t(s) - \Gamma_t(s'))^{\perp} \quad \forall (s, s') \in \mathbb{S}^1 \times \mathbb{S}^1 .$$

Then

$$\frac{\partial}{\partial t}\|\Gamma_t(s) - \Gamma_t(s')\|^2 = 2(\Phi_t(s) - \Phi_t(s')) \cdot (\Gamma_t(s) - \Gamma_t(s')) = 0 \quad \forall (s, s') \in \mathbb{S}^1 \times \mathbb{S}^1$$

which implies (4.2). □

Note that for numerical simulations, one replaces the continuous PDE (4.1) by a flow discretized at times $t_k = k\tau$ for some step size $\tau > 0$ and $k \in \mathbb{N}$,

$$\Gamma_{k+1} = \Gamma_k + \tau\Phi_k \quad \text{where} \quad \Phi_k \in T_{\Gamma_k}\mathcal{B}.$$

This is for instance the case of a gradient flow such as (2.7) where $\Phi_k = -\nabla_{R_{\Gamma_k}} E(\Gamma_k)$. In this discretized setting, imposing $\Phi_k \in \mathcal{R}_{\Gamma_k}$ only guarantees that rigidity (4.2) holds approximately and for small enough times t_k.

The following proposition describes this set of tangent fields in an intrinsic manner (using only derivative along the curve Γ), and is pivotal to extend $\mathcal{R}_\Gamma$ to piecewise-rigid tangent fields.

Proposition 4.2. For a C^2-curve Γ, one has $\Phi \in \mathcal{R}_\Gamma$ if and only if Φ is C^2 and satisfies $L_\Gamma(\Phi) = 0$ and $H_\Gamma(\Phi) = 0$, where L_Γ and H_Γ are the following linear operators

$$L_\Gamma(\Phi) \;=\; \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{t}_\Gamma \quad \text{and} \quad H_\Gamma(\Phi) \;=\; \frac{\mathrm{d}^2\Phi}{\mathrm{d}\Gamma(s)^2} \cdot \mathbf{n}_\Gamma . \tag{4.10}$$

Proof. Using the parameterization of Γ, any such deformation Φ satisfies

$$\exists!\,(a,b)\in\mathbb{R}\times\mathbb{R}^2,\quad \forall\, s\in[0,1],\quad \Phi(s)\;=\;a\Gamma(s)^{\perp}+b\,. \tag{4.11}$$

By differentiation with respect to s, this is equivalent to

$$\exists!\,a\in\mathbb{R},\quad \forall\, s\in[0,1],\quad \frac{\mathrm{d}\Phi}{\mathrm{d}s}(s)\;=\;a\|\Gamma'(s)\|\;\mathbf{n}_\Gamma(s)$$

which can be rewritten as $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}(s)=a\mathbf{n}_\Gamma(s)$ by differentiating with respect to the length element $\mathrm{d}\Gamma(s)=\|\Gamma'(s)\|\,\mathrm{d}s$, or simply as $\frac{\mathrm{d}\Phi}{\mathrm{d}s}(s)=a\mathbf{n}_\Gamma(s)$ by considering an arc-length parameterization. This is equivalent to

$$\exists!\,a\in\mathbb{R},\quad \forall\, s\in[0,1],\quad \begin{cases}\dfrac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{t}_\Gamma(s)\;=\;0\\[2ex] \dfrac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma(s)\;=\;a\end{cases}$$

which is equivalent to

$$\begin{cases}\dfrac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{t}_\Gamma\;=\;0\\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}\Gamma(s)}\left(\dfrac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right)\;=\;0\end{cases}$$

and, using that

$$\frac{\mathrm{d}}{\mathrm{d}\Gamma(s)}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right)\;=\;\frac{\mathrm{d}^2\Phi}{\mathrm{d}\Gamma(s)^2}\cdot\mathbf{n}_\Gamma\;-\;\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\kappa_\Gamma\,\mathbf{t}_\Gamma,$$

where κ_Γ is the curvature of Γ, we obtain the desired characterization. □

4.2 BV^2-piecewise rigid deformations

This section extends the globally rigid evolution considered in the previous section to piecewise-rigid evolution.

In the smooth case considered in the previous section, this corresponds to imposing that an instantaneous deformation $\Phi\in T_\Gamma\mathcal{B}$ satisfies (4.10) piecewisely for possibly different couples (a,b) on each piece. To generate a piecewise-smooth Finsler gradient $\Phi=\nabla_{R_\Gamma}E(\Gamma)$ (as defined in (2.3)) that is a piecewise rigid deformation, one should design a penalty R_Γ that satisfies this property. This is equivalent to imposing $L_\Gamma(\Phi)=0$ and $H_\Gamma(\Phi)=0$ for all $s\in[0,1]$ except for a finite number of points (the articulations between the pieces). In particular, note that $L_\Gamma(\Phi)$ is undefined at these points, while $H_\Gamma(\Phi)$ is the sum of Dirac measures concentrated at the articulation points (due to the variations of a). This suggests that, in the smooth case, we can favor piecewise rigidity by minimizing $\|H_\Gamma(\Phi)\|_{L^1(\Gamma)}$ under the constraint $L_\Gamma(\Phi)=0\;\;a.e.$, so that we control the jumps of the second derivative without setting in advance the articulation points. Note also that the minimization of the L^1-norm favors the sparsity and, in contrast to the L^2-norm, it enables the emergence of Dirac measures.

In order to extend such an idea to the BV^2-framework we remind that

$$\|H_\Gamma(\Phi)\|_{L^1(\Gamma)} = TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) \quad \forall\,\Phi\in C^2(\mathbb{S}^1,\mathbb{R}^2),\ \Gamma\in C^2(\mathbb{S}^1,\mathbb{R}^2)$$

which defines a suitable penalty in the BV^2 setting. Moreover, since we are interested in piecewise rigid motions, we deal with curves that could be not C^1 at some points s. It is useful to introduce the following operators

$$L^+_\Gamma(\Phi)(s) = \lim_{\substack{t\to s\\ t\in(s,s+\varepsilon)}} \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}(t)\cdot\,\mathbf{t}_\Gamma(t)\,, \tag{4.12}$$

$$L^-_\Gamma(\Phi)(s) = \lim_{\substack{t\to s\\ t\in(s-\varepsilon,s)}} \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}(t)\cdot\,\mathbf{t}_\Gamma(t)\,. \tag{4.13}$$

Of course if Γ and Φ are C^1 at s we have $L^+_\Gamma(\Phi)(s) = L^-_\Gamma(\Phi)(s) = L_\Gamma(\Phi)(s)$.

Next definition introduces a penalty for piecewise rigid evolution in $\mathcal{B}$.

Definition 4.3 (BV^2 **Piecewise-rigid penalty**)**.** For $\Gamma\in\mathcal{B}$ and $\Phi\in T_\Gamma\mathcal{B} = BV^2(\Gamma)$, we define

$$R_\Gamma(\Phi) = TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) + \iota_{\mathcal{C}_\Gamma}(\Phi) \tag{4.14}$$

where $\iota_{\mathcal{C}_\Gamma}$ is the indicator function of $\mathcal{C}_\Gamma$

$$\iota_{\mathcal{C}_\Gamma}(\Phi) = \begin{cases} 0 & \text{if } \Phi\in\mathcal{C}_\Gamma \\ +\infty & \text{otherwise} \end{cases}$$

and

$$TV_\Gamma\,(f) = \sup\left\{\int_{\mathbb{S}^1} f(s)\cdot\frac{\mathrm{d}g}{\mathrm{d}\Gamma(s)}(s)\,\mathrm{d}\Gamma(s)\ ;\ g\in \mathrm{C}^1_c(\mathbb{S}^1,\mathbb{R}^2), \|g\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)}\leqslant 1\right\} \tag{4.15}$$

for every $f\in L^1(\mathbb{S}^1,\mathbb{R}^2)$. Note that (4.15) is the total variation of f with respect to the measure $\mathrm{d}\Gamma(s) = \|\Gamma'(s)\|\mathrm{d}s$. We point out that $\frac{\mathrm{d}g}{\mathrm{d}\Gamma(s)}(s)\,\mathrm{d}\Gamma(s) = g'(s)\,\mathrm{d}s$ which implies that $TV_\Gamma(f) = |Df|(\mathbb{S}^1)$ for every $f\in L^1(\mathbb{S}^1,\mathbb{R}^2)$.

The set $\mathcal{C}_\Gamma$ is defined as follows

$$\mathcal{C}_\Gamma = \left\{\Phi\in T_\Gamma\mathcal{B}\ ;\ \ L^+_\Gamma(\Phi) = L^-_\Gamma(\Phi) = 0\right\}\ . \tag{4.16}$$

In order to define the Finsler gradient we consider a constraint on the normal component of the deformation field.

Definition 4.4 (**Deviation constraint**)**.** For $\Gamma\in\mathcal{B}$, we define

$$\mathcal{L}_\Gamma = \left\{\Phi\in T_\Gamma\mathcal{B}\ ;\ \left\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma)-\Phi)\right\|_{L^2(\Gamma)}\leqslant\rho\left\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\right\|_{L^2(\Gamma)}.\right\} \tag{4.17}$$

Here, $\rho \in (0,1)$ is called the rigidification parameter, and controls the deviation of the Finsler gradient from the L^2 gradient. Π_Γ is the projector introduced in equation (3.5).

We point out that in the applications studied in this paper we consider an intrinsic energy E (i.e., it does not depend on reparameterization). In this case the L^2-gradient of E is normal to the curve, so that $\mathcal{L}_\Gamma$ satisfies condition (2.2) in the case of an intrinsic energy and it can be used to define a valid Finsler gradient.

Using these specific instantiations for R_Γ and $\mathcal{L}_\Gamma$, Definition 2.1 reads in this BV^2 framework as follows.

Definition 4.5 (BV^2 **Piecewise-rigid Finsler gradient).** We define

$$\nabla_{R_\Gamma} E(\Gamma) \in \operatorname*{argmin}_{\Phi \in \mathcal{L}_\Gamma} \; R_\Gamma(\Phi). \tag{4.18}$$

The following proposition shows ensures the existence of a Finsler gradient.

Theorem 4.6. Problem (4.18) admits at least a solution.

In order to prove the theorem, we need the following lemmas.

Lemma 4.7. Let $\Gamma \in \mathcal{B}$ be an injective curve. We suppose that Γ is different from a circle. Then there exists a constant $C(\Gamma)$ depending on Γ such that

$$\|\Phi\|_{W^{2,1}(\Gamma)} \leqslant C(\Gamma)\left((1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)} + R_\Gamma(\Phi)\right) \tag{4.19}$$

for every $\Phi \in \mathcal{L}_\Gamma \cap \mathcal{C}_\Gamma$.

Proof. (of Lemma 4.7) The proof is essentially based on estimation (4.21) giving a bound for the L^∞-norms of the deformation Φ and its first derivative. We also remark that, as $\Phi \in \mathcal{L}_\Gamma$, we have

$$\|\Pi_\Gamma(\Phi)\|_{L^2(\Gamma)} \leqslant (1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)}\,. \tag{4.20}$$

In the following we denote by $l(\Gamma)$ the length of the curve Γ.

Bound on the first derivative. In this section we prove the following estimate for the L^∞-norms of $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma$ and Φ:

$$\max\left\{\|\Phi\|_{L^\infty(\Gamma)}\;;\;\left\|\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right\|_{L^\infty(\Gamma)}\right\} \leqslant C_0(\Gamma)\left((1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)} + R_\Gamma(\Phi)\right) \tag{4.21}$$

where $C_0(\Gamma)$ depends on Γ.

Let $s_0 \in \mathbb{S}^1$, we can write

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma = u + a$$

where $u \in BV(\Gamma)$ such that $u(s_0) = 0$ and $a = \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}(s_0) \cdot \mathbf{n}_\Gamma(s_0) \in \mathbb{R}$. As $\Phi \in \mathcal{C}_\Gamma$ we have $L^+_\Gamma(\Phi) = L^-_\Gamma(\Phi) = 0$, which implies

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} = \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right)\mathbf{n}_\Gamma$$

and

$$\Phi(s) = \Phi(s_0) + a[\Gamma(s) - \Gamma(s_0)]^{\perp} + \int_{s_0}^{s} u\mathbf{n}_\Gamma \mathrm{d}\Gamma(s) \quad \forall\, s \in \mathbb{S}^1 . \tag{4.22}$$

Now, by projecting on the normal to Γ, we can write

$$\Pi_\Gamma(\Phi) = \Pi_\Gamma(\Phi(s_0) + a[\Gamma(s) - \Gamma(s_0)]^{\perp}) + \Pi_\Gamma\left(\int_{s_0}^{s} u\mathbf{n}_\Gamma \mathrm{d}\Gamma(s)\right) . \tag{4.23}$$

In particular, by the properties of good representatives for BV-functions of one variable (see [3] p. 19), we have

$$|u(s)| = |u(s) - u(s_0)| \leqslant TV_\Gamma(u) \quad \forall s \in \mathbb{S}^1$$

which implies that

$$\left\| \int_{s_0}^{s} u\mathbf{n}_\Gamma \mathrm{d}\Gamma(s) \right\|_{L^\infty(\Gamma)} \leqslant l(\Gamma) TV_\Gamma(u) = l(\Gamma) R_\Gamma(\Phi) \tag{4.24}$$

and

$$\left\| \int_{s_0}^{s} u\mathbf{n}_\Gamma \mathrm{d}\Gamma(s) \right\|_{L^2(\Gamma)} \leqslant l(\Gamma)^{3/2} R_\Gamma(\Phi) \tag{4.25}$$

Thus, by (4.20), (4.25), and (4.23) it follows that

$$\|\Pi_\Gamma(\Phi(s_0) + a[\Gamma(s) - \Gamma(s_0)]^{\perp})\|_{L^2(\Gamma)} \leqslant (1+\rho)\|\Pi_\Gamma(\nabla_{L^2} E(\Gamma))\|_{L^2(\Gamma)} + l(\Gamma)^{3/2} R_\Gamma(\Phi) . \tag{4.26}$$

We remark now that $\|\Pi_\Gamma(\Phi(s_0) + a[\Gamma(s) - \Gamma(s_0)]^{\perp})\|^2_{L^2(\Gamma)}$ can be written as

$$\|\Pi_\Gamma(\Phi(s_0) + a[\Gamma(s) - \Gamma(s_0)]^{\perp})\|^2_{L^2(\Gamma)} = (\|\Phi(s_0)\|, a) \cdot A\left(\frac{\Phi(s_0)}{\|\Phi(s_0)\|}, s_0\right) \begin{pmatrix} \|\Phi(s_0)\| \\ a \end{pmatrix} \tag{4.27}$$

where, for any $e \in \mathbb{S}^1 \subset \mathbb{R}^2$ and $s_0 \in \mathbb{S}^1$, the matrix $A(e, s_0)$ is defined by

$$\begin{pmatrix} \int_{\mathbb{S}^1} (e \cdot \mathbf{n}_\Gamma)^2 \,\mathrm{d}\Gamma(s) & \int_{\mathbb{S}^1} \left([\Gamma(s) - \Gamma(s_0)]^{\perp} \cdot \mathbf{n}_\Gamma\right)(e \cdot \mathbf{n}_\Gamma)\,\mathrm{d}\Gamma(s) \\ \int_{\mathbb{S}^1} \left([\Gamma(s) - \Gamma(s_0)]^{\perp} \cdot \mathbf{n}_\Gamma\right)(e \cdot \mathbf{n}_\Gamma)\,\mathrm{d}\Gamma(s) & \int_{\mathbb{S}^1} \left([\Gamma(s) - \Gamma(s_0)]^{\perp} \cdot \mathbf{n}_\Gamma\right)^2 \mathrm{d}\Gamma(s) \end{pmatrix} . \tag{4.28}$$

Note that the bilinear form defined by $A(e, s_0)$ is degenerate if and only if the determinant of $A(e, s_0)$ is zero which means that there exists $\alpha \in \mathbb{R}$ such that $(e - \alpha[\Gamma(s) - \Gamma(s_0)]^{\perp}) \cdot \mathbf{n}_\Gamma = 0$ for every $s \in \mathbb{S}^1$. Note that this implies that Γ is either a circle or a line. Now, as we work with closed injective curves Γ is different from a line. Then, because of the hypothesis on Γ, we get that for every $s_0 \in \mathbb{S}^1$ the bilinear form associated with $A(e, s_0)$ is not degenerate.

In particular the determinant of A is positive which means that the bilinear form is positive-definite. This implies that its smallest eigenvalue is positive and in particular, by a straightforward calculation, it can be written as $\lambda\,(e, s_0)$ where $\lambda : \mathbb{S}^1 \to \mathbb{R}$ is a positive continuous function. Then, we have

$$\inf_{e, s_0 \in \mathbb{S}^1} \lambda(e, s_0)(\|\Phi(s_0)\|^2 + a^2) \leqslant \lambda\left(\frac{\Phi(s_0)}{\|\Phi(s_0)\|}, s_0\right)(\|\Phi(s_0)\|^2 + a^2) \leqslant \|\Pi_\Gamma(\Phi(s_0) + a\Gamma(s)^{\perp})\|^2_{L^2(\Gamma)} \tag{4.29}$$

where the infimum of λ on $\mathbb{S}^1 \times \mathbb{S}^1$ is a positive constant depending only on Γ and denoted by λ_Γ.

Previous relationship and (4.26) prove that, for every $s_0 \in \mathbb{S}^1$, we have

$$\max\left\{\|\Phi(s_0)\|\,,a\right\} \leqslant C_0(\Gamma)\left((1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)} + R_\Gamma(\Phi)\right) \tag{4.30}$$

where $C_0(\Gamma) = \max\{1/\lambda_\Gamma, l(\Gamma)^{3/2}/\lambda_\Gamma\}$ depends only on Γ.

Then, because of the arbitrary choice of s_0 and the definition of a ($a = \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}(s_0)\cdot \mathbf{n}_\Gamma(s_0)$), (4.30) implies (4.21).
In particular (4.21) gives a bound for the $W^{1,1}(\Gamma)$-norm of Φ.

Bound on the second variation. In our setting the second variation is defined by

$$TV^2_\Gamma(\Phi) := \sup\left\{\int_{\mathbb{S}^1}\Phi\cdot\frac{\mathrm{d}^2 g}{\mathrm{d}\Gamma(s)^2}\,\mathrm{d}\Gamma(s)\ ;\ g\in \mathrm{C}^2_c(\mathbb{S}^1,\mathbb{R}^2), \|g\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)}\leqslant 1\right\}. \tag{4.31}$$

By Proposition 9.3 we get

$$TV^2_\Gamma(\Phi) \leqslant TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right).$$

Now, $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} = \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right)\mathbf{n}_\Gamma \in BV(\Gamma)$ and, by the generalization of the product rule to BV-functions (see Theorem 3.96, Example 3.97, and Remark 3.98 in [3]), we get

$$TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right) \leqslant 2\left(TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) + \left\|\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right\|_{L^\infty(\Gamma)} TV_\Gamma(\mathbf{n}_\Gamma)\right).$$

The constant 2 in the previous inequality comes from the calculation of the total variation on the intersection of the jump sets of $\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right)$ and $\mathbf{n}_\Gamma$ (see Example 3.97 in [3]). Note also that $TV_\Gamma(\mathbf{n}_\Gamma) = |D\mathbf{n}_\Gamma|(\mathbb{S}^1)$.

Then, by (3.3) and (3.4), we get

$$\begin{aligned} TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right) &\leqslant 2\left(TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) + |\mathrm{curv}_\Gamma|(\mathbb{S}^1)\left\|\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right\|_{\mathrm{L}^\infty(\Gamma)}\right) \\ &\leqslant 2\left(TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) + 2|D^2\Gamma|(\mathbb{S}^1)\left\|\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right\|_{L^\infty(\Gamma)}\right) \end{aligned} \tag{4.32}$$

which implies that

$$TV^2_\Gamma(\Phi) \leqslant C_1(\Gamma)\left(R_\Gamma(\Phi) + \left\|\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right\|_{L^\infty(\Gamma)}\right). \tag{4.33}$$

where $C_1(\Gamma)$ is a constant depending on Γ.

The Lemma follows from (4.21) and (4.33). □

Next lemma gives an similar result in the case where Γ is a circle.

Lemma 4.8. Let $\Gamma \in \mathcal{B}$ be a circle. Then there exists a constant $C(\Gamma)$ depending on Γ such that

$$\|\Phi\|_{W^{2,1}(\Gamma)} \leqslant C(\Gamma)\left((1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)} + R_\Gamma(\Phi)\right) \tag{4.34}$$

for every $\Phi \in \mathcal{L}_\Gamma \cap \mathcal{C}_\Gamma$ such that $\Phi(s_0)\cdot\mathbf{t}_\Gamma(s_0) = 0$ for some $s_0 \in \mathbb{S}^1$.

Proof. The proof is based on the same arguments used to prove the previous lemma. We denote by r the radius of the circle.

As $\Phi(s_0)\cdot\mathbf{t}_\Gamma(s_0)=0$, by the properties of good representatives for BV-functions of one variable (see [3] p. 19), we have

$$|\Phi\cdot\mathbf{t}_\Gamma| = |\Phi(s)\cdot\mathbf{t}_\Gamma(s)-\Phi(s_0)\cdot\mathbf{t}_\Gamma(s_0)| \leqslant TV_\Gamma(\Phi\cdot\mathbf{t}_\Gamma) \quad \forall s\in\mathbb{S}^1\,. \tag{4.35}$$

Now, as $L^+_\Gamma(\Phi)=L^-_\Gamma(\Phi)=0$ and the curvature is equal to $1/r$ at each point, we get

$$\frac{\mathrm{d}(\Phi\cdot\mathbf{t}_\Gamma)}{\mathrm{d}\Gamma(s)} = \Phi\cdot\frac{\mathbf{n}_\Gamma}{r}$$

and from (4.35) it follows

$$\|\Phi\cdot\mathbf{t}_\Gamma\|_{L^\infty(\Gamma)} \leqslant \frac{\|\Phi\cdot\mathbf{n}_\Gamma\|_{L^1(\Gamma)}}{r}\,. \tag{4.36}$$

Now, as $\Phi\in\mathcal{L}_\Gamma$, we have

$$\|\Phi\cdot\mathbf{n}_\Gamma\|_{L^2(\Gamma)} \leqslant (1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)} \tag{4.37}$$

and, from (4.36) and (4.37), it follows

$$\|\Phi\|_{L^1(\Gamma)} \leqslant \sqrt{2\pi r}(2\pi+1)(1+\rho)\|\Pi_\Gamma(\nabla_{L^2}E(\Gamma))\|_{L^2(\Gamma)}\,. \tag{4.38}$$

Concerning the first derivative we remark that, as Φ is periodic, the mean value of its first derivative is equal to zero. Then, by Poincaré's inequality (see Theorem 3.44 in [3]), we have

$$\left\|\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right\|_{L^1(\Gamma)} \leqslant C_0(\Gamma)TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right) \tag{4.39}$$

where $C_0(\Gamma)$ is a constant depending on Γ. Moreover, by Proposition 9.3 we get

$$TV^2_\Gamma(\Phi) \leqslant TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right)\,. \tag{4.40}$$

So, in order to prove the lemma it suffices to prove a bound for $TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right)$.

Now, as $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} = \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right)\mathbf{n}_\Gamma$, by the generalization of the product rule to BV-functions (see Theorem 3.96, Example 3.97, and Remark 3.98 in [3]), we get

$$TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\right) \leqslant \left(1+\frac{1}{r}\right)TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) = \left(1+\frac{1}{r}\right)R_\Gamma(\Phi)\,, \tag{4.41}$$

where we used the fact that $\mathbf{n}_\Gamma$ has no jumps (see Example 3.97 in [3]).

The lemma follows from (4.38), (4.39), (4.40), and (4.41). □

We can now prove Theorem 4.6.

Proof. (of **Theorem 4.6**) The proof is based on Lemma 4.7 and Lemma 4.8, so we distinguish two cases: Γ is a circle and it is not.

We suppose that Γ is different from a circle. Let $\{\Phi_h\} \subset \mathcal{L}_\Gamma \cap \mathcal{C}_\Gamma$ be a minimizing sequence of R_Γ. We can also suppose $\sup_h R_\Gamma(\Phi_h) < +\infty$. From Lemma 4.7 it follows that

$$\sup_h \|\Phi_h\|_{BV^2(\Gamma)} \leqslant C(\Gamma) \left((1+\rho)\|\Pi_\Gamma(\nabla_{L^2} E(\Gamma))\|_{L^2(\Gamma)} + \sup_h R_\Gamma(\Phi_h) \right)$$

where $C(\Gamma)$ depends only on Γ. This gives a uniform bound for the $BV^2(\Gamma)$-norms of Φ_h and implies that $\{\Phi_h\}$ converges (up to a subsequence) toward some $\Phi \in BV^2(\Gamma)$ with respect to the $BV^2(\Gamma)$-weak* topology (see Theorem 3.23 in [3]).

In particular $\Phi_h \to \Phi$ in $W^{1,1}(\Gamma)$ which proves that $\Phi \in \mathcal{C}_\Gamma$, and, by the lower semi-continuity of the L^2-norm, we also get $\Phi \in \mathcal{L}_\Gamma$.

Now, as R_Γ is lower semi-continuous with respect to the $BV^2(\Gamma)$-weak* topology, the theorem ensues.

In the case where Γ is a circle, for every minimizing sequence $\{\Phi_h\} \subset \mathcal{L}_\Gamma \cap \mathcal{C}_\Gamma$, we consider the sequence

$$\Psi_h(s) = \Phi_h(s) - (\Phi_h(s_0) \cdot \mathbf{t}_\Gamma(s_0))\mathbf{t}_\Gamma \tag{4.42}$$

for some $s_0 \in \mathbb{S}^1$. We remark that $\{\Psi_h\} \subset \mathcal{L}_\Gamma$. Moreover

$$\frac{\mathrm{d}\Psi_h}{\mathrm{d}\Gamma(s)} = \left(\frac{\mathrm{d}\Phi_h}{\mathrm{d}\Gamma(s)} - \frac{\Phi_h(s_0) \cdot \mathbf{t}_\Gamma(s_0)}{r} \right) \mathbf{n}_\Gamma \tag{4.43}$$

which implies that, for every h, $\Psi_h \in \mathcal{C}_\Gamma$ and

$$R_\Gamma(\Psi_h) = R_\Gamma(\Phi_h)\,. \tag{4.44}$$

Then the sequence $\{\Psi_h\}$ is a minimizing sequence of Problem (4.18) such that $\Psi(s_0) \cdot \mathbf{t}_\Gamma(s_0) = 0$. We can also suppose $\sup_h R_\Gamma(\Psi_h) < +\infty$.

Then, by Lemma 4.8 we get

$$\sup_h \|\Psi_h\|_{BV^2(\Gamma)} \leqslant C(\Gamma) \left((1+\rho)\|\Pi_\Gamma(\nabla_{L^2} E(\Gamma))\|_{L^2(\Gamma)} + \sup_h R_\Gamma(\Psi_h) \right)$$

where $C(\Gamma)$ depends only on Γ.

This proves an uniform bound for $\|\Psi_h\|_{BV^2(\Gamma)}$ which implies that the minimizing sequence $\{\Psi_h\}$ converges (up to a subsequence) with respect to the BV^2-weak* topology. Then we can conclude as in the previous case. □

We point out that, as showed in the previous proof, when Γ is a circle the Finsler gradient is defined up to a tangential translation. This was actually expected because such a tangential translation is a rotation of the circle.

We have defined a penalty for BV^2-piecewise rigid deformations for curves belonging to $\mathcal{B}$. In the next section we use the Finsler descent method with respect to such a penalty to solve curve matching problems.

5 Application to Curve Matching

This section shows an application of the Finsler descent method to the curve matching problem.

5.1 The Curve Matching Problem

Given two curves Γ_0 and Λ in $\mathcal{B}$, the curve matching problem (also known as the registration problem) seeks for an (exact or approximate) bijection between their geometric realizations $[\Gamma_0]$ and $[\Lambda]$ (as defined in Section 3.2). One thus looks for a matching (or correspondence) $f : [\Gamma_0] \to \mathbb{R}^2$ such that $f([\Gamma_0])$ is equal or close to $[\Lambda]$.

There exists a variety of algorithms to compute a matching with desirable properties, that are reviewed in Section 1.1. A simple class of methods consists in minimizing an intrinsic energy $E(\Gamma)$ (i.e., E only depends on $[\Gamma]$), and to track the points of the curve, thus establishing a matching during the minimization flow. We suppose that $E(\Gamma) > 0$ if $[\Gamma] \neq [\Lambda]$ and $E(\Lambda) = 0$, so that the set of global minimizers of E is exactly $[\Lambda]$. This is for instance the case if $E(\Gamma)$ is a distance between $[\Gamma]$ and $[\Lambda]$. A gradient descent method (such as (2.7)) defines a set of iterates Γ_k, so that Γ_0 is the curve to be matched to Λ. The iterates Γ_k (or at least a sub-sequence) converges to Γ_∞, and the matching is simply defined as

$$\forall\, s \in \mathbb{S}^1, \quad f(\Gamma_0(s)) = \Gamma_\infty(s).$$

If the descent method succeeds in finding a global minimizer of E, then f is an exact matching, i.e. $f([\Gamma_0]) = [\Lambda]$. This is however not always the case, and the iterates Γ_k can converge to a local minimum. It is thus important to define a suitable notion of gradient to improve the performance of the method. The next sections describe the use of the Finsler gradient to produce piecewise rigid matching.

5.2 Matching Energy

The matching accuracy depends on the energy and on the kind of descent used to define the flow. In this paper we are interested in studying the Finsler descent method rather than designing novel energies. For the numerical examples, we consider an energy based on reproducing kernel Hilbert space (r.k.h.s.) theory [38, 5]. These energies have been introduced for curve matching in [36, 17]. For an overview on other types of energies we refer the reader to the bibliography presented in Section 1.1.

We consider a positive-definite kernel k in the sense of the r.k.h.s theory [38, 5]. Following [36], we define a distance between curves as

$$\mathrm{dist}(\Gamma, \Lambda)^2 = H(\Gamma, \Gamma) + H(\Lambda, \Lambda) - 2H(\Gamma, \Lambda)\ , \quad \forall\, \Gamma, \Lambda \in \mathcal{B} \tag{5.1}$$

where

$$H(\Gamma, \Lambda) = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathbf{n}_\Gamma(s) \cdot \mathbf{n}_\Lambda(t)\ k\left(\Gamma(s), \Lambda(t)\right)\ \mathrm{d}\Gamma(s)\mathrm{d}\Lambda(t)\,. \tag{5.2}$$

As the kernel k is positive-definite in the sense of r.k.h.s. theory, it can be shown that dist defined in (5.1) is a distance between the geometric realizations $([\Gamma], [\Lambda])$ (up to change in orientation) of the

curves. In our numerical tests, we define k as a sum of two Gaussian kernels with standard deviation $\sigma > 0$ and $\delta > 0$

$$k(v,w) = e^{-\frac{\|v-w\|^2}{2\sigma^2}} + e^{-\frac{\|v-w\|^2}{2\delta^2}}, \qquad \forall\, v,\, w \in \mathbb{R}^2, \tag{5.3}$$

which can be shown to be a positive-definite kernel. We use a double kernel to better capture features at different scales in the curves to be matched. This has been shown to be quite efficient in practice in a different context in [28]. This energy takes into account the orientation of the normals along the shape in order to stress the difference between the interior and the exterior of closed shapes. Remark that, to obtain interesting numerical results, both Γ and Λ have to be parameterized with the same orientation (clockwise or counter-clockwise).

Given a target curve $\Lambda \in \mathcal{B}$, we consider the following energy

$$E : \mathcal{B} \to \mathbb{R}, \qquad E(\Gamma) = \frac{1}{2}\mathrm{dist}(\Gamma,\Lambda)^2. \tag{5.4}$$

Remark that, as dist is a distance then $[\Lambda]$ is equal to the set of global minimizers of E. The gradient of E at Γ with respect to $L^2(\Gamma)$-topology is given by the following proposition.

Proposition 5.1. The gradient of E at Γ with respect to the $L^2(\Gamma)$-topology is given by

$$\nabla_{L^2} E(\Gamma)(s) = \mathbf{n}_\Gamma(s)\Big[\int_{\mathbb{S}^1} \mathbf{n}_\Gamma(t)\cdot\nabla_1 k(\Gamma(s),\Gamma(t)))\mathrm{d}\Gamma(t) - \int_{\mathbb{S}^1} \mathbf{n}_\Lambda(t)\cdot\nabla_1 k(\Gamma(s),\Lambda(t))\mathrm{d}\Lambda(t)\Big] \tag{5.5}$$

for every $s \in \mathbb{S}^1$, where $\nabla_1 k$ represents the derivative with respect to the first variable.

Proof. In order to prove (5.5) we calculate the gradient for $H(\Gamma,\Lambda)$ with respect to Γ. We rewrite H as

$$H(\Gamma,\Lambda) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Gamma'(s)\cdot\Lambda'(t)\; k\left(\Gamma(s),\Lambda(t)\right)\; \mathrm{d}t\; \mathrm{d}s$$

and we consider a smooth variation of the curve Γ, denoted by $\delta\Gamma$. Then, for h small, we have

$$\begin{aligned} I(h) = \frac{H(\Gamma+h\delta\Gamma,\Lambda) - H(\Gamma,\Lambda)}{h} = & \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} (\Gamma'(s)\cdot\Lambda'(t))(\nabla_1 k(\Gamma(s),\Lambda(t))\cdot\delta\Gamma(s))\,\mathrm{d}t\;\mathrm{d}s \\ & + \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \delta\Gamma'(s)\cdot\Lambda'(t)\; k\left(\Gamma(s),\Lambda(t)\right)\; \mathrm{d}t\;\mathrm{d}s\; + o(h) \end{aligned}$$

and integrating by parts we obtain

$$\begin{aligned} I(h) \;&= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} (\Gamma'(s)\cdot\Lambda'(t))(\nabla_1 k(\Gamma(s),\Lambda(t))\cdot\delta\Gamma(s))\,\mathrm{d}t\;\mathrm{d}s \\ &- \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} (\delta\Gamma(s)\cdot\Lambda'(t))(\nabla_1 k(\Gamma(s),\Lambda(t))\cdot\Gamma'(s))\,\mathrm{d}t\;\mathrm{d}s + o(h) \end{aligned}$$

which can be written as

$$I(h) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} [\nabla_1 k(\Gamma(s),\Lambda(t))^t(\delta\Gamma(s)\otimes\Gamma'(s) - \Gamma'(s)\otimes\delta\Gamma(s))\Lambda'(t)]\,\mathrm{d}t\;\mathrm{d}s + o(h) \tag{5.6}$$

$$\text{where} \quad v\otimes w = vw^t, \qquad \forall\, v,w \in \mathbb{R}^2.$$

Now, writing $\delta\Gamma(s)$ with respect to the basis $\{\mathbf{t}_\Gamma(s), \mathbf{n}_\Gamma(s)\}$ and reminding that $\Gamma'(s) = |\Gamma'(s)|\mathbf{t}_\Gamma(s)$, we can show that the matrix

$$M(s) = \delta\Gamma(s) \otimes \Gamma'(s) - \Gamma'(s) \otimes \delta\Gamma(s) = |\Gamma'(s)|(\delta\Gamma(s) \cdot \mathbf{n}_\Gamma(s))(\mathbf{n}_\Gamma(s) \otimes \mathbf{t}_\Gamma(s) - \mathbf{t}_\Gamma \otimes \mathbf{n}_\Gamma(s))$$

acts as

$$M(s)(v) = -|\Gamma'(s)|(\delta\Gamma(s) \cdot \mathbf{n}_\Gamma(s))v^\perp, \quad \forall\, v \in \mathbb{R}^2. \tag{5.7}$$

Then, by (5.6) and (5.7), we obtain

$$I(h) \;\; = -\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} (\delta\Gamma(s) \cdot \mathbf{n}_\Gamma(s))(\nabla_1 k(\Gamma(s), \Lambda(t)) \cdot \Lambda'(t)^\perp)\,\mathrm{d}t\; |\Gamma'(s)|\mathrm{d}s + o(h)\;\; .$$

Finally, as $h \to 0$, we obtain the gradient of $H(\Gamma, \Lambda)$ with respect to Γ which is given by

$$-\mathbf{n}_\Gamma(s)\int_{\mathbb{S}^1} \mathbf{n}_\Lambda(t) \cdot \nabla_1 k(\Gamma(s), \Lambda(t))\mathrm{d}\Lambda(t)$$

that represents the second term in (5.5). For the first term we need to apply the same argument to calculate the gradient of $H(\Gamma, \Gamma)$. □

5.3 Matching Flow

In order to minimize E on $\mathcal{B}$ we consider the scheme (2.7), that defines $\{\Gamma_k\}_k$ for $k > 0$ as

$$\Gamma_{k+1} = \Gamma_k - \tau_k \nabla_{R_{\Gamma_k}} E(\Gamma_k) \tag{5.8}$$

where Γ_0 is the input curve to be matched to Λ, where $\nabla_{R_{\Gamma_k}} E$ is defined by (4.18) and $\tau_k > 0$ is a step size, that satisfies the Wolfe rule (2.8). We prove the following convergence theorem for the method:

Theorem 5.2. Every accumulation point of $\{\Gamma_k\}_k$ in $\mathcal{B} = BV^2(\mathbb{S}^1, \mathbb{R}^2)$ is a critical point of E.

Proof. We prove that the energy E satisfies the hypothesis of Theorem 2.5. We first remind that, if $\Gamma_k \to \Gamma$ in $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ then $\Gamma_k \to \Gamma$ in $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$. Moreover, in dimension one, $BV(\mathbb{S}^1, \mathbb{R}^2)$ and $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ are continuously embedded in $L^\infty(\mathbb{S}^1, \mathbb{R}^2)$. Then we can suppose

$$\sup_k \|\Gamma_k'\|_{W^{1,\infty}(\mathbb{S}^1,\mathbb{R}^2)} \leqslant M \tag{5.9}$$

for some constant $M > 0$, because every converging subsequence of $\{\Gamma_k\}$ is uniformly bounded in $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ and $W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$. We rewrite H as

$$H(\Gamma, \Lambda) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Gamma'(s) \cdot \Lambda'(t)\; k\left(\Gamma(s), \Lambda(t)\right)\; \mathrm{d}s\; \mathrm{d}t\,,$$

so, by the dominated convergence theorem, we have that H and E are continuous with respect to the BV^2-topology. Note also that

$$\begin{aligned}\langle \nabla_{L^2} E(\Gamma), \Phi\rangle_{L^2(\Gamma)} = &\int_{\mathbb{S}^1} \Gamma'(s)^\perp \cdot \Phi(s) \int_{\mathbb{S}^1} \Gamma'(t)^\perp \cdot \nabla_1 k(\Gamma(s), \Gamma(t)))\mathrm{d}t\,\mathrm{d}s \\ &- \int_{\mathbb{S}^1} \Gamma'(s)^\perp \cdot \Phi(s) \int_{\mathbb{S}^1} \Lambda'(t)^\perp \cdot \nabla_1 k(\Gamma(s), \Lambda(t))\mathrm{d}t\;\mathrm{d}s\end{aligned}$$

where $(x,y)^\perp = (-y,x)$ for every $(x,y) \in \mathbb{R}^2$. This shows that $\nabla_{L^2} E$ is $BV^2(\mathbb{S}^1,\mathbb{R}^2)$-continuous. Then E is non-negative and C^1 with respect to the BV^2 topology on $\mathcal{B}$. In order to prove the convergence using Theorem 2.5, we have to prove that the gradient (5.5) is uniformly Lipschitz-continuous. In particular, because of (5.9), we prove that the gradient is uniformly Lipschitz-continuous in the intersection of $\mathcal{B}$ with any ball of $W^{1,\infty}(\mathbb{S}^1,\mathbb{R}^2)$.

We show that the term of the gradient depending on both Γ and Λ is uniformly Lipschitz-continuous. For the other term the proof is similar. For every couple of curves (Γ,Λ), we introduce the following function

$$\mathcal{I}(\Gamma,\Lambda)(s) = \int_{\mathbb{S}^1} \mathbf{n}_\Lambda(t)\cdot\nabla_1 k(\Gamma(s),\Lambda(t))\,\mathrm{d}\Lambda(t) = \int_{\mathbb{S}^1} \Lambda'(t)^\perp\cdot\nabla_1 k(\Gamma(s),\Lambda(t))\,\mathrm{d}t\,.$$

We must show that there exists $L>0$ such that

$$\|{\Gamma_1'}^\perp\mathcal{I}(\Gamma_1,\Lambda) - {\Gamma_2'}^\perp\mathcal{I}(\Gamma_2,\Lambda)\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)} \leqslant L\|\Gamma_1-\Gamma_2\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)}$$

for every couple of curves $(\Gamma_1,\Gamma_2)\in\mathcal{B}$. We have

$$\|{\Gamma_1'}^\perp\mathcal{I}(\Gamma_1,\Lambda) - {\Gamma_2'}^\perp\mathcal{I}(\Gamma_2,\Lambda)\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)} = \|\Gamma_1'\mathcal{I}(\Gamma_1,\Lambda) - \Gamma_2'\mathcal{I}(\Gamma_2,\Lambda)\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)}$$

and

$$\begin{aligned}\|\Gamma_1'\mathcal{I}(\Gamma_1,\Lambda) - \Gamma_2'\mathcal{I}(\Gamma_2,\Lambda)\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)} \leqslant &\|\Gamma_1'\mathcal{I}(\Gamma_1,\Lambda) - \Gamma_1'\mathcal{I}(\Gamma_2,\Lambda)\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)} &(5.10)\\ &+\|\Gamma_1'\mathcal{I}(\Gamma_2,\Lambda) - \Gamma_2'\mathcal{I}(\Gamma_2,\Lambda)\|_{L^2(\mathbb{S}^1,\mathbb{R}^2)}\,. &(5.11)\end{aligned}$$

Note that

$$\|\mathcal{I}(\Gamma,\Lambda)\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)} \leqslant \|\Lambda\|_{BV^2(\mathbb{S}^1,\mathbb{R}^2)}\,. \tag{5.12}$$

Now, we have

$$\begin{aligned}\|\Gamma_1'[\mathcal{I}(\Gamma_1,\Lambda)-\mathcal{I}(\Gamma_2,\Lambda)]\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^2)} &\leqslant M\|\mathcal{I}(\Gamma_1,\Lambda)-\mathcal{I}(\Gamma_2,\Lambda)\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^2)}\\ &\leqslant M\|\Lambda'\|^2_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\|\nabla_1 k(\Gamma_1(s),\Lambda(t)) - \nabla_1 k(\Gamma_2(s),\Lambda(t))\|^2\,\mathrm{d}s\,\mathrm{d}t\\ &\leqslant \frac{M\|\Lambda'\|^2_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)}}{\sigma^2}\int_{\mathbb{S}^1}\|\Gamma_1(s)-\Gamma_2(s)\|^2\,\mathrm{d}s,\end{aligned}$$

where we used the fact that re^{-r^2} is 1-Lipschitz continuous (given by a straightforward derivative calculation). Then, as $BV(\mathbb{S}^1,\mathbb{R}^2)$ is continuously embedded in $L^\infty(\mathbb{S}^1,\mathbb{R}^2)$, we get

$$\|\Gamma_1'[\mathcal{I}(\Gamma_1,\Lambda)-\mathcal{I}(\Gamma_2,\Lambda)]\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^2)} \leqslant C_1\|\Lambda\|^2_{BV^2(\mathbb{S}^1,\mathbb{R}^2)}\|\Gamma_1-\Gamma_2\|^2_{BV^2(\mathbb{S}^1,\mathbb{R}^2)} \tag{5.13}$$

where $C_1 = MC_0^2/\sigma^2$ (C_0 denotes here the constant of the embedding of $BV(\mathbb{S}^1,\mathbb{R}^2)$ in $L^\infty(\mathbb{S}^1,\mathbb{R}^2)$ so that $\|\Gamma\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)} \leqslant C_0\|\Gamma\|_{BV(\mathbb{S}^1,\mathbb{R}^2)}$). Moreover, by (5.12), we have

$$\|\Gamma_1'\mathcal{I}(\Gamma_2,\Lambda) - \Gamma_2'\mathcal{I}(\Gamma_2,\Lambda)\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^2)} \leqslant \|\Lambda\|_{BV^2(\mathbb{S}^1,\mathbb{R}^2)}\|\Gamma_1'-\Gamma_2'\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^2)}$$

which implies

$$\|\Gamma_1'\mathcal{I}(\Gamma_2,\Lambda) - \Gamma_2'\mathcal{I}(\Gamma_2,\Lambda)\|^2_{L^2(\mathbb{S}^1,\mathbb{R}^2)} \leqslant C_2\|\Gamma_1-\Gamma_2\|^2_{BV^2(\mathbb{S}^1,\mathbb{R}^2)} \tag{5.14}$$

where $C_2 = C_0^2\|\Lambda\|_{BV^2(\mathbb{S}^1,\mathbb{R}^2)}$. Then, by (5.10), (5.13) and (5.14), the L^2 gradient is Lipschitz continuous on every ball of $W^{1,\infty}(\mathbb{S}^1,\mathbb{R}^2)$ with respect to the topology of $\mathcal{B}$. □

6 Discretization

This section discretizes Problem (4.18) using finite elements in order to calculate numerically the Finsler gradient flow. We define a finite n-dimensional sub-space $\mathcal{B}_n \subset \mathcal{B}$ of piecewise linear curves. The embedding $\mathcal{B}_n \subset \mathcal{B}$ defines a natural finite dimensional Riemannian and Finsler structure on $\mathcal{B}_n$ inherited from the ones of $\mathcal{B}$. This allows us to apply our Finsler gradient flow in finite dimension to approximate the original infinite dimensional Finsler flow.

6.1 Finite Elements Spaces

Notations. In the following, to ease the notation, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and $\mathbb{S}^1$ with $[0,1]$ using periodic boundary conditions. The canonical inner produced on $\mathbb{C}^n$ is

$$\langle \tilde{f},\, \tilde{g}\rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} \langle \tilde{f}_i,\, \tilde{g}_i\rangle = \sum_{i=1}^{n} \mathrm{Real}(\tilde{f}_i\, \overline{\tilde{g}_i}), \quad \forall\, \tilde{f}, \tilde{g} \in \mathbb{C}^n\,, \tag{6.1}$$

where we denote by $\overline{\tilde{g}_i}$ the conjugate of $\tilde{g}_i$.

Piecewise affine finite elements. We consider the space $\mathbb{P}_{1,n}$ of the finite elements on $[0,1]$ (with periodic boundary conditions) of order one with n knots. A basis of $\mathbb{P}_{1,n}$ is defined as

$$\begin{aligned} \xi_i(s) &= \max\left\{0, 1 - n\left|s - \frac{i}{n}\right|\right\} \quad s \in [0,1],\ \forall\, i = 1, ..., n-1 \\ \xi_n(s) &= \max\{0, 1 - n\,|s|\} + \max\{0, 1 - n\,|s-1|\}\,, \quad s \in [0,1]. \end{aligned}$$

Every $f \in \mathbb{P}_{1,n}$ can be written as

$$f = \sum_{i=1}^{n} \tilde{f}_i\, \xi_i\,, \quad \tilde{f}_i \in \mathbb{C} \tag{6.2}$$

with $\tilde{f}_i = f(i/n) \in \mathbb{C}$ for every i. We denote by $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n) \in \mathbb{C}^n$ the coordinates of f with respect to the basis $\{\xi_i\}_{i=1,...,n}$. Remark that there exists a bijection between $\mathbb{P}_{1,n}$ and $\mathbb{C}^n$, defined by the following operator

$$P_1\ : \tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n) \in \mathbb{C}^n \ \ \mapsto\ \ P_1(\tilde{f}) = \ f \in \mathbb{P}_{1,n}\ \text{ s.t. }\ f = \sum_{i=1}^{n} \tilde{f}_i\, \xi_i\,. \tag{6.3}$$

The forward and backward finite differences operators are defined as

$$\begin{aligned} \Delta^+ &: \mathbb{C}^n \to \mathbb{C}^n\,, \qquad \Delta^+(\tilde{f})_i = \tilde{f}_{i+1} - \tilde{f}_i\,, \\ \Delta^- &: \mathbb{C}^n \to \mathbb{C}^n\,, \qquad \Delta^-(\tilde{f})_i = \tilde{f}_i - \tilde{f}_{i-1}\,, \end{aligned} \tag{6.4}$$

Piecewise constant finite elements. For every $f \in \mathbb{P}_{1,n}$, (6.2) implies that first derivative $\frac{\mathrm{d}f}{\mathrm{d}s}$ belongs to $\mathbb{P}_{0,n} \subset BV([0,1],\mathbb{R}^2)$, where $\mathbb{P}_{0,n}$ is the class of the piecewise constant functions with n knots. A basis of $\mathbb{P}_{0,n}$ is defined by

$$\zeta_i(s) = \mathbb{I}_{[\frac{i}{n},\frac{i+1}{n}]}(s) \quad \forall\, i = 1, ..., n-1\,, \quad \zeta_n(s) = \mathbb{I}_{[0,\frac{1}{n}]}(s)\,,$$

where $\mathbb{I}_A$ is the indicator function of a set A, and with $s \in [0,1]$. Then, the first derivative of f can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \sum_{i=1}^{n} \Delta^+(\tilde{f})_i \zeta_i\,. \tag{6.5}$$

We finally define the following bijection between $\mathbb{P}_{0,n}$ and $\mathbb{C}^n$:

$$P_0 \;:\; \tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n) \in \mathbb{C}^n \;\; \mapsto \;\; P_0(\tilde{f}) = f \in \mathbb{P}_{0,n} \;\text{ s.t. }\; f = \sum_{i=1}^{n} \tilde{f}_i \zeta_i\,. \tag{6.6}$$

6.2 Finite Element Spaces of Curves

Discretized curves. The discrete space of curves is defined as $\mathcal{B}_n = \mathbb{P}_{1,n} \subset \mathcal{B}$ and every curve $\Gamma \in \mathcal{B}_n$ can be written as

$$\Gamma = \sum_{i=1}^{n} \tilde{\Gamma}_i \xi_i\,, \quad \tilde{\Gamma}_i = \Gamma(i/n) \in \mathbb{C} \tag{6.7}$$

where the vector $\tilde{\Gamma} = P_1^{-1}(\Gamma) = (\tilde{\Gamma}_1, ..., \tilde{\Gamma}_n) \in \mathbb{C}^n$ are the coefficients of Γ in the finite element basis. By (6.5) the tangent and normal vectors (3.1) to $\Gamma \in \mathcal{B}_n$ are computed as

$$\mathbf{t}_\Gamma = \sum_{i=1}^{n} \frac{\Delta^+(\tilde{\Gamma})_i}{|\Delta^+(\tilde{\Gamma})_i|} \zeta_i\,, \;\; \mathbf{n}_\Gamma(i) = \mathbf{t}_\Gamma(i)^\perp\,, \tag{6.8}$$

where $(x,y)^\perp = (-y,x)$ for all $(x,y) \in \mathbb{R}^2$. In particular we have

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}s} = \sum_{i=1}^{n} \Delta^+(\tilde{\Gamma})_i \zeta_i\,. \tag{6.9}$$

Discretized tangent spaces. For every $\Gamma \in \mathcal{B}_n$, the discrete tangent space to $\mathcal{B}_n$ at Γ is defined as $T_\Gamma \mathcal{B}_n = \mathcal{B}_n$ equipped with the inner product $\langle \cdot,\, \cdot \rangle_{L^2(\Gamma)}$. Every vector field $\Phi \in T_\Gamma \mathcal{B}_n$ can be written as

$$\Phi = \sum_{i=1}^{n} \tilde{\Phi}_i \xi_i\,, \quad \tilde{\Phi}_i = \Phi(i/n) \in \mathbb{C} \tag{6.10}$$

where $\tilde{\Phi} = (\tilde{\Phi}_1, ..., \tilde{\Phi}_n) \in \mathbb{C}^n$ are the coordinates of Φ with respect to the basis of $\mathbb{P}_{1,n}$.

By identifying every vector field $\Phi \in T_\Gamma \mathcal{B}_n$ with its coordinates $\tilde{\Phi}$, the tangent space can be identified with $\mathbb{C}^n$. Moreover, $\mathbb{C}^n$ can be equipped with the following Riemannian metric:

Definition 6.1 (Discrete inner product). We define $\ell^2(\tilde{\Gamma})$ as the set $\mathbb{C}^n$ equipped the following inner product

$$\langle \tilde{\Phi},\, \tilde{\Psi} \rangle_{\ell^2(\tilde{\Gamma})} = \langle P_1(\tilde{\Phi}),\, P_1(\tilde{\Psi}) \rangle_{L^2(\Gamma)}\,. \tag{6.11}$$

We now give an explicit formula for the product (6.11), which is useful for computational purposes. More precisely, Proposition (6.2) details the relationship between the product (6.11) and the canonical inner product on $\mathbb{C}^n$ defined by (6.1). For this purpose, we define the stiffness matrix $M_{\tilde{\Gamma}} \in \mathbb{R}^{n\times n}$ as

$$M_{\tilde{\Gamma}} = \sum_{i=1}^{n} |\Delta^+(\tilde{\Gamma})_i| M^i \quad \text{where} \quad M^i_{h,j} = \int_{i/n}^{(i+1)/n} \xi_h \xi_j \,. \tag{6.12}$$

The elements of the matrices $M^i \in \mathbb{R}^{n\times n}$ for $i = 1, ..., n$ are equal to zero excepted for the following block:

$$\begin{pmatrix} M^i_{i,i} & M^i_{i,i+1} \\ M^i_{i+1,i} & M^i_{i+1,i+1} \end{pmatrix} = \frac{1}{6n} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

where the indices $i-1$ and $i+1$ should be understood modulo n.

Proposition 6.2. For all $\tilde{\Psi}, \tilde{\Phi}$ in $\mathbb{C}^n$, one has

$$\langle \tilde{\Phi},\, \tilde{\Psi} \rangle_{\ell^2(\tilde{\Gamma})} = \langle \tilde{\Phi},\, M_{\tilde{\Gamma}} \tilde{\Psi} \rangle_{\mathbb{C}^n}, \tag{6.13}$$

where $M_{\tilde{\Gamma}}$ is the stiffness matrix defined in (6.12).

Proof. Denoting $\Phi = P_1(\tilde{\Phi})$ and $\Psi = P_1(\tilde{\Psi})$, (6.10) and (6.9) imply that

$$\langle \Phi,\, \Psi \rangle_{L^2(\Gamma)} = \int_0^1 \Phi \cdot \Psi d\,\Gamma(s) = \sum_{i=1}^{n} |\Delta^+(\tilde{\Gamma})_i| \int_{i/n}^{(i+1)/n} \left(\sum_{j=1}^{n} \tilde{\Phi}_j \xi_j \cdot \sum_{h=1}^{n} \tilde{\Psi}_h \xi_h \right) \mathrm{d}s \,.$$

Then we have

$$\langle \tilde{\Phi},\, \tilde{\Psi} \rangle_{\ell^2(\tilde{\Gamma})} = \sum_{i=1}^{n} |\Delta^+(\tilde{\Gamma})_i| \langle \tilde{\Phi},\, M^i \tilde{\Psi} \rangle_{\mathbb{C}^n} = \langle \tilde{\Phi},\, M_{\tilde{\Gamma}} \tilde{\Psi} \rangle_{\mathbb{C}^n} \tag{6.14}$$

where M is rigidity matrix (6.12). □

6.3 Discrete Finsler Flow

The initial optimization (2.1) is discretized by restricting the minimization to the space $\mathcal{B}_n$, which corresponds to the following finite dimensional optimization

$$\min_{\tilde{\Gamma} \in \mathbb{C}^n} \tilde{E}(\tilde{\Gamma}) \,, \tag{6.15}$$

where $\tilde{E}(\tilde{\Gamma})$ approximates $E(P_1(\tilde{\Gamma}))$.

The discrete Finsler gradient is obtained in a similar way by restricting the optimization (2.3) to $\mathcal{B}_n$

$$\nabla_{\tilde{R}_{\tilde{\Gamma}}} \tilde{E}(\tilde{\Gamma}) \in \operatorname*{argmin}_{\tilde{\Phi} \in \tilde{\mathcal{L}}_{\tilde{\Gamma}}} \tilde{R}_{\tilde{\Gamma}}(\tilde{\Phi}) \,, \tag{6.16}$$

where the discrete penalty reads

$$\tilde{R}_{\tilde{\Gamma}}(\tilde{\Phi}) = R_{P_1(\tilde{\Gamma})}(P_1(\tilde{\Phi})) \tag{6.17}$$

and, as discrete constraint, we set

$$\tilde{\mathcal{L}}_{\tilde{\Gamma}} = \left\{ \tilde{\Phi} \in \mathbb{C}^n \,;\, \left\| \tilde{\Pi}_{\tilde{\Gamma}}(\nabla_{\ell^2} \tilde{E}(\tilde{\Gamma}) - \tilde{\Phi}) \right\|_{\ell^2(\tilde{\Gamma})} \leqslant \rho \left\| \tilde{\Pi}_{\tilde{\Gamma}}(\nabla_{\ell^2} \tilde{E}(\tilde{\Gamma})) \right\|_{\ell^2(\tilde{\Gamma})} \right\} \,. \tag{6.18}$$

The Finsler flow discretizing the original one (2.2) reads

$$\tilde{\Gamma}_{k+1} = \tilde{\Gamma}_k - \tau_k \nabla_{\tilde{R}_{\tilde{\Gamma}_k}} \tilde{E}(\tilde{\Gamma}_k). \tag{6.19}$$

where $\tau_k > 0$ is chosen following the Wolfe rule (2.8).

The following sections detail how to compute this flow for the particular case of the curve matching energy introduced in Section 5.

6.4 Discrete Energy

Exact energy for piecewise affine curves. For curves $\Gamma = P_1(\tilde{\Gamma})$ and $\Lambda = P_1(\tilde{\Lambda})$ in $\mathcal{B}_n$, the energy $E(\Gamma)$ defined in (5.4)

$$E(\Gamma) = \frac{1}{2}\mathcal{H}(\Gamma,\Gamma) - \mathcal{H}(\Gamma,\Lambda) + \frac{1}{2}\mathcal{H}(\Lambda,\Lambda)$$

can be computed as

$$\mathcal{H}(\Gamma,\Lambda) = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \Delta^+(\tilde{\Gamma})_i,\, \Delta^+(\tilde{\Lambda})_j \rangle T(\tilde{\Gamma},\tilde{\Lambda})_{i,j}$$

$$\text{where} \quad T(\tilde{\Gamma},\tilde{\Lambda})_{i,j} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} k(\Gamma(s),\Lambda(t))\, \mathrm{d}\Gamma(s)\mathrm{d}\Lambda(t)\,.$$

Approximate energy for piecewise affine curves. In general there is no closed form expression for the operator T, so that, to enable a direct computation of the energy and its gradient, we use a first order approximation with a trapezoidal quadrature formula

$$\tilde{T}(\tilde{\Gamma},\tilde{\Lambda})_{i,j} = \frac{1}{4}\big(k(\tilde{\Gamma}_i,\tilde{\Lambda}_j) + k(\tilde{\Gamma}_{i+1},\tilde{\Lambda}_j) + k(\tilde{\Gamma}_i,\tilde{\Lambda}_{j+1}) + k(\tilde{\Gamma}_{i+1},\tilde{\Lambda}_{j+1})\big)\,.$$

One thus has the approximation

$$\tilde{T}(\tilde{\Gamma},\tilde{\Lambda})_{i,j} = T(\tilde{\Gamma},\tilde{\Lambda})_{i,j} + O(1/n^2).$$

This defines the discrete energy $\tilde{E}$ on $\mathbb{C}^n$ as

$$\tilde{E}(\tilde{\Gamma}) = \frac{1}{2}\tilde{\mathcal{H}}(\tilde{\Gamma},\tilde{\Gamma}) - \tilde{\mathcal{H}}(\tilde{\Gamma},\tilde{\Lambda}) + \frac{1}{2}\tilde{\mathcal{H}}(\tilde{\Lambda},\tilde{\Lambda}) \tag{6.20}$$

$$\text{where} \quad \tilde{\mathcal{H}}(\tilde{\Gamma},\tilde{\Lambda}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \Delta^+(\tilde{\Gamma})_i,\, \Delta^+(\tilde{\Lambda})_j \rangle \tilde{T}(\tilde{\Gamma},\tilde{\Lambda})_{i,j}$$

Discrete ℓ^2-gradient. The following proposition gives the formula to calculate the gradient of $\tilde{E}$ with respect to inner product (6.11).

Proposition 6.3. The gradient of $\tilde{E}$ at $\tilde{\Gamma}$ with respect to the metric defined by the inner product (6.11) is

$$\nabla_{\ell^2}\tilde{E}(\tilde{\Gamma}) = M_{\tilde{\Gamma}}^{-1}\nabla\tilde{E}(\tilde{\Gamma})$$

where $M_{\tilde{\Gamma}}$ is the rigidity matrix (6.12) and $\nabla\tilde{E}$ the gradient of $\tilde{E}$ for the canonical inner product of $\mathbb{C}^n$ (6.1), which is given by

$$\begin{aligned}\nabla\tilde{E}(\tilde{\Gamma})_i &= \frac{1}{4}\sum_{j=1}^{n}(\tilde{\Gamma}_{i+1}-\tilde{\Gamma}_{i-1})(\tilde{\Lambda}_{j+1}-\tilde{\Lambda}_j)[\nabla_1 k(\tilde{\Gamma}_i,\tilde{\Lambda}_j)+\nabla_1 k(\tilde{\Gamma}_i,\tilde{\Lambda}_{j+1})] \\ &\quad + \sum_{j=1}^{n}(\tilde{\Lambda}_{j+1}-\tilde{\Lambda}_j)[T(\tilde{\Gamma},\tilde{\Lambda})_{i-1,j}-T(\tilde{\Gamma},\tilde{\Lambda})_{i,j}]\,.\end{aligned} \tag{6.21}$$

Proof. The gradient (6.21) of $\tilde{E}$ for the canonical inner product of $\mathbb{C}^n$ can be computed by a straightforward calculation. For every $\tilde{\Phi}\in\mathbb{C}^n$ we have the following expression for the derivative of $\tilde{E}$

$$D\tilde{E}(\tilde{\Gamma})(\tilde{\Phi}) = \langle\tilde{\Phi},\,\nabla_{\ell^2}\tilde{E}(\tilde{\Gamma})\rangle_{\ell^2(\tilde{\Gamma})} = \langle\tilde{\Phi},\,\nabla\tilde{E}(\tilde{\Gamma})\rangle_{\mathbb{C}^n}$$

and, by (6.13), we get

$$\nabla_{\ell^2}\tilde{E}(\tilde{\Gamma}) = M_{\tilde{\Gamma}}^{-1}\nabla\tilde{E}(\tilde{\Gamma})\,.$$

□

6.5 Discrete Piecewise-rigid Curve Matching

This section first describes in a general setting the discrete Finsler gradient over finite-element spaces, then specializes it to the piecewise rigid penalty for the matching problem, and lastly gives the explicit formula of the corresponding functionals to be minimized numerically.

Discrete piecewise-rigid penalty. To re-write conveniently the discrete Finsler gradient optimization (6.16), we introduce the following finite-dimensional operators.

Definition 6.4 (**Discrete operators**)**.** For all $\Gamma = P_1(\tilde{\Gamma}), \Phi = P_1(\tilde{\Phi})$ we define

$$\begin{aligned}\tilde{V}_{\tilde{\Gamma}}(\tilde{\Phi}) &= TV_{\Gamma}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_{\Gamma}\right) \\ \tilde{L}_{\tilde{\Gamma}}(\tilde{\Phi}) &= P_0^{-1}(L_{\Gamma}^{+}(\Phi)), \qquad \tilde{L}_{\tilde{\Gamma}}:\mathbb{C}^n\to\mathbb{R}^n \\ \tilde{\Pi}_{\tilde{\Gamma}}(\tilde{\Phi}) &= P_0^{-1}(\Pi_{\Gamma}(\Phi)), \qquad \tilde{\Pi}_{\tilde{\Gamma}}:\mathbb{C}^n\to\mathbb{C}^n\end{aligned}$$

The following proposition uses this discrete operators to compute the discrete Finsler penalty and constraint defined in (6.17).

Proposition 6.5. One has

$$\tilde{R}_{\tilde{\Gamma}}(\tilde{\Phi}) = \tilde{V}_{\tilde{\Gamma}}(\tilde{\Phi}) + \iota_{\tilde{\mathcal{C}}_{\tilde{\Gamma}}}(\tilde{\Phi}) \quad \text{where} \quad \tilde{\mathcal{C}}_{\tilde{\Gamma}} = \left\{\tilde{\Phi}\in\mathbb{C}^n\,;\ \tilde{L}_{\tilde{\Gamma}}(\tilde{\Phi})=0\right\}. \tag{6.22}$$

Proof. Denoting $\Gamma = P_1(\tilde{\Gamma}), \Phi = P_1(\tilde{\Phi})$, by (6.17), we have

$$\tilde{R}_{\tilde{\Gamma}}(\tilde{\Phi}) = R_\Gamma(\Phi) = TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma\right) + \iota_{\mathcal{C}_\Gamma}(\Phi) = \tilde{V}_{\tilde{\Gamma}}(\tilde{\Phi}) + \iota_{\tilde{\mathcal{C}}_{\tilde{\Gamma}}}(\tilde{\Phi})$$

where

$$\tilde{\mathcal{C}}_{\tilde{\Gamma}} = \left\{\tilde{\Phi} \in \mathbb{C}^n \; ; \; \tilde{L}_{\tilde{\Gamma}}(\tilde{\Phi}) = 0\right\}.$$

□

The following proposition gives explicit formula for the discrete operators introduced in Definition 6.4.

Proposition 6.6. For every $\tilde{\Gamma}, \tilde{\Phi} \in \mathbb{C}^n$, we consider $\Gamma = P_1(\tilde{\Gamma}) \in \mathcal{B}_n$, $\Phi = P_1(\tilde{\Phi}) \in T_\Gamma\mathcal{B}_n$. One has

$$\tilde{L}_{\tilde{\Gamma}}(\tilde{\Phi})_i = \langle \frac{\Delta^+(\tilde{\Phi})_i}{|\Delta^+(\tilde{\Gamma})_i|}, \frac{\Delta^+(\tilde{\Gamma})_i}{|\Delta^+(\tilde{\Gamma})_i|} \rangle, \tag{6.23}$$

$$\tilde{\Pi}_{\tilde{\Gamma}}(\tilde{\Phi}) = \sum_{i=1}^n \langle \tilde{\Phi}_i, (\tilde{\mathbf{n}}_\Gamma)_i \rangle (\tilde{\mathbf{n}}_\Gamma)_i \zeta_i \tag{6.24}$$

$$\tilde{V}_{\tilde{\Gamma}}(\tilde{\Phi}) = \|\Delta^-(\tilde{H}_{\tilde{\Gamma}}(\tilde{\Phi}))\|_{\ell^1} = \sum_{i=1}^n \left|\tilde{H}_{\tilde{\Gamma}}(\tilde{\Phi})_i - \tilde{H}_{\tilde{\Gamma}}(\tilde{\Phi})_{i-1}\right| \tag{6.25}$$

$$\text{where} \;\; \tilde{H}_{\tilde{\Gamma}}(\tilde{\Phi})_i := \langle \frac{\Delta^+(\tilde{\Phi})_i}{|\Delta^+(\tilde{\Gamma})_i|}, (\tilde{\mathbf{n}}_\Gamma)_i \rangle \tag{6.26}$$

and where $\tilde{\mathbf{n}}_\Gamma$ denotes the vector of the coordinates of $\mathbf{n}_\Gamma$ with respect to the basis of $\mathbb{P}_0$.

Proof. **(Proof of** (6.23)**)** Using (6.5) the first derivative of Φ with respect to the basis of $\mathbb{P}_{0,n}$ can be written as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} = \sum_{i=1}^n \frac{\Delta^+(\tilde{\Phi})_i}{|\Delta^+(\tilde{\Gamma})_i|}\zeta_i$$

which implies that

$$L_\Gamma^+(\Phi) = \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{t}_\Gamma = \sum_{i=1}^n \langle \frac{\Delta^+(\tilde{\Phi})_i}{|\Delta^+(\tilde{\Gamma})_i|}, \frac{\Delta^+(\tilde{\Gamma})_i}{|\Delta^+(\tilde{\Gamma})_i|} \rangle \zeta_i \,.$$

Then, by the definitions of $L_\Gamma^{+(-)}$, conditions $L_\Gamma^{+(-)}(\Phi) = 0$ become

$$\langle \frac{\Delta^+(\tilde{\Phi})_i}{|\Delta^+(\tilde{\Gamma})_i|}, \frac{\Delta^+(\tilde{\Gamma})_i}{|\Delta^+(\tilde{\Gamma})_i|} \rangle = 0 \qquad \forall\, i = 1, ..., n,$$

which is equivalent to $\tilde{L}_{\tilde{\Gamma}}(\tilde{\Phi}) = 0$.
(Proof of (6.24)**)** By (6.10) and (6.8), we get

$$\begin{aligned}\Pi_\Gamma(\Phi) &= \langle \sum_{i=1}^n \tilde{\Phi}_i \xi_i, \sum_{i=1}^n (\tilde{\mathbf{n}}_\Gamma)_i \zeta_i \rangle \sum_{i=1}^n (\tilde{\mathbf{n}}_\Gamma)_i \zeta_i \\ &= \sum_{i=1}^n \langle \tilde{\Phi}_i, (\tilde{\mathbf{n}}_\Gamma)_i \rangle (\tilde{\mathbf{n}}_\Gamma)_i \zeta_i\end{aligned}$$

which proves the result.
(Proof of (6.25)**)** By (6.5) and (6.8), we get

$$TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}\cdot\mathbf{n}_\Gamma\right) = TV_\Gamma\left(\sum_{i=1}^n \langle\frac{\Delta^+(\tilde\Phi)_i}{|\Delta^+(\tilde\Gamma)_i|}, (\tilde{\mathbf{n}}_\Gamma)_i\rangle\zeta_i\right)$$
$$= \sum_{i=1}^n \left|\tilde H_{\tilde\Gamma}(\tilde\Phi)_i - \tilde H_{\tilde\Gamma}(\tilde\Phi)_{i-1}\right|$$

where we used the fact that the total variation for piecewise constant functions coincides with the sum of jumps sizes. □

6.6 Calculation of the Discrete Finsler Gradient

One can minimize the matching energy $\tilde E$ defined in (6.20) using the Finsler flow $\{\tilde\Gamma_k\}$ of (6.19). This requires computing at each step k the Finsler gradient (6.16) for the piecewise-rigid penalty $\tilde R_{\tilde\Gamma}$ defined in (6.22). Solving (6.16) at each step in turn requires the resolution of a finite dimensional convex problem, and the functional to be minimized is explicitly given with closed form formula in Proposition 6.6.

Several convex optimization algorithms can be used to solve (6.16). A convenient method consists in recasting the problem into a second order cone program by introducing additional auxiliary variables $(\tilde\Phi, \tilde S, \tilde U, \tilde T)$ as follow

$$\operatorname*{Min}_{(\tilde\Phi,\tilde S,\tilde U,\tilde T)\in\mathbb{C}^{2n}\times\mathbb{R}^{2n}} \langle\tilde U,\, \mathbf{1}\rangle_{\mathbb{C}^n} \quad \text{where} \quad \mathbf{1} = (1,...,1)\in\mathbb{R}^n$$

where the minimum is taken under the following set of affine and conic constraints

$$\begin{aligned}
&-\tilde U_i \leqslant \tilde V_{\tilde\Gamma}(\tilde\Phi)_i \leqslant \tilde U_i\,, \qquad \forall\, i = 1,...,n\\
&\tilde L_{\tilde\Gamma}(\tilde\Phi) = 0\\
&\tilde S + \tilde\Pi_{\tilde\Gamma}(\tilde\Phi) = \tilde\Pi_{\tilde\Gamma}(\nabla_{\ell^2}\tilde E(\tilde\Gamma))\\
&\langle T, \mathbf{1}\rangle \leqslant \rho^2\left\|\tilde\Pi_{\tilde\Gamma}(\nabla_{\ell^2}\tilde E(\tilde\Gamma))\right\|^2_{\ell^2(\tilde\Gamma)}\\
&(\tilde S_i, \tilde T_i)\in\{(s,t)\in\mathbb{C}\times\mathbb{R}\,;\ |s|\leqslant t\}\,, \quad \forall\, i = 1,...,n.
\end{aligned}$$

For the numerical simulation, we use an interior point solver, see [7]. These interior points algorithms are powerful methods to solve medium scale SOCP problems, and work remarkably well for n up to several thousands, which is typically the case for the curve matching problem.

7 Numerical examples

In this section we give some numerical examples to point out the properties of the piecewise rigid Finsler evolution. These numerical experiments are performed using the discretization detailed in Section 6 with $n = 1280$ points.

7.1 Influence of ρ

To exemplify the main properties of the piecewise rigid Finsler flow, we introduce a synthetic example where we replace in the definition (4.18) of the Finsler gradient $\nabla_{R_\Gamma} E(\Gamma)$ (more precisely in the definition (4.17) of the constraint $\mathcal{L}_\Gamma$) the L^2 gradient $\nabla_{L^2} E(\Gamma)$ by the vector field $F(\Gamma) \in T_\Gamma \mathcal{B}$ defined as

$$F(\Gamma) : s \in \mathbb{S}^1 \mapsto -\left(5\Gamma_1(s), 1000(\Gamma_2(s) - 1/2)^2\right) \in \mathbb{R}^2 \tag{7.1}$$
$$\text{where} \quad \Gamma(s) = (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2.$$

The initial flow associated to this vector field reads

$$\Gamma_{k+1} = \Gamma_k - \tau_k F(\Gamma_k) \tag{7.2}$$

for some small enough time step $\tau_k > 0$.

Figure 1 shows the impact of the parameter ρ on this evolution. As ρ increases, the evolution becomes increasingly piecewise rigid. For ρ large enough, it is globally rigid, i.e. satisfies (4.2) and $\nabla_{R_{\Gamma_k}} E(\Gamma_k) \in \mathcal{R}_{\Gamma_k}$ for all k, where $\mathcal{R}_\Gamma$ is defined in (4.3).

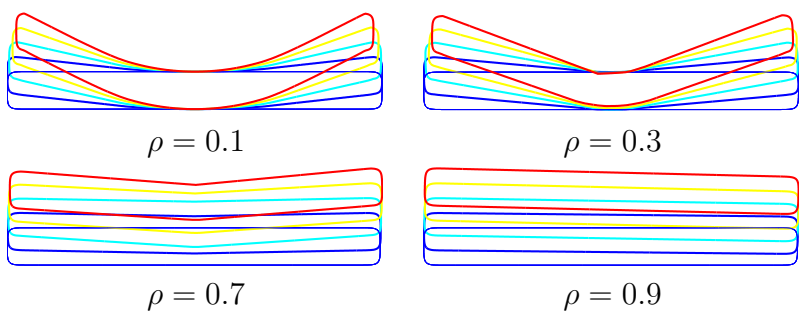

Figure 1: Evolution for different values of ρ.

7.2 Curve Registration

We now give an example of application of the Finsler flow to the curve matching problem described in Section 5. Figure 2 compares the results of the piecewise-rigid Finsler gradient with the Sobolev Riemannian gradient detailed in Remark (2.3) which is very similar to the one introduced in [32, 12].

In order to obtain good matching results, it is important to select the parameters (τ, σ, δ) (see (5.3) and (7.2)) in accordance to the typical size of the features of the curves to be matched. For each method, we have manually tuned the parameters (τ, σ, δ) in order to achieve the best matching results. Choosing a large value of σ and a smaller value for δ is useful to capture shapes with features at different scales, which is the case in our examples.

The piecewise rigid gradient is particularly efficient in this setting where the curves to be matched are naturally obtained by approximate articulations, which are well approximated by piecewise rigid deformations. Note however that our method does not necessitate a prior segmentation of the shape

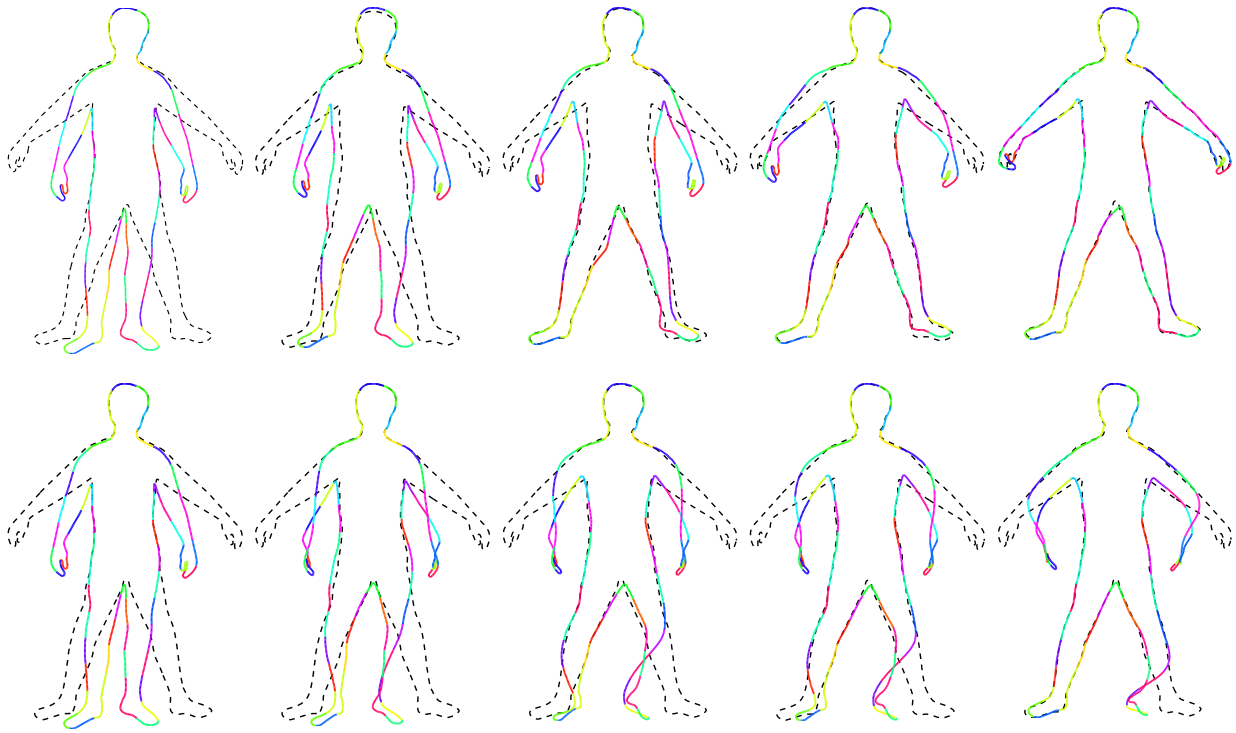

Figure 2: Finsler evolution (top) with $\rho = 0.8$ and Sobolev evolution (bottom) for different step k of the flow. Each image displays the target curve Λ (dash line) and the current curve Γ_k (solid line). The energy is computed using $\sigma = 0.8$, $\delta = 0.04$.

into disjoint areas undergoing rigid motions, i.e. the location of the articulations does not need to be known beforehand. The piecewise rigid matching is obtained solely by minimizing the distance energy $E(\Gamma)$ to the target curve Λ.

The Finsler gradient thus allows to avoid poor local minima and perform an overall good global matching. In contrast the Sobolev gradient flow is trapped in a poor local minimum and the matching has failed. Note, however, that the matching achieved by the Finsler gradient is not perfect. Some local defects near corners are mostly due to the strong constraint $L_\Gamma(\Phi) = 0$ which enforces the exact conservation of the length of the curve. This constraint is alleviated in Section 8, which presents a piecewise similarity Finsler gradient which leads to better matching results.

8 BV^2-piecewise Similarity Motions

In order to improve the matching results we augment our model by allowing the curve to shrink or to lengthen during the evolution. The idea consists in considering evolution by piecewise similarity motions instead of the rigid deformations considered in Section 4.

8.1 Similarity Curve Deformations

We extends the rigid deformations considered in Section 4.1 to smooth evolutions $t \mapsto \Gamma_t$ following a PDE (4.1) that includes also a global scaling of the space. This evolution is said to obey a

similarity transform if there exists a smooth function $\lambda : \mathbb{R} \to \mathbb{R}^+$ such that

$$\forall\,(s,s') \in \mathbb{S}^1 \times \mathbb{S}^1, \quad \|\Gamma_t(s) - \Gamma_t(s')\| = \lambda(t)\|\Gamma_0(s) - \Gamma_0(s')\|. \tag{8.1}$$

The following proposition shows that the set of instantaneous motion Φ_t giving rise to a similarity evolution is, at each time, a linear sub-space of dimension 4 of $T_{\Gamma_t}\mathcal{B}$.

Proposition 8.1. The evolution (4.1) satisfies (8.1) if and only if, for all $t \in \mathbb{R}$, $\Phi_t \in \mathcal{S}_{\Gamma_t}$ where

$$\mathcal{S}_\Gamma = \left\{\Phi \in T_\Gamma\mathcal{B}\ ;\ \forall\, s \in \mathbb{S}^1,\ \Phi(s) = A\Gamma(s) + b :\ \text{for } A \in \mathcal{S}_{2\times 2},\ b \in \mathbb{R}^2\right\} \tag{8.2}$$

$$\text{where} \quad \mathcal{S}_{2\times 2} = \left\{\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \in \mathbb{R}^{2\times 2}\ ;\ (\alpha,\beta) \in \mathbb{R}^2\right\}.$$

Proof. By the definition of evolution, we have

$$\Gamma_t(s) = \Gamma_0(s) + \int_0^t \Phi_\tau(s)\mathrm{d}\tau \quad \forall s \in \mathbb{S}^1\,, \tag{8.3}$$

If the evolution verifies (8.1), previous relationship defines a similarity transformation between $\Gamma_0(\mathbb{S}^1)$ and $\Gamma_t(\mathbb{S}^1)$. By the same argument used in the proof of Proposition 4.1, such a transformation can be extended to a similarity transformation of the plane. Then, by the characterization of the similarity transformations of the plane, we get that

$$\Gamma_t(s) = S(t)\Gamma_0(s) + c(t) \quad \forall s \in \mathbb{S}^1 \tag{8.4}$$

for some vector $c(t)$ and matrix $S(t) \in \mathcal{S}_{2\times 2}$

$$S(t) = \begin{pmatrix} \alpha(t) & -\beta(t) \\ \beta(t) & \alpha(t) \end{pmatrix}.$$

Now, by differentiating (8.3) and (8.4) with respect to t, we get

$$\Phi_t(s) = S'(t)\Gamma_0(s) + c'(t) \quad \forall s \in \mathbb{S}^1 \tag{8.5}$$

where $(\cdot)'$ denotes here the derivation with respect to t. Then by (8.4) and (8.5) we get

$$\Phi_t(s) = S'(t)S(t)^{-1}\Gamma_t(s) + c'(t) - S'(t)S(t)^{-1}c(t) \quad \forall s \in \mathbb{S}^1$$

which implies that $\Phi_t \in \mathcal{S}_{\Gamma_t}$ with $A = S'(t)S(t)^{-1} \in \mathcal{S}_{2\times 2}$ and $b = c'(t) - S'(t)S(t)^{-1}c(t)$.

Conversely, if $\Phi_t \in \mathcal{S}_{\Gamma_t}$, we have

$$(\Phi_t(s) - \Phi_t(s')) \cdot (\Gamma_t(s) - \Gamma_t(s')) = \alpha(t)\|\Gamma_t(s) - \Gamma_t(s')\|^2\,, \quad \forall\,(s,s') \in \mathbb{S}^1 \times \mathbb{S}^1$$

for some $\alpha(t) \in \mathbb{R}$. This means that

$$\frac{\partial}{\partial t}\|\Gamma_t(s) - \Gamma_t(s')\|^2 = \frac{\alpha(t)}{2}\|\Gamma_t(s) - \Gamma_t(s')\|^2\,, \quad \forall\,(s,s') \in \mathbb{S}^1 \times \mathbb{S}^1$$

which implies that

$$\|\Gamma_t(s) - \Gamma_t(s')\|^2 = e^{\int_0^t \alpha(\tau)/2\mathrm{d}\tau}\|\Gamma_0(s) - \Gamma_0(s')\|^2\,, \quad \forall\,(s,s') \in \mathbb{S}^1 \times \mathbb{S}^1$$

and this proves that the Φ_t verifies (8.1) with $\lambda(t) = e^{\int_0^t \alpha(\tau)/2\mathrm{d}\tau}$. □

Analogously to Proposition 4.2, the next proposition characterizes in an intrinsic manner the set $\mathcal{S}_\Gamma$.

Proposition 8.2. For a C^2-curve Γ, one has $\Phi \in \mathcal{S}_\Gamma$ if and only if Φ is C^2 and satisfies

$$\frac{\mathrm{d}K_\Gamma(\Phi)}{\mathrm{d}\Gamma(s)} = 0 \tag{8.6}$$

where we have introduced the following linear operator

$$K_\Gamma(\Phi) = \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{t}_\Gamma, \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma \right), \qquad \forall\, \Phi \in T_\Gamma \mathcal{B}\,.$$

Proof. Given a curve $\Gamma \in C^2(\mathbb{S}^1, \mathbb{R}^2)$, every deformation Φ of Γ which is the restriction, to the curve Γ, of a instantaneous similarity motion, can be written as

$$\Phi(s) \;=\; A\Gamma(s) + b, \qquad \forall\, s \in \mathbb{S}^1$$

for some matrix $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ and a vector b. Now, differentiating with respect to $\mathrm{d}\Gamma(s)$ we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}(s) = A\mathbf{t}_\Gamma$$

which is equivalent to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{t}_\Gamma \;=\; \alpha \quad \text{and} \quad \frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma \;=\; \beta \tag{8.7}$$

for every $s \in \mathbb{S}^1$. Remark that similarity is the only affine motion verifying (8.7) and this is due to the form of the matrix A. In fact, if $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)}$ verifies $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{t}_\Gamma = \alpha$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma = \beta$ then

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} = \alpha\mathbf{t}_\Gamma + \beta\mathbf{n}_\Gamma = \alpha\mathbf{t}_\Gamma + \beta\mathbf{t}_\Gamma^\perp = A\mathbf{t}_\Gamma\,.$$

In particular, if $\alpha = 0$ then Φ is a rigid motion and (8.7) coincides with the characterization proved in Proposition 4.2.

Then, differentiating again with respect to $d\Gamma(s)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\Gamma(s)} \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{t}_\Gamma \right) \;=\; 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\Gamma(s)} \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma \right) \;=\; 0$$

which is equivalent to (8.6). □

8.2 Piecewise Similarity Deformations

Similarly to the Finsler penalty introduced in 4.2, we define a penalty that favors piecewise similarity transformations by minimizing the L^1-norm of the first derivative of K_Γ. To control the piecewise rigid transformation part, we relax the equality constraint $L_\Gamma(\Phi) = 0$ defined as $\mathcal{C}_\Gamma$ in (4.16) to a constraint $\mathcal{C}_\Gamma^\lambda$ on the L^1 norm of $L_\Gamma(\Phi)$.

Definition 8.3 (**Piecewise-similarity penalty**). For $\lambda \geqslant 0$ and $\Gamma \in \mathcal{B}$, we define for all $\Phi \in T_\Gamma \mathcal{B}$

$$R_\Gamma^\lambda(\Phi) = TV_\Gamma(K_\Gamma(\Phi)) + \iota_{\mathcal{C}_\Gamma^\lambda} \quad \text{where} \quad \mathcal{C}_\Gamma^\lambda = \left\{ \Phi \in T_\Gamma \mathcal{B} \; ; \; \|L_\Gamma(\Phi)\|_{L^1(\Gamma)} \leqslant \lambda \right\} \tag{8.8}$$

where L_Γ is either L_Γ^+ or L_Γ^- as defined in (4.12) and TV_Γ is defined in (4.15).

The piecewise similarity Finsler gradient $\nabla_{R_\Gamma^\lambda} E(\Gamma)$ is defined by minimizing (2.3) with the penalty R_Γ^λ defined in (8.8) with the constraint set $\mathcal{L}_\Gamma$ defined in (4.17). The following proposition shows that, as λ tends to 0, the set of piecewise similarity Finsler gradients tends to the set of piecewise-rigid Finsler gradients.

Proposition 8.4. One has $R_\Gamma^0 = R_\Gamma$ where R_Γ is defined in (4.14).

Proof. One has $\mathcal{C}_\Gamma^0 = \mathcal{C}_\Gamma$. If $R_\Gamma^0(\Phi) \neq +\infty$, one has $L_\Gamma^+(\Phi) = L_\Gamma^-(\Phi) = 0$ a.e., so that in this case

$$TV_\Gamma(K_\Gamma(\Phi)) = TV_\Gamma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\Gamma(s)} \cdot \mathbf{n}_\Gamma\right).$$

□

The following theorem extends Theorem 4.6 to the piecewise similarity penalty and ensures existence of the corresponding Finsler gradient.

Theorem 8.5. The function R_Γ^λ defined in (8.8) admits at least a minimum on $\mathcal{L}_\Gamma$.

Proof. It suffices to adapt the proof of Theorem 4.6 by using the new constraint on the L^1-norm of $L_\Gamma(\Phi)$.

□

8.3 Numerical examples

We now show some numerical examples for the piecewise similarity Finsler gradient. The computation is performed with the discretization detailed in Section (6), which is extended in a straightforward manner to handle the piecewise similarity model.

Influence of λ. We first re-use the synthetic example introduced in Section 7.1 to illustrate the influence of the parameter λ. We thus use an evolution driven by the flow $F(\Gamma) \in T_\Gamma \mathcal{B}$ defined in (7.1). Figure 3 shows how λ allows one to interpolate between the piecewise rigid model (when $\lambda = 0$) to a piecewise similarity model when λ increases. For large value of λ, one clearly sees the global scaling introduced by the model which is helpful to better follow the flow of F.

Figure 4 compares the evolution obtained with the initial flow (7.2) (corresponding to $(\rho, \lambda) = (0, 0)$, i.e. the Finsler gradient is equal to $F(\Gamma)$), the piecewise rigid flow (corresponding to $\rho > 0$ and $\lambda = 0$) and the piecewise similarity flow (corresponding to $\rho > 0$ and $\lambda > 0$).

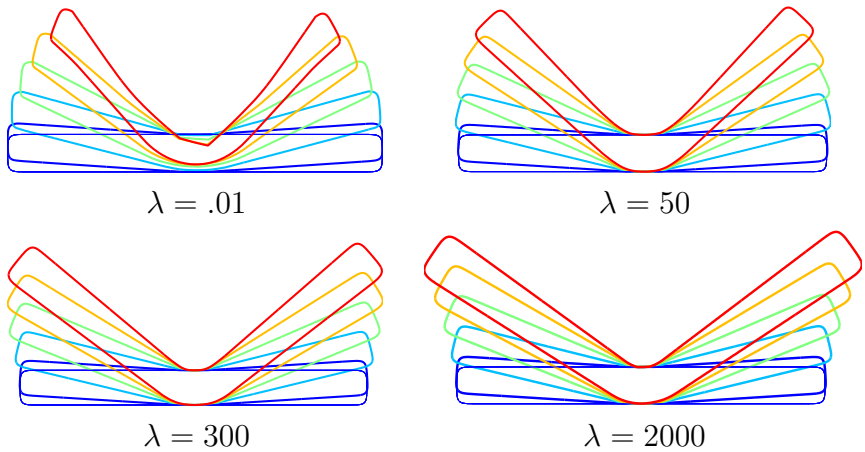

Figure 3: Piecewise similarity Finsler flow evolutions for $\rho = 0.3$ and for different values of λ.

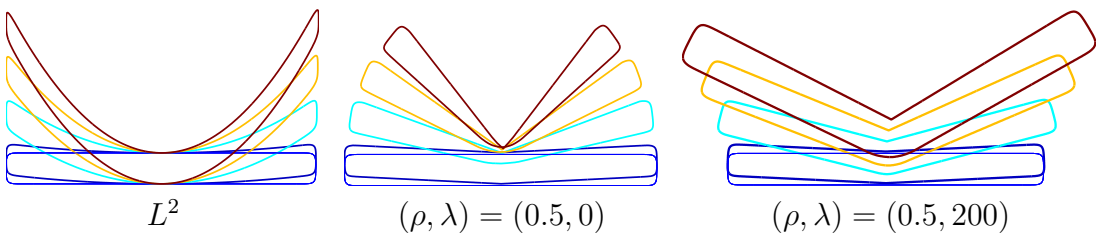

Figure 4: From left to right: evolutions by using the L^2 gradient, piecewise rigid Finsler gradient, piecewise similarity Finsler gradient.

Application to the matching problem. We now show an application of the Finsler descent method to the curve matching problem, by minimizing the energy E defined in (5.4). Figure 5 shows the results obtained with the piecewise similarity penalty for well chosen parameters $(\sigma, \delta, \lambda, \rho)$. These evolution should be compared with the ones reported in Section 7.2. Allowing the length of the curve to vary using a parameter $\lambda > 0$ allows the evolutions to better capture the geometry of the target shape and thus leads to better matchings.

Conclusion

This paper has presented a novel way to encode piecewise regular constraints for curve evolutions. This is achieved by designing Finsler penalties in the tangent space of curves. This method offers a unifying treatment of this class of evolutions. A distinctive feature of this approach is that it uses a convex modeling of the geometric constraints. For the particular case of piecewise rigid and piecewise similarity transforms, this avoids the need to specify the location of the articular points, which is a difficult problem. Instead, these articulations are obtained, at each iteration, through the resolution of a convex program. This novel method opens the doors to many fascinating theoretical questions, such as the definition of a continous flow when letting $\tau_k \to 0$, and the properties of Finslerian spaces of curves.

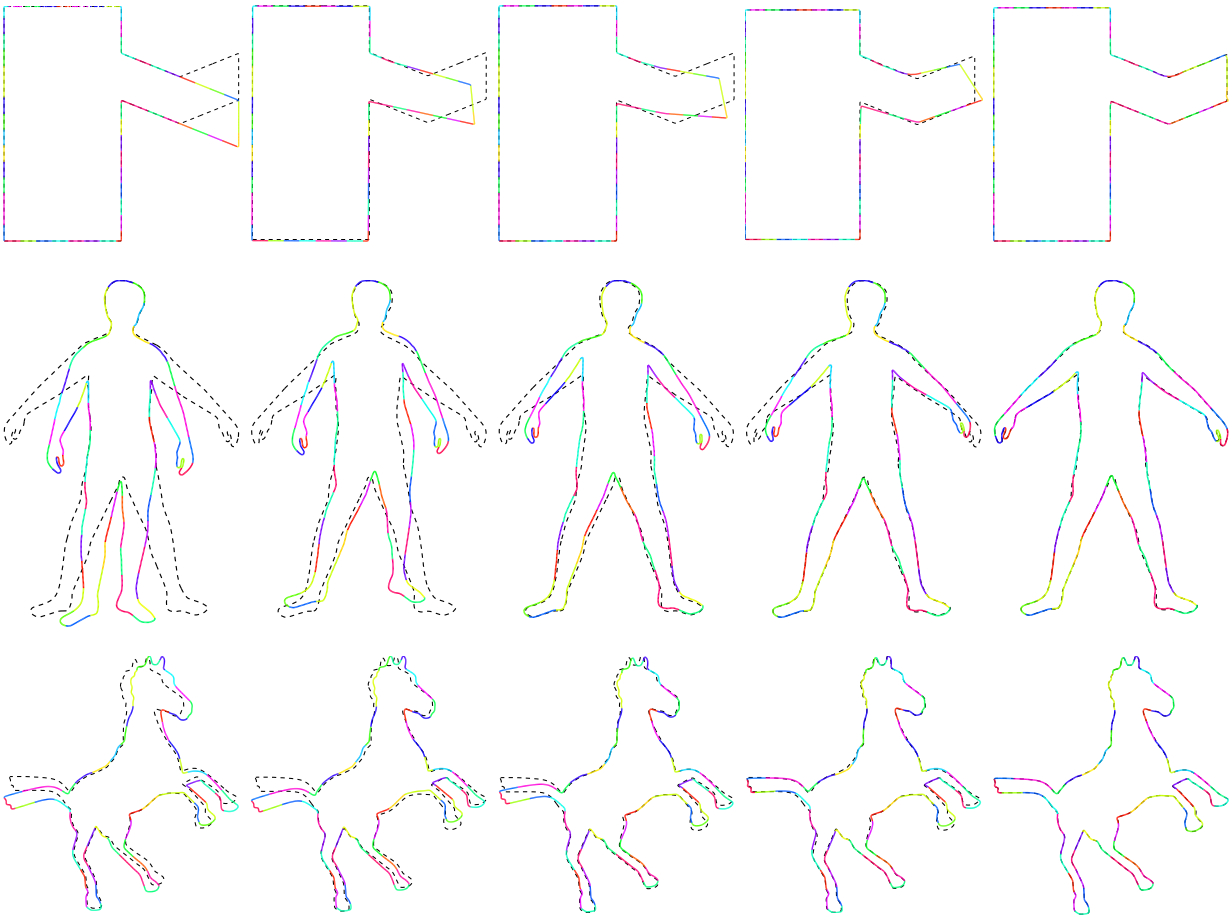

Figure 5: Curve matching by piecewise similarity motions. Each image displays the target curve Λ (dash line) and the current curve Γ_k (solid line). We used the following parameters: top row: $\sigma = 0.8$, $\delta = 0.04$, $\lambda = 2000$, $\rho = 0.85$; middle row: $\sigma = 0.8$, $\delta = 0.08$, $\lambda = 2000$, $\rho = 0.95$; bottom row: $\sigma = 0.9$, $\delta = 0.03$, $\lambda = 2000$, $\rho = 0.87$.

Acknowledgment

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9 Appendix: BV and BV^2 functions

In this section Ω denotes an open set of $\mathbb{R}^n$.

Definition 9.1. Let $u \in L^1(\Omega)$. We say that u is a function of bounded variation in Ω if

$$|Du|(\Omega) = \sup \left\{ \int_\Omega u \,\mathrm{div} g \,\mathrm{d}x \;;\; g \in \mathrm{C}^1_c(\Omega), \|g\|_{L^\infty(\Omega)} \leqslant 1 \right\} < \infty \,. \tag{9.1}$$

By Riesz's representation theorem this is equivalent to state that there exists a unique finite Radon measure, denoted by Du, such that

$$\int_\Omega u \,\mathrm{div} g \,\mathrm{d}x = - \int_\Omega g \,\mathrm{d}Du \quad \forall\, g \in \mathrm{C}^1_c(\Omega) \,.$$

Clearly the total variation of the measure Du on Ω, i.e., $|Du|(\Omega)$, coincides with the quantity defined in (9.1) and this justifies our notations. We denote the space of functions of bounded variation in Ω by $BV(\Omega)$. The space $BV(\Omega)$ equipped with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega)$$

is a Banach space. We say that $\{u_h\}$ weakly* converges in $BV(\Omega)$ to u if

$$u_h \xrightarrow{L^1} u \quad \text{and} \quad Du_h \overset{*}{\rightharpoonup} Du\,, \quad \text{as} \quad h \to \infty\,.$$

We now define the set of BV^2-functions as the functions whose second derivatives are Radon measures:

Definition 9.2. Let $u \in W^{1,1}(\Omega)$. We define the following operator:

$$H : g \in C^2(\Omega, \mathbb{R}^{n^2}) \mapsto \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}\,.$$

We say that u belongs to $BV^2(\Omega)$ if

$$|D^2u|(\Omega) := \sup\left\{\int_\Omega u\, H(g)\,\mathrm{d}x\ ;\ g \in \mathrm{C}^2_c(\Omega, \mathbb{R}^{n^2}), \|g\|_{L^\infty(\Omega,\mathbb{R}^{n^2})} \leqslant 1\right\} < \infty\,. \tag{9.2}$$

$BV^2(\Omega)$ equipped with the norm

$$\|u\|_{BV^2} = \|u\|_{BV} + |D^2u|(\Omega) \tag{9.3}$$

is a Banach space. In particular we have $W^{2,1}(\Omega) \subset BV^2(\Omega)$. We say that $\{u_h\}$ weakly* converges in $BV^2(\Omega)$ to u if

$$u_h \xrightarrow{W^{1,1}} u \quad \text{and} \quad D^2u_h \overset{*}{\rightharpoonup} D^2u\,, \quad \text{as} \quad h \to \infty\,.$$

We remind that if $\{u_h\} \subset BV^2(\Omega)$ is such that $\sup_h \|u_h\|_{BV^2} < M$ then there exists $u \in BV^2(\Omega)$ such that $\{u_h\}$ weakly* converges in $BV^2(\Omega)$ toward u and

$$|D^2u|(\Omega) \leqslant \liminf_{h\to\infty} |D^2u_h|(\Omega)\,.$$

Moreover we have the following proposition showing the link between BV and BV^2 functions:

Proposition 9.3. A function u belongs to $BV^2(\Omega)$ if and only if $u \in W^{1,1}(\Omega)$ and $\frac{\partial u}{\partial x_i} \in BV(\Omega)$, for every $i = 1, ..., n$. Moreover,

$$|D^2u|(\Omega) \leqslant \sum_{i=1}^{n} \left| D\left(\frac{\partial u}{\partial x_i}\right)\right|(\Omega) \leqslant n|D^2u|(\Omega)\,.$$

In dimension $n = 1$, $BV^2(\Omega)$ is embedded in $W^{1,\infty}(\Omega)$ so BV^2 functions are locally Lipschitz continuous (see Theorem 5, [16] pag. 131). Then, in particular, if $\Omega \subset \mathbb{R}$ is bounded, $BV^2(\Omega)$ is embedded $W^{1,p}(\Omega)$, for every $p \geqslant 1$.

A vector field $\mathbf{u}$ belongs to $BV(\Omega)(BV^2(\Omega))$ if every component of $\mathbf{u}$ belongs to $BV(\Omega)(BV^2(\Omega))$. In this case the $BV(BV^2)$-norm of $\mathbf{u}$ is equal to the sum of the respective norms of its components. We refer to [3] and [15] for more properties of these spaces.

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