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► **To cite this version:**

Laurence Cherfils, Hussein Fakih, Alain Miranville. Finite-dimensional attractors for the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation. 2013. hal-00849773

HAL Id: hal-00849773

<https://hal.science/hal-00849773>

Preprint submitted on 31 Jul 2013

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Finite-dimensional attractors for the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation in image inpainting

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Abstract

In this article, we are interested in the study of the asymptotic behavior, in terms of finite-dimensional attractors, of a generalization of the Cahn-Hilliard equation with a fidelity term (integrated over $\Omega \setminus D$ instead of the entire domain Ω , $D \subset\subset \Omega$). Such a model has, in particular, applications in image inpainting. The difficulty here is that we no longer have the conservation of mass, i.e. of the spatial average of the order parameter u , as in the Cahn-Hilliard equation. Instead, we prove that the spatial average of u is dissipative. We finally give some numerical simulations which confirm previous ones on the efficiency of the model.

Keywords: Cahn–Hilliard equation, fidelity term, image inpainting, global attractor, exponential attractor, simulations

AMS Subject Classification: 35K55; 35B40

1 Introduction

The Cahn–Hilliard equation,

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0, \quad (1.1)$$

is very important in materials science. This equation is a simple model for phase separation processes of a binary alloy at a fixed temperature. We refer the reader to [5, 6] for more details. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is of “bistable type” with three simple zeros and is the derivative of a double-well potential F whose wells correspond to the phases of the material. A typical model nonlinearity is given by

$$F(s) = \frac{1}{4}(s^2 - 1)^2,$$

i.e.

$$f(s) = s^3 - s.$$

The function $u(x, t)$ represents the concentration of one of the metallic components of the alloy.

It is interesting to note that the Cahn–Hilliard equation is also relevant in other phenomena than phase separation. We can mention, for instance, population dynamics [10], bacterial films [20], biology [8, 18, 22], thin films [25, 28], image processing [7], shape recovery in computer vision [11], and even the rings of Saturn [29].

We are interested in this article in the following generalization of the Cahn–Hilliard equation introduced in [1] in view of applications in image inpainting:

$$\frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta f(u) + \lambda_0 1_{\Omega \setminus D}(x)(u - h) = 0, \quad \varepsilon, \lambda_0 > 0, \quad (1.2)$$

where $h(x)$ is a given binary image, $D \subset\subset \Omega$ is the inpainting domain, and the last term on the left-hand side is added to keep the solution constructed close to the given image $h(x)$ in the complement of the inpainting domain $(\Omega \setminus D)$, where there is image information available. The idea here is to solve the equation up to equilibrium to have an inpainted version $u(x)$ of $h(x)$.

Image inpainting involves filling in parts of an image or video using information from the surrounding area. Its applications include restoration of old paintings by museum artists [16], removing scratches from old photographs [3], altering scenes in photographs [19], and restoration of motion pictures [21].

Well-posedness results for (1.2) have been obtained in [1] (see also [4] for the study of the stationary problem).

Equation (1.1) is endowed with Neumann boundary conditions,

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

In particular, this yields the conservation of mass, i.e. of the spatial average of the order parameter u ,

$$\langle u(t) \rangle = \langle u(0) \rangle, \quad \forall t \geq 0,$$

where

$$\langle . \rangle = \frac{1}{|\Omega|} \int_{\Omega} . dx.$$

Then assuming that $|\langle u(0) \rangle|$ is bounded, we can prove the existence of finite-dimensional global attractors (see, e.g., [24] and [27]).

On the contrary, equation (1.2), which is also endowed with Neumann boundary conditions, does not satisfy this conservation property. We prove instead that $\langle u \rangle$ is dissipative, which then allows us to prove the existence of finite-dimensional attractors.

We also give numerical simulations which show that a dynamic one step scheme involving the diffuse interface ε (we note that, in [1] and [2], the authors first consider a larger value of ε and then a smaller one in order to obtain their numerical simulations) allows us to connect regions across large inpainting domains. While the simulations in [1] and [2] are programmed in MATLAB, we use FreeFem++. These simulations confirm the ones performed in [1] and [2] on the efficiency of the model.

2 Setting of the problem

Let Ω be an open bounded domain of \mathbb{R}^n , $n = 1, 2$, or 3 , with a smooth boundary Γ , and D be an open bounded subset of Ω with a smooth boundary ∂D such that $D \subset\subset \Omega$. The unknown function is a scalar $u = u(x, t)$, $x \in \Omega$, $t \in \mathbb{R}$, and the equation reads (for simplicity, we set ε equal to 1)

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \lambda_0 1_{\Omega \setminus D}(x)(u - h) = 0, \quad (2.3)$$

where f is the cubic function

$$f(s) = s^3 - s \quad (2.4)$$

and $h \in L^2(\Omega)$.

We denote by F the antiderivative of f vanishing at $s = 0$,

$$F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2. \quad (2.5)$$

The equation is associated with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \Gamma. \quad (2.6)$$

We finally supplement the equation with the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (2.7)$$

We denote by $\|\cdot\|$ the L^2 -norm (with associated scalar product $((\cdot, \cdot))$) and set

$$V = \left\{ \phi \in H^2(\Omega), \quad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \Gamma \right\}. \quad (2.8)$$

The space V in (2.8) makes sense by the trace theorem. Furthermore, V is a closed subspace of $H^2(\Omega)$ and is equipped with the norm induced by $H^2(\Omega)$ denoted by $\|\cdot\|_2$.

We denote by $\langle \phi \rangle$ the average over Ω of a function ϕ in $L^1(\Omega)$,

$$\langle \phi \rangle = \frac{1}{|\Omega|} \int_{\Omega} \phi(x) dx, \quad (2.9)$$

and we write $\bar{\phi} = \phi - \langle \phi \rangle$.

We set $\dot{L}^2(\Omega) = \{\phi \in L^2(\Omega), \langle \phi \rangle = 0\}$.

For ϕ given in $\dot{L}^2(\Omega)$, we denote by $\Psi = N(\phi)$ the solution to the poisson equation

$$-\Delta \Psi = \phi$$

associated with the Neumann boundary condition

$$\frac{\partial \Psi}{\partial \nu} = 0 \quad \text{on } \Gamma.$$

It is easily seen that $\{((N(\phi), \phi))\}^{1/2}$ is a continuous norm on $\dot{L}^2(\Omega)$; we denote it by $\|\phi\|_{-1}$. Similarly, $((\phi_1, \phi_2))_{-1} = ((N(\phi_1), \phi_2)) = ((\phi_1, N(\phi_2)))$ is a (pre-Hilbertian) continuous scalar product on $\dot{L}^2(\Omega)$.

We note that

$$v \rightarrow (\|v - \langle v \rangle\|_{-1}^2 + \langle v \rangle^2)^{\frac{1}{2}},$$

$$v \rightarrow (\|v - \langle v \rangle\|^2 + \langle v \rangle^2)^{\frac{1}{2}},$$

$$v \rightarrow (\|\nabla v\|^2 + \langle v \rangle^2)^{\frac{1}{2}}$$

and

$$v \rightarrow (\|\Delta v\|^2 + \langle v \rangle^2)^{\frac{1}{2}},$$

are norms in $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual ones.

We finally set $\dot{H}^{-1}(\Omega) = \{\phi \in H^{-1}(\Omega), \langle \phi, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)} = 0\}$.

Throughout this article, the same letter c (and, sometimes, c' and c'') denotes constants which may vary from line to line, or even in a same line. Similarly, the same letter Q denotes monotone increasing functions which may vary from line to line, or even in a same line.

3 A priori estimates

The weak formulation of the problem is obtained by multiplying (2.3) by a test function $v \in V$, integrating over Ω , and using the Green formula and the boundary conditions. We find

$$\begin{aligned} \frac{d}{dt}((u, v)) + ((\Delta u, \Delta v)) + ((f'(u) \nabla u, \nabla v)) + \lambda_0((1_{\Omega \setminus D}(x)(u - h), v)) \\ = 0, \quad \forall v \in V. \end{aligned} \quad (3.10)$$

Now, we replace v by 1 in (3.10) to have

$$\frac{d}{dt} \int_{\Omega} u(x, t) dx = -\lambda_0 \int_{\Omega} 1_{\Omega \setminus D}(x)(u(x, t) - h(x)) dx, \quad (3.11)$$

hence

$$\frac{d}{dt} \langle u \rangle = -\frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u(x, t) - h(x)) dx. \quad (3.12)$$

Owing to (3.12), we can rewrite (2.3) in the form

$$\frac{\partial \bar{u}}{\partial t} + \Delta^2 \bar{u} - \Delta f(u) + \lambda_0 1_{\Omega \setminus D}(x)(u - h) - \frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u - h) dx = 0. \quad (3.13)$$

Recalling that $u = \bar{u} + \langle u \rangle$, we have

$$\frac{\partial \bar{u}}{\partial t} + \Delta^2 \bar{u} - \Delta f(\bar{u} + \langle u \rangle) + \lambda_0 1_{\Omega \setminus D}(x)(u - h) - \frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u - h) dx = 0, \quad (3.14)$$

which is equivalent to

$$\begin{aligned} \frac{\partial}{\partial t} N(\bar{u}) - \Delta \bar{u} + f(\bar{u} + \langle u \rangle) - \langle f(\bar{u} + \langle u \rangle) \rangle + N\left(\lambda_0 1_{\Omega \setminus D}(x)(u - h) \right. \\ \left. - \frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u - h) dx\right) = 0. \end{aligned} \quad (3.15)$$

We take the scalar product of this equation by \bar{u} in $L^2(\Omega)$ and obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{-1}^2 - ((\Delta \bar{u}, \bar{u})) + ((f(\bar{u} + \langle u \rangle) - f(\langle u \rangle), \bar{u})) \\ + ((\lambda_0 1_{\Omega \setminus D}(x)(u - h) - \frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u - h) dx, N(\bar{u}))) = 0. \end{aligned} \quad (3.16)$$

We have

$$\begin{aligned} ((f(\bar{u} + \langle u \rangle) - f(\langle u \rangle), \bar{u})) \\ = \int_{\Omega} (\bar{u}^4 + 3\bar{u}^3 \langle u \rangle + 3\bar{u}^2 \langle u \rangle^2) dx - \|\bar{u}\|^2 \\ \geq \int_{\Omega} (\bar{u}^4 + 3\bar{u}^2 \langle u \rangle^2) dx - 3 \int_{\Omega} |\bar{u}|^3 |\langle u \rangle| dx - \|\bar{u}\|^2 \\ \geq c_0 \int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx - \|\bar{u}\|^2, \quad c_0 > 0, \end{aligned} \quad (3.17)$$

owing to Young's inequality (e.g., $3ab \leq \frac{7}{8}a^2 + \frac{18}{7}b^2$, $a, b \geq 0$; here, $a = \bar{u}^2$ and $b = |\bar{u} \langle u \rangle|$), and

$$\begin{aligned} |((\lambda_0 1_{\Omega \setminus D}(x)(u(x, t) - h(x)), N(\bar{u})))| &\leq c \|u - h\| \|\bar{u}\| \\ &\leq c(\|\bar{u}\|^2 + |\langle u \rangle| \|\bar{u}\|) + c' \|h\|^2 \\ &\leq \frac{c_0}{4} \int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx + c(\|h\|^2 + 1). \end{aligned} \quad (3.18)$$

It follows from (3.16), (3.17) and (3.18) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{-1}^2 + \|\nabla \bar{u}\|^2 + c_0 \int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx &\leq \frac{c_0}{4} \int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx \\ &+ \|\bar{u}\|^2 + c(\|h\|^2 + 1), \end{aligned}$$

hence

$$\frac{d}{dt} \|\bar{u}\|_{-1}^2 + \|\nabla \bar{u}\|^2 + c_0 \int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx \leq c. \quad (3.19)$$

It thus follows from (3.19) that

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{-1}^2 + c \|\bar{u}\|_{-1}^2 \leq c', \quad c > 0. \quad (3.20)$$

By Gronwall's lemma, we find

$$\|\bar{u}(t)\|_{-1}^2 \leq e^{-ct} \|\bar{u}_0\|_{-1}^2 + c', \quad c > 0, \quad t \geq 0. \quad (3.21)$$

Let B be a bounded subset of $\dot{H}^{-1}(\Omega)$ and t_0 be such that $\bar{u}_0 \in B$ and $t \geq t_0$ implies $\bar{u}(t) \in \mathcal{B}_0$, where $\mathcal{B}_0 = \{\phi \in \dot{H}^{-1}(\Omega), \|\phi\|_{-1}^2 \leq 2c'\}$, c' being the constant in (3.21). We then deduce from (3.19) that, for $t \geq t_0$,

$$\int_t^{t+r} \|\nabla \bar{u}\|^2 ds \leq c(r), \quad \int_t^{t+r} ds \int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx \leq c(r), \quad (3.22)$$

for $r > 0$ fixed.

We then multiply (3.14) by \bar{u} and find, noting that

$$f' \geq -c_1, \quad c_1 > 0, \quad (3.23)$$

the inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{u}\|^2 + \|\Delta \bar{u}\|^2 + ((\lambda_0 1_{\Omega \setminus D}(x)(u-h) - \frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u-h) dx, \bar{u})) \\ \leq c_1 \|\nabla \bar{u}\|^2. \end{aligned} \quad (3.24)$$

We have

$$\begin{aligned} |((\lambda_0 1_{\Omega \setminus D}(x)(u(x,t) - h(x)) - \frac{\lambda_0}{|\Omega|} \int_{\Omega} 1_{\Omega \setminus D}(x)(u(x,t) - h(x)) dx, \bar{u}))| \\ = \lambda_0 |((1_{\Omega \setminus D}(u-h), \bar{u}))| \\ \leq c \|u-h\| \|\bar{u}\| \\ \leq c(\|\bar{u}\|^2 + |\langle u \rangle| \|\bar{u}\|) + c' \|h\|^2 \\ \leq c \left(\int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx + \|h\|^2 + 1 \right). \end{aligned} \quad (3.25)$$

Therefore, we obtain

$$\frac{d}{dt} \|\bar{u}\|^2 + \|\Delta \bar{u}\|^2 \leq c \|\nabla \bar{u}\|^2 + c' \left(\int_{\Omega} (\bar{u}^4 + \bar{u}^2 \langle u \rangle^2) dx + \|h\|^2 + 1 \right). \quad (3.26)$$

We finally deduce from (3.22), (3.26) and the uniform Gronwall's lemma that

$$\|\bar{u}(t)\|^2 \leq c, \quad t \geq t_0 + r, \quad (3.27)$$

where the constant c is independent of \bar{u}_0 and t , hence

$$\|\bar{u}(t)\|^2 \leq Q(\|\bar{u}_0\|), \quad t \geq 0, \quad (3.28)$$

for some monotone increasing function Q .

Now, setting $u = \langle u \rangle + \bar{u}$ in (3.11), we have

$$\frac{d}{dt} \langle u \rangle + \frac{\lambda_0}{|\Omega|} \int_{\Omega \setminus D} (\langle u \rangle + \bar{u} - h) dx = 0.$$

Therefore,

$$\frac{d}{dt} \langle u \rangle + c \langle u \rangle = -\frac{\lambda_0}{|\Omega|} \int_{\Omega \setminus D} (\bar{u} - h) dx,$$

where $c = \frac{\lambda_0 |\Omega \setminus D|}{|\Omega|}$, hence

$$\frac{d}{dt} (e^{ct} \langle u \rangle) = -\frac{\lambda_0}{|\Omega|} e^{ct} \int_{\Omega \setminus D} (\bar{u} - h) dx$$

and

$$\langle u \rangle = e^{-ct} \langle u_0 \rangle - \frac{\lambda_0}{|\Omega|} e^{-ct} \int_0^t e^{cs} \int_{\Omega \setminus D} (\bar{u} - h) dx ds.$$

Thus,

$$|\langle u \rangle| \leq e^{-ct} |\langle u_0 \rangle| + c' e^{-ct} \int_0^t e^{cs} (\|\bar{u}\| + \|h\|) ds, \quad \forall t \geq 0,$$

where $c' = \frac{\lambda_0}{|\Omega|^{\frac{1}{2}}}$. Here,

$$\begin{aligned} c' e^{-ct} \int_0^t e^{cs} \|h\| ds &\leq c' e^{-ct} \|h\| e^{ct} \\ &\leq c' \|h\| \\ &\leq c''. \end{aligned}$$

Furthermore, for $t \geq t_0 + r$,

$$\begin{aligned} c' e^{-ct} \int_0^t e^{cs} \|\bar{u}\| ds &= c' e^{-ct} \int_0^{t_0+r} e^{cs} \|\bar{u}\| ds + c' e^{-ct} \int_{t_0+r}^t e^{cs} \|\bar{u}\| ds \\ &\leq (\text{use (3.28)}) \\ &\leq Q(\|\bar{u}_0\|) e^{-ct} e^{c(t_0+r)} + c' e^{-ct} \int_{t_0+r}^t e^{cs} \|\bar{u}\| ds \\ &\leq Q(\|\bar{u}_0\|) e^{-ct} + c' e^{-ct} \int_{t_0+r}^t e^{cs} \|\bar{u}\| ds \\ &\leq (\text{use (3.27)}) \\ &\leq Q(\|\bar{u}_0\|) e^{-ct} + c'' e^{-ct} (e^{ct} - e^{c(t_0+r)}) \\ &\leq Q(\|\bar{u}_0\|) e^{-ct} + c''. \end{aligned}$$

Finally, we obtain

$$|\langle u \rangle| \leq (Q(\|\bar{u}_0\|) + |\langle u_0 \rangle|)e^{-ct} + c'', \quad \forall t \geq 0, \quad (3.29)$$

where c and c'' are two constants which are nonnegative and independent of t and u_0 .

4 Further a priori estimates

We multiply (2.3) by $\Delta^2 u$ and have

$$\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \|\Delta^2 u\|^2 - ((\Delta f(u), \Delta^2 u)) + ((\lambda_0 1_{\Omega \setminus D}(x)(u-h), \Delta^2 u)) = 0. \quad (4.30)$$

Noting that

$$\begin{aligned} |((\lambda_0 1_{\Omega \setminus D}(x)(u-h), \Delta^2 u))| &\leq c \|u-h\| \|\Delta^2 u\| \\ &\leq c' \|u-h\|^2 + \frac{1}{4} \|\Delta^2 u\|^2 \\ &\leq c' \|u\|^2 + c' \|h\|^2 + \frac{1}{4} \|\Delta^2 u\|^2, \end{aligned} \quad (4.31)$$

we find

$$\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \|\Delta^2 u\|^2 \leq ((\Delta f(u), \Delta^2 u)) + c' \|u\|^2 + c' \|h\|^2 + \frac{1}{4} \|\Delta^2 u\|^2, \quad (4.32)$$

where

$$\begin{aligned} ((\Delta f(u), \Delta^2 u)) &\leq \|\Delta f(u)\| \|\Delta^2 u\| \\ &\leq c \|\Delta f(u)\|^2 + \frac{1}{8} \|\Delta^2 u\|^2. \end{aligned} \quad (4.33)$$

We further have (see, e.g., [27])

$$\|\Delta f(u)\|^2 \leq \frac{1}{8} \|\Delta^2 u\|^2 + c'',$$

hence, owing to (4.32) and (4.33),

$$\frac{d}{dt} \|\Delta u\|^2 + \|\Delta^2 u\|^2 \leq c \|u\|^2 + c' \quad \forall t \geq 0. \quad (4.34)$$

Noting that

$$\begin{aligned} \frac{d}{dt} |\langle u \rangle|^2 &\leq 2 |\langle u \rangle| \left| \frac{d}{dt} \langle u \rangle \right| \\ &\leq c |\langle u \rangle| \left| \int_{\Omega \setminus D} (u-h) dx \right| \\ &\leq c' (|\langle u \rangle|^2 + \|u\|^2 + 1), \end{aligned}$$

owing to (3.12), we find

$$\frac{d}{dt} (\|\Delta u\|^2 + |\langle u \rangle|^2) + \|\Delta^2 u\|^2 \leq c (\|u\|^2 + |\langle u \rangle|^2) + c' \quad \forall t \geq 0, \quad (4.35)$$

and, owing to (3.27) and (3.29), $\exists t'_0$ such that, $\forall t \geq t'_0$, we have

$$|\langle u(t) \rangle| \leq c \quad \text{and} \quad \|\bar{u}(t)\|^2 \leq c, \quad (4.36)$$

where the constant c is independent of u_0 and t , hence

$$\frac{d}{dt}(\|\Delta u\|^2 + |\langle u \rangle|^2) + \|\Delta^2 u\|^2 \leq c \quad \forall t \geq t'_0. \quad (4.37)$$

We note that, integrating (3.26) over $(t, t+r)$, for $0 < r < 1$ fixed and owing to (3.22), we have, for $t \geq t'_0$,

$$\int_t^{t+r} \|\Delta u\|^2 ds \leq c(r).$$

Finally, using the uniform Gronwall's lemma in (4.37), we deduce that

$$\|u(t)\|_{H^2(\Omega)}^2 \leq c \quad \forall t \geq t'_0 + r, \quad (4.38)$$

where c is independent of u_0 and t , for $0 < r < 1$ fixed.

Multiplying (3.14) by $\frac{\partial \bar{u}}{\partial t}$, we have, for $t \geq t'_0 + 1$,

$$\left\| \frac{\partial \bar{u}}{\partial t} \right\|^2 + \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 \leq \left| \left(\Delta f(u), \frac{\partial \bar{u}}{\partial t} \right) \right| + \left| \lambda_0 \left(\left(\overline{1_{\Omega \setminus D}(x)(u-h)}, \frac{\partial \bar{u}}{\partial t} \right) \right) \right|. \quad (4.39)$$

Here,

$$\begin{aligned} \left| \lambda_0 \left(\left(\overline{1_{\Omega \setminus D}(x)(u-h)}, \frac{\partial \bar{u}}{\partial t} \right) \right) \right| &= \left| \lambda_0 \left(\left(1_{\Omega \setminus D}(x)(u-h), \frac{\partial \bar{u}}{\partial t} \right) \right) \right| \\ &\leq c \|u-h\| \left\| \frac{\partial \bar{u}}{\partial t} \right\| \\ &\leq c(\|u\|^2 + \|h\|^2) + \frac{1}{4} \left\| \frac{\partial \bar{u}}{\partial t} \right\|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| \left(\Delta f(u), \frac{\partial \bar{u}}{\partial t} \right) \right| &\leq \|\Delta f(u)\| \left\| \frac{\partial \bar{u}}{\partial t} \right\| \\ &\leq c \|\Delta f(u)\|^2 + \frac{1}{4} \left\| \frac{\partial \bar{u}}{\partial t} \right\|^2, \end{aligned}$$

and, owing to (4.38), we note that (see, e.g., [26])

$$\|\Delta f(u)\|^2 \leq c \|u\|_{H^2(\Omega)}^2.$$

Thus, owing to (4.38), (4.39) and by the above estimate, we have

$$\frac{d}{dt} \|u\|_{H^2(\Omega)}^2 + \left\| \frac{\partial \bar{u}}{\partial t} \right\|^2 \leq c \|u\|_{H^2(\Omega)}^2 + c', \quad \forall t \geq t'_0 + 1, \quad (4.40)$$

and deduce that

$$\int_t^{t+r} \left\| \frac{\partial \bar{u}}{\partial t} \right\|^2 d\tau \leq c, \quad \forall t \geq t'_0 + 1. \quad (4.41)$$

Differentiating (3.14) with respect to time, we rewrite the resulting equation as, setting $\theta = \frac{\partial \bar{u}}{\partial t}$,

$$\frac{\partial \theta}{\partial t} + \Delta^2 \theta - \Delta \left(f'(u) \frac{\partial u}{\partial t} \right) + \overline{\lambda_0 1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t}} = 0. \quad (4.42)$$

We multiply (4.42) by θ and have, for $t \geq t'_0 + 1$,

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \|\Delta \theta\|^2 \leq \left| \left(\left(f'(u) \frac{\partial u}{\partial t}, \Delta \theta \right) \right) \right| + \left| \lambda_0 \left(\overline{1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t}}, \theta \right) \right|. \quad (4.43)$$

Here,

$$\begin{aligned} \left| \lambda_0 \left(\overline{1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t}}, \theta \right) \right| &= \left| \lambda_0 \left(1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t}, \theta \right) \right| \\ &\leq c \left\| \frac{\partial u}{\partial t} \right\| \|\theta\| \\ &\leq c(\|\theta\| + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right) \|\theta\| \\ &\leq c(\|\theta\|^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2). \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| \left(\left(f'(u) \frac{\partial u}{\partial t}, \Delta \theta \right) \right) \right| &\leq \left\| f'(u) \frac{\partial u}{\partial t} \right\| \|\Delta \theta\| \\ &\leq (\text{thanks to (4.38) and the continuous} \\ &\quad \text{embedding } H^2(\Omega) \subset C(\bar{\Omega})) \\ &\leq c \left\| \frac{\partial u}{\partial t} \right\| \|\Delta \theta\| \\ &\leq c(\|\theta\|^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2) + \frac{1}{4} \|\Delta \theta\|^2, \end{aligned}$$

which yields

$$\frac{d}{dt} \|\theta\|^2 + \|\Delta \theta\|^2 \leq c(\|\theta\|^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2), \quad \forall t \geq t'_0 + 1. \quad (4.44)$$

Noting that

$$\begin{aligned} \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2 &= \left| \frac{d}{dt} \langle u, \theta \rangle \right|^2 \\ &\leq (\text{use (3.12)}) \\ &\leq c \|u - h\|^2 \\ &\leq c(\|u\|^2 + \|h\|^2) \\ &\leq c(\|u\|_{H^1(\Omega)}^2 + \|h\|^2) \\ &\leq (\text{thanks to (4.38)}) \\ &\leq c, \end{aligned} \quad (4.45)$$

we have, owing to (4.44),

$$\frac{d}{dt} \|\theta\|^2 + \|\Delta \theta\|^2 \leq c \|\theta\|^2 + c'. \quad (4.46)$$

Thus, by (4.41), (4.46), and the uniform Gronwall's lemma,

$$\left\| \frac{\partial \bar{u}}{\partial t} \right\|^2 \leq c, \quad \forall t \geq t'_0 + 1 + r, \quad 0 < r < 1. \quad (4.47)$$

We now rewrite (2.3) in the form

$$\Delta^2 u = h_u, \quad \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad (4.48)$$

where

$$h_u = -\frac{\partial u}{\partial t} + \Delta f(u) - \lambda_0 1_{\Omega \setminus D}(x)(u - h) \quad (4.49)$$

satisfies, for $t \geq t'_0 + 2$,

$$\|h_u\| \leq c. \quad (4.50)$$

It then follows from (4.50) that

$$\|u\|_{H^4(\Omega)}^2 \leq c, \quad \forall t \geq t'_0 + 2. \quad (4.51)$$

We note that, integrating (4.46) over $(t, t+r)$, for $0 < r < 1$ fixed, we have, for $t \geq t'_0 + 2$,

$$\int_t^{t+r} \|\Delta \theta\|^2 ds \leq c(r). \quad (4.52)$$

Differentiating (2.3) with respect to time, we rewrite the resulting equation as, setting $\Psi = \frac{\partial u}{\partial t}$,

$$\frac{\partial \Psi}{\partial t} + \Delta^2 \Psi - \Delta(f'(u)\Psi) + \lambda_0 1_{\Omega \setminus D}(x)\Psi = 0. \quad (4.53)$$

We multiply (4.53) by $\Delta \Psi$ and have, for $t \geq t'_0 + 2$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \Psi\|^2 + \|\nabla \Delta \Psi\|^2 &\leq \left| ((\Delta(f'(u)\Psi), \Delta \Psi)) \right| \\ &+ \left| \lambda_0 ((1_{\Omega \setminus D}(x)\Psi, \Delta \Psi)) \right|. \end{aligned} \quad (4.54)$$

Here,

$$\begin{aligned} \left| \lambda_0 ((1_{\Omega \setminus D}(x)\Psi, \Delta \Psi)) \right| &\leq c \|\Psi\| \|\Delta \Psi\| \\ &\leq c \|\Psi\|^2 + \frac{1}{2} \|\Delta \Psi\|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| ((\Delta(f'(u)\Psi), \Delta \Psi)) \right| &= \left| ((\nabla(f'(u)\Psi), \nabla \Delta \Psi)) \right| \\ &\leq \|\nabla(f'(u)\Psi)\| \|\nabla \Delta \Psi\| \\ &\leq c \|\nabla(f'(u)\Psi)\|^2 + \frac{1}{2} \|\nabla \Delta \Psi\|^2, \end{aligned}$$

and

$$\begin{aligned} \|\nabla(f'(u)\Psi)\|^2 &= \|f''(u)\Psi \nabla u + f'(u)\nabla \Psi\|^2 \\ &\leq (\text{thanks to (4.51)}) \\ &\leq c \|\Psi\|^2 + c' \|\nabla \Psi\|^2, \end{aligned}$$

which yields

$$\frac{d}{dt} \|\nabla \Psi\|^2 + \|\nabla \Delta \Psi\|^2 \leq c(\|\Psi\|^2 + \|\nabla \Psi\|^2) + \|\Delta \Psi\|^2 + c''. \quad (4.55)$$

Thus, by (4.45), (4.52), (4.55) and the uniform Gronwall's lemma,

$$\left\| \nabla \frac{\partial u}{\partial t} \right\|^2 \leq c, \quad \forall t \geq t'_0 + 2 + r, \quad 0 < r < 1. \quad (4.56)$$

5 Existence of the global attractor

We first have the

Proposition 5.1. *For every $u_0 \in L^2(\Omega)$ and every $T > 0$, the initial-boundary value problem (3.10) has a unique solution u which belongs to $C([0, T], L^2(\Omega)) \cap L^2(0, T, V) \cap L^4(0, T, L^4(\Omega))$.*

Proof. See [1], [27], and [24]. \square

Proposition 5.2. *We have the continuous (with respect to the H^{-1} -norm) semigroup $S(t)$ defined as*

$$S(t) : L^2(\Omega) \rightarrow L^2(\Omega), \quad u_0 \rightarrow u(t), \quad t \geq 0.$$

Proof. Let u_1 and u_2 be two solutions to (2.3)–(2.6) with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$ and have

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) + \lambda_0 1_{\Omega \setminus D}(x)u = 0, \quad (5.57)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad (5.58)$$

$$u|_{t=0} = u_0. \quad (5.59)$$

Integrating (5.57) over Ω , we have

$$\frac{d}{dt} \langle u \rangle = -\lambda_0 \int_{\Omega \setminus D} u dx,$$

which yields

$$\frac{\partial \bar{u}}{\partial t} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) + \lambda_0 \overline{1_{\Omega \setminus D}(x)u} = 0. \quad (5.60)$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial t} N(\bar{u}) - \Delta u + f(u_1) - f(u_2) - \langle f(u_1) - f(u_2) \rangle \\ + \lambda_0 N(\overline{1_{\Omega \setminus D}(x)u}) = 0. \end{aligned} \quad (5.61)$$

We multiply (5.61) by \bar{u} and have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{-1}^2 + \|\nabla u\|^2 + ((f(u_1) - f(u_2)), u) - ((f(u_1) - f(u_2)), \langle u \rangle) \\ + \lambda_0 ((N(\overline{1_{\Omega \setminus D}(x)u}), \bar{u})) = 0. \end{aligned}$$

Here,

$$\begin{aligned}
|\lambda_0((N(\overline{1_{\Omega \setminus D}(x)u}), \bar{u}))| &= \lambda_0|(1_{\Omega \setminus D}(x)u, N(\bar{u}))| \\
&\leq c\|u\|\|\bar{u}\| \\
&\leq c(\|\bar{u}\|^2 + |\langle u \rangle| \|\bar{u}\|) \\
&\leq c\|\bar{u}\|^2 + c'|\langle u \rangle|^2.
\end{aligned}$$

Furthermore,

$$((f(u_1) - f(u_2), u)) \geq -c_1\|u\|^2$$

and

$$\begin{aligned}
|((f(u_1) - f(u_2), \langle u \rangle))| &= |\langle u \rangle \int_{\Omega} u \int_0^1 f'(u_1 + s(u_2 - u_1)) ds dx| \\
&\leq c|\langle u \rangle| \int_{\Omega} (|u_1|^2 + |u_2|^2 + 1)|u| dx \\
&\leq c|\langle u \rangle| (\|u_1\|_{L^4(\Omega)}^2 + \|u_2\|_{L^4(\Omega)}^2 + 1)\|u\| \\
&\leq c(\|u_1\|_{L^4(\Omega)}^2 + \|u_2\|_{L^4(\Omega)}^2 + 1)(|\langle u \rangle|^2 + \|\bar{u}\|^2),
\end{aligned}$$

which yields

$$\frac{d}{dt}\|\bar{u}\|_{-1}^2 + \|\nabla u\|^2 \leq c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 + 1)(\|\bar{u}\|_{-1}^2 + |\langle u \rangle|^2), \quad (5.62)$$

owing to the interpolation inequality $\|\bar{u}\|^2 \leq c\|\bar{u}\|_{-1}\|\nabla u\|$. Noting then that

$$\begin{aligned}
\frac{d}{dt}|\langle u \rangle|^2 &\leq 2|\langle u \rangle| \left| \frac{d}{dt} \langle u \rangle \right| \\
&\leq c|\langle u \rangle| \left\| \int_{\Omega \setminus D} u dx \right\| \\
&\leq c|\langle u \rangle| \|u\| \\
&\leq c(|\langle u \rangle|^2 + \|\bar{u}\|^2),
\end{aligned} \quad (5.63)$$

it follows that

$$\begin{aligned}
\frac{d}{dt}(\|\bar{u}\|_{-1}^2 + |\langle u \rangle|^2) + \frac{1}{2}\|\nabla u\|^2 &\leq c(\|u_1\|_{L^4(\Omega)}^4 + \|u_2\|_{L^4(\Omega)}^4 \\
&\quad + 1)(\|\bar{u}\|_{-1}^2 + |\langle u \rangle|^2).
\end{aligned} \quad (5.64)$$

We deduce from (5.64), Proposition 5.1, and Gronwall's lemma that

$$\begin{aligned}
\|u_1(t) - u_2(t)\|_{H^{-1}(\Omega)} &\leq Q(T, \|u_{0,1}\|, \|u_{0,2}\|)\|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)} \\
&\quad 0 \leq t \leq T.
\end{aligned} \quad (5.65)$$

□

It follows from (4.51) that $S(t)$ possesses a bounded absorbing set \mathcal{B}'_0 which is compact in $L^2(\Omega)$ and bounded in $H^4(\Omega)$. We thus deduce from standard results (see, e.g., [23, 27]) the following theorem.

Theorem 5.3. *The semigroup $S(t)$ possesses the connected global attractor \mathcal{A} such that \mathcal{A} is compact in $L^2(\Omega)$ and bounded in $H^4(\Omega)$.*

Remark 5.4. *It is easy to see that we can assume, without loss of generality, that \mathcal{B}'_0 is positively invariant by $S(t)$, i.e. $S(t)\mathcal{B}'_0 \subset \mathcal{B}'_0$, $\forall t \geq 0$.*

6 Existence of exponential attractors

Let u_1 and u_2 be two solutions of (2.3)–(2.6) with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We again set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$ and have

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u_1) - f(u_2)) + \lambda_0 1_{\Omega \setminus D}(x)u = 0, \quad (6.66)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad (6.67)$$

$$u|_{t=0} = u_0. \quad (6.68)$$

Furthermore, it is sufficient here to take initial data belonging to the bounded absorbing set \mathcal{B}'_0 defined in the previous section.

We rewrite (6.66) as

$$\begin{aligned} \frac{\partial}{\partial t} N(\bar{u}) - \Delta u + f(u_1) - f(u_2) - \langle f(u_1) - f(u_2), \bar{u} \rangle \\ + \lambda_0 N(\overline{1_{\Omega \setminus D}(x)u}) = 0. \end{aligned} \quad (6.69)$$

Multiplying (6.69) by $t \frac{\partial \bar{u}}{\partial t}$, we have

$$\begin{aligned} t \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 + \frac{t}{2} \frac{d}{dt} \|\nabla u\|^2 + t \left(\left(f(u_1) - f(u_2), \frac{\partial \bar{u}}{\partial t} \right) \right) \\ + \lambda_0 t \left(\left(N(\overline{1_{\Omega \setminus D}(x)u}), \frac{\partial \bar{u}}{\partial t} \right) \right) = 0. \end{aligned} \quad (6.70)$$

Here,

$$\begin{aligned} \left| \lambda_0 \left(\left(N(\overline{1_{\Omega \setminus D}(x)u}), \frac{\partial \bar{u}}{\partial t} \right) \right) \right| &= \left| \lambda_0 \left(\left(1_{\Omega \setminus D}(x)u, N\left(\frac{\partial \bar{u}}{\partial t}\right) \right) \right) \right| \\ &\leq c \|u\| \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}, \end{aligned}$$

where the constant c depends only on \mathcal{B}'_0 . Furthermore,

$$\begin{aligned} \left| \left(\left(f(u_1) - f(u_2), \frac{\partial \bar{u}}{\partial t} \right) \right) \right| &\leq c \|\nabla(f(u_1) - f(u_2))\| \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1} \\ &\leq c \left\| \nabla \left(\int_0^1 f'(u_1 + s(u_2 - u_1)) ds u \right) \right\| \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1} \\ &\leq c \left\| \int_0^1 f'(u_1 + s(u_2 - u_1)) ds \nabla u \right\| \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1} \\ &\quad + c \left\| u \int_0^1 f''(u_1 + s(u_2 - u_1)) (\nabla u_1 + s(\nabla u_2 - \nabla u_1)) ds \right\| \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1} \\ &\leq c (\|\nabla u\| + \|u \nabla u_1\| + \|u \nabla u_2\|) \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1} \\ &\leq c \|u\|_{H^1(\Omega)} \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}, \end{aligned}$$

where the constant c depends only on \mathcal{B}'_0 , which yields

$$t \frac{d}{dt} \|\nabla u\|^2 + ct \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 \leq c't \|u\|_{H^1(\Omega)}^2, \quad (6.71)$$

and, owing to (5.63), we find

$$\begin{aligned} \frac{d}{dt} (t \|\nabla u\|^2 + t | \langle u(t) \rangle |^2) + ct \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 &\leq c't (\|u\|_{H^1(\Omega)}^2 \\ &+ | \langle u(t) \rangle |^2) + c'' (\|\nabla u\|^2 + | \langle u(t) \rangle |^2), \end{aligned}$$

hence

$$\begin{aligned} \frac{d}{dt} (t \|u\|_{H^1(\Omega)}^2) + ct \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 &\leq c't \|u\|_{H^1(\Omega)}^2 \\ &+ c'' \|u\|_{H^1(\Omega)}^2. \end{aligned} \quad (6.72)$$

We note that, integrating (5.64) over $(0, t)$, we have

$$\int_0^t \|\nabla u\|^2 ds \leq ce^{\alpha t} \|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)}^2, \quad (6.73)$$

where the constant α depends only on \mathcal{B}'_0 , hence

$$\int_0^t \|u\|_{H^1(\Omega)}^2 ds \leq ce^{\alpha t} \|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)}^2. \quad (6.74)$$

By (6.72), (6.74), and Gronwall's lemma, we obtain

$$\|u_1 - u_2\|_{H^1(\Omega)}^2 \leq \frac{c}{t} e^{\alpha t} \|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)}^2 \quad \forall t > 0. \quad (6.75)$$

Now multiplying (6.69) by $\frac{\partial \bar{u}}{\partial t}$, we obtain, proceeding as above,

$$\frac{d}{dt} (\|\nabla u\|^2 + | \langle u \rangle |^2) + \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 \leq c (\|\nabla u\|^2 + | \langle u \rangle |^2), \quad (6.76)$$

where the constant c depends only on \mathcal{B}'_0 . Therefore, integrating (6.76) over $(1, t)$ and owing to (6.75) (for $t = 1$), we have

$$\int_1^t \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{H^{-1}(\Omega)}^2 d\tau \leq ce^{\alpha t} \|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)}^2, \quad (6.77)$$

where the constant c depends only on \mathcal{B}'_0 .

We note that, integrating (6.66) over $(0, t)$, we easily obtain

$$\int_0^t \left\langle \frac{\partial u}{\partial t} \right\rangle^2 d\tau \leq ce^{\alpha t} \|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)}^2, \quad (6.78)$$

where the constant c depends only on \mathcal{B}'_0 .

Differentiating (6.69) with respect to time, we find

$$\begin{aligned} \frac{\partial}{\partial t} N(\theta) - \Delta \theta + l(t) \frac{\partial u}{\partial t} + l'(t) u - \left\langle \frac{\partial}{\partial t} l(t) u \right\rangle \\ + \lambda_0 N \left(\overline{1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t}} \right) = 0, \end{aligned} \quad (6.79)$$

where $l(t) = \int_0^1 f'(u_1 + s(u_2 - u_1))ds$ and $\theta = \frac{\partial \bar{u}}{\partial t}$.

We multiply (6.79) by $(t-1)\theta$ and have

$$\begin{aligned} \frac{t-1}{2} \frac{d}{dt} \|\theta\|_{-1}^2 + (t-1) \|\nabla \theta\|^2 + (t-1) \left(l(t) \frac{\partial u}{\partial t}, \theta \right) + (t-1) \left(l'(t)u, \theta \right) \\ + \lambda_0(t-1) \left(\overline{N \left(1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t} \right)}, \theta \right) = 0. \end{aligned} \quad (6.80)$$

Here,

$$\begin{aligned} \left| \lambda_0 \left(\overline{N \left(1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t} \right)}, \theta \right) \right| &= \left| \lambda_0 \left(1_{\Omega \setminus D}(x) \frac{\partial u}{\partial t}, N(\theta) \right) \right| \\ &\leq c \left\| \frac{\partial u}{\partial t} \right\| \|\theta\|_{-1} \\ &\leq c \left(\|\theta\|^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2 \right) + c' \|\theta\|_{-1}^2 \\ &\leq c \left(\|\theta\|_{-1}^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2 \right) + c' \|\nabla \theta\|^2, \end{aligned}$$

owing to the above estimates and a proper interpolation inequality. Furthermore,

$$\begin{aligned} \left\| \left(l(t) \frac{\partial u}{\partial t}, \theta \right) \right\| &\leq c \left\| \nabla \left(l(t) \frac{\partial u}{\partial t} \right) \right\| \|\theta\|_{-1} \\ &\leq c (\|l\|_{L^\infty(\Omega)} + \|\nabla l\|_{L^4(\Omega)}) \left\| \frac{\partial u}{\partial t} \right\|_{H^1(\Omega)} \|\theta\|_{-1} \\ &\leq (\text{thanks to (4.51)}) \\ &\leq c \left(\|\nabla \theta\| + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right| \right) \|\theta\|_{-1} \end{aligned}$$

and

$$\begin{aligned} \|l'(t)u\| &= \left\| \int_0^1 f''(u_1 - s(u_2 - u_1)) \left(\frac{\partial u_1}{\partial t} + s \left(\frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial t} \right) \right) ds u \right\| \\ &\leq c \left(\left\| \frac{\partial u_1}{\partial t} \right\|_{L^4(\Omega)} + \left\| \frac{\partial u_2}{\partial t} \right\|_{L^4(\Omega)} \right) \|u\|_{L^4(\Omega)} \\ &\leq c \left(\left\| \frac{\partial u_1}{\partial t} \right\|_{H^1(\Omega)} + \left\| \frac{\partial u_2}{\partial t} \right\|_{H^1(\Omega)} \right) \|u\|_{H^1(\Omega)} \\ &\leq (\text{thanks to (4.47) and (4.56)}) \\ &\leq c \|u\|_{H^1(\Omega)}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{d}{dt} \left((t-1) \left(\|\theta\|_{-1}^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2 \right) \right) + (t-1) \|\theta\|_{H^1(\Omega)}^2 &\leq c(t-1) \left(\|\theta\|_{-1}^2 + \right. \\ &\left. \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2 \right) + \|\theta\|_{-1}^2 + \left| \left\langle \frac{\partial u}{\partial t}, \theta \right\rangle \right|^2 + c'(t-1) \|u\|_{H^1(\Omega)}^2. \end{aligned} \quad (6.81)$$

We thus deduce from (6.74), (6.75), (6.77), (6.78), and Gronwall's lemma that

$$\|\theta(t)\|_{H^{-1}(\Omega)}^2 \leq \frac{c}{t-1} e^{\alpha t} \|u_0\|_{H^{-1}(\Omega)}^2, \quad \forall t > 1. \quad (6.82)$$

We finally rewrite (6.69) in the form

$$-\Delta u = \tilde{h}_u, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad (6.83)$$

where

$$\begin{aligned} \tilde{h}_u = & -N\left(\frac{\partial \bar{u}}{\partial t}\right) - (f(u_1) - f(u_2)) + \langle f(u_1) - f(u_2) \rangle \\ & - \lambda_0 N\left(\overline{1_{\Omega \setminus D}(x)u}\right) \end{aligned} \quad (6.84)$$

satisfies

$$\|\tilde{h}_u\| \leq c\left(\left\|\frac{\partial \bar{u}}{\partial t}\right\|_{-1} + \|u\|_{H^1(\Omega)}\right), \quad (6.85)$$

where the constant c depends only on \mathcal{B}'_0 . It then follows from (6.75), (6.82), (6.85), and standard elliptic regularity results that

$$\|u_1(t) - u_2(t)\|_{H^2(\Omega)} \leq \frac{c}{\sqrt{t-1}} e^{c't} \|u_{0,1} - u_{0,2}\|_{H^{-1}(\Omega)}, \quad c, c' \geq 0, \quad t > 1, \quad (6.86)$$

where the constant c depends only on \mathcal{B}'_0 .

Next, we derive a Hölder (both with respect to space and time) estimate. Actually, owing to (5.65), it suffices to prove the Hölder continuity with respect to time. We have

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{H^{-1}(\Omega)} &= \left\| \int_{t_1}^{t_2} \frac{\partial u}{\partial t} d\tau \right\|_{H^{-1}(\Omega)} \leq \left| \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{H^{-1}(\Omega)} d\tau \right| \\ &\leq |t_1 - t_2|^{\frac{1}{2}} \left| \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{H^{-1}(\Omega)}^2 d\tau \right|^{\frac{1}{2}}, \end{aligned} \quad (6.87)$$

where u is solution of (2.3)–(2.6)–(2.7).

We note that, owing to (4.40),

$$\left| \int_{t_1}^{t_2} \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{H^{-1}(\Omega)}^2 d\tau \right| \leq c, \quad (6.88)$$

where the constant c depends only on \mathcal{B}'_0 and T such that $t_1, t_2 \in [0, T]$, so that

$$\|u(t_1) - u(t_2)\|_{H^{-1}(\Omega)} \leq c|t_1 - t_2|^{\frac{1}{2}}, \quad (6.89)$$

where the constant c depends only on \mathcal{B}'_0 and T such that $t_1, t_2 \in [0, T]$.

We finally deduce from (5.65), (6.86), and (6.89) the following result (see, e.g., [13, 12]).

Theorem 6.1. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}'_0$, i.e.*

- (i) \mathcal{M} is compact in $H^{-1}(\Omega)$;
- (ii) \mathcal{M} is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$;
- (iii) \mathcal{M} has finite fractal dimension in $H^{-1}(\Omega)$;

(iv) \mathcal{M} attracts exponentially fast the bounded subsets of Φ ,

$$\forall B \subset \Phi \text{ bounded, } \text{dist}_{H^{-1}(\Omega)}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{H^2(\Omega)})e^{-ct},$$

$$c > 0, \quad t \geq 0,$$

where the constant c is independent of B and $\text{dist}_{H^{-1}(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$\text{dist}_{H^{-1}(\Omega)}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{H^{-1}(\Omega)}.$$

Remark 6.2. Setting $\tilde{\mathcal{M}} = S(1)\mathcal{M}$, we can prove that $\tilde{\mathcal{M}}$ is an exponential attractor for $S(t)$, but now in the topology of $H^2(\Omega)$ (see, e.g., [14]).

Since \mathcal{M} (or $\tilde{\mathcal{M}}$) is a compact attracting set, we deduce from Theorem 6.1 the following corollary.

Corollary 6.3. The semigroup $S(t)$ possesses the finite-dimensional global attractor $\mathcal{A} \subset \mathcal{B}'_0$.

Remark 6.4. We can more generally consider a nonlinear term f of the form

$$f(s) = \sum_{k=0}^{2p+1} a_k s^k, \quad a_{2p+1} > 0$$

(see [8]).

7 Numerical simulations

As far as the numerical simulations are concerned, we rewrite the problem in the form

$$\frac{\partial u}{\partial t} + \Delta \mu + \lambda_0 \mathbf{1}_{\Omega \setminus D}(x)(u - h) = 0 \quad \text{in } \Omega, \quad (7.90)$$

$$\mu = \varepsilon \Delta u - \frac{1}{\varepsilon} f(u) \quad \text{in } \Omega, \quad (7.91)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad (7.92)$$

$$u|_{t=0} = u_0, \quad (7.93)$$

which has the advantage of splitting the fourth-order (in space) equation into a system of two second-order ones (see [15] and [9]). Consequently, we use a P1-finite element for the space discretization, together with a semi-implicit Euler time discretization (i.e. implicit for the linear terms and explicit for the nonlinear ones). The numerical simulations are performed with the software Freefem++ [17].

In the numerical results presented below, Ω is a $(0, 0.5) \times (0, 0.5)$ -square. The triangulation is obtained by dividing Ω into 120×120 rectangles and by dividing every rectangle along the same diagonal.

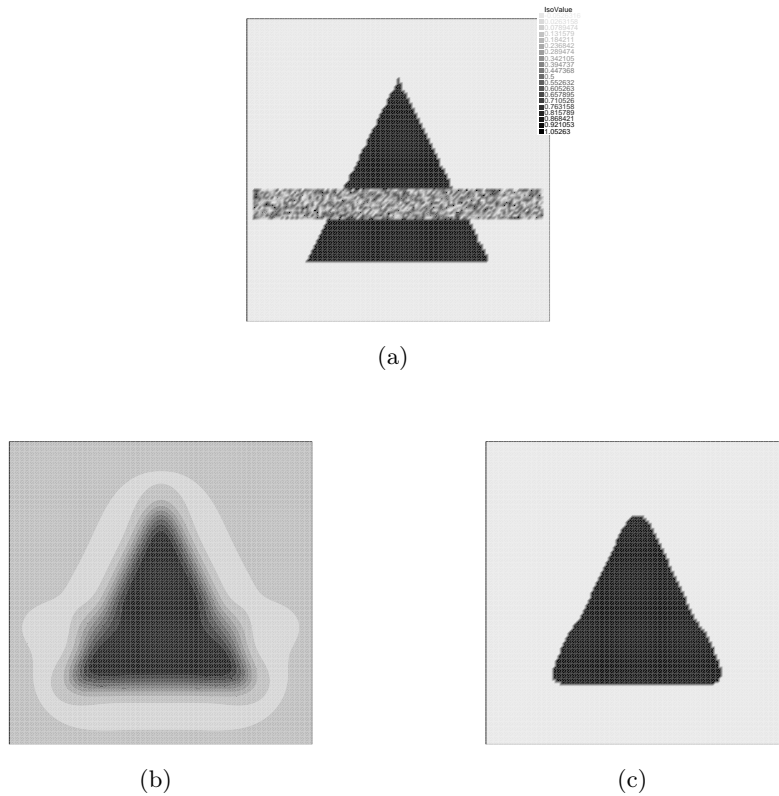


Figure 1: (a) Inpainting region in gray, random initial datum between 0 and 1 in inpainting region, $\varepsilon = 0.03$, $f(s) = s^3 - s$. (b) Solution at $t = 1$. (c) Replacing the values larger than $\frac{1}{2}$ by 1 and those smaller than $\frac{1}{2}$ by 0.

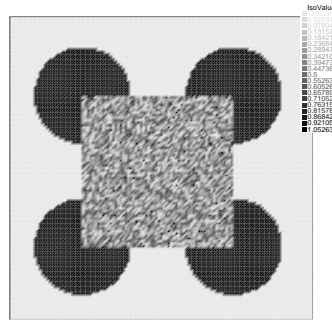
7.1 Inpainting of a triangle

The gray region in Figure 1(a) denotes the inpainting region. We run the modified Cahn-Hilliard equation with $f(s) = s^3 - s$, $\varepsilon = 0.03$ and, at $t = 1$, we come close to a steady state, shown in Figure 1(b). We finally replace all the values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting result in Figure 1(c). The parameters are $\Delta t = 0.05$, $\lambda_0 = 900000$.

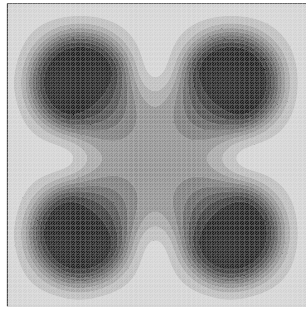
7.2 Inpainting of four 3/4 circles

In Figure 2(a), the gray region denotes the region to be inpainted. The modified Cahn-Hilliard equation is run close to a steady state with $\varepsilon = 0.05$ and $f(s) = s^3 - s$, resulting in Figure 2(b) at $t = 1.25$. We replace all the values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting result in Figure 2(c).

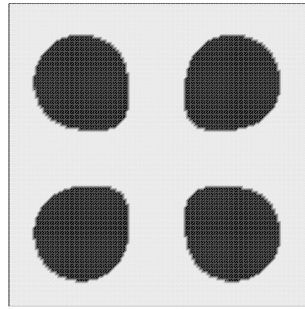
Furthermore, we run again the modified Cahn-Hilliard equation with the same initial datum as in Figure 2(a) and the same $\varepsilon = 0.05$, but we now take $f(s) = 4s^3 - 6s^2 + 2s$ as in [1]. We are close to a steady state at $t = 1.25$,



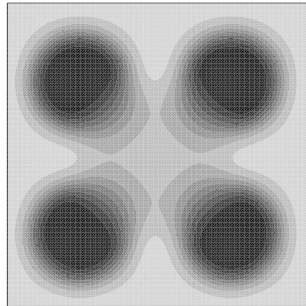
(a)



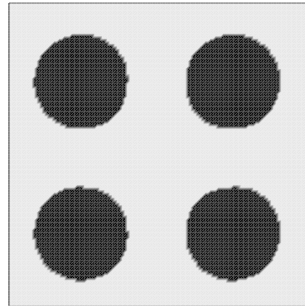
(b)



(c)



(d)



(e)

Figure 2: (a) Inpainting region in gray, random initial datum between 0 and 1 in inpainting region, $\varepsilon = 0.05$, $f(s) = s^3 - s$. (b) Solution at $t = 1.25$. (c) Replacing the values larger than $\frac{1}{2}$ by 1 and those smaller than $\frac{1}{2}$ by 0. (d) Solution at $t = 1.25$ when $f(s) = 4s^3 - 6s^2 + 2s$. (e) Replacing the values larger than $\frac{1}{2}$ by 1 and those smaller than $\frac{1}{2}$ by 0.

as shown in Figure 2(d). As above, we replace all the values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting in Figure 2(e). We finally deduce that, in the inpainting of a circle, the result obtained is better when considering the function $f(s) = 4s^3 - 6s^2 + 2s$ than $f(s) = s^3 - s$. In this test, $\Delta t = 0.05$, $\lambda_0 = 900000$.

Remark 7.1. *In the examples of the four circles, the choice $f(s) = s^3 - s$ gives a bad inpainting result. We note that, if we take $f(s) = \frac{1}{4}(s^3 - s)$ in*

the example of the circles, the inpainting result is better.

Acknowledgments: The authors wish to thank S. Zelik for several useful comments.

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