# Front propagation in a kinetic reaction-transport equation

Emeric Bouin*†, Vincent Calvez‡, Grégoire Nadin§

July 31, 2013

**Abstract**

In this paper, we study the existence and stability of travelling wave solutions of a kinetic reaction-transport equation. The model describes particles moving according to a velocity-jump process, and proliferating thanks to a reaction term of monostable type. The boundedness of the velocity set appears to be a necessary and sufficient condition for the existence of positive travelling waves. The minimal speed of propagation of waves is obtained from an explicit dispersion relation. We construct the waves using a technique of sub- and super- solutions and prove their stability in a weighted $L^2$ space. In case of an unbounded velocity set, we prove a superlinear spreading and give partial results concerning the rate of spreading associated to particular initial data. It appears that the rate of spreading depends strongly on the decay at infinity of the stationary Maxwellian.

**Key-words:** Kinetic equations, travelling waves, dispersion relation, superlinear spreading.

## 1 Introduction

We address the issue of front propagation in a reaction-transport equation of kinetic type,

$$\begin{cases} \partial_t g + v\partial_x g = \rho_g \left(M(v) - g\right) + r\left(M(v)\rho_g - g\right) , & (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V , \\ g(0,x,v) = g^0(x,v) , & (x,v) \in \mathbb{R} \times V . \end{cases} \tag{1.1}$$

Here, the density $g(t,x,v)$ describes a population of individuals in a continuum setting, and $\rho_g(t,x) = \int_V g(t,x,v)\,dv$ is the macroscopic density. The subset $V \subset \mathbb{R}$ is the set of all possible velocities. Individuals move following a velocity-jump process: they run with speed $v \in V$, and change velocity at rate 1. They instantaneously choose a new velocity following the probability distribution $M(v)\,dv$. Unless otherwise stated, we assume in this paper that $V$ is symmetric and $M$ satisfies the following properties: $M \in L^1(V) \cap \mathcal{C}^0(V)$, and

$$\int_V M(v)dv = 1 , \quad \int_V vM(v)dv = 0 , \quad \int_V v^2 M(v)dv = D < +\infty . \tag{1.2}$$

In addition, individuals are able to reproduce, with rate $r > 0$. New individuals start with a random velocity chosen with the same probability distribution $M(v)\,dv$. We could have chosen a different distribution without changing the main results, but we do not for the sake of clarity of the presentation. Finally, we include a quadratic saturation term, which accounts for local competition between individuals, regardless of their speed.

The main motivation for this work comes from the study of pulse waves in bacterial colonies of *Escherichia coli* [1, 28, 41, 42]. Kinetic models have been proposed to describe the run-and-tumble motion of individual

*Corresponding author

†Ecole Normale Supérieure de Lyon, UMR CNRS 5669 'UMPA', and INRIA Alpes, project-team NUMED, 46 allée d'Italie, F-69364 Lyon cedex 07, France. E-mail: emeric.bouin@ens-lyon.fr

‡Ecole Normale Supérieure de Lyon, UMR CNRS 5669 'UMPA', and INRIA Alpes, project-team NUMED, 46 allée d'Italie, F-69364 Lyon cedex 07, France. E-mail: vincent.calvez@ens-lyon.fr

§Université Pierre et Marie Curie-Paris 6, UMR CNRS 7598 'LJLL', BC187, 4 place de Jussieu, F-75252 Paris cedex 05, France. E-mail: nadin@ann.jussieu.fr

bacteria at the mesoscopic scale [2, 40]. Several works have been dedicated to derive macroscopic equations from those kinetic models in the diffusion limit [26, 15, 10, 41]. Recently it has been shown that for some set of experiments, the diffusion approximation is not valid, so one has to stick to the kinetic equation at the mesoscopic scale to carefully compare with data [42].

There is one major difference between this motivation and model (1.1). Pulse waves in bacterial colonies of *E. coli* are mainly driven by chemotaxis which create macroscopic fluxes. Growth of the population can be merely ignored in such models. In model (1.1) however, growth and dispersion are the main reasons for front propagation, and there is no macroscopic flux due to the velocity-jump process since the distribution $M$ satisfies $\int_V vM(v)dv = 0$. For the sake of applications, we also refer to the growth and branching of the plant pathogen *Phytophthora* by mean of a reaction-transport equation similar to (1.1) [25].

There is a strong link between (1.1) and the classical Fisher-KPP equation [18, 30]. In case of a suitable balance between scattering and growth (more scattering than growth), we can perform the parabolic rescaling $(r, t, x) \mapsto \left(\varepsilon^2 r, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ in (1.1),

$$\varepsilon^2 \partial_t g_\varepsilon + \varepsilon v \partial_x g_\varepsilon = (M(v)\rho_{g_\varepsilon} - g_\varepsilon) + \varepsilon^2 r \rho_{g_\varepsilon} (M(v) - g_\varepsilon) \, . \tag{1.3}$$

The diffusion limit yields $g_\varepsilon \to M(v)\rho_0$, where $\rho_0$ is solution to the Fisher-KPP equation (see [12] for example),

$$\partial_t \rho_0 - D\partial_{xx}\rho_0 = r\rho_0 (1 - \rho_0) \, . \tag{1.4}$$

We recall that for nonincreasing initial data decaying sufficiently fast at $x = +\infty$, the solution of (1.4) behaves asymptotically as a travelling front moving at the minimal speed $c^* = 2\sqrt{rD}$ [30, 3]. In addition, this front is stable in some weighted $L^2$ space [29, 20]. Therefore it is natural to address the same questions for (1.1). We give below the definition of a travelling wave for equation (1.1).

**Definition 1.1.** *We say that a function $g(t, x, v)$ is a smooth travelling front solution of speed $c \in \mathbb{R}_+$ of equation* (1.1) *if it can be written $g(t, x, v) = f(x - ct, v)$, where the profile $f \in \mathcal{C}^2(\mathbb{R} \times V)$ satisfies*

$$\forall (z, v) \in \mathbb{R} \times V \, , \quad 0 \leq f(z, v) \leq M(v) \, , \quad \lim_{z \to -\infty} f(z, v) = M(v) \, , \quad \lim_{z \to +\infty} f(z, v) = 0 \, . \tag{1.5}$$

In fact the profile $f$ is a solution to the stationary equation in a moving frame,

$$(v - c)\partial_z f = (M(v)\rho_f - f) + r\rho_f (M(v) - f) \, , \quad (z, v) \in \mathbb{R} \times V \, . \tag{1.6}$$

The existence of travelling waves in reaction-transport equations has been adressed by Schwetlick [43, 44] for a similar class of equations. First, the set $V$ is bounded and $M$ is the uniform distribution over $V$. Second, the nonlinearity can be chosen more generally (either monostable as here, or bistable), but it depends only on the macroscopic density $\rho_g$ [43, Eq. (4)]. For the monostable case, using a quite general method he was able to prove existence of travelling waves of speed $c$ for any $c \in [c^*, \sup V)$, a result very similar to the Fisher-KPP equation. We emphasize, that although the equations differ between Schwetlick's work and ours, they coincide as far as the linearization in the regime of low density $g \ll 1$ is concerned. On the contrary to Schwetlick, we do not consider a general nonlinearity and we restrict to the logistic case, but we consider general velocity kernels $M(v)$.

More recently, the rescaled equation (1.3) has been investigated by Cuesta, Hittmeir and Schmeiser [12] in the parabolic regime $\varepsilon \ll 1$. Using a micro-macro decomposition, they construct possibly oscillatory travelling waves of speed $s \geq 2\sqrt{rD}$ for $\varepsilon$ small enough (depending on $s$). In addition, when the set of admissible speeds $V$ is bounded, and $s > 2\sqrt{rD}$ they prove that the travelling wave constructed in this way is indeed nonnegative.

Lastly, when $M$ is the measure $M = \frac{1}{2}(\delta_{-\nu} + \delta_\nu)$ for some $\nu > 0$, equation (1.1) is analogous to the reaction-telegraph equation for the macroscopic density $\rho_g$ (up to a slight change in the nonlinearity however). This equation has been the subject of a large number of studies in the applied mathematics community [14, 23, 27, 21, 36, 16, 17, 19, 39, 38]. Recently, the authors prove the existence of a minimal speed $c^*$ such that travelling waves exist for all speed $c \geq c^*$ [5]. Moreover these waves are stable in some $L^2$ weighted space, with a weight which differs from the classical exponential weight arising for the Fisher-KPP equation. As the reaction-telegraph equation involves both parabolic and hyperbolic contributions, the

smoothness of the wave depends on the balance between these contributions. In fact there is a transition between a parabolic (smooth waves) and a hyperbolic regime (discontinuous waves), see Remark 1.3 below. The authors also prove the existence of supersonic waves, having speed $c > \nu$ (see Remark 1.4).

The aim of the present paper is to investigate the existence and stability of travelling waves for equation (1.1) for arbitrary kernels $M$ satisfying (1.2). For the existence part, we shall use the method of sub- and supersolutions, which do not rely on a perturbation argument. The stability part relies on the derivation of a suitable weight from which we can build a Lyapunov functional for the linearized version of (1.1). The crucial assumption for the existence of travelling waves is the boundedness of $V$. We prove in fact that in the case $V = \mathbb{R}$ there exists no (positive) travelling wave. We then investigate the spreading rate for some particular choices of $M$ (Gaussian distribution, Cauchy's distribution). Unfortunately we are only able to give partial answer to this last question. In the last stage of writting of this paper, we realized that this issue was already addressed by Méndez, Campos and Gómez-Portillo for a slightly different equation admitting the same linearization near the front edge [37]. Our results are in agreement with their predictions.

**Existence of travelling waves when the velocity set is bounded.**

**Theorem 1.2.** *Assume that the set $V$ is compact, and that $M \in \mathcal{C}^0(V)$ satisfies (1.2). Let $v_{\max} = \sup V$. There exists a speed $c^* \in (0, v_{\max})$ such that there exists a travelling wave $f$ solution of (1.6) of speed $c$ for all $c \in [c^*, v_{\max})$. The travelling wave is nonincreasing with respect to the space variable: $\partial_z f \leq 0$. Moreover, if $\inf_V M > 0$ then there exists no travelling wave of speed $c \in [0, c^*)$.*

The minimal speed $c^*$ is given through the following implicit dispersion relation: for each $\lambda > 0$ there is a unique $c(\lambda) \in (v_{\max} - \lambda^{-1}, v_{\max})$ such that

$$(1 + r) \int_V \frac{M(v)}{1 + \lambda(c(\lambda) - v)}\, dv = 1\,. \tag{1.7}$$

Then we have the formula

$$c^* = \inf_{\lambda > 0} c(\lambda)\,.$$

**Remark 1.3.** *In the special case of two possible velocities only [5], corresponding to $M(v) = \frac{1}{2}\left(\delta_{-v_{\max}} + \delta_{v_{\max}}\right)$, two regimes have to be distinguished, namely $r < 1$ and $r \geq 1$. In the case $r \geq 1$ the travelling wave with minimal speed vanishes on a half-line. There, the speed of the wave is not characterized by the linearized problem for $f \ll 1$. Note that this case is not contained in the statement of Theorem 1.2 since it is assumed that $M \in \mathcal{C}^0(V)$. This makes a clear difference between the case of integrable $M$ and the case of a measure with atoms.*

**Remark 1.4.** *We expect that travelling waves exist for any $c \geq c^*$, although this seems to contradict the finite speed of propagation when $c > v_{\max}$. In fact* supersonic *waves corresponding to $c > v_{\max}$ should be driven by growth mainly, as it is the case in a simplified model with only two speeds [5]. A simple argument to support the existence of such waves consists in eliminating the transport part, and seeking waves driven by growth only, $-c\partial_z f = M(v)\rho_f - f + r\rho_f(M - f)$. Integrating with respect to $v$ yields a logistic equation for $\rho_f$, $-c\partial_z \rho_f = r\rho_f(1 - \rho_f)$, which as a solution connecting 1 and 0 for any positive $c$. However these waves are purely artificial and we do not address this issue further.*

We now define $c^* = c^*(M)$ and investigate the dependence of the minimal speed with respect to the velocity kernel $M = M(v)$. In the following Proposition, we give some general bounds on the minimal speed.

**Proposition 1.5.** *Assume that $V$ is symmetric and that $M(v) = M(-v)$ for all $v \in V$. The minimal speed satisfies the following properties,*

*a-* [Scaling] *For $\sigma > 0$, define $M_\sigma(v) = \sigma^{-1} M\left(\sigma^{-1} v\right)$, then*

$$c^*(M_\sigma) = \sigma c^*(M).$$

*b-* [Rearrangement] *Denote $M^\star$ the Schwarz decreasing rearrangement of the function $M$ (see [32] for a definition of this notion) and $M_\star = -(-M)^\star$ the Schwarz increasing rearrangement of the density distribution $M$, then*

$$c^*(M^\star) \le c^*(M) \le c^*(M_\star).$$

*c-* [Comparison] *If $r < 1$ then*

$$\frac{2\sqrt{rD}}{1+r} \le c^*(M) \le \frac{2\sqrt{r}}{1+r} v_{\max}\,,$$

*On the other hand, if $r \ge 1$ then*

$$\sqrt{D} \le c^*(M) \le v_{\max}\,,$$

*d-* [Diffusion limit] *In the diffusion limit $(r,t,x) \mapsto \left(\varepsilon^2 r, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ we recover the KPP speed of the wave,*

$$\lim_{\varepsilon \to 0} c^*_\varepsilon = 2\sqrt{rD}$$

**Spreading of the front.**

In the case of a bounded set of velocities, we prove that for suitable initial data $g(0,x,v)$, the front spreads asymptotically with speed $c^*$, in a weak sense.

**Proposition 1.6.** *Assume that $V$ is bounded and that $\inf_V M > 0$. Let $g^0 \in L^\infty(\mathbb{R}\times V)$ such that $0 \le g^0(x,v) \le M(v)$ for all $(x,v) \in \mathbb{R}\times V$. Let $g$ be the solution of the Cauchy problem* (1.1)*. Then*

1. *if there exists $x_R$ such that $g^0(x,v) = 0$ for all $x \ge x_R$ and $v \in V$, then for all $c > c^*$,*

$$(\forall v \in V) \quad \lim_{t\to+\infty} \sup_{x \ge ct} g(t,x,v) = 0\,,$$

2. *if there exists $x_L$ and $\delta \in (0,1)$ such that $g^0(x,v) \ge \delta M(v)$ for all $x \le x_L$ and $v \in V$, then for all $c < c^*$,*

$$(\forall v \in V) \quad \lim_{t\to+\infty} \sup_{x \le ct} |M(v) - g(t,x,v)| = 0\,,$$

*where $c^*$ is the minimal speed of existence of travelling waves given by Theorem 1.2.*

**Stability of the travelling waves.**

We also establish linear and nonlinear stability in suitable weighted $L^2$ spaces. The keypoint is to derive a correct weight which enables to build a Lyapunov functional for the linear problem. We construct a semi-explicit weight $\phi(z,v)$, but we believe it is not the optimal one in some sense (see Remark 5.3). Let $f$ be a travelling wave (1.6) of speed $c$, and let $u = g - f$ the perturbation of $f$ in the moving frame. Neglecting the nonlinear contributions, we are led to investigate the linear equation

$$\partial_t u + (v-c)\partial_z u + (1 + r\rho_f)\, u = ((1+r)M - rf)\,\rho_u\,. \tag{1.8}$$

**Proposition 1.7** (Linear stability)**.** *There exists a weight $\phi$ such that the travelling front of speed $c \in [c^*, v_{\max})$ is linearly stable in the weighted space $L^2\left(e^{-2\phi(z,v)}dzdv\right)$. More precisely, the following Lyapunov identity holds true for any solution $u$ of the linear equation* (1.8)*,*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V} |u|^2\, e^{-2\phi(z,v)} dzdv\right) + \int_{\mathbb{R}\times V} \frac{r}{2}\left(\rho_f + \frac{f}{M(v) + r\,(M(v) - f)}\right) |u|^2\, e^{-2\phi(z,v)} dzdv \le 0\,. \tag{1.9}$$

The weight $\phi$ is explicitly given in Definition 5.1.

Using a comparison argument, in the spirit of [12], together with the explicit formula of the dissipation for the linearized system (1.9), we prove a nonlinear stability result.

**Theorem 1.8** (Nonlinear stability). *Let $\gamma \in \left(\frac{1}{2}, 1\right]$. Let $f$ be a travelling wave with speed $c \in [c^*, v_{\max})$. Let $g$ be a solution of* (1.1). *Suppose that the initial data satisfies*

$$\forall\, (x,v) \in \mathbb{R} \times V, \qquad g^0(x,v) \geq \gamma f(x,v)\,. \tag{1.10}$$

*Then the following Lyapunov identity holds true for the perturbation $u = g - f$,*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V} |u|^2\, e^{-2\phi(z,v)} dzdv\right) + \int_{\mathbb{R}\times V} \frac{r}{2}\left[(2\gamma-1)\rho_f + \frac{f}{M(v) + r\,(M(v)-f)}\right] |u|^2\, e^{-2\phi(z,v)} dzdv \leq 0\,, \tag{1.11}$$

*where $\phi$ denotes the same weight as in Proposition 1.7.*

We expect that nonlinear stability holds true for any $\gamma \in (0,1]$. However this would require to redefine the weight $\phi$, since we believe it is not the optimal one.

**Superlinear propagation when the admissible speeds are unbounded.**

Boundedness of $V$ is a crucial hypothesis in order to build the travelling waves. We believe that it is a necessary and sufficient condition. We make a first step to support this conjecture by investigating the case $V = \mathbb{R}$. We first prove infinite speed of spreading of the front under the natural assumption ($\forall v \in \mathbb{R}$) $M(v) > 0$. As a corollary there cannot exist travelling wave in the sense of Definition 1.1. Note that there exist travelling waves with less restrictive conditions than Definition 1.1, at least in the diffusive regime [12]. These fronts are expected to oscillate as $x \to +\infty$. We expect that such oscillating fronts do exist far from the diffusive regime. In the case where $V = \mathbb{R}$ and $M$ is Gaussian, we plotted the dispersion relation (1.7) in the complex plane $\lambda \in \mathbb{C}$, for an arbitrary given $c > 0$. We observed that it selects two complex conjugate roots, supporting the fact that oscillating fronts should exist (results not shown).

**Proposition 1.9.** *Assume that $M(v) > 0$ for all $v \in \mathbb{R}$. Let $g^0 \in L^\infty(\mathbb{R}\times V)$ such that $0 \leq g^0(x,v) \leq M(v)$ for all $(x,v) \in \mathbb{R}\times V$ and there exists $x_L$ and $\delta \in (0,1)$ such that $g^0(x,v) \geq \delta M(v)$ for all $x \leq x_L$ and $v \in V$. Let $g$ be the solution of the Cauchy problem* (1.1). *Then for all $c > 0$,*

$$(\forall v \in V) \quad \lim_{t\to+\infty} \sup_{x\leq ct} |M(v) - g(t,x,v)| = 0\,.$$

We can immediately deduce from this result the non-existence of travelling waves when $V = \mathbb{R}$, by taking such a travelling wave as an initial datum $g^0$ in order to reach a contradiction.

**Corollary 1.10.** *Assume that $M(v) > 0$ for all $v \in \mathbb{R}$. Then equation* (1.1) *does not admit any travelling wave solution.*

Next we investigate specific choices for the distribution $M$, both numerically and theoretically. In the case of a Gaussian distribution, we expect a spreading rate following the power law $\langle x\rangle \sim t^{3/2}$ (see also [37]). To support this guess, we prove in fact that spreading occurs *at most* with this rate. For this purpose we build a supersolution which is spreading with this rate.

This issue has been addressed by Méndez, Campos and Gómez-Portillo [37] in a physical paper for a slightly different equation, where the nonlinearity lies in the diffusion kernel instead of the growth rate. They conjectured that, as for the KPP equation, the front speed is determined through the linearization of the equation near the unstable steady state 0. We believe that the linearization should give the power law of the propagation, but it is not clear to us that it will give the exact location of the transition. However, it turns out that the linearized equations in [37] and in the present paper are the same. Then, performing some Fourier-Laplace transform of the solution, Mendez et al derived heuristically the power law of the propagation. In particular, for a Gaussian kernel, they found out that the spreading rate is $\langle x\rangle \sim t^{3/2}$.

**Theorem 1.11.** *Let $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$ for all $v \in \mathbb{R}$. Let $g^0 \in L^\infty(\mathbb{R}\times V)$ such that $0 \leq g^0(x,v) \leq M(v)$ for all $(x,v) \in \mathbb{R}\times V$. Assume that there exists $a \geq b \geq 1$ such that*

$$(\forall (x,v) \in \mathbb{R}\times V) \quad g^0(x,v) \leq \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra}\,.$$

*Let $g$ be the solution of the Cauchy problem* (1.1). *Then for all $\varepsilon > 0$, one has*

$$\lim_{t\to+\infty} \sup_{|x|\geq(1+\varepsilon)\sigma\sqrt{2r}(t+a)^{3/2}} \rho_g(t,x) \to 0\,.$$

In the case of a Cauchy distribution $M(v) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+v^2}$, we obtain faster spreading rate under similar assumptions (see Proposition 7.3), namely for all $\varepsilon > 0$, one has

$$\lim_{t\to+\infty} \sup_{|x|\geq\sqrt{\frac{\sigma}{\pi}}(t+a)\exp\left(\frac{(1+\varepsilon)r(t+a)}{2}\right)} \rho_g(t,x) \longrightarrow 0\,.$$

The phenomenon of accelerating fronts have raised a lot of attention in the literature of reaction-diffusion equations. This phenomenon occurs for the Fisher-KPP equation (1.4) when the initial datum decays more slowly than any exponential [24]; for a variant of the Fisher-KPP equation where the diffusion operator is replaced by a nonlocal dispersal operator with fat tails [31, 35, 22], or by a nonlocal fractional diffusion operator [8, 9]. Recently, accelerating fronts have been conjectured to occur in a reaction-diffusion-mutation model which extends the Fisher-KPP equation to a population with heterogeneous diffusion coefficient [6].

**Remark 1.12** (Front propagation and diffusive limit)**.** *There is some subtlety hidden behind this phenomenon of infinite speed of spreading. In fact the diffusion limit of the scattering equation (namely $r = 0$) towards the heat equation makes no difference between bounded or unbounded velocity sets (see [13] and the references therein). However very low densities behave quite differently, which can be measured in the setting of large deviations or WKB limit. This can be observed even in the case of a bounded velocity set. In [4] the large deviation (WKB) limit of the scattering equation is performed. It differs largely from the classical eikonal equation obtained from the heat equation. The case of unbounded velocities is even more complicated [7]. To conclude, let us emphasize that low densities are the one that drive the front here (pulled front). So the diffusion limit is irrelevant in the case of unbounded velocities, since very low density of particles having very large speed makes a big difference.*

# 2 Preliminary results

We first recall some useful results concerning the Cauchy problem associated with (1.1): well-posedness and a strong maximum principle. These statements extend some results given in [12]. They do not rely on the boundedness of $V$.

**Proposition 2.1** (Global existence: Theorem 4 in [12])**.** *Let $g^0$ a measurable function such that $0 \leq g^0(x,v) \leq M(v)$ for all $(x,v) \in \mathbb{R}\times V$. Then the Cauchy problem* (1.1) *has a unique solution $g \in \mathcal{C}^0_b(\mathbb{R}_+\times\mathbb{R}\times V)$ in the sense of distributions, satisfying*

$$(\forall (t,x,v) \in \mathbb{R}_+\times\mathbb{R}\times V) \quad 0 \leq g(t,x,v) \leq M(v)\,.$$

The next result refines the comparison principle of [12] in order to extend it to sub and supersolutions in the sense of distributions and to state a strong maximum principle. Its proof is given in Appendix.

**Proposition 2.2** (Comparison principle)**.** *Assume that $u_1, u_2 \in \mathcal{C}(\mathbb{R}_+, L^\infty(\mathbb{R}\times V))$ are respectively a super- and a subsolution of* (1.1), i.e.

$$\partial_t g_1 + v\partial_x g_1 \geq (M(v)\rho_{g_1} - g_1) + r\rho_{g_1}\left(M(v) - g_1\right)\,,$$
$$\partial_t g_2 + v\partial_x g_2 \leq (M(v)\rho_{g_2} - g_2) + r\rho_{g_2}\left(M(v) - g_2\right)\,,$$

*in the sense of distributions. Assume in addition that $g_2$ satisfies $g_2(t,x,v) \leq M(v)$ for all $(t,x,v) \in \mathbb{R}_+\times\mathbb{R}\times V$. Then $g_2(t,x,v) \leq g_1(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+\times\mathbb{R}\times V$.*

*Assume in addition that $V$ is an interval, and that $\inf_V M > 0$. If there exists $(x_0,v_0)$ such that $g_2(0,x_0,v_0) > g_1(0,x_0,v_0)$, then one has $g_1(t,x,v) > g_2(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+\times\mathbb{R}\times V$ such that $|x - x_0| < v_{\max}t$.*

**Remark 2.3.** *If $V = \mathbb{R}$, then this statement reads as in the parabolic framework: if $g_2 \geq g_1$ and $g_2 \not\equiv g_1$ at $t = 0$, then $g_2 > g_1$ for all $t > 0$. In the case $V = [-v_{\max}, v_{\max}]$ we have to take into account finite speed of propagation, obviously.*

# 3 Existence and construction of travelling wave solutions

We assume throughout this Section that $V = \operatorname{supp} M$ is compact. We construct the travelling waves for $c \in [c^*, v_{\max})$. The proof is divided into several steps. It is based on a sub and supersolutions method.

## 3.1 The linearized problem.

The aim of this first step is to solve the linearized equation of (1.6) at $+\infty$, in the regime of low density $f \ll 1$. Such an achievement gives information about the speed and the space decreasing rate of a travelling wave solution of the nonlinear problem, as for the Fisher-KPP equation. The linear problem for (1.6) writes

$$(v-c)\partial_x f = (M(v)\rho_f - f) + rM(v)\rho_f\,, \tag{3.12}$$

We seek a solution having exponential decay at $+\infty$. More specifically we separate the variables in our ansatz: $f(x,v) = e^{-\lambda x}F(v)$, with $\int_V F(v)dv = 1$. The next Proposition gathers the results concerning the linear problem.

**Proposition 3.1** (Existence of a minimal speed for the linearized equation)**.** *There exists a minimal speed $c^*$ such that for all $c \in [c^*, v_{\max})$, there exists $\lambda > 0$ such that $f_\lambda(x,v) = e^{-\lambda x}F_\lambda(v)$ is a nonnegative solution of* (3.12)*. The profile $F_\lambda$ is explicitely given by*

$$F_\lambda(v) = \frac{(1+r)M(v)}{1+\lambda(c-v)} \geq 0\,.$$

*The admissible $(\lambda, c)$ are solutions of the following dispersion relation,*

$$\int_V \frac{(1+r)M(v)}{1+\lambda(c-v)}\,dv = 1\,. \tag{3.13}$$

*Moreover, among all possible $\lambda$ for a given $c$, the minimal one $\lambda_c$ is well defined and isolated.*

**Remark 3.2.** *Here appears the crucial assumption on the boundedness of $V$. If this condition is not fulfilled, it is never possible to ensure that the profile $F_\lambda$ is nonnegative since the denominator is linear with respect to $v$.*

**Proof of Proposition 3.1.** # **Step 1.** Plugging the ansatz $f_\lambda(x,v) = e^{-\lambda x}F_\lambda(v)$ into (3.12) yields

$$(c-v)\lambda F_\lambda(v) = (M(v) - F_\lambda(v)) + rM(v)\,. \tag{3.14}$$

The profile is given by

$$F_\lambda(v) = \frac{(1+r)M(v)}{1+\lambda(c-v)}.$$

The dispersion relation reads $\int_V F_\lambda(v)dv = 1$, or equivalently (3.13). Moreover, we require the profile $F_\lambda$ to be nonnegative, which gives the condition $1+\lambda(c-v) > 0$ for all $v \in V$, which implies $\lambda < \frac{1}{v_{\max}-c}$.

From now on, we focus on the existence of solutions $(\lambda, c)$ of (3.13), with $c \in [0, v_{\max})$ and $\lambda \in \left[0, \frac{1}{v_{\max}-c}\right)$. Let us denote

$$I(\lambda; c) = \int_V \frac{(1+r)M(v)}{1+\lambda(c-v)}\,dv. \tag{3.15}$$

so that we look for solutions of $I(\lambda; c) = 1$.

# **Step 2.** Technically speaking, for all $c \in [0, v_{\max})$, the function $\lambda \mapsto I(\lambda; c)$ is analytic over $\left[0, \frac{1}{v_{\max}-c}\right)$ Indeed, as $v \mapsto v^n M(v)$ is integrable for all $n$, it is clear that

$$I(\lambda; c) = \sum_{n\geq 0}(1+r)\lambda^n \int_V M(v)(v-c)^n dv$$

is the analytic development of $I$ for $\lambda \in \left[0, \frac{1}{v_{\max}-c}\right)$. Next we observe that $c \mapsto I(\lambda; c)$ is decreasing for all $\lambda \in \left(0, \frac{1}{v_{\max}-c}\right)$, and that $\lambda \mapsto I(\lambda; c)$ is strictly convex. Moreover, the function $I$ satisfies the following properties:

$$
\begin{aligned}
&I(0; c) = 1 + r > 1\,,\\
&I(\lambda; 0) = (1+r)\int_V \frac{M(v)}{1-\lambda v}\,dv > 1, \quad \text{for all } \lambda \in \left[0, \frac{1}{v_{\max}}\right)\\
&I\left(\lambda; v_{\max}\right) = (1+r)\int_V \frac{M(v)}{1+\lambda(v_{\max}-v)}\,dv \xrightarrow[\lambda\to+\infty]{} 0\,.
\end{aligned}
$$

The last property relies on the Lebesgue's dominated convergence theorem since $M \in L^1(V)$.

\# **Step 3.** Assume first that $\frac{M(v)}{v_{\max}-v} \notin L^1(V)$. Then Fatou's lemma gives

$$
\liminf_{\lambda \nearrow \frac{1}{v_{\max}-c}} I(\lambda; c) = \liminf_{\lambda \nearrow \frac{1}{v_{\max}-c}} \int_V \frac{M(v)}{1+\lambda(c-v)} dv \geq \int_V \liminf_{\lambda \nearrow \frac{1}{v_{\max}-c}} \frac{M(v)}{1+\lambda(c-v)} dv = \int_V \frac{M(v)}{1-\frac{v-c}{v_{\max}-c}} dv = +\infty.
$$

As a consequence, $\theta(c) = \min\left\{I(\lambda; c) : \lambda \in \left[0, \frac{1}{v_{\max}-c}\right)\right\}$ is well defined and finite for all $c \in [0, v_{\max})$. It follows from the earlier properties that $\theta(0) > 1$ and $\theta(v_{\max}) = 0$. Moreover, the regularity and monotonicity properties of $I$ guarantee that $\theta$ is continuous and decreasing. Hence, there exists $c^*$ such that $\theta(c^*) = 1$ and there exists $\lambda_{c^*}$ such that $I(\lambda_{c^*}; c^*) = 1$.

Next, for all $c \in (c^*, v_{\max})$, as $c \mapsto I(\lambda; c)$ is decreasing, one has $I(\lambda_{c^*}; c) < 1$ for all $c > c^*$. Thus, as $I(0; c) > 1$, there exists $\lambda$ such that $I(\lambda; c) = 1$ for all $c > c^*$.

Second, consider a general $M \in \mathcal{C}^0(V)$ possibly vanishing at $v = v_{\max}$. To recover the first step, we define for $n \in \mathbb{N}^*$ a new Maxwellian by $M_n = \frac{M+1/n}{1+|V|/n}$ over $V$ (and 0 outside of $V$), where $|V|$ is the measure of $V$. Then $\frac{M_n(v)}{v_{\max}-v} \notin L^1(V)$ since $M_n(v_{\max}) \geq \frac{1/n}{1+|V|/n} > 0$, and thus the earlier step yields that there exists a sequence $c_n^*$ of minimal speeds associated with $(M_n)_n$. We also associate $I_n$ with $M_n$ through (3.15). We define

$$
c^* = \limsup_{n\to\infty} c_n^*,
$$

and we now show that it is the minimal speed.

- Take $c < c^*$. Then for all $\lambda \in \left(0, \frac{1}{v_{\max}-c}\right)$ and for some arbitrarily large $n$ so that $\lambda \in \left(0, \frac{1}{v_{\max}-c_n^*}\right)$, one has

  $$
  I_n(\lambda; c) = I_n(\lambda; c_n^*) - \int_c^{c_n^*} \partial_c I_n(\lambda, c')dc' \geq 1 - \int_c^{c_n^*} \partial_c I_n(\lambda, c')dc' \geq 1 + \frac{(1+r)\lambda}{\left(1+\lambda(c_n^*+v_{\max})\right)^2}(c_n^* - c).
  $$

  Because $I_n(\lambda; c) \underset{n\to+\infty}{\to} I(\lambda; c)$ as $n \to +\infty$, we get

  $$
  I(\lambda; c) \geq 1 + \frac{(1+r)\lambda}{1+\lambda(c^*+v_{\max})}(c^* - c) > 1.
  $$

  Thus $I(\lambda; c) = 1$ has no solution for $\lambda \in \left(0, \frac{1}{v_{\max}-c}\right)$ if $c < c^*$.

- Assume that $c > c^*$. Then one has $c > c_n^*$ when $n$ is large enough and thus for all $n$ sufficiently large, there exists $\lambda_n \in \left(0, \frac{1}{v_{\max}-c}\right)$ such that $I_n(\lambda_n; c) = 1$. Up to extraction, one may assume that $(\lambda_n)_n$ converges to some $\lambda_\infty \in \left[0, \frac{1}{v_{\max}-c}\right]$. Fatou's lemma yields $I(\lambda_\infty; c) \leq 1$. Hence, there exists a solution $\lambda \in \left[0, \frac{1}{v_{\max}-c}\right]$ of $I(\lambda; c) = 1$ and obviously $\lambda \neq 0$ since $I(0; c) > 1$.

- Lastly, if $c = c^*$, we know that for all $k \in \mathbb{N}^*$, there exists $\lambda_k \in \left(0, \frac{1}{v_{\max} - (c^* + 1/k))}\right]$ such that $I(\lambda_k; c^* + 1/k) = 1$. Assuming that $\lambda_k \to \lambda \in \left[0, \frac{1}{v_{\max} - c}\right]$ as $k \to +\infty$, we get $I(\lambda; c^*) = 1$.

□

**Lemma 3.3** (Spatial decay rate)**.** *For all $c \in [c^*, v_{\max})$, the quantity*

$$\lambda_c = \min\{f > 0 : I(f; c) = 1\}.$$

*is well-defined. Moreover, for all $c \in (c^*, v_{\max})$, if $\gamma > 0$ is small enough, then $I(\lambda_c + \gamma; c) < 1$.*

**Proof of Lemma 3.3.** We know from the definition of $c^*$ that for all $c \in [c^*, v_{\max})$, the set $\Lambda_c = \{f > 0 : I(f; c) = 1\}$ is not empty. Thus, we can take a minimizing sequence $\lambda_n$ which converges towards the infimum of $\Lambda_c$. As this sequence is bounded, one can assume, up to extraction, that $\lambda_n \to \lambda_c \geq 0$. Then Lebesgue's dominated convergence theorem gives $I(\lambda_c; c) = 1$. Hence $\lambda_c = \min \Lambda_c$.

Next, we have already noticed in the proof of Proposition 3.1 that $I(\lambda_{c^*}, c) < 1$ for all $c > c^*$. As $I(0, c) = 1 + r > 1$, the definition of $\lambda_c$ yields $\lambda_c < \lambda_{c^*}$. The conclusion follows from the strict convexity of the function $\lambda \mapsto I(\lambda; c)$. □

## 3.2 Construction of sub and supersolutions when $c \in (c^*, v_{\max})$.

In this step we construct sub and supersolutions for (1.1). We fix $c \in (c^*, v_{\max})$ and we denote $\lambda = \lambda_c$ without ambiguity.

**Lemma 3.4** (Supersolution)**.** *Let*

$$\overline{f}(x, v) = \min\left\{M(v), e^{-\lambda x} F_\lambda(v)\right\}.$$

*Then $\overline{f}$ is a supersolution of (1.6), that is, it satisfies in the sense of distributions:*

$$(v - c)\partial_x \overline{f} \geq \left(M(v)\rho_{\overline{f}} - \overline{f}\right) + r\rho_{\overline{f}}\left(M(v) - \overline{f}\right), \qquad (x, v) \in \mathbb{R} \times V. \tag{3.16}$$

**Lemma 3.5** (Subsolution)**.** *There exist $A > 0$ and $\gamma > 0$ such that if*

$$\underline{f}(x, v) = \max\left\{0, e^{-\lambda x} F_\lambda(v) - A e^{-(\lambda+\gamma)x} F_{\lambda+\gamma}(v)\right\},$$

*then $\underline{f}$ is a subsolution of (1.6), that is satisfies in the sense of distributions:*

$$(v - c)\partial_x \underline{f} \leq \left(M(v)\rho_{\underline{f}} - \underline{f}\right) + r\rho_{\underline{f}}\left(M(v) - \underline{f}\right), \qquad (x, v) \in \mathbb{R} \times V. \tag{3.17}$$

**Proof of Lemma 3.4.** First, $(x, v) \mapsto e^{-\lambda x} F_\lambda(v)$ and $(x, v) \mapsto M(v)$ both clearly satisfy (3.16) since $\overline{f} \geq 0$. Next, as $\overline{f}$ is continuous, it immediately follows from the jump formula that, as a minimum of two supersolutions, it is a supersolution of (3.16) in the sense of distributions. □

**Proof of Lemma 3.5.** The same arguments as in the proof of Lemma 3.4 yield that it is enough to prove that (3.17) is satisfied by $\underline{f}$ over the open set $\{\underline{f} > 0\}$. As $c > c^*$, Proposition 3.1 gives $\gamma \in (0, \lambda)$ small enough such that $I(\lambda + \gamma; c) < 1$ and $F_{\lambda+\gamma}(v) > 0$. We compute the linear part:

$$(v - c)\partial_x \underline{f} - \left(M(v)\rho_{\underline{f}} - \underline{f}\right) - r\rho_{\underline{f}} M(v) = A\left(I(\lambda + \gamma, c) - 1\right)(1 + r)e^{-(\lambda+\gamma)x} M(v).$$

To prove the Lemma, we now have to choose a relevant $A$ such that

$$r\underline{f}\rho_{\underline{f}} \leq A(1 + r)M(v)\left(1 - I(\lambda + \gamma, c)\right)e^{-(\lambda+\gamma)x}. \tag{3.18}$$

holds for all $(x,v) \in \mathbb{R} \times V$. As $\underline{f}(x,v) \le e^{-\lambda x} F_\lambda(v)$ for all $(x,v) \in \mathbb{R} \times V$, one has $\rho_{\underline{f}}(x) \le e^{-\lambda x}$ and thus it is enough to choose $A$ such that

$$\begin{aligned} & re^{-2\lambda x} F_\lambda(v) \le A(1+r) M(v) \left(1 - I(\lambda+\gamma, c)\right) e^{-(\lambda+\gamma)x}, \\ & \frac{re^{-(\lambda-\gamma)x}}{1 - I(\lambda+\gamma, c)} \left( \frac{1}{1+\lambda(c-v)} \right) \le A. \end{aligned} \tag{3.19}$$

On the other hand for all $(x,v) \in \mathbb{R} \times V$ such that $\underline{f}(x,v) > 0$, we have $F_\lambda(v) > Ae^{-\gamma x} F_{\lambda+\gamma}(v)$, meaning that

$$e^{-\gamma x} < \frac{1}{A} \left( \frac{1 + (\lambda+\gamma)(c-v)}{1+\lambda(c-v)} \right).$$

Plugging this estimate into (3.19), it is enough to choose $A$ such that

$$\begin{aligned} & \left( \frac{1}{A} \left( \frac{1 + (\lambda+\gamma)(c-v)}{1+\lambda(c-v)} \right) \right)^{\frac{\lambda-\gamma}{\gamma}} \frac{r}{1 - I(\lambda+\gamma, c)} \left( \frac{1}{1+\lambda(c-v)} \right) \le A \\ & \sup_{v \in V} \left\{ \left( \frac{1 + (\lambda+\gamma)(c-v)}{1+\lambda(c-v)} \right)^{\frac{\lambda-\gamma}{\gamma}} \frac{r}{1 - I(\lambda+\gamma, c)} \left( \frac{1}{1+\lambda(c-v)} \right) \right\} \le A^{\frac{\lambda}{\gamma}}. \end{aligned}$$

This concludes the proof since such a $A$ obviously exists. □

## 3.3 Construction of the travelling waves in the regime $c \in (c^*, v_{\max})$.

Let $c \in (c^*, v_{\max})$, where $c^*$ denotes the minimal speed of Proposition 3.1. In order to prove the existence of travelling waves, we will prove that the solution of the following evolution equation, corresponding to equation (1.1) in the moving frame at speed $c$, converges to a travelling wave as $t \to +\infty$:

$$\begin{cases} \partial_t g + (v-c)\partial_x g = M(v)\rho_g - g + r\rho_g \left(M(v) - g\right) \text{ in } \mathbb{R} \times V, \\ g(0,x,v) = \overline{f}(x,v) \text{ for all } (x,v) \in \mathbb{R} \times V. \end{cases} \tag{3.20}$$

The well-posedness of equation (3.20) immediately follows from Proposition 2.1. Let now derive some properties of the function $g$ from Proposition 2.2.

**Lemma 3.6.** *For all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$, one has $\underline{f}(x,v) \le g(t,x,v) \le \overline{f}(x,v)$.*

**Proof of Lemma 3.6.** As $\underline{f}$ is a subsolution of (3.20) and $\overline{f}$ is a supersolution of (3.20), with $\underline{f}(x,v) \le \overline{f}(x,v)$ for all $(x,v) \in \mathbb{R} \times V$, this result is an immediate corollary of Proposition 2.2. □

**Lemma 3.7.** *For all $(t,v) \in \mathbb{R}_+ \times V$, the function $x \in \mathbb{R} \mapsto g(t,x,v)$ is nonincreasing.*

**Proof of Lemma 3.7.** Take $h \ge 0$ and define $g_h(t,x,v) = g(t,x+h,v)$. Then as $\overline{f}$ is nonincreasing in $x$, one has $g_h(0,x,v) \le g(0,x,v)$ for all $(x,v) \in \mathbb{R} \times V$. Proposition 2.2 yields that $g_h(t,x,v) \le g(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$. □

**Lemma 3.8.** *For all $(x,v) \in \mathbb{R} \times V$, the function $t \in \mathbb{R}_+ \mapsto g(t,x,v)$ is nonincreasing.*

**Proof of Lemma 3.8.** Take $\tau \ge 0$ and define $g_\tau(t,x,v) = g(t+\tau,x,v)$. Then Lemma 3.6 yields that $g_\tau(0,x,v) \le \overline{f}(x,v) = g(0,x,v)$ for all $(x,v) \in \mathbb{R} \times V$. Hence, Proposition 2.2 gives $g_\tau(t,x,v) \le g(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$. □

**Lemma 3.9.** *The family $(g(t,\cdot,\cdot))_{t \ge 0}$ is uniformly continuous with respect to $(x,v) \in \mathbb{R} \times V$. Moreover, for any $A \in (c^*, v_{\max})$, the continuity constants does not depend on $c \in (c^*, A)$.*

**Proof of Lemma 3.9.** We begin with the space regularity. Let $|h| < 1$. The function $g(0,x,v) = \overline{f}(x,v) = \min\{M(v), e^{-\lambda x} F_\lambda(v)\}$ is such that $\log g(0,x,v)$ is Lipschitz-continuous with respect to $x$. Therefore there exists a constant $C_0 > 0$ such that for all $(x,v) \in \mathbb{R} \times V$, we have $g(0,x+h,v) \leq (1 + C_0|h|) g(0,x,v)$. As $1 + C_0|h| > 1$, it is easily checked that $(t,x,v) \mapsto (1 + C_0|h|) g(t,x-h,v)$ is a supersolution of (3.20). Hence Proposition 2.2 yields that

$$g(t,x,v) \leq (1 + C_0|h|) g(t,x-h,v) \quad \text{for all } (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V\,.$$

Hence the function $\log g$ is Lipschitz continuous with respect to $x$. Since the function $\log g$ is bounded from above, $g = \exp(\log g)$ is also Lipschitz continuous with respect to $x$. The Lipschitz constant is uniform with respect to $c \in (c^*, A)$ and $\lambda \in (0, 1/(v_{\max} - c))$.

We now come to the velocity regularity. For the sake of clarity we first consider the case where $M$ is $\mathcal{C}^1$ on $V$. The function $v \mapsto g(0,x,v)$ is $\mathcal{C}^1$ too. We introduce $g_v = \partial_v g$. It satisfies the following equation

$$\partial_t g_v + (v-c)\partial_x g_v + (1 + r\rho_g) g_v = (1+r) M'(v) \rho_g - \partial_x g \text{ in } \mathbb{R} \times V\,.$$

Multiplying the equation by $\operatorname{sign} g_v$ we obtain

$$\partial_t |g_v| + (v-c)\partial_x |g_v| + (1 + r\rho_g)|g_v| \leq (1+r)|M'(v)|\rho_g + |\partial_x g| \text{ in } \mathbb{R} \times V\,.$$

The l.h.s. is linear with respect to $|g_v|$ and satisfies the maximum principle. The r.h.s. is uniformly bounded since $0 \leq \rho_g \leq 1$ and $g$ is uniformly Lipschitz with respect to $x$. Obviously the constant $(1+r)\sup_V |M'(v)| + \sup_{\mathbb{R}_+ \times \mathbb{R} \times V} |\partial_x g|$ is a supersolution. We deduce that $g_v$ is uniformly bounded over $\mathbb{R}_+ \times \mathbb{R} \times V$.

In the case where $M$ is only continuous over the compact set $V$, thus uniformly continuous, we shall use the method of translations again. However we have to be careful since $V$ is bounded. Let $0 < h < 1$. We introduce $H(v) = \max(v+h, v_{\max}) - v$. The function $g_H(t,x,v) = g(t,x,v + H(v)) - g(t,x,v)$ satisfies the following equation

$$\partial_t g_H + (v-c)\partial_x g_H + (1 + r\rho_g) g_H = (1+r)(M(v+H(v)) - M(v))\rho_g - H(v)\partial_x g(t,x,v+H(v))\,.$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $0 < h < \delta$ we have $|g_H(0,x,v)| \leq \delta$ and $|M(v+H(v)) - M(v)| < \delta$. Moreover we have obviously $0 < H(v) < \delta$. We conclude using the same argument as in the $\mathcal{C}^1$ case. The modulus of uniform continuity is uniform with respect to $c \in (c^*, A)$ and $\lambda \in (0, 1/(v_{\max} - c))$. □

We are now in position to prove the existence of travelling waves of speed $c$, except for the minimal speed $c^*$.

**Proof of the existence in Theorem 1.2 when $c > c^*$.** Gathering Lemmas 3.6, 3.7 and 3.8, we know that

$$f(x,v) = \lim_{t \to +\infty} g(t,x,v),$$

is well-defined for all $(x,v) \in \mathbb{R} \times V$, that $f(\cdot,v)$ is nonincreasing in $x$ for all $v$ and that $\underline{f} \leq f \leq \overline{f}$.

Let now prove that $f$ defines a travelling wave solution of (1.3). As $g$ satisfies (3.20), converges pointwise and is bounded by the locally integrable function $\overline{f}$, it follows from Lebesgue's dominated convergence theorem that $f$ satisfies (1.6) in the sense of distributions. Moreover, Lemma 3.9 ensures that $f$ is continuous with respect to $(x,v)$.

We next check the limits towards infinity. Let $f^\pm(v) = \lim_{x \to \pm\infty} f(x,v)$. Thanks to $f \leq \overline{f}$, the Lebesgue dominated convergence theorem gives $\rho_{f^\pm} = \int_V f^\pm(v)dv \leq 1$. On the other hand, we get

$$\left(M(v)\rho_{f^\pm} - f^\pm(v)\right) + r\rho_{f^\pm}\left(M(v) - f^\pm(v)\right) = 0 \tag{3.21}$$

in the sense of distributions. Integrating (3.21) over the compact set $V$, we deduce that $\rho_{f^\pm}\left(1 - \rho_{f^\pm}\right) = 0$, *i.e.* that $\rho_{f^\pm} = 0$ or 1. As $f$ is nonincreasing and $\underline{f} \leq f \leq \overline{f}$, one necessarily has $\rho_{f^+} = 0$ and $\rho_{f^-} = 1$. Finally, (3.21) gives $f^+(v) = 0$ and $f^-(v) = M(v)$ for all $v \in V$. This gives the appropriate limits. □

### 3.4 Construction of the travelling waves with minimal speed $c^*$.

**Proof of the existence in Theorem 1.2 when $c = c^*$.** Consider a decreasing sequence $(c_n)$ converging towards $c^*$. We already know that for all $n$, equation (1.1) admits a travelling wave solution $u_n(t, x, v) = f_n(x - c_n t, v)$, with $f_n(-\infty, v) = M(v)$ and $f_n(+\infty, v) = 0$, and $z \mapsto f_n(z, v)$ is nonincreasing. Up to translation, we can assume that $\rho_{f_n}(0) = 1/2$. Moreover, Lemma 3.9 ensures that the functions $(f_n)_n$ are uniformly continuous with respect to $(x, v) \in \mathbb{R} \times V$ since the continuity stated in Lemma 3.9 is uniform with respect to $c \in (c^*, A)$ for any $A \in (c^*, v_{\max})$. Thanks to the Ascoli theorem and a diagonal extraction process, we can assume that the sequence $(f_n)_n$ converges locally uniformly in $(x, v) \in \mathbb{R} \times V$ to a function $f$. Clearly $f$ satisfies (1.6) in the sense of distributions. Moreover, as $f$ is nonincreasing with respect to $x$, one could recover the appropriate limits at infinity with the same arguments as in the proof of the existence of travelling waves with speeds $c > c^*$. □

### 3.5 Non-existence of travelling wave solutions in the subcritical regime $c \in [0, c^*)$.

**Lemma 3.10.** *Assume that* $\inf_V M(v) > 0$. *For all* $0 \le c < c^*$ *there exists* $c < c_0 < c^*$ *and a nonnegative, arbitrarily small, compactly supported function* $h(x, v)$ *which is a subsolution of*

$$(v - c^0)\partial_x f = M(v)\rho_f - f + r\rho_f\,(M(v) - f) \qquad \text{in } \mathbb{R} \times V\,. \tag{3.22}$$

**Proof of Lemma 3.10.** For the sake of clarity we emphasize the dependence of the function $I$ (3.15) upon the growth rate $r > 0$:

$$I_r(\lambda; c) = \int_V \frac{(1 + r)M(v)}{1 + \lambda(c - v)}\,dv\,.$$

We denote by $c_r^*$ the smallest speed such that there exists a solution $\lambda > 0$ of $I_r(\lambda, c) = 1$ (see Proposition 3.1).

Let $\delta > 0$. By continuity we can choose $\delta$ so small that $c < c^*_{r-\delta}$. We claim that there exists $(c^0, \lambda^0)$ such that $I_{r-\delta}(\lambda^0; c^0) = 1$, with $c < c^0 < c^*_{r-\delta}$ and $\lambda^0 \in \mathbb{C} \setminus \mathbb{R}$. Indeed we know from the proof of Proposition 3.1 [Step 3] that $\lambda_r^* < 1/(v_{\max} - c_r^*)$ under the assumption $v \mapsto M(v)/(v_{\max} - v) \notin L^1(V)$. Using a continuity argument we also have the strict inequality $\lambda^*_{r-\delta} < 1/(v_{\max} - c^*_{r-\delta})$, uniformly with respect to $\delta$. The complex function $\lambda \mapsto I_{r-\delta}(\lambda; c^*_{r-\delta})$ is analytic in a neighborhood of $\lambda^*_{r-\delta}$. Hence, the Rouché theorem yields that there exists $c^0 < c^*_{r-\delta}$ such that the equation $I_{r-\delta}(\lambda; c^0) = 1$ has a solution $\lambda^0 \in \mathbb{C}$ with $\lambda^0$ arbitrarily close to $\lambda^*_{r-\delta}$. We denote by $F^0(v)$ the corresponding velocity profile,

$$F^0(v) = \frac{(1 + r - \delta)M(v)}{1 + \lambda^0(c^0 - v)}\,, \quad \int_V F^0(v)\,dv = 1\,.$$

By continuity we can choose $c^0$ and $\lambda^0$ such that $\mathrm{Re}\left(F^0(v)\right) > 0$ holds for all $v \in V$. By the very definition of $c^*_{r-\delta}$, we have $\lambda^0 \notin \mathbb{R}$. We denote $\lambda^0 = \lambda_R + i\lambda_I$. Recall that we have the strict inequality $\lambda^*_{r-\delta} < 1/(v_{\max} - c^*_{r-\delta})$, uniformly with respect to $\delta$. Using a continuity argument we can impose that $\lambda_R < 1/(v_{\max} - c^0)$.

Now define the real function $h^0$ by

$$h^0(x, v) = \mathrm{Re}\left(e^{-\lambda^0 x}F^0(v)\right) = e^{-\lambda_R x}\left[\mathrm{Re}\left(F^0(v)\right)\cos(\lambda_I x) + \mathrm{Im}\left(F^0(v)\right)\sin(\lambda_I x)\right], \tag{3.23}$$

One has $h^0(0, v) > 0$ and $h^0(\pm\pi/\lambda_I, v) < 0$ for all $v \in V$. Thus, there exists an interval $[b_1, b_2] \subset \mathbb{R}$ and a bounded domain $D \subset [b_1, b_2] \times V$ such that:

$$\begin{cases} h^0(x, v) > 0 & \text{for all } (x, v) \in D, \\ h^0(x, v) = 0 & \text{for } (x, v) \in \partial D. \end{cases}$$

On the other hand, as $\lambda_R < 1/(v_{\max} - c^0)$, there exists a constant $C(\delta)$ such that

$$(\forall v \in V) \quad |h^0(x, v)| \le e^{-\lambda_R b_1}|F^0(v)| = e^{-\lambda_R b_1}\frac{(1 + r - \delta)M(v)}{|1 + \lambda^0(c^0 - v)|} \le C(\delta)M(v)\,.$$

Hence, one can choose $\overline{\kappa} > 0$ small enough such that

$$r\overline{\kappa}h^0(x,v) \le \frac{\delta}{2}M(v) \quad \text{for all } (x,v) \in \mathbb{R}\times V\,.$$

For all $\kappa \in (0,\overline{\kappa})$ we deduce from $I_{r-\delta}(\lambda^0;c^0)=1$ the following identities,

$$\begin{aligned}\kappa(v-c^0)\partial_x\left(e^{-\lambda^0 x}F^0(v)\right)+\kappa\left(e^{-\lambda^0 x}F^0(v)\right) &= \kappa e^{-\lambda^0 x}(1+r-\delta)M(v)\\ &= \kappa(1+r-\delta)M(v)\int_V e^{-\lambda^0 x}F^0(v')\,dv'\,.\end{aligned}$$

Taking real part on both sides, we get for $(x,v)\in D$,

$$\begin{aligned}(v-c)\partial_x\left(\kappa h^0\right)+\kappa h^0 &= (1+r-\delta)M(v)\int_V \kappa h^0(x,v')\,dv'\\ &= M(v)\rho_{\kappa h^0}+rM(v)\rho_{\kappa h^0}-\delta M(v)\rho_{\kappa h^0}\\ &\le M(v)\rho_{\kappa h^0}+r\left(M(v)-\kappa h^0\right)\rho_{\kappa h^0}\,.\end{aligned}$$

Hence $\kappa h^0$ is a subsolution of (3.22) for all $\kappa\in(0,\overline{\kappa})$ on $D$. We deduce that the truncated function $h=\max(0,\kappa h^0)$ is a subsolution of (3.22) over $\mathbb{R}\times V$. □

**Proof of the non-existence in Theorem 1.2.** Assume that $f\in\mathcal{C}^0(\mathbb{R}\times V)$ is a travelling wave solution of (1.6) of speed $c\in(0,c^*)$. According to Lemma 3.10, there exists $c<c^0<c^*$ and a nonnegative compactly supported subsolution $h$ of (1.6) with speed $c^0$. As $f$ is positive and continuous, we can decrease $h$ so as to obtain $f\ge h$. Let $g_1(t,x,v)=f(x-ct,v)$ and $g_2(t,x,v)=h(x-c^0t,v)$. These two functions are respectively a solution and a subsolution of (1.1). As $g_1(0,x,v)=f(x,v)\ge h(x,v)=g_2(0,x,v)$ for all $(x,v)\in\mathbb{R}\times V$, Proposition 2.2 implies

$$g_1(t,x,v)=f(x-ct,v)\ge h(x-c^0t,v)=g_2(t,x,v) \quad \text{for all } (t,x,v)\in\mathbb{R}_+\times\mathbb{R}\times V.$$

Taking $x=c^0t$ and letting $t\to+\infty$, we get

$$0=\lim_{t\to+\infty} f\left((c^0-c)t,v\right)\ge h(0,v)\,.$$

This is a contradiction. □

## 3.6 Proof of the spreading properties

**Proof of Proposition 1.6.** 1. Let $c>c^*$. Consider first the initial datum

$$\widetilde{g}^0(x,v)=\begin{cases} M(v) & \text{if} \quad x<x_R\,,\\ 0 & \text{if} \quad x\ge x_R\,,\end{cases}$$

and let $\widetilde{g}$ the solution of the Cauchy problem (2.1). Denote by $f$ a travelling wave of minimal speed $c^*$. There exists $\kappa>1$ such that $\widetilde{g}^0(x,v)\le\kappa f(x,v)$ for all $(x,v)\in\mathbb{R}\times V$. It is straightforward to check that $g_1(t,x,v)=\kappa f(x-c^*t,v)$ is a supersolution of (1.1). Hence, the comparison principle of Proposition 2.2 implies that $\widetilde{g}(t,x,v)\le g_1(t,x,v)$ for all $(t,x,v)\in\mathbb{R}_+\times\mathbb{R}\times V$. In particular we have,

$$\widetilde{g}(t,ct,v)\le g_1(t,ct,v)=\kappa f\left((c-c^*)t,v\right) \quad \text{for all } (t,v)\in\mathbb{R}_+\times V.$$

As $f(+\infty,v)=0$ for all $v\in V$ and $c>c^*$, we get $\lim_{t\to+\infty}\widetilde{g}(t,ct,v)=0$ for all $v\in V$.

On the other hand, as $\widetilde{g}^0$ is nonincreasing with respect to $x\in\mathbb{R}$ it follows from the comparison principle that $x\mapsto\widetilde{g}(t,x,v)$ is nonincreasing (see Lemma 3.7). Thus $\widetilde{g}(t,x,v)\le\widetilde{g}(t,ct,v)$ for all $x\ge ct$ and the conclusion follows.

For a general initial datum $g^0$ satisfying the hypotheses of Proposition 1.6, one has $g^0(x,v) \le \widetilde{g}^0(x,v)$ for all $(x,v) \in \mathbb{R} \times V$ and thus $g(t,x,v) \le \widetilde{g}(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$, from which the conclusion follows.

2. Let $c < c^*$. The same arguments as in the first step yield that we can assume that

$$g^0(x,v) = \begin{cases} \delta M(v) & \text{if} \quad x < x_L\,, \\ 0 & \text{if} \quad x \ge x_L\,. \end{cases}$$

Let $h$ a compactly supported subsolution of (3.22) associated with a speed $c^0 \in (c, c^*)$. Since $h$ can be chosen arbitrarily small, up to translation of $h$, we can always assume that $h(x,v) \le g^0(x,v)$. Let $g_2$ the solution of the Cauchy problem (2.1) associated with the initial datum $g_2(0,x,v) = h(x,v)$. The comparison principle yields $g(t,x,v) \ge g_2(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$.

Let $w(t,x,v) = g_2(t, x + c^0 t, v)$. This function satisfies

$$\begin{cases} \partial_t w + (v - c^0)\partial_x w = M(v)\rho_w - w + r\rho_w\,(M(v) - w) & \text{in } \mathbb{R}_+ \times \mathbb{R} \times V \\ w(0,x,v) = g(x,v) & \text{in } \mathbb{R} \times V. \end{cases} \tag{3.24}$$

Clearly $h$ is a (stationary) subsolution of this equation. The comparison principle Proposition 2.2 yields that $t \mapsto w(t,x,v)$ is nondecreasing for all $(x,v) \in \mathbb{R} \times V$ (see also Lemma 3.8 for a similar proof).

Let $w_\infty(x,v) = \lim_{t \to +\infty} w(t,x,v)$. This function is clearly a weak solution of

$$(v - c^0)\partial_x w_\infty = M(v)\rho_{w_\infty} - w + r\rho_{w_\infty}\,(M(v) - w_\infty) \qquad \text{in } \mathbb{R} \times V.$$

Moreover, we have $w_\infty(x,v) \ge w(0,x,v) = h(x,v)$ and $w_\infty(x,v) \le M(v)$.

**Lemma 3.11** (Sliding lemma)**.** *We have $w_\infty \equiv M$.*

**Proof of Lemma 3.11.** # **Step 1.** First we prove that $w_\infty$ is positive everywhere.

Take $(x_0, v_0) \in \mathbb{R} \times V$ such that $w_\infty(x_0, v_0) > 0$. As $\widetilde{w}(t,x,v) = w_\infty(x - c^0 t, v)$ satisfies (1.1), Proposition 2.2 yields $\widetilde{w}(t,x,v) > 0$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$ such that $|x - x_0| < v_{\max} t$. As $c^0 < c^* \le v_{\max}$, for all $(x,v) \in \mathbb{R} \times V$ one can take $t > 0$ large enough so that $|x + ct - x_0| < v_{\max} t$. Therefore $w(x,v) = \widetilde{w}(t, x + ct, v) > 0$. We thus conclude that $w_\infty$ is positive over $\mathbb{R} \times V$.

# **Step 2.** Next we prove that $\inf w_\infty > 0$.

Let $y \in \mathbb{R}$. Define $h_y(x,v) = h(x - y, v)$, and

$$\kappa_y = \sup\{\kappa \in (0,1),\ w_\infty \ge \kappa h_y \text{ in } \mathbb{R} \times V\}\,.$$

As $h_y$ is compactly supported and $w_\infty$ is positive over $\mathbb{R} \times V$ and continuous, we have $w_\infty \ge \kappa h_y$ when $\kappa > 0$ is small enough. Therefore $\kappa_y > 0$.

We argue by contradiction. Assume that $\kappa_y < 1$. The definition of $\kappa_y$ yields that $u = w_\infty - \kappa_y h_y \ge 0$ and that $\inf_{\mathbb{R} \times V} u = 0$. As $h_y$ is compactly supported, this infimum is indeed reached: there exists $(x_y, v_y) \in \mathbb{R} \times V$ such that $u(x_y, v_y) = 0$. Assume that $u \not\equiv 0$ and take $(x'_y, v'_y) \in \mathbb{R} \times V$ such that $w_\infty(x'_y, v'_y) > \kappa_y h_y(x'_y, v'_y)$.

We introduce $w_1(t,x,v) = w_\infty(x - c^0 t, v)$ and $w_2(t,x,v) = \kappa_y h_y(x - c^0 t, v)$. As $w_1(0, x'_y, v'_y) > w_2(0, x'_y, v'_y)$, Proposition 2.2 gives $w_1(t,x,v) > w_2(t,x,v)$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$ such that $|x - x'_y| < v_{\max} t$, that is:

$$w_\infty(x - c^0 t, v) > \kappa_y h_y(x - c^0 t, v) \quad \text{ if } |x - x'_y| < v_{\max} t\,.$$

As $c^0 < c^* \le v_{\max}$, for all $x \in \mathbb{R}$, one can take $t > 0$ large enough so that $|x + c^0 t - x'_y| < v_{\max} t$, leading to $w_\infty(x,v) > \kappa_y h_y(x,v)$ for all $(x,v) \in \mathbb{R} \times V$, a contradiction since equality holds at $(x_y, v_y)$.

Hence, $w_\infty \equiv \kappa_y h_y$, which is also a contradiction since $w_\infty$ is positive while $h_y$ is compactly supported. We conclude that $\kappa_y = 1$, namely $w_\infty \ge h_y$. Evaluating this inequality at $x = y$, one gets $w_\infty(y,v) \ge h(0,v)$ for all $(y,v) \in \mathbb{R} \times V$. As $\inf_V g(0,v) > 0$ under the assumption $\inf_V M > 0$, we have proved in fact that

$$\inf_{\mathbb{R} \times V} w_\infty > 0\,.$$

# **Step 3.** As $\inf_V M > 0$, we can define

$$\kappa^* = \sup\{\kappa \in (0,1),\ w_\infty(x,v) \geq \kappa M(v) \text{ for all } (x,v) \in \mathbb{R} \times V\}.$$

We know from the previous step that this quantity is positive. If $\kappa^* < 1$, then the same types of arguments as in Step 2 lead to a contradiction. Hence $\kappa^* = 1$, meaning that $w_\infty \geq M(v)$. As $w_\infty \leq M(v)$, we conclude that $w_\infty \equiv M(v)$. □

As a consequence of Lemma 3.11 we obtain

$$\lim_{t \to +\infty} g_2(t, x + c^0 t, v) = M(v) \quad \text{for all } (x,v) \in \mathbb{R} \times V.$$

This implies in particular that $\lim_{t\to+\infty} g(t, x + c^0 t, v) = M(v)$ for all $(x,v)$ by a sandwiching argument. Moreover, as $g^0$ is nonincreasing with respect to $x$, $x \mapsto g(t,x,v)$ is nonincreasing and thus $g(t,x,v) \geq g(t, c^0 t, v)$ for all $x \leq c^0 t$, from which the conclusion follows since $c^0 > c$. □

# 4 Proof of the dependence results

**Proof of Proposition 1.5(a).** Recall that the dispersion relation giving the speed $c$ as a function of the exponential decay $\lambda$ is $I(\lambda; c) = 1$, where $I(\lambda; c)$ is defined in (3.15). Let introduce $I_\sigma$ the function associated with the dilated velocity profile $M_\sigma$. The function $I_\sigma$ clearly satisfies the scaling relation $I_\sigma(\lambda; c) = I(\sigma\lambda; \sigma^{-1} c)$, therefore we get $c^*(M_\sigma) = \sigma c^*(M)$ from the very definition of $c^*$. □

**Proof of Proposition 1.5(b).** We use the symmetry of the kernel $M(v) = M(-v)$ to write

$$I(\lambda; c) = \int_0^{v_{\max}} \frac{(1+r)(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 v^2} 2M(v)\, dv\,.$$

Let define

$$f(v) = \frac{(1+r)(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 v^2}.$$

It is an increasing function over $(0, v_{\max})$, thus $f_\star = f$. The Hardy-Littlewood inequality [32, Chap. 3] yields

$$\int_0^{v_{\max}} M^\star(v) f_\star(v) dv \leq \int_0^{v_{\max}} M(v) f(v) dv \leq \int_0^{v_{\max}} M_\star(v) f_\star(v) dv\,.$$

The dispersion relation is nonincreasing with respect to $c$. It follows immediately that

$$c^*(M^\star) \leq c^*(M) \leq c^*(M_\star).$$

□

**Proof of Proposition 1.5(c).** We use the symmetry of the kernel $M(v) = M(-v)$. For $\lambda > 0$ the dispersion relation writes

$$(1+r) \int_0^{v_{\max}} \frac{(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 v^2} 2M(v)\, dv = 1\,. \tag{4.25}$$

Since the function $X \mapsto \big((1+\lambda c)^2 - \lambda^2 X\big)^{-1}$ is convex on its domain of definition, Jensen's inequality yields

$$(1+r) \frac{(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 \left(2\int_0^{v_{\max}} v^2 M(v)\, dv\right)} \leq 1\,.$$

We recognize the dispersion relation associated with the two-speed model [5]. We deduce

$$\lambda^2 c^2 + (1-r)\lambda c - D\lambda^2 - r \geq 0\,.$$

This second-order polynomial has a negative value at $c = 0$, therefore $c$ is necessarily greater than the vanishing value,

$$c \geq \frac{(r-1) + \sqrt{(r-1)^2 + 4(D\lambda^2 + r)}}{2\lambda}\,.$$

Minimizing the right-hand-side with respect to $\lambda$, we deduce that $I(\lambda, c) = 1$ implies:

$$\begin{cases} c \geq \dfrac{2\sqrt{rD}}{1+r} & \text{if } r < 1\,, \\ c \geq \sqrt{D} & \text{if } r \geq 1\,. \end{cases}$$

On the other hand we clearly obtain from (4.25),

$$(1+r)\frac{(1+\lambda c)}{(1+\lambda c)^2 - \lambda^2 v_{\max}^2} \geq 1\,.$$

By comparison of the relations, as in the proof of Proposition 1.5(b), we deduce that the speed corresponding to a given Maxwellian $M(v)$ supported on $(-v_{\max}, v_{\max})$ is smaller than the speed corresponding to $\frac{1}{2}(\delta_{-v_{\max}} + \delta_{v_{\max}})$. This peculiar case is analysed in [5]. The minimal speed in this case is

$$c^*\left(\frac{1}{2}(\delta_{-v_{\max}} + \delta_{v_{\max}})\right) = \begin{cases} \dfrac{2\sqrt{r}}{1+r} v_{\max} & \text{if } r < 1\,, \\ v_{\max} & \text{if } r \geq 1\,. \end{cases}$$

□

**Proof of Proposition 1.5(d).** The dispersion relation for the rescaled equation (1.3) reads

$$I_\varepsilon(\lambda; c) = (1 + \varepsilon^2 r) \int_V \frac{1}{1 + \varepsilon^2 \lambda (c - v/\varepsilon)} M(v)\, dv\,. \tag{4.26}$$

The previous result guarantees that $c_\varepsilon^*$ is bounded from above for $\varepsilon^2 r < 1$,

$$c_\varepsilon^* \leq \frac{2\sqrt{\varepsilon^2 r}}{1+\varepsilon^2 r}\left(\frac{v_{\max}}{\varepsilon}\right) \leq 2\sqrt{r} v_{\max}\,.$$

For a given $\lambda > 0$, we perform a Taylor expansion of (4.26) up to second order,

$$I_\varepsilon(\lambda; c) = 1 + \varepsilon^2(r - \lambda c + \lambda^2 D) + O(\varepsilon^3)\,,$$

uniformly for $c \in [0, 2\sqrt{r} v_{\max}]$, since $V$ is bounded. Therefore, solving the relation dispersion for the minimal speed boils down to solving

$$r - \lambda c_\varepsilon(\lambda) + \lambda^2 D + O(\varepsilon) = 0\,.$$

We deduce

$$\lim_{\varepsilon \to 0} c_\varepsilon(\lambda) = \frac{r}{\lambda} + \lambda D\,.$$

Therefore the minimal speed verifies $\lim_{\varepsilon\to 0} c_\varepsilon^* = 2\sqrt{rD}$. □

# 5 Stability of the travelling waves

## 5.1 Linear stability

In this Subsection, we focus on the linearized problem around some travelling wave moving at speed $c \in [c^*, v_{\max})$. We recall that we consider a solution $u$ of the equation associated with the linearization around a travelling wave:

$$\partial_t u + (v - c)\partial_z u + (1 + r\rho_f)\, u = ((1+r)M - rf) \int_V u' dv'. \tag{5.27}$$

where the notation $'$ always stands in the sequel for a function of the $(t, z, v')$ variable. We shall prove stability of the wave in a suitable $L^2$ framework, inspired by [29, 20, 21, 5].

**Proof of Proposition 1.7.** We search for an ansatz $u = we^{\phi}$. The function $w$ satisfies:

$$\partial_t w + (v-c)\partial_z w + ((v-c)\partial_z \phi + 1 + r\rho_f)\, w = ((1+r)M - rf) \int_V e^{\phi' - \phi} w' dv', \tag{5.28}$$

From (5.28), we shall derive the dissipation inequality stated in Proposition (1.7). We test (5.28) against $w$ to obtain the kinetic energy estimate:

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}\times V} |w|^2\, dzdv\right) + \int_{\mathbb{R}\times V} ((v-c)\partial_z\phi + 1 + r\rho_f)\, |w|^2\, dzdv \\ = \int_{\mathbb{R}\times V\times V'} ((1+r)M - rf)\, e^{\phi'-\phi} w w' dv dv' dz. \tag{5.29}$$

We aim at choosing a weight $\phi$ such that the dissipation is coercive in $L^2$ norm. Let define the symmetric kernel $K$ as follows

$$K(v,v') = ((v-c)\partial_z\phi + 1 + r\rho_f)\, \delta_{v=v'} - \frac{1}{2}\left(((1+r)M - rf)\, e^{\phi'-\phi} + ((1+r)M' - rf')\, e^{\phi-\phi'}\right), \tag{5.30}$$

we seek a function $\phi$ such that

$$K(v,v') \geq A(z,v)\delta_{v=v'},$$

for a suitable positive function $A$, in the sense of kernel operators. For this purpose we focus on the eigenvalues of the kernel operator $A(z,v)\delta_{v=v'} - K(v,v')$.

**Definition 5.1** (Weight $\phi$)**.** *We introduce $\Lambda(z) \in \left[0, \frac{1}{v_{\max}-c}\right)$ the smallest solution of the following dispersion relation*

$$\int_V \frac{(1+r)M(v) - rf(z,v)}{1 + \Lambda(z)(c-v)}\, dv = 1\,, \tag{5.31}$$

*and we define $\Gamma(z)$ through the differential equation*

$$\frac{1}{2}\frac{\Gamma'(z)}{\Gamma(z)} = \Lambda(z)\,, \quad \Gamma(0) = 1\,. \tag{5.32}$$

*Finally we define*

$$\phi(z,v) = \frac{1}{2}\ln\left(\frac{(1+r)M(v) - rf(z,v)}{\Gamma(z)}\right), \tag{5.33}$$

Recall that $0 \leq f \leq M$, so that the weight $\phi$ is well-defined as soon as $\Lambda$ is well-defined. A small argumentation is required to prove that $\Lambda(z)$ is well-defined too. For a given $c$ and $z$, define

$$G(\Lambda) = \int_V \frac{(1+r)M(v) - rf(z,v)}{1 + \Lambda(c-v)}\, dv\,, \quad \Lambda \in \left[0, \frac{1}{v_{\max} - c}\right).$$

The function $G$ is continuous, and satisfies the following properties,

$$G(0) = (1+r) - r\rho_f(z) = (1+r)\,(1-\rho_f(z)) + \rho_f(z) \in [1, 1+r]\,, \\ G(\lambda) = \int_V \frac{(1+r)M(v) - rf(z,v)}{1+\lambda(c-v)} dv = 1 - \int_V \frac{rf(z,v)}{1+\lambda(c-v)} dv \leq 1\,,$$

where $\lambda$ is chosen such that $I(\lambda; c) = 1$. Thus we can define the smallest $\Lambda(z) \in [0, \lambda]$ such that $G(\Lambda(z)) = 1$.

**Lemma 5.2.** *Let $A$ be defined as*

$$A(z,v) = \frac{r}{2}\left(\rho_f(z) + \frac{f(z,v)}{(1+r)M(v) - rf(z,v)}\right),$$

*and $\mathbf{T}$ be the operator associated with the symmetric kernel $T(v,v') = A(z,v)\delta_{v=v'} - K(v,v')$. The operator $\mathbf{T}$ is nonpositive.*

**Proof of Lemma 5.2.** We shall prove that 0 is the Perron eigenvalue of the operator $\mathbf{T}$. For that purpose we shall exhibit a positive eigenvector in the nullset of $\mathbf{T}$. The equation $\mathbf{T}(W) = 0$ reads

$$(\forall v \in V) \quad \int_V \left(A(z,v)\delta_{v=v'} - K(v,v')\right) W(v')dv' = 0\,.$$

Plugging the formula for $K(v,v')$ (5.30) into this expression we get,

$$\begin{aligned}\left(A(z,v) - (v-c)\partial_z\phi(z,v) - 1 - r\rho_f(z)\right) W(v) + \frac{1}{2}\left((1+r)M(v) - rf(z,v)\right)\left(\int_V e^{\phi(z,v')-\phi(z,v)}W(v')\,dv'\right)\\ + \frac{1}{2}\int_V \left((1+r)M(v') - rf(z,v')\right)e^{\phi(z,v)-\phi(z,v')}W(v')\,dv' = 0\,.\end{aligned}$$

From the Definitions (5.31)-(5.33) we have,

$$\partial_z\phi(z,v) = -\frac{r}{2}\frac{\partial_z f(z,v)}{(1+r)M(v) - rf(z,v)} - \Lambda(z)\,.$$

The weight $\phi$ and the function $A$ are chosen such that

$$\begin{aligned}&A(z,v) - (v-c)\partial_z\phi(z,v) - 1 - r\rho_f(z)\\ &= \frac{r}{2}\left(\rho_f(z) + \frac{f(z,v)}{(1+r)M(v) - rf(z,v)} + (v-c)\frac{\partial_z f(z,v)}{(1+r)M(v) - rf(z,v)}\right) + (v-c)\Lambda(z) - 1 - r\rho_f(z)\\ &= \frac{r}{2}\left(2\rho_f(z)\right) + (v-c)\Lambda(z) - 1 - r\rho_f(z)\\ &= (v-c)\Lambda(z) - 1\,.\end{aligned}$$

Therefore the equation $\mathbf{T}(W) = 0$ is equivalent to

$$W(v) = \frac{1}{1+\Lambda(z)(c-v)}\left(\frac{1}{2}\left((1+r)M(v) - rf(z,v)\right)e^{-\phi(z,v)}X_1(z) + \frac{1}{2}e^{\phi(z,v)}X_2(z)\right)\,, \tag{5.34}$$

where the macroscopic quantities $X_1$ and $X_2$ are defined as follows,

$$X_1(z) = \int_V e^{\phi(z,v')}W(v')\,dv'\,, \quad X_2(z) = \int_V \left((1+r)M(v') - rf(z,v')\right)e^{-\phi(z,v')}W(v')\,dv'\,.$$

To resolve this eigenvalue problem, we seek proper values for $X_1$ and $X_2$. From (5.34) we deduce that they are solution of a $2\times 2$ closed linear system, namely

$$\begin{cases} X_1(z) = \frac{1}{2}\left(\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\,dv\right)X_1(v) + \frac{1}{2}\left(\int_V \frac{e^{2\phi(z,v)}}{1+\Lambda(z)(c-v)}\,dv\right)X_2(z)\\ X_2(z) = \frac{1}{2}\left(\int_V \frac{\left((1+r)M(v) - rf(z,v)\right)^2 e^{-2\phi(z,v)}}{1+\Lambda(z)(c-v)}\,dv\right)X_1(v) + \frac{1}{2}\left(\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\,dv\right)X_2(z)\end{cases}$$

This system simplifies thanks to the choice of $\Lambda(z)$ (5.31). Indeed we have

$$\begin{aligned}&\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\,dv = 1\\ &\int_V \frac{e^{2\phi(z,v)}}{1+\Lambda(z)(c-v)}\,dv = \left(\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\,dv\right)\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)}\\ &\int_V \frac{\left((1+r)M(v) - rf(z,v)\right)^2 e^{-2\phi(z,v)}}{1+\Lambda(z)(c-v)}\,dv = \left(\int_V \frac{(1+r)M(v) - rf(z,v)}{1+\Lambda(z)(c-v)}\,dv\right)\Gamma(z) = \Gamma(z)\,.\end{aligned}$$

We are reduced to the following eigenvalue problem,

$$\begin{pmatrix} X_1(z) \\ X_2(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \Gamma(z)^{-1} \\ \Gamma(z) & 1 \end{pmatrix} \begin{pmatrix} X_1(z) \\ X_2(z) \end{pmatrix} .$$

Clearly, $(X_1(z), X_2(z)) = (1, \Gamma(z))$ is the unique solution up to multiplication. We obtain eventually that $W$ is given (up to a multiplicative factor) by

$$\begin{aligned} W(v) &= \frac{1}{2} \frac{\left((1+r)M(v) - rf(z,v)\right) e^{-\phi(z,v)} + e^{\phi(z,v)}\Gamma(z)}{1 + \Lambda(z)(c-v)} \\ &= \frac{\left[\left((1+r)M(v) - rf(z,v)\right)\Gamma(z)\right]^{1/2}}{1 + \Lambda(z)(c-v)} > 0 . \end{aligned}$$

As a consequence, we have found that the symmetric operator $\mathbf{T}$, which is nonnegative out of the diagonal $v = v'$, possesses a positive eigenvector $W$ associated with the eigenvalue 0. Therefore it is a nonpositive operator. This ends the proof of the Lemma. □

We can now conclude the proof of Proposition 1.7. Lemma 5.2 claims that for all $w \in L^2(\mathbb{R})$ such that $u = e^{\phi} w$ is solution to the linearized equation, we have

$$\frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}\times V} |w|^2 \, dzdv \right) + \int_{\mathbb{R}\times V} A(z,v) \, |w|^2 \, dzdv \leq 0 .$$

which proves the Proposition.

□

**Remark 5.3** (Non optimality of the weight)**.** *We believe that the weight* $\exp(\phi(z,v))$ *proposed in Definition 5.1 is not optimal with respect to the spectral property of the linearized operator* (5.27)*. Indeed the dissipation factor* $A(z,v)$ *is equivalent in the diffusion limit* ($r \to r\varepsilon^2$) *to* $r\varepsilon^2 \rho_f(z)$*, although we expect* $2r\varepsilon^2\rho_f(z)$ *[29, 12]. The missing factor* 2 *is responsible for the restriction* $\gamma > 1/2$ *in our nonlinear stability result, Theorem 1.8.*

*Let us recall how to derive the spectral properties of the linearized equation in the diffusive limit, namely the linearized Fisher-KPP equation,*

$$\partial_t u - c\partial_z u - D\partial_{zz} u = r(1 - 2\rho_f)u , \tag{5.35}$$

*where* $\rho_f(z)$ *is the profile of the travelling wave in the frame* $z = x - ct$*. Applying the same procedure as in the proof of Proposition 1.7, we shall derive an equation for the weighted perturbation* $w = e^{-\phi}u$*, and optimize the dissipation with respect to the weight* $\phi$ *(see also [5]), as follows*

$$\begin{aligned} &\partial_t w - c\partial_z w - D\partial_{zz} w - 2D\partial_z\phi\partial_z w - D\partial_{zz}\phi w - (c\partial_z\phi + D|\partial_z\phi|^2)w = r(1-2\rho_f)w \\ &\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} |w|^2 \, dz\right) + D\int_{\mathbb{R}} |\partial_z w|^2 \, dz + \int_{\mathbb{R}} \left(2r\rho_f - r - c\partial_z\phi - D|\partial_z\phi|^2\right)|w|^2 \, dz = 0 \end{aligned}$$

*The best choice is achieved when* $\partial_z\phi$ *is constant and minimizes* $r + c\lambda + D\lambda^2$*,* i.e. $\partial_z\phi = -c/(2D)$*. In the case of the minimal speed* $c = c^* = 2\sqrt{rD}$*, we obtain the following dissipation formula for the linearized operator,*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} |w|^2 \, dz\right) + D\int_{\mathbb{R}} |\partial_z w|^2 \, dz + \int_{\mathbb{R}} 2r\rho_f |w|^2 \, dz = 0 . \tag{5.36}$$

*Notice the factor* 2 *which is apparently missing in the dissipation term* (1.9)*.*

*A systematic way to find the correct weight is to derive the eigenvectors of the operator and its dual, then to use the framework of relative entropy (see [34] for a general presentation). This was done by Kirchgässner [29] who derived the so-called* eichform *for* (5.35)*. The linearized operator* $\mathcal{L}(u) = -c\partial_z u - \partial_{zz} u - (1 - 2\rho_f)u$ *possesses obviously the nonpositive eigenvector* $\eta = \partial_z\rho_f$*,* $\mathcal{L}(\eta) = 0$*. The dual operator* $\mathcal{L}^*(\varphi) = +c\partial_z\varphi - \partial_{zz}\varphi - (1 - 2\rho_f)\varphi$ *possesses the nonpositive eigenvector* $\psi = \partial_z\rho_f e^{cz}$*,* $\mathcal{L}^*(\psi) = 0$*, as can be checked by*

*direct calculation. Therefore the relative entropy identity for the convex function $H(p) = \frac{1}{2}|p|^2$ writes for the linearized system as follows,*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}\psi(z)\left(\frac{u(t,z)}{\eta(z)}\right)^2\eta(z)\,dz\right)+\int_{\mathbb{R}}\psi(z)\left|\frac{\partial}{\partial z}\left(\frac{u(t,z)}{\eta(z)}\right)\right|^2\eta(z)\,dz=0\,,$$

*which is equivalent to* (5.36) *after straightforward computation (recall $w = e^{(c/2D)z}u$).*

*A similar strategy could be performed here: the linearized operator $\mathcal{L}(u) = (v-c)\partial_z u + (1+r\rho_f)\,u - ((1+r)M - rf)\int_V u'dv'$ possesses the nonpositive eigenvector $\eta = \partial_z f$, $\mathcal{L}(\eta) = 0$ (recall $z \mapsto f(z,v)$ is nonincreasing). To derive the corresponding relative entropy identity, we should find an eigenvector $\psi$ in the nullset of the dual operator*

$$\mathcal{L}^*(\varphi) = -(v-c)\partial_z\varphi + (1+r\rho_f)\,\varphi - \int_V((1+r)M' - rf')\,\varphi' dv'\,.$$

*Existence of such an eigenvector would follow from the Krein-Rutman Theorem. However we were not able to find an explicit formulation of $\psi$, and thus of the dissipation, which is necessary to derive a quantitative nonlinear stability estimate such as Proposition 1.8. This is the reason why we stick to the weight proposed in Definition 5.1 although we believe it is not the optimal one.*

## 5.2 Nonlinear stability by a comparison argument.

**Proof of Proposition 1.8.** First, the comparison principle of Proposition 2.2 and (1.10) yield

$$\rho_u(t,z) \geq (\gamma-1)\,\rho_f(t,z), \qquad \forall (t,z)\in\mathbb{R}_+\times\mathbb{R}. \tag{5.37}$$

Now, we write the nonlinear equation verified by the weighted perturbation $w = e^{-\phi}u$,

$$\partial_t w + (v-c)\partial_z w + ((v-c)\partial_z\phi + 1 + r\rho_f)\,w = ((1+r)M - rf)\int_V e^{\phi'-\phi}w'dv' - rw\rho_u\,, \tag{5.38}$$

and as for the linear stability problem we test (5.38) against $w$:

$$\frac{d}{dt}\left(\int_{\mathbb{R}\times V}\frac{|w|^2}{2}dzdv\right)+\int_{\mathbb{R}\times V}((v-c)\partial_z\phi+1+r\rho_f)\,|w|^2\,dzdv$$
$$=\int_{\mathbb{R}\times V\times V'}w\,((1+r)M-rf)\,e^{\phi'-\phi}w'dzdvdv' - \int_{\mathbb{R}\times V} r|w|^2\rho_u dzdv\,.$$

Using (5.37) we deduce

$$\frac{d}{dt}\left(\int_{\mathbb{R}\times V}\frac{|w|^2}{2}dzdv\right)+\int_{\mathbb{R}\times V}((v-c)\partial_z\phi+1+\gamma r\rho_f)\,|w|^2\,dzdv$$
$$\leq\int_{\mathbb{R}\times V\times V'}w\,((1+r)M-rf)\,e^{\phi'-\phi}w'dvdv'dz.$$

This last equation is very similar to (5.29). Following the same steps as in the proof of Proposition 1.7, we find that using the same weight $\phi$ and setting

$$A(z,v) = \frac{r}{2}\left((2\gamma-1)\rho_f + \frac{f}{(1+r)M(v)-rf}\right)\,,$$

we obtain the estimate (1.11).

□

# 6 Numerics

In this Section, we provide some numerical simulations to illustrate our results, and to introduce the last Section about accelerating fronts. We used a very basic explicit numerical scheme for approximating (1.1). The transport part is approximated by a upwind scheme.

We show in Figure 1 the expected asymptotic behavior when the velocity space is bounded, namely convergence towards a travelling front with minimal speed.

Next we have investigated the case of $V = \mathbb{R}$. Of course, numerical simulations require that the support of $M$ is truncated. We opted for the following strategy: we truncated the velocity set to a finite interval $V_A = [-A, A]$ and we renormalized the distribution $M$. For any $A > 0$ we observed the asymptotic regime of a travelling front with finite speed, as expected. However, as $A \to +\infty$, we observed no stabilization of the asymptotic spreading speed. In fact, we observed that the envelope of the spreading speed scales approximately as $\langle c \rangle \sim t^{1/2}$. Hence the front is accelerating with the approximate power law $\langle x \rangle \sim t^{3/2}$. We expect that the scaling strongly depends on the decay of $M$ as $|v| \to +\infty$.

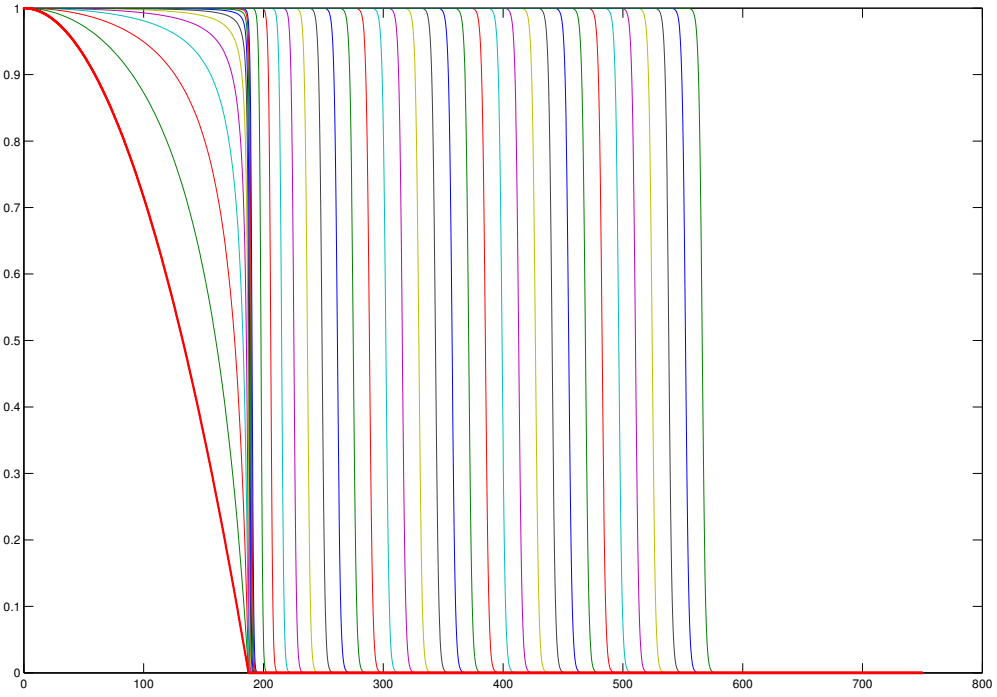


Figure 1: Numerical simulation of equation (1.1) with initial datum being defined as $g^0(x < 0, v) = M(v)$ and $g^0(x > 0, v) = M(v)\left(1 - \alpha x^2\right)_+$. The density distribution $M(v)$ is a truncated Gaussian on a compact velocity set. We plot the evolution of the macroscopic density $\rho_g$ (initial condition in red bold). After short time the density has accumulated towards a steep profile. Then the front starts to propagate with constant speed.

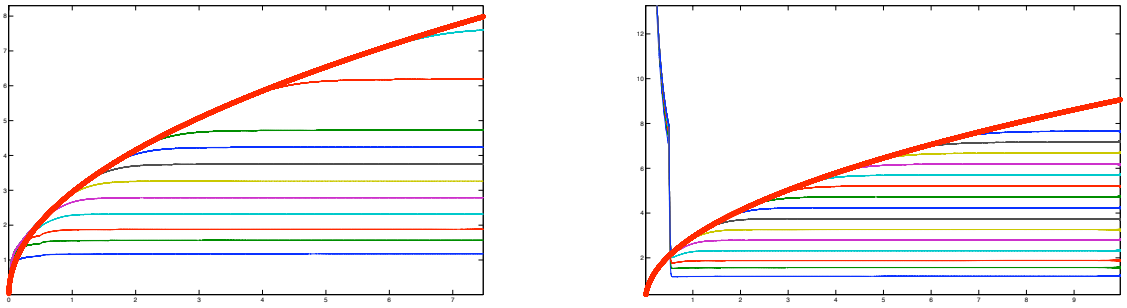

Figure 2: Numerical simulations of equation (1.1) with initial datum being defined as (left) $g^0(x<0,\cdot) = M(\cdot)$ and $g^0(x>0,\cdot)=0$, and (right) the same initial condition as in Figure 1. The distribution $M$ is a Gaussian. Each plot corresponds to the evolution of speed of the front for some truncation $V=[-A,A]$, for (left) $A=[(1:9),15,20]$, and (right) $A=(1:15)$. The curves are ordered from bottom to top: the speed of the front increases with $A$. We plot in red bold the function $t\mapsto t^{1/2}$. We observe that it fits very well with the envelop of the family of curves. As a consequence, the front propagation scales approximately as $x\sim t^{\frac{3}{2}}$.

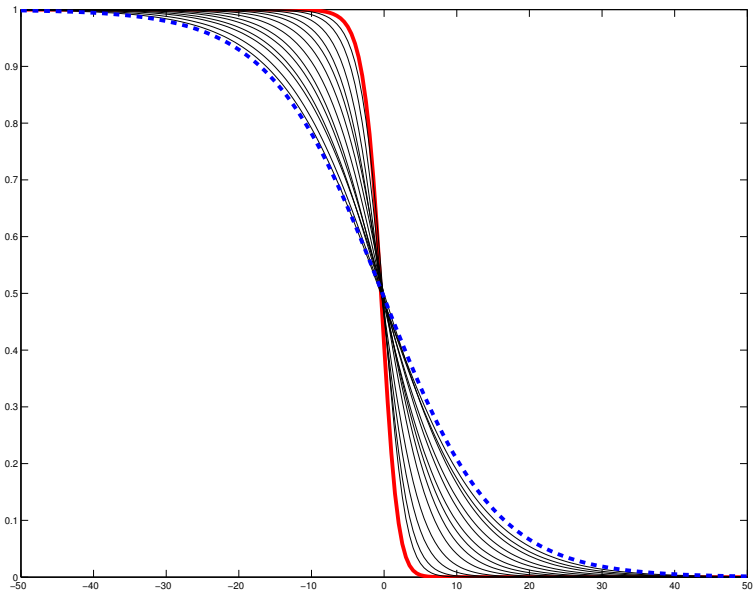

Figure 3: Same numerical simulation as Figure 6 with the same initial datum as in Figure 6. We superpose various macroscopic profiles $\rho_g$ obtained as long time simulations of the scheme, for different truncation levels $A=(1:15)$. Time $T$ is the same for all profiles, and sufficiently large to guarantee that we have reached the asymptotic regime (Figure 6, right). All profiles are translated such that $\rho_g(T,0)=\frac{1}{2}$. We observe that the exponential decay is monotonically decreasing with $A$. This indicates that the solution corresponding to $V=\mathbb{R}$ should flatten when $t\to\infty$.

# 7 The case of an unbounded velocity space ($V=\mathbb{R}$)

We assume in this Section that $V=\mathbb{R}$ and that $M(v)>0$ for all $v\in\mathbb{R}$. We prove superlinear spreading. We deduce as a Corollary that there cannot exist a travelling wave solution of (1.1). We also give some

quantitative features about the spreading of the density in two cases: the case where $M$ is a Gaussian, and the case where it is a Cauchy's Law. We construct explicit supersolutions from which we estimate the spreading from above. We expect those estimates to be sharp. In fact they are in accordance with numerical simulations. As we are not able to construct suitable subsolutions, we leave it as an open problem to prove the exact spreading rate, at least for these two specific cases (Gaussian and Cauchy's law).

Before we go to the proof, let us give some heuristics concerning the superlinear spreading rate. Reaction-diffusion fronts with KPP nonlinearity are *pulled fronts*: the spreading rate is determined by the dynamics of small populations at the far edge of the front. In the kinetic model with unbounded velocities, individuals with arbitrary large speeds go at the far edge of the front. No matter their low density, they yield exponential growth of the population and pull the accelerating front. Of course we expect the acceleration to depend on the specific tail of the distribution $M(v)$. Actually we conjecture that the spreading rates for a Gaussian and a Cauchy's law scales respectively as $t^{3/2}$, and as $te^{rt}$.

Last, let us emphasize that the diffusive limit of (1.1) leads to the classical Fisher-KPP equation, under the assumption that $M$ has some finite moments which is obviously the case for a Gaussian. The Fisher-KPP equation exhibits linear spreading whereas the kinetic equation may exhibit superlinear spreading. The point is that small populations does not show the same scaling at the far edge of the front. To summarize we shall say that asymptotics of large deviations are very much different in the two cases: reaction-diffusion as opposed to kinetic transport-reaction.

## 7.1 Unexistence of travelling waves and superlinear spreading

**Proof of Proposition 1.9.** Let $\underline{A} > 0$ so that $(1+r)\int_{-\underline{A}}^{\underline{A}} M(v)dv = 1$. For all $A > \underline{A}$ we define the renormalized truncated kernel and the associated growth rate,

$$M_A(v) = \frac{1_{[-A,A]}(v)}{\int_{-A}^{A} M} M(v) \quad \text{and} \quad r_A = (1+r)\int_{-A}^{A} M(v)dv - 1 \in (0,r)\,.$$

As $M_A$ is compactly supported, we can apply the results proved when $V$ is bounded in order to construct appropriate subsolutions.

Before we proceed with subsolutions we investigate the dispersion relation in the limit $A \to +\infty$. Define for all $c \in (0,A)$ and $\lambda \in (0, 1/(A-c))$:

$$I_A(\lambda; c) = (1+r_A)\int_{\mathbb{R}} \frac{M_A(v)}{1+\lambda(c-v)}dv = (1+r)\int_{-A}^{A} \frac{M(v)}{1+\lambda(c-v)}dv$$

and $c_A^*$ the corresponding minimal speed defined in Lemma 3.3.

**Lemma 7.1.** *One has* $\lim_{A\to+\infty} c_A^* = +\infty$.

**Proof of Lemma 7.1.** For all $A > \underline{A}$, let $\lambda_A \in (0, 1/(A-c_A^*))$ such that

$$I_A(\lambda_A; c_A^*) = (1+r)\int_{-A}^{A} \frac{M(v)}{1+\lambda_A(c_A^*-v)}dv = 1. \tag{7.39}$$

If $c_A^*$ does not diverge to $+\infty$ as $A \to +\infty$, then it is bounded along a sequence $(A_n)_n$ and one has $\lim \lambda_{A_n} = 0$ simply by comparison $\lambda_{A_n} \le 1/(A_n - c_{A_n}^*)$. Applying Fatou's lemma to (7.39), one gets

$$(1+r)\int_{\mathbb{R}} \liminf_{n\to+\infty} \frac{M(v)1_{(-A_n,A_n)}(v)}{1+\lambda_{A_n}(c_{A_n}^*-v)}dv = (1+r)\int_{\mathbb{R}} M(v)dv = 1+r \le \liminf_{n\to+\infty} I_{A_n}(\lambda_{A_n}, c_{A_n}^*) = 1\,,$$

a contradiction. □

Let $g_A$ the solution of

$$\begin{cases} \partial_t g_A + v\partial_x g_A = M_A(v)\rho_{g_A} - g_A + r_A\rho_{g_A}\left(M_A(v) - g_A\right) & \text{in } \mathbb{R}_+ \times \mathbb{R} \times [-A,A], \\ g_A(0,x,v) = g^0(x,v) & \text{in } \mathbb{R} \times [-A,A], \end{cases} \tag{7.40}$$

and $\widetilde{g}_A = \frac{r_A}{r} g_A$. Clearly, $M_A(v) \le \frac{M(v)}{\int_{-A}^{A} M}$ for all $v \in V$. Hence, multiplying (7.40) by $\frac{r_A}{r}$, we get

$$\begin{aligned}
\partial_t \widetilde{g}_A + v\partial_x \widetilde{g}_A &\le \frac{M(v)}{\int_{-A}^{A} M} \rho_{\widetilde{g}_A} - \widetilde{g}_A + r_A \rho_{\widetilde{g}_A} \left( \frac{M(v)}{\int_{-A}^{A} M} - g_A \right) \\
&\le (1 + r_A) \frac{M(v)}{\int_{-A}^{A} M} \rho_{\widetilde{g}_A} - \widetilde{g}_A - r_A \rho_{\widetilde{g}_A} g_A \\
&= (1 + r) M(v) \rho_{\widetilde{g}_A} - \widetilde{g}_A - r_A \rho_{\widetilde{g}_A} g_A \\
&= (1 + r) M(v) \rho_{\widetilde{g}_A} - \widetilde{g}_A - r \rho_{\widetilde{g}_A} \widetilde{g}_A \\
&= M(v) \rho_{\widetilde{g}_A} - \widetilde{g}_A + r \rho_{\widetilde{g}_A} \left( M(v) - \widetilde{g}_A \right).
\end{aligned}$$

Extending $\widetilde{g}_A$ by 0 outside of $\mathbb{R}_+ \times \mathbb{R} \times [-A, A]$, as $\widetilde{g}_A(0, x, v) = \frac{r_A}{r} g^0(x, v) \le g^0(x, v)$, we get that $\widetilde{g}_A$ is a subsolution of (1.1) and it follows from the maximum principle stated in Proposition 2.2 that $g(t, x, v) \ge \widetilde{g}_A(t, x, v)$ for all $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$.

On the other hand, we know from Proposition 1.6 that for all $c < c_A^*$:

$$\lim_{t \to +\infty} \sup_{x \le ct} |M_A(v) - \widetilde{g}_A(t, x, v)| = \lim_{t \to +\infty} \sup_{x \le ct} \left( M_A(v) - \widetilde{g}_A(t, x, v) \right) = 0.$$

Hence, as $M(v) \ge g(t, x, v) \ge \widetilde{g}_A(t, x, v)$ and $M_A(v) \ge M(v)$ for all $v \in [-A, A]$, one gets for all $v \in [-A, A]$: $0 \le \lim_{t \to +\infty} \sup_{x \le ct} \left( M(v) - g(t, x, v) \right) \le 0$. Therefore we conclude

$$\lim_{t \to +\infty} \sup_{x \le ct} |M(v) - g(t, x, v)| = 0 \quad \text{for all } c < c_A^* \text{ and } A > \underline{A}.$$

the conclusion follows from the fact that $\lim_{A \to +\infty} c_A^* = +\infty$.

□

## 7.2 Upper bound for the spreading rate

We construct below supersolutions for (1.1) when $V = \mathbb{R}$ and the distribution $M$ is a Gaussian.

**Proposition 7.2.** *Let $V = \mathbb{R}$ and $M(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$. For $1 \le b \le a$ define*

$$\rho(t, x) = M\left(\frac{x}{t + a}\right) e^{r(t+a)} \quad \text{and} \quad g^0(x, v) = \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra}. \tag{7.41}$$

*Let $g$ be defined by*

$$g(t, x, v) = g^0(x - vt, v) e^{-t} + \int_0^t (1 + r) M(v) \rho(s, x - v(t - s)) e^{-(t-s)} ds.$$

*Then $\overline{g}(t, x, v) = \min\left\{M(v), g(t, x, v)\right\}$ is a supersolution of (1.1), that is:*

$$\partial_t \overline{g} + v\partial_x \overline{g} \ge (M(v) \rho_{\overline{g}} - \overline{g}) + r \rho_{\overline{g}} \left( M(v) - \overline{g} \right), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R} \times V.$$

**Proof of Proposition 7.2.** We shall prove that $g$ is a supersolution of (1.1). Indeed, it will follow that $\overline{g}$ is a supersolution since it is the minimum of two supersolutions.

From the Duhamel formula, we deduce that

$$\partial_t g + v\partial_x g + g = (1 + r) M(v) \rho,$$

To prove that $g$ is a subsolution we must prove in fact that

$$(1 + r) M(v) \rho \ge (1 + r) M(v) \rho_g - r \rho_g g.$$

This is sufficient to prove that the inequality $\rho \geq \rho_g$ holds true. Computing the expression of $\rho_g$ we obtain

$$\rho_g(t,x) = \underbrace{\int_V g^0(x-vt,v)e^{-t}dv}_{=A} + \underbrace{\int_0^t (1+r)e^{-(t-s)+r(s+a)} \int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right) dvds}_{=B}$$

We first deal with the estimate of $B$. We claim the following inequality holds true: for all $x \in \mathbb{R}$ and $s \in [0,t]$,

$$\int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right) dv \leq M\left(\frac{x}{t+a}\right) \tag{7.42}$$

In fact one has

$$\begin{aligned}
\int_V M(v)M\left(\frac{x-v(t-s)}{s+a}\right) dv &= \int_V \frac{1}{2\pi\sigma^2} \exp\left(-\frac{v^2}{2\sigma^2} - \frac{\left(\frac{x-v(t-s)}{s+a}\right)^2}{2\sigma^2}\right) dv \\
&= \frac{1}{\sqrt{2\pi}\sigma} \frac{s+a}{[(s+a)^2+(t-s)^2]^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2}\frac{x^2}{(s+a)^2+(t-s)^2}\right) \\
&\leq \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}\frac{x^2}{(t+a)^2}\right),
\end{aligned}$$

since

$$\forall s \in [0,t], \quad (t+a)^2 \geq (s+a)^2 + (t-s)^2 \geq (s+a)^2.$$

This yields

$$\begin{aligned}
B(t,x) &\leq (1+r)\left(\int_0^t e^{-(t-s)+r(s+a)}ds\right) M\left(\frac{x}{t+a}\right) \\
&= e^{ra-t}\left(e^{(1+r)t}-1\right) M\left(\frac{x}{t+a}\right) \\
&= \left(e^{(1+r)t}-1\right) e^{-t+ra}\rho(t,x)e^{-r(t+a)} \\
&= \left(1-e^{-(1+r)t}\right)\rho(t,x)
\end{aligned}$$

To estimate $A$, we plug in the formula for $\rho$ (7.41),

$$\left(\frac{A}{\rho}\right)(t,x) = \sqrt{2\pi}\sigma \exp\left(\frac{x^2}{2\sigma^2(t+a)^2} - (1+r)t - ra\right) \int_V g^0(x-vt,v)dv\,.$$

We compute the last integral using the formula for the initial condition $g^0$ (7.41),

$$\int_V g^0(x-vt,v)dv = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{(t^2+b^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2}\frac{x^2}{t^2+b^2}\right) e^{ra},$$

Thus, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\left(\frac{A}{\rho}\right)(t,x) = \frac{1}{(t^2+b^2)^{\frac{1}{2}}} \exp\left(-\frac{x^2}{2\sigma^2(t+a)^2}\left[\frac{(t+a)^2}{t^2+b^2}-1\right]\right) \exp\left(-(1+r)t\right) \leq \exp\left(-(1+r)t\right) \tag{7.43}$$

as long as $b \geq 1$ and $(t+a)^2 \geq t^2+b^2$, that is $a \geq b \geq 1$. This concludes the proof.

□

**Proof of Theorem 1.11.** Let $\varepsilon > 0$. For all $t \geq 0$, we define the zone $\Gamma_t = \left\{x \in \mathbb{R} \,|\, |x| \geq \sigma\,(1+\varepsilon)\,\sqrt{2r}(t+a)^{3/2}\right\}$. From the definition of $\overline{g}$, we deduce that $\overline{g}$ is a supersolution such that $\rho_{\overline{g}} \leq \min\,(1, \rho_g) \leq \min\,(1, \rho)$. However, for all $t > 0$ and $x \in \Gamma_t$, we have

$$\rho(t,x) \leq \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{2r\sigma^2\,(1+\varepsilon)^2\,(t+a)^3}{2\sigma^2(t+a)^2} + r(t+a)\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-r(t+a)\left((1+\varepsilon)^2 - 1\right)\right).$$

It yields that

$$\lim_{t\to+\infty} \sup_{x\in\Gamma_t} \rho(t,x) \longrightarrow 0.$$

□

Computations are made easier above since the Gaussian distribution is stable by convolution. This is also the case for the Cauchy distribution. Therefore we are able to derive an inequality similar to (7.42). Let us comment that specific case before we give the spreading estimate. Because the distribution $M$ has an infinite variance, we learn from [33] that the correct macroscopic limit leads to a nonlocal fractional Laplacian operator. On the other hand, we expect from [8, 9, 11] an exponentially fast propagation in the diffusion regime. Similarly as for our previous results, we can reasonably expect that the spreading rate is faster in the kinetic model than in the macroscopic limit. Therefore we expect a spreading rate faster than exponential. In fact the supersolution that we are able to derive confirms this expectation.

**Proposition 7.3.** *Let $V = \mathbb{R}$ and $M(v) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+v^2}$. For $a \geq \frac{5}{4}$ and $b \in \left[1, a - \frac{1}{4}\right]$, define*

$$\rho(t,x) = M\left(\frac{x}{t+a}\right) e^{r(t+a)} \quad \text{and} \quad g^0(x,v) = \frac{1}{b} M\left(\frac{x}{b}\right) M(v) e^{ra}\,. \tag{7.44}$$

*Let $g$ be defined by*

$$g(t,x,v) = g^0(x - vt, v)e^{-t} + \int_0^t (1+r)M(v)\rho(s, x - v(t-s))e^{-(t-s)}ds\,.$$

*Then $\overline{g}(t,x,v) = \min\left\{M(v)\,, g(t,x,v)\right\}$ is a supersolution of (1.1), that is:*

$$\partial_t \overline{g} + v\partial_x \overline{g} \geq (M(v)\rho_{\overline{g}} - \overline{g}) + r\rho_{\overline{g}}\,(M(v) - \overline{g})\,, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V\,.$$

**Proof of Proposition 7.3.** The proof is the same as for Proposition 7.2. We just show the main computations in the case of the Cauchy distribution. To prove (7.42) we use the residue method as follows,

$$\begin{aligned}
\int_V M(v) M\left(\frac{x - v(t-s)}{s+a}\right) dv &= \int_V \frac{\sigma^2}{\pi^2}\frac{1}{\sigma^2+v^2} \cdot \frac{1}{\sigma^2 + \left(\frac{x-v(t-s)}{s+a}\right)^2} dv \\
&= \frac{\sigma}{\pi}\frac{(s+a)(t+a)}{x^2 + \sigma^2(t+a)^2} \\
&\leq M\left(\frac{x}{t+a}\right)
\end{aligned}$$

The analog computation for proving (7.43) goes as follows. First we have

$$\left(\frac{A}{\rho}\right)(t,x) = \pi\left(1 + \left(\frac{x}{t+a}\right)^2\right) \exp\left(-(1+r)t - ra\right) \int_V f^0(x - vt, v) dv$$

Thanks to the expression of the initial condition, we compute the latest integral:

$$\int_V f^0(x - vt, v) dv = \frac{1}{\pi}\frac{t+b}{x^2 + (t+b)^2} e^{ra},$$

Thus,

$$\left(\frac{A}{\rho}\right)(t,x) = \frac{t+b}{(t+a)^2}\frac{x^2 + (t+a)^2}{x^2 + (t+b)^2} \exp\left(-(1+r)t\right) \leq \exp\left(-(1+r)t\right)$$

which holds true if $b \geq 1$ and $a \geq b + \frac{1}{4}$. □

# Appendix

We give in this Section the proofs of Propositions 2.1 and 2.2. Well-posedness relies on a fixed point argument which is also used for the comparison principle. We first state two Lemmas.

**Lemma 7.4.** *Let $a, b \in \mathcal{C}_b^0(\mathbb{R}_+ \times \mathbb{R} \times V)$ and $g^0 \in \mathcal{C}_b^0\left(\mathbb{R}, L^1(V)\right)$. Then there exists a unique function $g \in \mathcal{C}_b^0\left(\mathbb{R}_+ \times \mathbb{R}, L^1(V)\right)$ such that*

$$\left\{ \begin{array}{lll} \partial_t g + v\partial_x g + a(t,x,v)g = b(t,x,v)\rho_g & in & \mathbb{R}_+ \times \mathbb{R} \times V, \\ g(0,x,v) = g^0(x,v) & in & \mathbb{R} \times V, \end{array} \right. \tag{7.45}$$

*in the sense of distributions. This solution also satisfy the Duhamel formula:*

$$\begin{aligned} g(t,x,v) = g^0(x-vt,v)e^{-\int_0^t a(s,x-(t-s)v,v)ds} \\ + \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s,x-v(t-s),v\right)\rho_g\left(s,x-v(t-s),v\right)\,ds\,. \end{aligned} \tag{7.46}$$

*Moreover, if $b \geq 0$ and $g^0 \geq 0$, then $g \geq 0$ in $\mathbb{R}_+ \times \mathbb{R} \times V$.*

**Proof of Lemma 7.4.** For $T > 0$ we define the operator

$$\begin{array}{rccc} A_T: & \mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right) & \to & \mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right) \\ & g & \mapsto & \widetilde{g} \end{array} \tag{7.47}$$

where

$$\begin{aligned} \widetilde{g}(t,x,v) = g^0(x-vt,v)e^{-\int_0^t a(s,x-(t-s)v,v)ds} \\ + \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s,x-v(t-s),v\right)\rho_g\left(s,x-v(t-s),v\right)\,ds\,. \end{aligned} \tag{7.48}$$

Take $g_1, g_2 \in \mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right)$ and define $\widetilde{g}_1 = A_T g_1$ and $\widetilde{g}_2 = A_T g_2$. Assume that $a \not\equiv 0$ over $(0,T) \times \mathbb{R} \times V$. For all $(t,x) \in (0,T) \times \mathbb{R}$, one has:

$$\begin{aligned} &\int_V |\widetilde{g}_1(t,x,v) - \widetilde{g}_2(t,x,v)|\,dv \\ &\leq \int_V \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s,x-v(t-s),v\right) \left|\rho_{g_1}\left(s,x-v(t-s),v\right) - \rho_{g_2}\left(s,x-v(t-s)\right)\right| dvds \\ &\leq \int_0^t e^{(t-s)\|a\|_{L^\infty}} \|b\|_{L^\infty} ds \times \sup_{(t,x)\in(0,T)\times\mathbb{R}} \int_V |g_1(t,x,v) - g_2(t,x,v)|\,dv \\ &\leq \frac{1}{\|a\|_{L^\infty}} \left(e^{T\|a\|_{L^\infty}} - 1\right) \|b\|_{L^\infty} \times \sup_{(t,x)\in(0,T)\times\mathbb{R}} \int_V |g_1(t,x,v) - g_2(t,x,v)|\,dv\,. \end{aligned}$$

Hence, there exists $T_0 > 0$ such that for all $T \in (0, T_0)$, $A_T$ is a contraction over $\mathcal{C}_b^0\left((0,T) \times \mathbb{R}, L^1(V)\right)$. If $a \equiv 0$ on $(0,T) \times \mathbb{R} \times V$, then such an estimate can be derived similarly. Hence, $A_T$ admits a unique fixed point, which satisfies (7.46) over $(0,T) \times \mathbb{R} \times V$. This gives the local existence and uniqueness of the solution of (7.46). Moreover, as $T_0$ does not depend on the initial datum $g^0$, the global existence follows.

If $b \geq 0$ and $g^0 \geq 0$, then $A_T$ preserves the cone of nonnegative functions and thus applying the fixed point theorem in this cone, we get the nonnegativity of $g$. □

**Lemma 7.5.** *Assume that $b$ is everywhere positive and that $V$ is an interval. Then if $g^0 \in \mathcal{C}_b^0(\mathbb{R}_+ \times \mathbb{R} \times V)$ is nonnegative and if there exists $(x_0, v_0) \in \mathbb{R} \times V$ such that $g^0(x_0, v_0) > 0$, letting $g$ the unique solution of (7.45), one has $g(t,x,v) > 0$ for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times V$ such that $|x - x_0| < v_{\max} t$.*

**Proof of Lemma 7.5.** First, assume by contradiction that there exists $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ such that $\rho_g(t,x) = 0$, with $|x - x_0| < v_{\max} t$. Then integrating (7.46) over $V$, one gets

$$0 = \rho_g(t,x) = \int_{v\in V} g^0(x-vt,v) e^{-\int_0^t a(s,x-(t-s)v,v)ds} dv \\ + \int_{v\in V}\int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s,x-v(t-s),v\right)\rho_g\left(s,x-v(t-s)\right) ds dv.$$

Hence, $\rho_g\left(s,x-v(t-s)\right) = 0$ for all $v \in V$ and $s \in (0,t)$. Letting $s \to 0$, one gets $\rho_g\left(0,x-vt\right) = 0$ for all $v \in V$. As $|x - x_0| < v_{\max} t$ and $V$ is an interval, one can take $v \in V$ such that $x - vt = x_0$, leading to $\rho_g(0,x_0) = 0$. This is a contradiction since, as $g$ is continuous, nonegative and $g(0,x_0,v_0) > 0$, one has $\rho_g(0,x_0) > 0$. Hence $\rho_g(t,x,v) > 0$ for all $(t,x,v) \in (0,T) \times R \times V$ such that $|x - x_0| < v_{\max} t$.

Next, as

$$g(t,x,v) = g^0(x-vt,v)e^{-\int_0^t a(s,x-(t-s)v,v)ds} + \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s,x-v(t-s),v\right)\rho_g\left(s,x-v(t-s)\right) ds\,,$$

it follows from the first step that $g(t,x,v) > 0$ as soon as there exists $s \in (0,t)$ such that $|x - x_0 - v(t-s)| < v_{\max} s$, which also reads: $|x - x_0| < v_{\max} t$. □

**Proof of Proposition 2.2.** Define $w = g_1 - g_2$. As in the proof of Lemma 6 in [12], we first remark that this function satisfies

$$\partial_t w + v\partial_x w + \left(1 + r\rho_{g_1}\right) w \geq \left(M(v) + r\left(M(v) - g_2\right)\right)\rho_w \text{ in } \mathbb{R}_+ \times \mathbb{R} \times V, \tag{7.49}$$

with $w(0,x,v) \geq 0$ for all $(x,v) \in \mathbb{R} \times V$. We define $a = 1 + r\rho_{g_1}$ and $b = M(v) + r\left(M(v) - g_2\right)$. Writing the integral formulation as in the proof of Lemma 7.4 gives

$$w\left(t,x,v\right) \geq \int_0^t e^{-\int_s^t a(\tau,x-(t-\tau)v,v)d\tau} b\left(s,x-v(t-s),v\right)\rho_w\left(s,x-v(t-s)\right) ds\,,$$

and thus $w \geq A_T w$ in $(0,T) \times \mathbb{R} \times V$ for some operator $A_T$ which is monotone and contractive when $T$ is small enough. It follows that $w \geq A_T^n w$ for all $n \geq 1$. Since $A_T$ is contractive the sequence $\left(A_T^n w\right)_n$ converges to 0. We conclude that $w \geq 0$, meaning that $g_1 \geq g_2$.

Next, assume that $\inf_V M > 0$, $V$ is an interval, and that there exists $(x_0,v_0)$ such that $g_2(0,x_0,v_0) > g_1(0,x_0,v_0)$. We can follow the proof of Lemma 7.5, where $b$ defined above is positive everywhere. We deduce that $w(t,x,v) > 0$ as soon as $|x - x_0| < v_{\max} t$. □

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