On the resurgent approach to Ecalle-Voronin's invariants

Artem Dudko, David Sauzin

On the resurgent approach to Écalle-Voronin's invariants

Artem Dudko, David Sauzin

Abstract

Given a holomorphic germ at the origin of $\mathbb{C}$ with a simple parabolic fixed point, the Fatou coordinates have a common asymptotic expansion whose formal Borel transform is resurgent. We show how to use Écalle's alien operators to study the singularities in the Borel plane and relate them to the horn maps, providing each of Écalle-Voronin's invariants in the form of a convergent numerical series. The proofs are original and self-contained, with ordinary Borel summability as the only prerequisite.

Keywords: Complex dynamics, Ecalle-Voronin invariants, Resurgent functions.

1 Alien operators for simple $2\pi\mathrm{i}\mathbb{Z}$-resurgent series

We first present an extension of the classical Borel-Laplace summation theory [Ram93], [Mal95], [Sau14b], to be applied to the formal Fatou coordinate of a simple parabolic germ in next section.

We denote by $\mathbb{C}_{\log}$ the Riemann surface of the logarithm viewed as the universal cover of $\mathbb{C}^*$ with base-point at 1. For $\beta_1, \beta_2 \in \mathbb{C}$ with $\operatorname{Re}\beta_1, \operatorname{Re}\beta_2 > -1$ and $\hat\phi_1 \in \zeta^{\beta_1}\mathbb{C}\{\zeta\}$, $\hat\phi_2 \in \zeta^{\beta_2}\mathbb{C}\{\zeta\}$, we extend the usual convolution and define $\hat\phi_1 * \hat\phi_2 \in \zeta^{\beta_1+\beta_2+1}\mathbb{C}\{\zeta\}$ by

$$\hat\phi_1 * \hat\phi_2(\zeta) \coloneqq \int_0^1 \hat\phi_1\big((1-t)\zeta\big)\hat\phi_2(t\zeta)\zeta\,\mathrm{d}t \quad \text{for } \zeta \in \mathbb{C}_{\log} \text{ with } |\zeta| \text{ small enough.}$$

For $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > -1$, we extend the classical formal Borel transform by defining

$$\mathscr{B}\colon\ \tilde\phi = \sum_{n\geq 0} c_n z^{-n-\beta-1} \in z^{-\beta-1}\mathbb{C}[[z^{-1}]] \mapsto \hat\phi \coloneqq \sum_{n\geq 0} c_n \tfrac{\zeta^{n+\beta}}{\Gamma(n+\beta+1)} \in \zeta^{\beta}\mathbb{C}[[\zeta]].$$

Observe that if $\mathscr{B}\tilde\phi_1 = \hat\phi_1 \in \zeta^{\beta_1}\mathbb{C}\{\zeta\}$ and $\mathscr{B}\tilde\phi_2 = \hat\phi_2 \in \zeta^{\beta_2}\mathbb{C}\{\zeta\}$ then $\mathscr{B}(\tilde\phi_1\tilde\phi_2) = \hat\phi_1 * \hat\phi_2$.

Definition 1.

- Given $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > -1$, if $\hat\phi \in \zeta^{\beta}\mathbb{C}\{\zeta\}$ converges for any ζ on $\mathbb{C}_{\log}$ with $|\zeta| < 2\pi$ and extends analytically along any path of $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ issuing from 1, then $\hat\phi$ and $\tilde\phi = \mathscr{B}^{-1}\hat\phi$ are said to be $2\pi\mathrm{i}\mathbb{Z}$-*resurgent*.

- Given $\omega \in 2\pi\mathrm{i}\mathbb{Z}$ and a path Γ in $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ going from 1 to $\omega+1$, if $\hat\phi$ is $2\pi\mathrm{i}\mathbb{Z}$-resurgent and if its analytic continuation along Γ, which is a holomorphic germ $\operatorname{cont}_\Gamma\hat\phi$ at $\omega+1$, takes the form
$$\operatorname{cont}_\Gamma\hat\phi(\omega+\zeta) = \frac{c}{2\pi\mathrm{i}\zeta} + \hat\chi(\zeta)\frac{\log\zeta}{2\pi\mathrm{i}} + \hat R(\zeta) \quad \text{for } \zeta > 0, \tag{1}$$
where $c \in \mathbb{C}$ and $\hat\chi, \hat R \in \mathbb{C}\{\zeta\}$, then $\hat\phi$ and $\tilde\phi = \mathscr{B}^{-1}\hat\phi$ are said to be (ω,Γ)-*simple*.

- In the above situation, we set $\mathscr{A}^\Gamma_\omega\,\tilde\phi \coloneqq \hat\chi$, which is the monodromy of $\zeta \mapsto \operatorname{cont}_\Gamma\hat\phi(\omega+\zeta)$ around 0, and $\mathscr{A}^\Gamma_\omega\,\tilde\phi \coloneqq c + \mathscr{B}^{-1}\hat\chi \in \mathbb{C}[[z^{-1}]]$. The operator $\mathscr{A}^\Gamma_\omega$ thus defined on the space of all (ω,Γ)-simple formal series is called the *alien operator associated with* (ω,Γ).

(Observe that $\mathscr{A}_\omega^\Gamma\,\tilde\phi$ is itself $2\pi\mathrm{i}\mathbb{Z}$-resurgent because $\mathrm{cont}_\Gamma\,\hat\phi(\omega+\zeta)$ extends analytically along any path of $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$.)

Definition 2. Let $\hat\phi$ be $2\pi\mathrm{i}\mathbb{Z}$-resurgent. We say that $\hat\phi$ is of *finite exponential type in non-vertical directions* if, for any path γ of $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ going from 1 to $\zeta_*\in\mathrm{i}\mathbb{R}$ and any $\delta_0\in(0,\pi/2)$, there exist $C_0,R_0>0$ such that

$$\left|\mathrm{cont}_\gamma\,\hat\phi\big(\zeta_*+t\,\mathrm{e}^{\mathrm{i}\theta}\big)\right|\le C_0\,\mathrm{e}^{R_0 t}\quad\text{for all } t\ge 0 \text{ and } \theta\in[-\delta_0,\delta_0]\cup[\pi-\delta_0,\pi+\delta_0]. \tag{2}$$

We omit the proof of the following technical statement.

Lemma 1. *Let $\tilde\phi_0\in z^{-1}\mathbb{C}[[z^{-1}]]$ be $2\pi\mathrm{i}\mathbb{Z}$-resurgent with $\hat\phi_0\coloneqq\mathscr{B}\tilde\phi_0$ of finite exponential type in non-vertical directions. Let $\tilde\phi\coloneqq z^{-\alpha}\tilde\phi_0\in z^{-\beta-1}\mathbb{C}[[z^{-1}]]$, where $\alpha,\beta\in\mathbb{C}$ and $\mathrm{Re}\,\beta>-1$. Then $\hat\phi\coloneqq\mathscr{B}\tilde\phi\in\zeta^\beta\mathbb{C}\{\zeta\}$. Moreover, for any simple curve $\gamma\colon(0,+\infty)\to\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ of the form $\gamma(t)=tu_0$ for $t<t_0$ and $\gamma(t)=c+tu_1$ for $t>t_1$, with $|u_0|=|u_1|=1$, $u_1\neq\pm\mathrm{i}$, $c\in\mathbb{C}$ and $0<t_0<t_1$, and for any lift $\tilde\gamma$ of γ in $\mathbb{C}_{\log}$, the germ $\hat\phi$ admits analytic continuation along $\tilde\gamma$ and*

$$\int_{\tilde\gamma}\mathrm{e}^{-z\zeta}\hat\phi(\zeta)\,\mathrm{d}\zeta=z^{-\alpha}\int_\gamma e^{-z\zeta}\hat\phi_0(\zeta)\,\mathrm{d}\zeta \tag{3}$$

for all $z\in\mathbb{C}_{\log}$ with $\arg z\in(-\theta-\frac{\pi}{2},-\theta+\frac{\pi}{2})$ and $\mathrm{Re}(ze^{\mathrm{i}\theta})$ large enough, where $\theta\in\mathbb{R}$ is the argument of u_1 which determines the asymptotic direction of $\tilde\gamma$.

In particular, $\mathcal{L}^\theta\hat\phi(z)\coloneqq\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty}\mathrm{e}^{-z\zeta}\hat\phi(\zeta)\,\mathrm{d}\zeta$ is well defined and coincides with $z^{-\alpha}\mathcal{L}^\theta\hat\phi_0(z)$ for $z\in\mathbb{C}_{\log}$ as above.

Lemma 2. *Let $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{2})$ and denote by $\mathcal{H}_\theta$ a θ-rotated Hankel contour,* i.e. *a contour in $\mathbb{C}_{\log}$ which goes along a ray from $\mathrm{e}^{\mathrm{i}(\theta-2\pi)}\infty$ to $\mathrm{e}^{\mathrm{i}(\theta-2\pi)}\varepsilon$ (with $0<\varepsilon<\pi$), circles counterclockwise 0 and then follows the ray from $\mathrm{e}^{\mathrm{i}\theta}\varepsilon$ to $\mathrm{e}^{\mathrm{i}\theta}\infty$. Suppose that $\hat\phi$ is (ω,Γ)-simple and $\zeta\mapsto\mathrm{cont}_\Gamma\,\hat\phi(\omega+\zeta)$ has at most exponential growth along $\mathcal{H}_\theta$, then*

$$\int_{\mathcal{H}_\theta}\mathrm{e}^{-z\zeta}\,\mathrm{cont}_\Gamma\,\hat\phi(\omega+\zeta)\,\mathrm{d}\zeta=c+\mathcal{L}^\theta\hat\chi(z),$$

where $\mathscr{A}_\omega^\Gamma\,\mathscr{B}^{-1}\hat\phi=c+\mathscr{B}^{-1}\hat\chi$, $\arg z\in(-\theta-\frac{\pi}{2},-\theta+\frac{\pi}{2})$ and $\mathrm{Re}(ze^{\mathrm{i}\theta})>>0$.

Proof. The term $\frac{c}{2\pi\mathrm{i}\zeta}$ contributes c, the remainder contributes $\int_{\mathrm{e}^{\mathrm{i}\theta}\varepsilon}^{\mathrm{e}^{\mathrm{i}\theta}\infty}\mathrm{e}^{-z\zeta}\hat\chi(\zeta)\,\mathrm{d}\zeta+O(\varepsilon|\ln\varepsilon|)$. □

Lemma 3. *If $b_0\in\mathbb{C}\{z^{-1}\}$ and $\tilde\phi$ is $2\pi\mathrm{i}\mathbb{Z}$-resurgent, then the formal series $b_0\tilde\phi$ and $C_{\mathrm{id}+b_0}\tilde\phi\coloneqq\tilde\phi\circ(\mathrm{id}+b_0)$ are $2\pi\mathrm{i}\mathbb{Z}$-resurgent. If moreover $\tilde\phi$ is (ω,Γ)-simple, then they are also (ω,Γ)-simple, with*

$$\mathscr{A}_\omega^\Gamma(b_0\tilde\phi)=b_0\,\mathscr{A}_\omega^\Gamma\,\tilde\phi,\qquad \mathscr{A}_\omega^\Gamma\,C_{\mathrm{id}+b_0}\tilde\phi=\mathrm{e}^{-\omega b_0}C_{\mathrm{id}+b_0}\,\mathscr{A}_\omega^\Gamma\,\tilde\phi.$$

Idea of the proof. Start with $b_0\in z^{-1}\mathbb{C}\{z^{-1}\}$, hence $\hat b_0(\zeta)$ is entire; $\tilde\psi_1\coloneqq b_0\tilde\phi$ and $\tilde\psi_2\coloneqq C_{\mathrm{id}+b_0}\tilde\phi-\tilde\phi$ have Borel images

$$\hat\psi_j(\zeta)=\int_0^\zeta K_j(\xi,\zeta)\hat\phi(\xi)\,\mathrm{d}\xi\quad\text{for }\zeta\in\mathbb{C}_{\log}\text{ with }|\zeta|<2\pi,\qquad j=1,2, \tag{4}$$

where $K_1(\xi,\zeta)=\hat b_0(\zeta-\xi)$ and $K_2(\xi,\zeta)=\sum_{k\ge1}\frac{(-\xi)^k}{k!}\hat b_0^{*k}(\zeta-\xi)$ are holomorphic in $\mathbb{C}\times\mathbb{C}$ (*cf.* the proof of Lemma 2 in [DS13]). The analytic continuation of $\hat\psi_1$ or $\hat\psi_2$ along a path $\gamma\colon[0,1]\to\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ is thus given by the same integral as (4), but integrating on the concatenation $[0,\gamma(0)]+\gamma+[\gamma(1),\zeta]$. It is then possible to analyze the singularities of $\hat\psi_1$ and $\hat\psi_2$ at ω. We omit the details. □

The above is inspired by Écalle's resurgence theory [Eca81]. In fact, $2\pi\mathrm{i}\mathbb{Z}$-resurgent series are stable under multiplication (see [Eca81], [Sau13] or [Sau14a]), and so are the (ω,Γ)-simple ones, but we shall not require these facts in this article, nor the far-reaching extension of the framework which allows to define alien operators or alien derivations in much more general situations.

2 The singular structure of the formal Borel transform of the formal Fatou coordinate

We use the same notations as in [DS13]: $f = (\mathrm{id}+b)\circ(\mathrm{id}+1) = z+1-\rho z^{-1}+O(z^{-2})$ is a simple parabolic germ at ∞; under the change of unknown $v(z) = z+\rho\log z+\varphi(z)$ the equation $v\circ f = v+1$ is transformed into

$$C_{\mathrm{id}-1}\varphi = C_{\mathrm{id}+b}\varphi + b_*, \qquad b_*(z) \coloneqq b(z) + \rho\log\tfrac{1+z^{-1}b(z)}{1-z^{-1}} \in z^{-2}\mathbb{C}\{z^{-1}\} \tag{5}$$

which has a unique solution $\tilde\varphi \in z^{-1}\mathbb{C}[[z^{-1}]]$, shown to be $2\pi\mathrm{i}\mathbb{Z}$-resurgent and Borel summable. Setting $\hat\varphi \coloneqq \mathscr{B}\tilde\varphi$, one gets two normalized Fatou coordinates $v_*^{\pm}(z) = z+\rho\log z+\mathcal{L}^{\pm}\hat\varphi(z)$, where $\mathcal{L}^{+}\hat\varphi(z)$ and $\mathcal{L}^{-}\hat\varphi(z)$ are Laplace transforms along $\mathbb{R}^+$ and $\mathbb{R}^-$, holomorphic for $\arg z\in(-\pi,\pi)$ and $\arg z\in(0,2\pi)$, and a pair of normalized lifted horn maps $h_*^{\mathrm{up/low}} = v_*^{+}\circ(v_*^{-})^{-1}$ (or Écalle-Voronin modulus—see [Eca81], [Vor81], [DS13]):

$$h_*^{\mathrm{up}}(Z) = Z + \sum_{m\geq 1} A_m \mathrm{e}^{2\pi\mathrm{i}mZ} \quad \text{for } \mathrm{Im}\, Z >> 0,$$

$$h_*^{\mathrm{low}}(Z) = Z - 2\pi\mathrm{i}\rho + \sum_{m\geq 1} A_{-m}\mathrm{e}^{-2\pi\mathrm{i}mZ} \quad \text{for } \mathrm{Im}\, Z << 0.$$

We are interested in the Écalle-Voronin invariants A_m, $m\in\mathbb{Z}^*$.

Let $\omega\in 2\pi\mathrm{i}\mathbb{Z}^*$ and Γ be a path in $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ going from 1 to $\omega+1$. In general $\tilde\varphi$ is not (ω,Γ)-simple, but we shall directly prove that, up to a linear combination of monomials, $z^{\rho\omega}\tilde\varphi$ is (ω,Γ)-simple. More precisely, we shall prove the (ω,Γ)-simplicity of $\tilde\Phi \coloneqq z^{\rho\omega}\{\tilde\varphi\}_N$, with the notation

$$\tilde\varphi = [\tilde\varphi]_N + \{\tilde\varphi\}_N, \qquad [\tilde\varphi]_N\in \mathrm{Span}(z^{-1},z^{-2},\dots,z^{-N}), \quad \{\tilde\varphi\}_N\in z^{-N-1}\mathbb{C}[[z^{-1}]],$$

where $N\geq\max\{0,-2\,\mathrm{Re}\,\alpha\}$ is integer, with $\alpha\coloneqq-\rho\omega$. Observe that $\tilde\Phi\in z^{-\beta-1}\mathbb{C}[[z^{-1}]]$ with $\beta\coloneqq\alpha+N$ and $\mathrm{Re}\,\beta\geq 0$. We shall also compute $\mathscr{A}_\omega^\Gamma\tilde\Phi$ and relate it to the lifted horn maps of f.

We first introduce an auxiliary sequence of formal series, using the same operator

$$E\colon\ z^{-2}\mathbb{C}[[z^{-1}]]\to z^{-1}\mathbb{C}[[z^{-1}]]$$

inverse of $C_{\mathrm{id}-1}-\mathrm{Id}$ as in [DS13], but replacing B with

$$B^\omega\colon\ \tilde\psi\mapsto \mathrm{e}^{-\omega b_*}C_{\mathrm{id}+b}\tilde\psi-\tilde\psi = \mathrm{e}^{-\omega b_*}B\tilde\psi + (\mathrm{e}^{-\omega b_*}-1)\tilde\psi.$$

Proposition. *The formal series $\tilde\Psi_k = (EB^\omega)^k 1\in z^{-k}\mathbb{C}[[z^{-1}]]$ yield a formally convergent series $\sum\tilde\Psi_k$ and $\sum_{k\geq 0}\tilde\Psi_k = \mathrm{e}^{-\omega\tilde\varphi}$. For each $k\geq 1$, $\tilde\Psi_k$ is $2\pi\mathrm{i}\mathbb{Z}$-resurgent and, for any $\varepsilon,L>0$, there exist $C_0,M>0$ such that*

$$|\mathrm{cont}_\gamma\,\mathscr{B}\tilde\Psi_k|\leq C_0\frac{M^k}{k!}, \qquad \textit{for all } k\geq 1 \textit{ and } \gamma\in\mathscr{R}_{\varepsilon,L},$$

where $\mathscr{R}_{\varepsilon,L}$ is the set of all paths of length $< L$ issuing from 1 and staying at distance $> \varepsilon$ from $2\pi\mathrm{i}\mathbb{Z}$.

Proof. Let $c^\omega := \mathrm{e}^{-\omega b_*} - 1$. The estimates are obtained by adapting the proof of Lemma 2 of [DS13], with a new kernel function $K^\omega(\xi,\zeta) = \hat{c}^\omega(\zeta-\xi) + \sum_{k\geq 1} \frac{(-\xi)^k}{k!}\hat{b}^{*k}(\zeta-\xi) + \sum_{k\geq 1}\frac{(-\xi)^k}{k!}(\hat{c}^\omega * \hat{b}^{*k})(\zeta-\xi)$. The formal series $\tilde{\Psi} := \sum_{k\geq 0}\tilde{\Psi}_k$ is the unique solution with constant term 1 of $C_{\mathrm{id}-1}\tilde{\Psi} = \mathrm{e}^{-\omega b_*} C_{\mathrm{id}+b}\tilde{\Psi}$—so is $\mathrm{e}^{-\omega\tilde{\varphi}}$, as can be seen by exponentiating (5). ☐

Since $\operatorname{Re}\beta \geq 0$, the operator $C_{\mathrm{id}-1} - \mathrm{Id}$ maps $z^{-\beta-1}\mathbb{C}[[z^{-1}]]$ to $z^{-\beta-2}\mathbb{C}[[z^{-1}]]$ bijectively; we denote by E_β the inverse, whose Borel counterpart is the multiplication operator

$$\hat{E}_\beta\colon\ \hat{\phi}(\zeta) \in \zeta^{\beta+1}\mathbb{C}[[\zeta]] \mapsto \frac{1}{\mathrm{e}^\zeta - 1}\hat{\phi}(\zeta) \in \zeta^\beta\mathbb{C}[[\zeta]].$$

We introduce an operator

$$B_\alpha\colon\ \tilde{\phi} \in z^{-\beta-1}\mathbb{C}[[z^{-1}]] \mapsto c_\alpha C_{\mathrm{id}+b}\tilde{\phi} - \tilde{\phi} \in z^{-\beta-2}\mathbb{C}[[z^{-1}]],$$

with $c_\alpha := \left(\frac{1+z^{-1}b}{1-z^{-1}}\right)^\alpha \in 1 + z^{-1}\mathbb{C}\{z^{-1}\}$.

Theorem 1. *Let $b_\alpha := z^{-\alpha}(1-z^{-1})^{-\alpha}b_N$, with $b_N := b_* - C_{\mathrm{id}-1}[\tilde{\varphi}]_N + C_{\mathrm{id}+b}[\tilde{\varphi}]_N$.*

- *The formal series $\tilde{\Phi}_k(z) := (E_\beta B_\alpha)^k E_\beta b_\alpha \in z^{-\beta-1}\mathbb{C}[[z^{-1}]]$, $k \geq 0$, are (ω,Γ)-simple, and so is $z^{-\alpha}\{\tilde{\varphi}\}_N$.*
- *For any $\varepsilon, L > 0$, there exist $C_0, \Lambda > 0$ with $\Lambda < 1$ such that, for each $\gamma \in \mathscr{R}_{\varepsilon,L}$, $|\mathrm{cont}_\gamma \mathscr{B}\tilde{\Phi}_k| \leq C_0\Lambda^k$. Moreover $\hat{\Phi} := \sum_{k\geq 0}\mathscr{B}\tilde{\Phi}_k$ coincides with the formal Borel transform of $z^{-\alpha}\{\tilde{\varphi}\}_N$.*
- *There exist complex numbers $S^\Gamma_{\omega,k}$, $k \geq 0$, which are $O(\Lambda^k)$ for some $\Lambda < 1$ and satisfy*

$$\mathscr{A}^\Gamma_\omega\,\tilde{\Phi}_k = \sum_{k_1+k_2=k} S^\Gamma_{\omega,k_1}\tilde{\Psi}_{k_2} \quad \text{for each } k \geq 0, \tag{6}$$

$$\mathscr{A}^\Gamma_\omega(z^{-\alpha}\{\tilde{\varphi}\}_N) = S^\Gamma_\omega\,\mathrm{e}^{-\omega\tilde{\varphi}} \quad \text{where } S^\Gamma_\omega = \sum_{k\geq 0} S^\Gamma_{\omega,k}. \tag{7}$$

Theorem 2. *For $m \in \mathbb{Z}^*$, let Γ_m denote the line segment going from 1 to $2\pi\mathrm{i}m + 1$. Then the Écalle-Voronin invariants of f are given by*

$$A_{-m} = S^{\Gamma_m}_{2\pi\mathrm{i}m}\,\mathrm{e}^{-4\pi^2 m\rho} \ \text{ if } m > 0, \qquad A_{-m} = -S^{\Gamma_m}_{2\pi\mathrm{i}m} \ \text{ if } m < 0.$$

Remark 1. When $\alpha \neq 0$, the series of formal series $\sum\tilde{\Phi}_k$ is not convergent for the topology of the formal convergence (but it converges for the product topology $z^{-\beta-1}\mathbb{C}[[z^{-1}]] \simeq \mathbb{C}^{\mathbb{N}}$) and, unless $-\alpha \in \mathbb{N}$ or $S^\Gamma_\omega = 0$, the singularity at ω of $\mathrm{cont}_\Gamma\mathscr{B}\tilde{\varphi}$ is not "simple": it is rather of the form $\mathrm{cont}_\Gamma\mathscr{B}\tilde{\varphi}(\omega+\zeta) = \frac{1}{\Gamma(-\alpha)}S^\Gamma_\omega\zeta^{-\alpha-1}(1+O(|\zeta|)) + O(1)$ if $\alpha \notin \mathbb{Z}$ or $\frac{(-1)^\alpha\alpha!}{2\pi\mathrm{i}}S^\Gamma_\omega\zeta^{-\alpha-1}(1+O(|\zeta|))$ if $\alpha \in \mathbb{N}$.

Remark 2. Theorem 2 can be extracted from [Eca81], which gives detailed proofs only for the case $\rho = 0$, and also the case $\Gamma = \Gamma_m$ of (7) with the equivalent formulation $\Delta^+_{\omega_m}\tilde{\varphi} = A_m z^{-\rho\omega_m}\mathrm{e}^{-\omega_m\tilde{\varphi}}$ ("Bridge equation"), where $\omega_m = 2\pi\mathrm{i}m$. The name "resurgence" evokes the fact that the singular behaviour near ω_m of the analytic continuation of a germ at 0 like $\hat{\Phi}$ can be explicitly expressed in terms of $\hat{\Phi}$ itself.

The representation of $\hat\Phi = \mathscr{B}(z^{-\alpha}\{\tilde\varphi\}_N)$ as the convergent series $\sum \hat\Phi_k = \sum \mathscr{B}\big((E_\beta B_\alpha)^k E_\beta b_\alpha\big)$ and the computation of the action of the alien operator $\mathscr{A}_\omega^\Gamma$ in (6)–(7) are new. The point is that $\mathscr{A}_\omega^\Gamma\, \tilde\Phi_k = S_{\omega,k}^\Gamma + O(z^{-1})$, so Equation (6) says that

$$\mathrm{cont}_\Gamma\, \hat\Phi_k(\omega+\zeta) = \frac{S_{\omega,k}^\Gamma}{2\pi\mathrm{i}\zeta} + \hat{\mathscr{A}}_\omega^\Gamma\, \tilde\Phi_k(\zeta)\frac{\log\zeta}{2\pi\mathrm{i}} + \text{regular germ}, \quad \hat{\mathscr{A}}_\omega^\Gamma\, \tilde\Phi_k = \sum_{k_1+k_2=k,\ k_2\geq 1} S_{\omega,k_1}^\Gamma\, \hat\Psi_{k_2} \in \mathbb{C}\{\zeta\}.$$

The numbers $S_{\omega,k}^\Gamma$ appear as generalized residua, for which we can give quite explicit formulas:

Addendum to Theorem 1. *Let $\hat c_* \coloneqq \mathscr{B}(c_\alpha - 1)$, which is an entire function, and*

$$K_\alpha(\xi,\zeta) = \hat c_*(\zeta-\xi) + \sum_{k\geq 1} \frac{(-\xi)^k}{k!}\hat b^{*k}(\zeta-\xi) + \sum_{k\geq 1}\frac{(-\xi)^k}{k!}(\hat c_* * \hat b^{*k})(\zeta-\xi), \tag{8}$$

which is holomorphic in $\mathbb{C}\times\mathbb{C}$. Let $\tilde\Gamma$ denote a parametrization of the path obtained by concatenating the line segment $(0,1]$, the path Γ and the line segment $[\omega+1,\omega]$. Then

$$S_{\omega,0}^\Gamma = 2\pi\mathrm{i}\,\mathrm{cont}_{\tilde\Gamma}\,\hat b_\alpha(\omega),$$

$$S_{\omega,k}^\Gamma = 2\pi\mathrm{i}\int_{\Delta_{\tilde\Gamma,k}} \mathrm{cont}_{\tilde\Gamma}\,\hat b_\alpha(\xi_1)\frac{K_\alpha(\xi_1,\xi_2)\cdots K_\alpha(\xi_{k-1},\xi_k)K_\alpha(\xi_k,\omega)}{(\mathrm{e}^{\xi_1}-1)\cdots(\mathrm{e}^{\xi_{k-1}}-1)(\mathrm{e}^{\xi_k}-1)}\,\mathrm{d}\xi_1\wedge\cdots\wedge\mathrm{d}\xi_k, \qquad k\geqslant 1,$$

with the notation $\Delta_{\tilde\Gamma,k} \coloneqq \big\{\, \big(\tilde\Gamma(s_1),\dots,\tilde\Gamma(s_k)\big) \mid s_1\leqslant\cdots\leqslant s_k \,\big\}$ for each positive integer k.

Observe that, according to (7), each coefficient S_ω^Γ, and in particular each Écalle-Voronin invariant, can be obtained as the convergent series of these "residua" $S_{\omega,k}^\Gamma$.

Proof of Addendum to Theorem 1. The normal convergence of the series (8) follows easily from the estimates available for the convolution of entire functions (see inequality (9) in [DS13]), and the Borel counterpart of B_α is the integral transform with kernel function K_α, hence

$$\mathrm{cont}_\gamma\, \hat B_\alpha\hat\phi\big(\gamma(s)\big) = \int_0^s K_\alpha\big(\gamma(\sigma),\gamma(s)\big)\,\mathrm{cont}_\gamma\,\hat\phi\big(\gamma(\sigma)\big)\gamma'(\sigma)\,\mathrm{d}\sigma \tag{9}$$

for any $2\pi\mathrm{i}\mathbb{Z}$-resurgent $\hat\phi$ and any path $\gamma\colon (0,\ell] \to \mathbb{C}_{\log}$ (with any $\ell>0$) whose projection onto $\mathbb{C}$ avoids $2\pi\mathrm{i}\mathbb{Z}$ and which starts as $\gamma(s) = su_0$ for $s>0$ small enough (with fixed $u_0\in\mathbb{C}^*$). Writing $\tilde\Phi_k = E_\beta(B_\alpha E_\beta)^k b_\alpha$, by repeated use of (9), we obtain an explicit formula for the analytic continuation of its Borel transform: for any ζ close enough to the endpoint of a path γ as above,

$$\mathrm{cont}_\gamma\,\hat\Phi_0(\zeta) = \frac{1}{\mathrm{e}^\zeta-1}\,\mathrm{cont}_\gamma\,\hat b_\alpha(\zeta),$$

$$\mathrm{cont}_\gamma\,\hat\Phi_k(\zeta) = \frac{1}{\mathrm{e}^\zeta-1}\int_{\Delta_{\gamma,k}} \mathrm{cont}_\gamma\,\hat b_\alpha(\xi_1)\frac{K_\alpha(\xi_1,\xi_2)\cdots K_\alpha(\xi_{k-1},\xi_k)K_\alpha(\xi_k,\zeta)}{(\mathrm{e}^{\xi_1}-1)\cdots(\mathrm{e}^{\xi_{k-1}}-1)(\mathrm{e}^{\xi_k}-1)}\,\mathrm{d}\xi_1\wedge\cdots\wedge\mathrm{d}\xi_k, \qquad k\geqslant 1,$$

with the notation $\Delta_{\gamma,k} \coloneqq \big\{\, \big(\gamma(s_1),\dots,\gamma(s_k)\big) \mid s_1\leqslant\cdots\leqslant s_k \,\big\}$. The conclusion follows. □

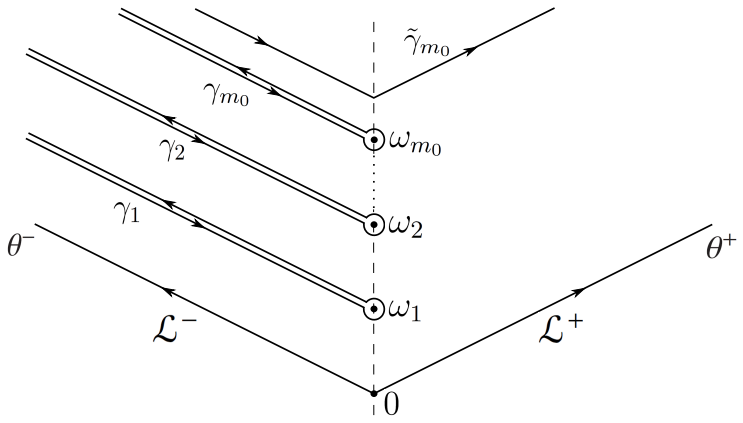

Figure 1: Deformation of the contour.

Theorem 1 implies Theorem 2. For all $m \geq 1$ we set $\omega_m \coloneqq 2\pi \mathrm{i} m$ and $\tilde\Phi^{(m)} \coloneqq z^{-\alpha_m}\{\tilde\varphi\}_{N_m}$ with $\alpha_m \coloneqq -\rho\omega_m$ and $N_m \coloneqq \max\{0, -2\operatorname{Re}\alpha_m\}$. By [DS13] we know that $\hat\varphi$ is of finite exponential type in non-vertical directions; we fix $\theta^+ \in (0, \frac{\pi}{2})$, $\theta^- \in (\frac{\pi}{2}, \pi)$ and $Q_R^{\mathrm{low}} \coloneqq \big\{ z \in \mathbb{C}_{\log} \mid \arg z \in (-\theta^+ - \frac{\pi}{2}, -\theta^- + \frac{\pi}{2}),\ \operatorname{Re}(z\,\mathrm{e}^{\mathrm{i}\theta^+}) > R,\ \operatorname{Re}(z\,\mathrm{e}^{\mathrm{i}\theta^-}) > R \big\}$, with R large enough. Identifying $z \in Q_R^{\mathrm{low}}$ with its projection in $\mathbb{C}$, we have $\operatorname{Im} z < 0$ and the branches of log used in v_*^+ and v_*^- differ by $-2\pi\mathrm{i}$; for $m_0 \geq 1$, deforming the contour of integration (see Figure 1), we get

$$\begin{aligned} v_*^+(z) - v_*^-(z) &= -2\pi\mathrm{i}\rho + (\mathcal{L}^+ - \mathcal{L}^-)\hat\varphi(z) \\ &= -2\pi\mathrm{i}\rho + \left(\sum_{m=1}^{m_0} \int_{\gamma_m} + \int_{\tilde\gamma_{m_0}} \right) \mathrm{e}^{-z\zeta}\hat\varphi(\zeta)\,\mathrm{d}\zeta = -2\pi\mathrm{i}\rho + \sum_{m=1}^{m_0} I_m + O(\mathrm{e}^{(2\pi m_0 + \pi)\operatorname{Im} z}), \end{aligned}$$

with $I_m = \int_{\gamma_m} \mathrm{e}^{-z\zeta} \operatorname{cont}_{\Gamma_m} \hat\varphi(\zeta)\,\mathrm{d}\zeta = \int_{\gamma_m} \mathrm{e}^{-z\zeta} \operatorname{cont}_{\Gamma_m}\{\hat\varphi(\zeta)\}_{N_m}\,\mathrm{d}\zeta$. Since each γ_m can be expressed as the difference of two paths to which Lemma 1 applies, we have

$$I_m = z^{\alpha_m} \int_{\gamma_m} \mathrm{e}^{-z\zeta} \operatorname{cont}_{\Gamma_m} \hat\Phi^{(m)}(\zeta)\,\mathrm{d}\zeta.$$

Denoting by $\mathcal{H}^-$ a θ^--rotated Hankel contour, we get $I_m = z^{\alpha_m}\,\mathrm{e}^{-\omega_m z} \int_{\mathcal{H}^-} \mathrm{e}^{-z\zeta} \operatorname{cont}_{\Gamma_m} \hat\Phi^{(m)}(\omega_m + \zeta)\,\mathrm{d}\zeta = S_{\omega_m}^{\Gamma_m}\,\mathrm{e}^{\alpha_m \log z - \omega_m z - \omega_m \mathcal{L}^- \hat\varphi}$ by Lemma 2 and (7) (Borel-Laplace summation commutes with exponentiation—see *e.g.* [Sau14a]). Thus

$$(h_*^{\mathrm{low}} - \mathrm{id}) \circ v_*^- = v_*^+ - v_*^- = -2\pi\mathrm{i}\rho + \sum_{m=1}^{m_0} S_{\omega_m}^{\Gamma_m}\,\mathrm{e}^{-\omega_m(-2\pi\mathrm{i}\rho + v_*^-)} + O(\mathrm{e}^{(2\pi m_0 + \pi)\operatorname{Im} z}) \quad \text{on } Q_R^{\mathrm{low}},$$

whence $A_{-m} = S_{\omega_m}^{\Gamma_m}\,\mathrm{e}^{2\pi\mathrm{i}\rho\omega_m}$ for $m > 0$. The case $m < 0$ is similar, except that, on the domain Q_R^{up} defined analogously with $-\pi < \theta^- < -\frac{\pi}{2} < \theta^+ < 0$, the two branches of log used in $v_*^{\pm}$ match and the orientation of the paths differs. □

Proof of Theorem 1. First observe that $b_\alpha \in z^{-\beta-2}\mathbb{C}\{z^{-1}\}$ because b_* and $[\tilde\varphi]_N \in \mathbb{C}\{z^{-1}\}$ hence $b_N \in \mathbb{C}\{z^{-1}\}$, and

$$b_N = C_{\mathrm{id}\,-1}\{\tilde\varphi\}_N - C_{\mathrm{id}\,+b}\{\tilde\varphi\}_N \in z^{-N-2}\mathbb{C}[[z^{-1}]].$$

Thus the formal series $\tilde\Phi_k$ are well defined in $z^{-\beta-1}\mathbb{C}[[z^{-1}]]$; they are $2\pi\mathrm{i}\mathbb{Z}$-resurgent because this property is preserved by E_β and B_α (Lemma 3) and we start with $\mathscr{B}\tilde\Phi_0 = \hat E_\beta \hat b_\alpha = \frac{\hat b_\alpha(\zeta)}{\mathrm{e}^\zeta - 1}$

meromorphic on $\mathbb{C}_{\log}$. Their (ω,Γ)-simplicity and the existence of contants $S^\Gamma_{\omega,k}$ such that (6) holds result by induction on k (using the fact that $B^\omega\tilde\Psi_k\in z^{-2}\mathbb{C}[[z^{-1}]]$ for all k) from

Lemma 4. *If $\tilde\phi$ is (ω,Γ)-simple, then so is $B_\alpha\tilde\phi$, with $\mathscr{A}^\Gamma_\omega\, B_\alpha\tilde\phi = B^\omega\,\mathscr{A}^\Gamma_\omega\,\tilde\phi$. If moreover $\mathscr{A}^\Gamma_\omega\,\tilde\phi\in z^{-2}\mathbb{C}[[z^{-1}]]$ (*i.e. $c=\hat\chi(0)=0$ *in (1)), then also $E_\beta\tilde\phi$ is (ω,Γ)-simple and $\mathscr{A}^\Gamma_\omega\,E_\beta\tilde\phi - E\,\mathscr{A}^\Gamma_\omega\,\tilde\phi$ is a constant.*

Proof of Lemma 4. The first identity results from Lemma 3 and $\mathrm{e}^{-\omega b}c_\alpha=\mathrm{e}^{-\omega b_*}$. For the second one, in view of (1), the analytic continuation of $\mathscr{B}E_\beta\tilde\phi$ is $\mathrm{cont}_\Gamma\,\hat E_\beta\hat\phi(\omega+\zeta)=\frac{\hat\chi(\zeta)}{\mathrm{e}^\zeta-1}\frac{\log\zeta}{2\pi\mathrm{i}}+\frac{\hat R(\zeta)}{\mathrm{e}^\zeta-1}$, hence $\mathscr{A}^\Gamma_\omega\,E_\beta\tilde\phi = E\mathscr{B}^{-1}\hat\chi+2\pi\mathrm{i}\hat R(0)$. □

We finish the proof of Theorem 1 by estimating $\hat\Phi_k\coloneqq\mathscr{B}\tilde\Phi_k$ by means of the recursive formula $\hat\Phi_k=\hat E_\beta\hat B_\alpha\hat\Phi_{k-1}$.

Lemma 5. *For $0<\varepsilon<1$, $L>1$, let $\tilde{\mathscr{R}}_{\varepsilon,L}$ denote the set of all naturally parametrised paths $\gamma\colon(0,\ell(\gamma)]\to\mathbb{C}_{\log}$, where $\ell(\gamma)<L$ is the length of γ, for which there exists $\theta\in[-\pi,3\pi]$ such that $\gamma(s)=s\,\mathrm{e}^{\mathrm{i}\theta}$ for $s\le\varepsilon$ and $\mathrm{dist}\big(\gamma(s),2\pi\mathrm{i}\mathbb{Z}\big)>\varepsilon$ for $s\ge\varepsilon$. For any integer $\ell\ge0$, let $V_{\varepsilon,L;\ell}$ denote the space of all $\hat\phi\in\zeta^{\beta+\ell}\mathbb{C}\{\zeta\}$ admitting analytic continuation along the paths of $\tilde{\mathscr{R}}_{\varepsilon,L}$ and such that $\|\hat\phi\|_{\varepsilon,L;\ell}\coloneqq\sup\big\{\,s^{-\mathrm{Re}\,\beta-\ell}|\mathrm{cont}_\gamma\,\hat\phi\big(\gamma(s)\big)|$, for $\gamma\in\tilde{\mathscr{R}}_{\varepsilon,L}$ and $s\in\big(0,\ell(\gamma)\big]\,\big\}$ is finite. Then the operator $\hat E_\beta\hat B_\alpha$ leaves invariant the spaces $V_{\varepsilon,L;\ell}$ and, for each ε and L, there exists $M(\varepsilon,L)>0$ such that*

$$\|\hat E_\beta\hat B_\alpha\hat\phi\|_{\varepsilon,L;\ell}\le\tfrac{M(\varepsilon,L)}{\mathrm{Re}\,\beta+\ell+1}\|\hat\phi\|_{\varepsilon,L;\ell}\quad\textit{for all }\hat\phi\in V_{\varepsilon,L;\ell}\textit{ and }\ell\ge0.$$

Proof of Lemma 5. One can find $\kappa=\kappa(\varepsilon,L)>0$ such that, for each $\gamma\in\tilde{\mathscr{R}}_{\varepsilon,L}$ and $s\in(0,\ell(\gamma)]$, $|\gamma(s)|\ge\kappa s$ (because $\arg\gamma(s)$ is uniformly bounded). Let $M_0=M_0(\varepsilon,L)\coloneqq\sup\big\{\,\big|\frac{\zeta}{\mathrm{e}^\zeta-1}\big|$, for $|\zeta|<L$, $\mathrm{dist}\big(\zeta,2\pi\mathrm{i}\mathbb{Z}^*\big)>\varepsilon\,\big\}$. From (9), we get $\|\hat B_\alpha\hat\phi\|_{\varepsilon,L;\ell+1}\le\frac{N_L}{\mathrm{Re}\,\beta+\ell+1}\|\hat\phi\|_{\varepsilon,L;\ell}$ with $N_L\coloneqq\sup_{|\xi|,|\zeta|<L}\{\,|K(\xi,\zeta)|\,\}$, and $\|\hat E_\beta\hat B_\alpha\hat\phi\|_{\varepsilon,L;\ell}\le\frac{N_LM_0}{\kappa(\mathrm{Re}\,\beta+\ell+1)}\|\hat\phi\|_{\varepsilon,L;\ell}$. □

We have $\big|\frac{\alpha}{\alpha+N+1}\big|<1$ because $\mathrm{Re}\,\alpha>-\frac{N+1}{2}$. Let us choose an integer $d\ge0$ so that $\frac{M(\varepsilon,L)}{\mathrm{Re}\,\beta+d+1}\le\Lambda\coloneqq\max\Big\{\big|\frac{\alpha}{\alpha+N+1}\big|,\frac12\Big\}$. For any $\hat\phi\in\zeta^\beta\mathbb{C}\{\zeta\}$, we use the notation

$$\hat\phi=\big[\hat\phi\big]_d+\big\{\hat\phi\big\}_d,\qquad\big[\hat\phi\big]_d\in\mathscr{E}_d\coloneqq\mathrm{Span}(\zeta^\beta,\zeta^{\beta+1},\dots,\zeta^{\beta+d-1}),\quad\big\{\hat\phi\big\}_d\in\zeta^{\beta+d}\mathbb{C}\{\zeta\}.$$

Now $\hat\phi\in\zeta^{\beta+d}\mathbb{C}\{\zeta\}\Rightarrow\hat B_\alpha\hat\phi=\alpha1*\hat\phi+O(\zeta^{\beta+d+2})\Rightarrow\hat E_\beta\hat B_\alpha\hat\phi=\frac{\alpha}{\zeta}1*\hat\phi+O(\zeta^{\beta+d+1})\in\zeta^{\beta+d}\mathbb{C}\{\zeta\}$, thus

$$\big[\hat\Phi_k\big]_d=A\big[\hat\Phi_{k-1}\big]_d,\qquad\big\{\hat\Phi_k\big\}_d=\big\{\hat E_\beta\hat B_\alpha\big[\hat\Phi_{k-1}\big]_d\big\}_d+\hat E_\beta\hat B_\alpha\big\{\hat\Phi_{k-1}\big\}_d,$$

where $A\in\mathrm{End}\,\mathscr{E}_d$ is defined by $A\hat\phi\coloneqq\big[\hat E_\beta\hat B_\alpha\hat\phi\big]_d$ and has a triangular matrix in the basis $(\zeta^\beta,\zeta^{\beta+1},\dots,\zeta^{\beta+d-1})$, with $A\zeta^{\beta+\ell}=\lambda_\ell\zeta^{\beta+\ell}\big(1+O(\zeta)\big)$, $\lambda_\ell\coloneqq\frac{\alpha}{\beta+\ell+1}$, $\ell=0,\dots,d-1$. By the choice of β, the eigenvalues have modulus $|\lambda_\ell|\le\Lambda$, hence $\|\big[\hat\Phi_k\big]_d\|=O(\Lambda^k)$ for any norm on $\mathscr{E}_d$. Since $\hat\phi\in\mathscr{E}_d\mapsto\big\{\hat E_\beta\hat B_\alpha\hat\phi\big\}_d\in V_{\varepsilon,L;d}$ is linear, we can find C_0' such that $\|\big\{\hat E_\beta\hat B_\alpha\big[\hat\Phi_{k-1}\big]_d\big\}_d\|_{\varepsilon,L;d}\le C_0'\Lambda^k$ for all $k\ge1$, hence $\|\big\{\hat\Phi_k\big\}_d\|_{\varepsilon,L;d}\le C_0'\Lambda^k+\Lambda\|\big\{\hat\Phi_{k-1}\big\}_d\|_{\varepsilon,L;d}$. We thus get $\|\big\{\hat\Phi_k\big\}_d\|_{\varepsilon,L;d}\le\big(\|\big\{\hat\Phi_0\big\}_d\|_{\varepsilon,L;d}+C_0'k\big)\Lambda^k$ for all $k\ge0$ and, increasing slightly Λ, the desired bounds for $\Big|\mathrm{cont}_\gamma\,\hat\Phi_k\Big|$.

By uniform convergence, we can now define a function $\hat{\Phi} = \sum_{k\geq 0} \hat{\Phi}_k \in \zeta^\beta \mathbb{C}\{\zeta\}$, whose inverse formal Borel transform $\tilde{\Phi} \in z^{-\beta-1}\mathbb{C}[[z^{-1}]]$ satisfies $\tilde{\Phi} = E_\beta b_\alpha + E_\beta B_\alpha \tilde{\Phi}$, whence

$$C_{\mathrm{id}\,-1}\tilde{\Phi} = c_\alpha C_{\mathrm{id}\,+b}\tilde{\Phi} + b_\alpha.$$

Multiplying both sides by $z^\alpha(1-z^{-1})^\alpha$, we find that $\tilde{\psi} \coloneqq z^\alpha\tilde{\Phi} \in z^{-N-1}\mathbb{C}[[z^{-1}]]$ satisfies $C_{\mathrm{id}\,-1}\tilde{\psi} = C_{\mathrm{id}\,+b}\tilde{\psi} + b_N$, an equation of which $\{\tilde{\varphi}\}_N$ is the unique solution in $z^{-1}\mathbb{C}[[z^{-1}]]$, hence $\tilde{\Phi} = z^{-\alpha}\{\tilde{\varphi}\}_N$.

The estimates for the $\hat{\Phi}_k$'s imply similar estimates for the monodromies $\hat{\chi}_k \coloneqq \hat{\mathscr{A}}^\Gamma_\omega \tilde{\Phi}_k$, and hence for the residues $S^\Gamma_{\omega,k}$ of the functions $\mathrm{cont}_\gamma \hat{\Phi}_k(\omega+\zeta) - \hat{\chi}(\zeta)\frac{\log\zeta}{2\pi\mathrm{i}}$. Equation (7) follows by uniform convergence. □

Acknowledgements. The authors are grateful to F. Fauvet and M. Yampolsky for fruitful discussions and encouragements. The first author expresses thanks to the Centro di Ricerca Matematica Ennio De Giorgi and Fibonacci Laboratory for hospitality. The second author acknowledges the support of the French National Research Agency under the reference ANR-12-BS01-0017.

References

[DS13] A. Dudko, D. Sauzin. The resurgent character of the Fatou coordinates of a simple parabolic germ. *Comptes Rendus Mathématique* **352**, 3 (2014) 255–261.

[Eca81] J. Écalle. *Les fonctions résurgentes*, Vol. 1 & 2. Publ. Math. d'Orsay 81-05 & 81-06, 1981.

[Mal95] B. Malgrange. Sommation des séries divergentes. *Exp. Math.* **13** (1995), 163–222.

[Ram93] J.-P. Ramis. *Séries divergentes et théories asymptotiques.* Bull. Soc. Math. France (suppl.), Panoramas et Synthèses, v. 121. Paris: Société Mathématique de France. 74 pp., 1993.

[Sau13] D. Sauzin. On the stability under convolution of resurgent functions. *Funkcialaj Ekvacioj Serio Internacia* **56**, 3 (2013) 397–413.

[Sau14a] D. Sauzin. Nonlinear analysis with resurgent functions. Preprint oai:hal.archives-ouvertes.fr:hal-00766749, 35 p., 2014. To appear in *Annales Scientifiques de l'École Normale Supérieure.*

[Sau14b] D. Sauzin. Introduction to 1-summability and resurgence. Preprint oai:hal.archives-ouvertes.fr:hal-00860032, 125 p., 2014.

[Vor81] S. Voronin. Analytic classification of germs of conformal mappings $(\mathbb{C},0) \to (\mathbb{C},0)$. *Funktsional. Anal. i Prilozhen* **15** (1981), no. 1, 1–17, 96 (in Russian). English translation: *Functional Anal. Appl.* **15** (1981), no. 1, 1–13.

Artem Dudko: Institute for Mathematical Sciences, University of Stony Brook, NY, USA.

David Sauzin: CNRS UMI 3483 - Laboratoire Fibonacci, Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, Italy.