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On the Entropies of Hypersurfaces with bounded mean curvature

S. Ilias, B.Nelli, M.Soret

Abstract

We are interested in the impact of entropies on the geometry of a hypersurface of a Riemannian manifold. In fact, we will be able to compare the volume entropy of a hypersurface with that of the ambient manifold, provided some geometric assumption are satisfied. This depends on the existence of an embedded tube around such hypersurface. Among the consequences of our study of the entropies, we point out some new answers to a question of do Carmo on stable Euclidean hypersurfaces of constant mean curvature.

1 Introduction

For a complete noncompact Riemannian manifold, there are many results relating the exponential volume growth of geodesic balls to some of its Riemannian invariants. An important result concerning this asymptotic invariant is that obtained by Brooks, which gives an upperbound of the bottom of the essential spectrum in terms of the exponential volume growth of geodesic balls (see [6] and Theorem 5.1). As we will introduce other growth functionals, we will call such exponential volume growth, *volume entropy*, even if this is commonly used for the exponential volume growth of the universal cover of a compact manifold.

The main purpose of the present paper is to give new results concerning various entropies for hypersurfaces of spaces of bounded geometry. A first immediate remark is an extrinsic generalization of the result of Brooks. In fact, we will introduce an extrinsic entropy for hypersurfaces and show that Brooks' upperbound of the bottom of its essential spectrum is still valid with this new extrinsic entropy (see Corollary 5.1).

Then, we prove a general embedded tube result concerning hypersurfaces with mean curvature bounded away from zero (see Theorem 3.1). More precisely, we prove the existence of an embedded tube of fixed radius around any hypersurface with bounded curvature and mean curvature bounded away from zero, properly embedded in a simply connected manifold with bounded geometry (see also [38]). The embeddedness of this tube, apart of its own interest, allows us to compare the volume of extrinsic balls of such hypersurfaces with that of the geodesic balls of the ambient space (Corollary 5.2). This gives, in particular, a comparison between the extrinsic volume entropy of the hypersurface and the (intrinsic) volume entropy of the ambient space. In order to relate the volume entropy to the total curvature entropy (see Section 5 for the Definition) of finite index hypersurfaces with constant mean curvature we use some Cacciopoli's inequalities (obtained in [28]). Finally, using these results about entropies, we obtain some relations between spectra and curvature and various nonexistence results concerning some classes of hypersurfaces, including

those of constant mean curvature stable or of finite index. Among the problems considered, we will be interested in a question of do Carmo about Euclidean stable hypersurfaces of constant mean curvature. In fact, M. do Carmo ([15]) asked the following : "Is a noncompact, complete, stable, constant mean curvature hypersurface of \mathbb{R}^{n+1} , $n \geq 3$, necessarily minimal? " The answer was already known to be positive for $n = 2$ [31], [14], [42]. Later, the answer was proved to be positive for $n = 3, 4$, by M.F. Elbert, the second author and H. Rosenberg [17] and independently by X. Cheng [11], using a Bonnet-Myers's type method. So far, stability alone - or more generally finite index - does not seem to yield the answer to do Carmo's question in higher dimensions. Note that do Carmo's question can also be asked for general ambient spaces. We observe also, that R. Schoen, L. Simon and S.T. Yau [44] proved that a properly immersed orientable stable minimal hypersurface of \mathbb{R}^{n+1} , $n \leq 5$, which has Euclidean volume growth - hence zero volume entropy - is a hyperplane.

Using our results about entropies, we give a positive answer to do Carmo's question if the volume entropy of M is zero (Corollary 7.1) (the ambient manifold being arbitrary) or if the total curvature entropy of M is zero and $n \leq 5$ (Theorem 7.3) (the ambient manifold being a space-form). In particular, the comparison between the volume entropy of the ambient space and that of the hypersurface will also gives a positive answer if the hypersurface M has bounded curvature and is properly embedded (Theorem 7.2) (the ambient manifold being a simply-connected manifold with bounded geometry and with zero volume entropy). In the same spirit, we obtain nonexistence results for finite index hypersurfaces.

The plan of the paper is as follows: In Section 2, we recall the notion of stability, finite indices, and establish a useful analytic Lemma (Lemma 2.1), depending on the maximum principle. The construction of embedded half-tubes around a complete hypersurface with bounded curvature and mean curvature bounded away from zero, that is properly embedded in a simply-connected manifold of bounded curvature, is carried out in Section 3. Some results from [28] about Caccioppoli's inequalities for constant mean curvature hypersurfaces of finite index are recalled in Section 4. Next, in Section 5, various notions of entropies are introduced and discussed. In Section 6, we apply Caccioppoli's inequalities in order to relate the volume entropy to the total curvature entropy. Finally, in Section 7 we answer positively to do Carmo's question in the cases described above. Some of the arguments used in the proof of the halftube Theorem 3.1 and which are of independent interest are detailed in Section 8 (Appendix).

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2 Stability of an operator L and of a manifold M

Consider a Riemannian manifold M and the operator $L = \Delta + V$ where $\Delta = tr \circ Hess$ on M and V is a smooth potential. Associated to L one has the quadratic form

$$Q(f, f) := - \int_M f L f$$

defined on $f \in C_0^\infty(M)$.

Let Ω be a relatively compact domain of M . Define $i_{L|\Omega}$ (respectively $Wi_{L|\Omega}$) the number of negative eigenvalues of the operator $-L$, for the Dirichlet problem on Ω :

$$\begin{aligned}
& -Lf = \lambda f, \quad f|_{\partial\Omega} = 0 \\
& \text{(resp. } -Lf = \lambda f, \quad f|_{\partial\Omega} = 0, \quad \int_{\Omega} f = 0\text{)}.
\end{aligned}$$

The $\text{Index}(L)$ (resp. $\text{WIndex}(L)$) is defined as follows

$$\begin{aligned}
& \text{Index}(L) := \sup\{i_{L|\Omega} \mid \Omega \subset M \text{ rel. comp.}\} \\
& \text{(resp. } \text{WIndex}(L) := \sup\{Wi_{L|\Omega} \mid \Omega \subset M \text{ rel. comp.}\})
\end{aligned}$$

The operator L is said *nonpositive* (respectively *weakly nonpositive*) if $\text{Index}(L) = 0$ (respectively $\text{WIndex}(L) = 0$).

We will look at the following situation in which the operator L has a geometric interpretation. Assume that M is a constant mean curvature hypersurface in a manifold \mathcal{N} . It is known that constant mean curvature hypersurfaces are critical for the area functional with respect to compact support deformations that keep the boundary fixed and whose algebraic volume swept in \mathcal{N} during deformation remains zero (see for instance [2] and [3]). The operator $L := \Delta + \text{Ric}(\nu, \nu) + |A|^2$ is called the *stability operator* of M . Here Ric , ν and A are respectively the Ricci curvature, the unit normal field and the second fundamental form of the hypersurface M . We define the $\text{Index}(M) := \text{Index}(L)$ and $\text{WIndex}(M) := \text{WIndex}(L)$. Moreover M is said to be *stable* (respectively *weakly stable*) if L is nonpositive (respectively *weakly nonpositive*).

Stability (respectively weak stability) of M means that

$$Q(f, f) \geq 0, \quad \forall f \in C_0^\infty(M) \quad \text{(resp. } \forall f \in C_0^\infty(M) \quad \int_M f = 0\text{)}$$

It is easy to see that $Q(f, f)$ is the second derivative of the volume in the direction of $f\nu$ (see [2]), then $\text{Index}(M)$ (respectively $\text{WIndex}(M)$) measures the number of linearly independent normal deformations with compact support of M , that decrease area (respectively that decrease area, leaving fixed a volume). When $H = 0$, one can drop the condition $\int_M f = 0$, then, for a minimal hypersurface one consider only the $\text{Index}(M)$. It is proved in [1] that $\text{Index}(M)$ is finite if and only if $\text{WIndex}(M)$ is finite. So, when we assume finite index we are referring to either of the indexes without distinction. Finally, we recall that M has finite index if and only if there exists a compact subset K of M such that $M \setminus K$ is stable (see Proposition 2.1 of [28]). Later on, we will need an analytic Lemma that summarizes some properties of the operator $L = \Delta + V$, where $V \in L_{loc}^1(M)$ is a suitable potential.

Lemma 2.1. *Let M be a complete manifold and $L = \Delta + V$, $V \in L_{loc}^1(M)$. Consider the following assertions:*

- (1) *For any compact domain $\Omega \subset M$, L satisfies the maximum principle: for $u, v \in W^{1,2}(\Omega)$ such that $u \leq v$ on $\partial\Omega$ and $Lu \geq Lv$, one has $u \leq v$ on Ω .*
- (2) *The Dirichlet problem for L on any compact domain $\Omega \subset M$ has a unique solution in $W^{1,2}(\Omega)$.*
- (3) *The operator L is nonpositive on M .*
- (4) *There exists a smooth positive ϕ on M such that $L\phi \leq 0$.*
- (5) *There exists a smooth positive ϕ on M such that $L\phi = 0$.*

Then (3),(4),(5) are equivalent and any of the latter assertions implies (1) which implies (2).

Proof. For the equivalence between (3), (4), (5) see [18], [19] and [37] (if M is an Euclidean space, such equivalence was originally due to Glazman (unpublished) and also to Moss & Pieperbrinck [36]).

(1) \implies (2): apply (1) to $u - v$ and $v - u$ and use linearity of L .

(5) \implies (1): by the linearity of L , it is enough to prove that if $w \in W^{1,2}(\Omega)$ such that $w \leq 0$ on $\partial\Omega$ and $Lw \geq 0$ on Ω , then $w \leq 0$ on Ω . Define $h := \frac{w}{\phi}$, where ϕ is as in (5). By a straightforward computation one has that

$$\Delta h + 2\langle \nabla h, \frac{\nabla \phi}{\phi} \rangle \geq 0.$$

By the maximum principle (see for instance Theorem 10.1 [21]), one has $h \leq 0$ i.e. $w \leq 0$ on Ω . \square

We need the following application of the maximum principle ((1) of Lemma 2.1).

Corollary 2.1. *Suppose L is nonpositive and there exists a nonnegative $v \in W^{1,2}(M)$ such that $Lv \leq -c$ for some positive constant c . If there exists a positive $u \in W_0^{1,2}(M)$ such that $Lu > -c$, then $v \geq u$ on M .*

Proof. Let Ω be the support of u , then $v \geq u$ on $\partial\Omega$. By hypothesis $Lv \leq -c \leq Lu$. Since L is nonpositive, the maximum principle ((1) of Lemma 2.1) yields $v \geq u$ on Ω and hence on M . \square

Remark 2.1. *Functions similar to functions u, v of Corollary 2.1 are studied in [39].*

3 Tubes around embedded hypersurfaces of mean curvature bounded from below

Throughout this section, the manifold \mathcal{N} is a simply-connected manifold with bounded curvature. Let M be an orientable hypersurface *properly embedded* in \mathcal{N} and assume that M has mean curvature function H bounded away from zero. Orient M by its mean curvature vector \vec{H} . Then $\vec{H} = H\vec{\nu}$, and H is positive. Moreover we assume that M has bounded second fundamental form. Since \mathcal{N} is simply connected, by Jordan-Brouwer Separation Theorem, M separates \mathcal{N} into two components (see for instance [32]). We call *mean convex* side of M the component towards which \vec{H} points. We consider the normal exponential map of M defined by

$$\begin{aligned} \exp : M \times \mathbb{R} &\longrightarrow \mathcal{N} \\ (p, r) &\longmapsto \exp_p(r\nu) \end{aligned} \tag{1}$$

As M and \mathcal{N} have bounded curvature, for any $p \in M$, there exists a geodesic ball $B_{r_0}(p)$ centered at p of radius $r_0 > 0$ such that $\exp|_{B_{r_0}(p) \times (-r_0, r_0)}$ is a diffeomorphism on its image.

Let

$$T^+(r_0) := \exp(M \times (0, r_0))$$

be the *half-tube* in \mathcal{N} around M of radius r_0 . The half-tube $T^+(r_0)$ is locally diffeomorphic to $M \times (0, r_0)$. In this section, we prove the following embeddness property of the half-tube $T^+(r_0)$ which will be crucial in the proof of one of the main results of this article.

Theorem 3.1. *Let M be a complete hypersurface properly embedded in a simply connected manifold \mathcal{N} . Assume that M and \mathcal{N} have bounded curvature and M has possibly non empty compact boundary. Assume that M has mean curvature function H bounded away from zero ($H \geq \frac{\epsilon}{n} > 0$). Then one has the following results.*

- (1) *If $\partial M = \emptyset$, there exists an embedded half-tube $T^+(\rho)$ contained in the mean-convex side of M , where the radius ρ depends on the curvatures of M and \mathcal{N} .*
- (2) *If ∂M is compact, then there exists a compact subset K of M and a half-tube $T^+(\rho')$ of $M \setminus K$ contained in the mean-convex side of M , where the radius ρ' depends on the compact K , and the curvatures of M and \mathcal{N} .*

Remark 3.1. • *In the case where M has no boundary and constant mean curvature, the existence of a tube of constant radius around M was stated in [38] (see also [45] and [40]). We believe that it is necessary and useful to give a detailed proof of this result, especially in the more general setting that we are treating.*

- *If \mathcal{N} is not simply-connected, assuming in addition that M is strongly Alexandroff embedded (see Definition 3.3 in [38]), one obtains the same conclusion as in Theorem 3.1.*

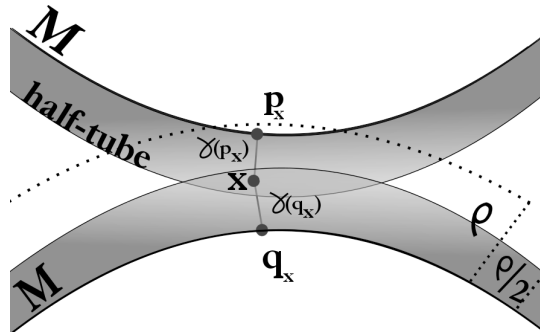


Figure 1: Self-intersection of a halftube of M in \mathcal{N}

Proof of Theorem 3.1. (1) See Figure 1. Let $r_1 > 0$ be such that for any $p \in M$, $exp|_{B_\rho(p) \times (0, r_1)}$ is an isometry on $B_\rho(p) \times (0, r_1)$ endowed with the pull-back metric. The proof is in two steps. In the first step, we will prove that if there is a positive $\rho \leq r_1$ such that $T^+(\rho) \cap M = \emptyset$, then $T^+(\frac{\rho}{2})$ is embedded. In the second step we prove that, for ρ sufficiently small, we have $T^+(\rho) \cap M = \emptyset$.

Proof of Step 1. Assume, by contradiction that $T^+(\frac{\rho}{2})$ is not embedded. Then there exist $x \in T^+(\frac{\rho}{2})$ and $p_x \neq q_x \in M$, such that x belongs to the geodesic $\gamma(p_x)$ starting at p_x , orthogonal to M and also belongs to the geodesic $\gamma(q_x)$ starting at q_x , orthogonal to M . Moreover $d(p_x, x) \leq \frac{\rho}{2}$ and $d(q_x, x) \leq \frac{\rho}{2}$ where d is the distance in \mathcal{N} . By the distance inequality, $d(p_x, q_x) \leq \rho$. Then q_x is contained in a geodesic ball in \mathcal{N} , centered at p_x , of radius ρ . As $\rho \leq r_1$, there are (p, t) , (q, s) , (x_0, l) , contained in a geodesic ball $B_{(p, t)}(\rho)$ of $M \times (0, \rho]$ centered at (p, t) of radius ρ and such that $exp(p, t) = p_x$, $exp(q, s) = q_x$, $exp(x_0, l) = x$. Without loss of generality, we can assume that

$t = 0$. If $s = 0$, then both geodesics $\exp^{-1}(\gamma(p_x))$, $\exp^{-1}(\gamma(q_x))$, would be orthogonal to $M \times \{0\}$ and would meet at $(x_0, l) \in B_{(p,0)}(\rho)$, which is contradiction since \exp is an isometry on $B_{(p,0)}(\rho)$. Then $s \neq 0$ and $q_x \notin M$. Contradiction. Therefore $T^+(\frac{\rho}{2})$ is embedded.

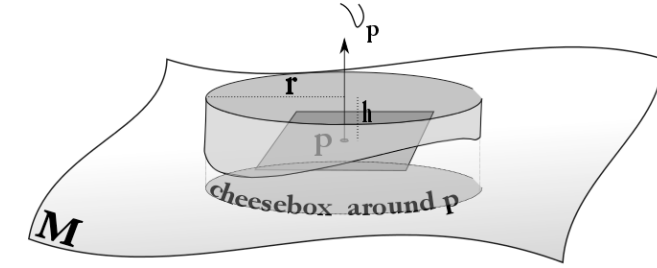


Figure 2: A cheesebox around $p \in M \subset \mathcal{N}$

Proof of Step 2. Assume by contradiction that for any sufficiently small $\rho > 0$, $T^+(\rho) \cap M \neq \emptyset$. Let \tilde{p} be a point in $T^+(\rho) \cap M$ and let $(p, t) \in M \times (0, \rho]$ be the nearest point to $M \times \{0\}$ in $\exp^{-1}(\tilde{p})$ which is on the geodesic starting at $(p, 0) \in M \times \{0\}$ and orthogonal to M . Define a function ϕ by $\phi(p) = t$ and let Ω be the set of points p of M such that $t \leq \rho$. Since M is properly embedded, $\phi(p) > 0$.

Let $P := \{(p, \phi(p)) \mid p \in \Omega\}$. The fact that M is embedded implies that $P \cap (M \times \{0\}) = \emptyset$.

We claim that the function ϕ is bounded uniformly in $C^1(\Omega)$ and $\|\phi\|_{C^1(\Omega)}$ goes to zero as $\rho \rightarrow 0$. We prove these facts by using the so called *cheesebox argument* (see the details in Appendix 8.1, see also [45]).

We recall the cheesebox argument: Let M be a complete hypersurface with bounded curvature in a manifold \mathcal{N} with bounded curvature. Then, there exists a positive ρ such that, for any point $p \in M$ (see Figure 2):

- (a) The hypersurface M is locally a graph G_p of a function ϕ defined on a ball $D(p, \rho)$ of $T_p M$, centered at p of radius ρ .
- (b) The graph G_p is contained in the cheesebox $D(p, \rho) \times [-h/2, h/2]$, of radius ρ and height $h = c\rho^2$ (where c is a constant depending on bounds of the curvatures of M and \mathcal{N}). Moreover G_p cuts the boundary of the cheesebox only on $\partial D(p, \rho) \times [-c\rho^2, c\rho^2]$.

It is proved in Appendix 8.1 that properties (a) and (b) imply the following estimate (see inequality (52)):

$$\|\phi\|_1 := \left(\sup_{p \in \Omega} |\phi| + \sup_{p \in \Omega} |\nabla \phi| \right) \leq O(\rho). \quad (2)$$

One concludes in particular that the tangent plane to $M \times \{0\}$ at p and the tangent plane to P at p' are as close as one wishes if ϕ is small. From Appendix 8.2, ϕ satisfies a uniformly elliptic quasilinear partial differential equation (see equation (57)). By a classical argument in elliptic theory, the derivatives of any order of ϕ are uniformly bounded (see for instance [21] Theorem 6.2

and Problem 6.1). Thus $\|\phi\|_{C^\infty(\Omega)}$ is uniformly bounded.

We claim that the mean curvature vector of P points towards $M \times \{0\}$. Indeed, let $(p, 0) \in \Omega \times \{0\}$ and $(p, \phi(p))$ the corresponding point on P . Let γ be the geodesic, orthogonal to $M \times \{0\}$ joining $(p, 0)$ to $(p, \phi(p))$. The geodesic γ in \mathcal{N} does not intersect M , hence it is contained in the mean convex side of M . This implies that the mean curvature of P at $\phi(p)$ points towards $M \times \{0\}$. From equation (57) in Appendix 8.2, the mean curvature H_P of P is given by

$$nH_P = nH + \Delta\phi + O(\rho^\alpha) \quad (3)$$

where the term $O(\rho^\alpha)$ (with $0 < \alpha < 1$) converges uniformly to zero as ρ tends to zero provided M and \mathcal{N} have uniform curvature bounds.

Since the mean curvature of P points towards $M \times \{0\}$, the hypothesis $H \geq \frac{\varepsilon}{n}$ implies $H_P \leq -\frac{\varepsilon}{n}$. Therefore, for ρ sufficiently small, one has

$$\Delta\phi \leq -2\varepsilon \quad (4)$$

Now we construct a function ψ_R in terms of the distance function from a fixed point p_0 of M , such that $\Delta\psi_R \geq -\varepsilon$ on $B_{p_0}(R) \cap \Omega$. Let r be the distance function from $p_0 \in M$, and consider the radial test function $\psi_R(x) = f_R \circ r(x)$ where

$$f_R(r) = \begin{cases} \beta \left(1 - \left(\frac{r}{R}\right)^2\right) & \forall r \leq R, \\ 0 & \forall r \geq R. \end{cases} \quad (5)$$

with $\beta = \rho - \delta$, for small positive δ . Notice that ψ_R vanishes on $\partial B_{p_0}(R) \cap \Omega$ and $\psi_R \leq \phi$ on $B_{p_0}(R) \cap \partial\Omega$, since $\phi|_{\partial\Omega} = \rho$. Therefore $\psi_R \leq \phi$ on $\partial(\Omega \cap B_{p_0}(R))$. Since M has bounded curvature, there exists $k > 0$, such that $\text{Ric}_M \geq -(n-1)k^2$. By standard comparison theorems (see for instance [46]) one has

$$\Delta r \leq \frac{n-1}{r}(1+kr) \quad (6)$$

where the inequality holds outside the cut-locus of M and holds in the weak sense at any point of M . Using inequality (6) and that $\Delta f_R(r) = f'_R(r)\Delta r + f''_R(r)|\nabla r|^2$, one has

$$\Delta\psi_R \geq -\frac{2\beta}{R^2}(n + (n-1)kR). \quad (7)$$

For R large, $\Delta\psi_R \geq -\varepsilon$, as we wished. This last inequality with inequality (4) yield $\Delta\phi \leq \Delta\psi_R$ on $B_{p_0}(R) \cap \Omega$, for R large. Then, by Corollary 2.1, $\phi \geq \psi_R$ on $B_{p_0}(R) \cap \Omega$, for R large. Letting $R \rightarrow \infty$, we obtain $\phi \geq \beta$ on Ω . Therefore $\phi \geq \rho - \delta$ in Ω for any $\delta > 0$. Thus $\phi \geq \rho$ in Ω . This is a contradiction, hence $T^+(\rho)$ is embedded.

(2) The proof is the same as in (1), except than for the choice of the test function ψ_R . Without loss of generality, we can assume that $\varepsilon < 1$ and $\rho < 1$. Let $R_1 > 1$ be such that $K \subset B_{p_0}(R_1 - 1)$ and let $\sigma := \inf_{p \in \partial B_{p_0}(R_1 - 1)} \phi(p)$ (notice that $\sigma \leq \rho \leq 1$). Let $\Omega' = \Omega \cap (M \setminus B_{p_0}(R_1))$ and let $r(x)$ be the distance function in M from any fixed point p_0 in Ω' . Define R to be the distance from $p_0 \in \Omega'$ to $B_{R_1 - 1}$ (notice that $R > 1$). We define a radial test function $\psi_R = g_R \circ r$ where

$$g_R(r) = \begin{cases} \beta' \left(1 - \left(\frac{r}{R}\right)^2\right) & \forall r \leq R, \\ 0 & \forall r \geq R. \end{cases} \quad (8)$$

where the constant β' equals $\frac{\varepsilon}{2(n+(n-1)k)}(\sigma - \delta)$, where δ is small positive number and k is such that $Ric_M \geq -(n-1)k^2$. Notice that ψ_R vanishes on $\partial B_{p_0}(R) \cap \Omega'$ and $\psi_R \leq \phi$ on $B_{p_0}(R) \cap \partial\Omega'$. Therefore $\psi_R \leq \phi$ on $\partial(\Omega' \cap B_{p_0}(R))$.

Using inequality (6), by a straightforward computation one has

$$\Delta\psi_R \geq -\varepsilon(\sigma - \delta) \quad (9)$$

Since $0 < \sigma - \delta < 1$, $\Delta\psi_R \geq -\varepsilon$, as we wished. This last inequality together with inequality (4) yield $\Delta\phi \leq \Delta\psi_R$ on $B_{p_0}(R) \cap \Omega'$. Then, by Corollary 2.1, $\phi \geq \psi_R$ on $B_{p_0}(R) \cap \Omega'$, for R large. In particular $\phi(p_0) \geq \beta'$ for any $p_0 \in \Omega \setminus B_{p_0}(R_1)$. This proves that the tube $T^+(\beta')(\Omega \setminus B_{p_0}(R_1))$ is embedded. \square

4 Caccioppoli's inequality for constant mean curvature hypersurfaces with finite index

In this section we recall some results obtained in [28] and needed in the sequel. We assume that \mathcal{N} is an orientable Riemannian manifold with bounded sectional curvature. Moreover let M be an orientable hypersurface immersed in \mathcal{N} and assume that M has constant mean curvature. When the mean curvature is non zero, we orient M by its mean curvature vector \vec{H} . Then $\vec{H} = H\vec{\nu}$ with H positive and $\vec{\nu}$ a unit normal vector. When the mean curvature is zero, we choose once for all an orientation $\vec{\nu}$ on M . We denote by φ the length of the traceless part of the second fundamental form A i.e. $\varphi := |A - Hg|$. In the present article we will need the following Caccioppoli's inequalities (Theorems 5.1 and 5.4 of [28]) and a reverse Hölder inequality (Theorem 4.2 [28]).

Theorem 4.1 (Caccioppoli's inequality of type I). *Let M be a complete hypersurface immersed with constant mean curvature H in a manifold \mathcal{N} . Assume M has finite index. Then, there exist a compact subset K of M (which is empty if M is stable) and constants $\beta_1, \beta_2, \beta_3$, such that for every $f \in C_0^\infty(M \setminus K)$ and $x \geq 1$*

$$\beta_1 \int_{M \setminus K} f^{2x+2} \varphi^{2x+2} \leq \beta_2 \int_{M \setminus K} |\nabla f|^{2x+2} + \beta_3 \int_{M \setminus K} f^{2x+2}. \quad (10)$$

Moreover the constant β_1 is positive if and only if $x \in [1, 1 + \sqrt{\frac{2}{n}})$,

Theorem 4.2. *Let M be a complete hypersurface immersed with constant mean curvature H in a manifold with constant sectional curvature c . Assume M has finite index. Then there exists a compact subset K of M (which is empty if M is stable) and a positive constant \mathcal{S} such that for any $x \in [1, 1 + \sqrt{\frac{2}{n}})$*

$$\int_{M \setminus K} \varphi^{2x+2} \leq \mathcal{S} \int_{M \setminus K} \varphi^{2x} \quad (11)$$

It is worthwhile to note that, if M is stable, using a suitable test function in inequality (58) of the proof of Theorem 4.2 in [28], one can deduce

$$\int_{B_{p_0}(R)} \varphi^{2x+2} \leq \mathcal{S} \int_{B_{p_0}(R+1)} \varphi^{2x} \quad (12)$$

where p_0 is a fixed point of M and \mathcal{S} is a positive constant.

Before stating the next Theorem, we need to define two new notations:

- For $\gamma = \frac{n-2}{n}$, $\mu = \frac{n^2}{4(n-1)}$, let g be the following function

$$g_n(x) = \frac{(2x - \gamma)^2 - x^4}{(2x - \gamma)^2 - \mu x^4} \quad (13)$$

- Let x_2 the following real number

$$x_2 = \frac{2\sqrt{n-1}}{n} \left(1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}} \right) \quad (14)$$

Theorem 4.3 (Caccioppoli's inequality of type III - $H \neq 0$). *Let M be a complete hypersurface immersed with constant mean curvature $H \neq 0$, in a manifold with constant curvature c . Assume M has finite index and $n \leq 5$. Then there exist a compact subset K in M and a constant γ such that, for any $f \in C_0^\infty(M \setminus K)$*

$$\gamma \int_{M \setminus K} f^2 \varphi^{2x} \leq \mathcal{D} \int_{M \setminus K} \varphi^{2x} |\nabla f|^2 \quad (15)$$

provided either (1) $c = 0$ or 1 , $x \in [1, x_2]$ or (2) $c = -1$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 \geq g_n(x)$.

5 Entropies

Let \mathcal{N} be a complete, noncompact Riemannian manifold. In this section, except when it is indicated, there is no hypothesis on the curvature of \mathcal{N} .

In this Section, we deal with the exponential growth of various functionals on \mathcal{N} . As we explained in the Introduction, we will call such exponential growths, *entropies*.

The most important one is the entropy associated to the volume of geodesic balls in \mathcal{N} (see for instance [6], [7]).

Definition 5.1. *Let w be a positive non-decreasing function. The entropy of w is by definition*

$$\mu_w := \limsup_{r \rightarrow \infty} \left(\frac{\ln w(r)}{r} \right). \quad (16)$$

We say that the function w has *subexponential growth* if its entropy is zero. It is worth noting that $\mu_w = 0$ is equivalent to

$$\limsup_{r \rightarrow \infty} \frac{w(r)}{e^{\alpha r}} = 0, \quad \forall \alpha > 0. \quad (17)$$

We say that w has *exponential growth* if its entropy is positive.

We observe that having a subexponential growth is a assumption weaker than being bounded by a polynomial of any degree (for instance $w(r) = e^{r^\beta}$, $\beta < 1$ has subexponential growth). Let $B_\sigma^{\mathcal{N}}(R)$ be a geodesic ball in \mathcal{N} , of radius R , centered at a fixed point $\sigma \in \mathcal{N}$ and denote by $|B_\sigma^{\mathcal{N}}(R)|$ its volume. When $w(R) = |B_\sigma^{\mathcal{N}}(R)|$, the entropy of w is called the *volume entropy* of \mathcal{N} and it is denoted by

$$\mu_{\mathcal{N}} := \limsup_{R \rightarrow \infty} \left(\frac{\ln |B_\sigma^{\mathcal{N}}(R)|}{R} \right). \quad (18)$$

Using the distance inequality, one can easily check that the volume entropy does not depend on the center σ of the balls.

Remark 5.1. *In the definition of entropies, one can take \liminf instead of \limsup .*

Let M be a complete, noncompact manifold immersed in \mathcal{N} . Let g (respectively A) be the induced metric (respectively the second fundamental form) of M and consider the traceless second fundamental form of M , i.e. $\phi = A - Hg$, where H is the mean curvature of M .

In addition to the volume entropy of M , we will be interested in the following entropies.

- *Extrinsic volume entropy* of M . It is denoted by $\mu_M^{\mathcal{N}}$ and is obtained by replacing, in the definition of the volume entropy, the volume of the intrinsic balls of M by the volume in M of $B_\sigma^{\mathcal{N}}(R) \cap M_\sigma$ where M_σ is the connected component of M containing σ .

As for the volume entropy, the extrinsic volume entropy doesn't depend on the choice of σ .

- *Total p -curvature entropy* of M . It is denoted by $\mu_{\mathcal{T}_p}$, for any $p > 0$ and it is the entropy of the total p -curvature \mathcal{T}_p defined by

$$\mathcal{T}_p(R) = \int_{B_\sigma(R)} |\phi|^p \quad (19)$$

There are important relations between the volume entropy of a manifold \mathcal{N} and the bottom of its *essential spectrum*. Let Δ be the Laplacian on \mathcal{N} , then the bottom of the spectrum $\sigma(\mathcal{N})$ of $-\Delta$ is

$$\lambda_0(\mathcal{N}) = \inf\{\sigma(\mathcal{N})\} = \inf_{\substack{f \in C_0^\infty(\mathcal{N}) \\ f \neq 0}} \left(\frac{\int_{\mathcal{N}} |\nabla f|^2}{\int_{\mathcal{N}} f^2} \right). \quad (20)$$

The bottom of the essential spectrum $\sigma_{ess}(\mathcal{N})$ of $-\Delta$ is

$$\lambda_0^{ess}(\mathcal{N}) = \inf\{\sigma_{ess}(\mathcal{N})\} = \sup_K \lambda_0(\mathcal{N} \setminus K) \quad (21)$$

where K runs through all compact subsets of \mathcal{N} .

Another invariant related to the spectrum is the Cheeger isoperimetric constant. Recall that the Cheeger constant $h_{\mathcal{N}}$ of a Riemannian manifold \mathcal{N} is defined as $h_{\mathcal{N}} = \inf_{\Omega} \frac{|\partial\Omega|}{|\Omega|}$, where Ω runs over all compact domains of \mathcal{N} with piecewise smooth boundary $\partial\Omega$.

J. Cheeger [9] and R. Brooks [6] proved the following important comparison result between the bottom of the essential spectrum, the volume entropy and the Cheeger constant:

Theorem 5.1 (Brooks-Cheeger's Theorem). *If \mathcal{N} has infinite volume, then*

$$\frac{h_{\mathcal{N}}^2}{4} \leq \lambda_0(\mathcal{N}) \leq \lambda_0(\mathcal{N} \setminus K) \leq \lambda_0^{\text{ess}}(\mathcal{N}) \leq \frac{\mu_{\mathcal{N}}^2}{4} \quad (22)$$

where K is any compact subset of \mathcal{N} and $h_{\mathcal{N}}$ is the Cheeger constant of \mathcal{N} .

Remark 5.2. • *The upper bound in (22) is still true if one replaces lim sup by lim inf in the definition of the entropy (see [27], [41]).*

• *When \mathcal{N} has finite volume, we have $h_{\mathcal{N}} = \mu_{\mathcal{N}} = \lambda_0(\mathcal{N}) = 0$ and Theorem 5.1 becomes trivial. In the finite volume case, R. Brooks introduced a modified h and modified μ and was able to obtain a similar but nontrivial result (see [7]).*

• *We observe that the infiniteness of the volume follows, for example, from the existence of a Sobolev inequality (see [8] and [25]). Another important case of infiniteness of volume, useful for us, is that where M is a complete noncompact submanifold with bounded mean curvature of a manifold with bounded geometry (see [20]).*

A result analogous to that of Theorem 5.1 for the p -Laplacian is known. In fact, B. P. Lima, J. F. Montenegro and N.L. Santos [33], adapting Theorem 2 in [6] proved a Brooks type result for the p -Laplacian, and the generalization of the lower bound in terms of the Cheeger constant, in the case of p -Laplacian, was obtained by D. Lefton and L. Wei [30] (see also [34] for the case where the manifold is compact).

It is easy to compare the volume entropy with the extrinsic volume entropy. Indeed we have the following result.

Proposition 5.1. *Let M be a complete, noncompact manifold. Then*

$$h_M \leq \mu_M \leq \inf_{\mathcal{N}} \mu_M^{\mathcal{N}} := j_M \quad (23)$$

where \mathcal{N} runs in the set of all manifolds \mathcal{N} in which M admits an isometric immersion.

Proof. The first inequality of (23) is an immediate consequence of Theorem 5.1. We wrote it, in order to emphasize that it is independent of the spectrum. For the sake of completeness, we give a very short proof of it. In fact, it suffices to observe, that for any $x \in M$ and for any $R > 0$, we have

$$\frac{|\partial B_x^M(R)|}{|B_x^M(R)|} = \frac{d}{dr} \Big|_{r=R} (\ln |B_x^M(R)|) \geq h_M$$

which gives after integration and after taking the limit for $R \rightarrow \infty$, the desired inequality $h_M \leq \mu_M$.

For the second inequality, it suffices to observe that for any $x \in M$, $R > 0$ and any \mathcal{N} satisfying the hypothesis, one has

$$B_x^M(R) \subset M \cap B_x^{\mathcal{N}}(R)$$

then

$$|B_x^M(R)|_M \leq |M \cap B_x^{\mathcal{N}}(R)|_M,$$

where $|\cdot|_M$ is the volume in M . Inequality (23) follows from the definition of the entropies. \square

Remark 5.3. *A comparison similar to (23) holds for the intrinsic and extrinsic entropies of the p -total curvatures, that can be obviously defined.*

It is worth stating the following immediate consequence of Theorems 5.1 and Proposition 5.1, that gives an extrinsic upper bound of $\lambda_0(M)$.

Corollary 5.1. *Let M be a complete, noncompact manifold with infinite volume, then*

$$\frac{h_M^2}{4} \leq \lambda_0(M) \leq \lambda_0^{ess}(M) \leq \frac{j_M^2}{4} \quad (24)$$

Remark 5.4. *All the previous results can be adapted to the case where M has finite volume, using the modified volume entropy and Cheeger constant introduced by Brooks [7].*

Now we recall some classical estimates for the volume entropy and the Cheeger constant in terms of bounds on curvatures. A first result is the following well known consequence of the Bishop volume comparison theorem.

Lemma 5.1. *(see [29]) Let M be a complete Riemannian manifold of dimension m and such that $Ric_M \geq -(m-1)k^2$ (for some constant k). Then we have:*

$$\mu_M \leq (m-1)k \quad (25)$$

The proof of next Lemma is based on the comparison theorems for the hessian of the distance function,

Lemma 5.2. *(see [48]) Let M be a complete simply connected Riemannian manifold of dimension m with sectional curvature bounded from above by $-k^2$. Then we have:*

$$h_M \geq (m-1)k \quad (26)$$

The result of Lemma 5.2 was extended to minimal submanifolds of such simply connected manifolds by J. Choe and R. Gulliver [12]. The proof in [12] (in particular Lemmas 7 and 8) can be easily adapted to submanifolds of bounded mean curvature in order to give the following result:

Lemma 5.3. *Let \mathcal{N} be an n -dimensional complete simply connected Riemannian manifold with sectional curvature bounded from above by $-k^2$ (for some positive constant k). Let M be a complete noncompact submanifold of \mathcal{N} with bounded mean curvature satisfying $|H| \leq a$, then we have:*

$$h_M + a \geq (n-1)k \quad (27)$$

Remark 5.5. • *The estimate given in (27) is nontrivial only for $(n-1)k > a$.*

• *By Lemma 5.1 and the upper bound in (23), if $Ric_M \geq -(m-1)k^2$, then we obtain the well known Mac Kean estimate:*

$$\lambda_0(M) \leq \frac{(m-1)^2 k^2}{4}.$$

Many recent results in the literature can be deduced or generalized immediately from our results. Let us give some examples of such consequences.

1. B. P. Lima, J.F. Montenegro and N.L. Santos (see [33]) obtain a generalization of Brooks upper bound for the p -Laplacian. More precisely, if one denotes by $\lambda_{0,p}(M) = \inf_{\substack{f \in C_0^\infty(M) \\ f \neq 0}} \left(\frac{\int_M |\nabla f|^p}{\int_M f^p} \right)$

and $\lambda_{0,p}^{\text{ess}}(M) = \sup_K \lambda_{0,p}(M \setminus K)$, where K runs through all compact subsets of \mathcal{N} , they prove that

$$\lambda_{0,p}(M) \leq \lambda_{0,p}^{\text{ess}}(M) \leq \left(\frac{\mu_M}{p} \right)^p.$$

Using inequality (23) in the previous inequality, we deduce immediately an upper bound for $\lambda_{0,p}(M)$ in terms of the extrinsic volume entropy, which gives a p -Laplace version of Corollary 5.1.

2. Under the same assumption as in Lemma 5.3, if $a < (n-1)k$, Corollary 5.1 yields $\lambda_0(M) \geq \frac{((n-1)k-a)^2}{4}$. Hence we obtain the result by L.F. Cheung and P.F. Leung [10] and G.P. Bessa and J. F. Montenegro [5].
3. In [30], L. Lefton and D. Wei obtained the following generalization of Cheeger inequality (the first inequality of (22)) to the first eigenvalue of p -Laplace operator, that is

$$\left(\frac{h_M^2}{p} \right)^p \leq \lambda_{0,p}(M)$$

The proof in [30] is done in the Euclidean case, but it can be easily adapted to a manifold. This inequality combined with Lemma 5.2 gives the generalization to p -Laplacian of the McKean inequality obtained in [33], and combined with Lemma 5.3 gives the inequality

$$\lambda_{0,p}(M) \geq \frac{((n-1)k-a)^p}{p^p}$$

which generalizes the aforementioned results of [10] and [5]

4. Theorem A in [23] and the main Theorem in [22] can be immediately deduced from our results. Let us be more precise. In [23], V. Gimeno and V. Palmer proved that if M is a minimal m -dimensional submanifold properly immersed in a simply connected manifold \mathcal{N} , such that

$$\text{sec}(\mathcal{N}) \leq b \leq 0, \quad \sup_R \frac{|B_\sigma^{\mathcal{N}}(R) \cap M|}{|\mathcal{B}_\sigma^m(R)|} < \infty \quad (28)$$

where $\mathcal{B}_\sigma^m(R)$ is the geodesic ball centered at σ of $\mathbb{H}^m(b)$ (the space form of sectional curvature b), then

$$h_M \leq (m-1)\sqrt{-b}. \quad (29)$$

As the volume entropy of $\mathbb{H}^m(b)$ is $(m-1)\sqrt{-b}$, the second hypothesis in (28) implies that $\mu_M^{\mathcal{N}} \leq (m-1)\sqrt{-b}$. Then, Corollary 5.1 yields (29) without the minimality hypothesis.

In [22], V. Gimeno proves that under the same assumptions (28) (as in [23]), one has in addition the equality $\lambda_0(M) = -\frac{(m-1)^2 b}{4}$. This is an immediate consequence of Corollary 5.1 and Lemma 5.3 which give in this case:

$$h_M = \lambda_0(M) = \lambda_0^{ess}(M) = -\frac{(m-1)^2 b}{4},$$

again without any minimality assumption.

It is possible to obtain a stronger result about the comparison between the volume of extrinsic balls and the volume of balls of the ambient space, by applying Proposition 3.1.

Corollary 5.2. *Let M be a noncompact, complete, hypersurface of bounded curvature, properly embedded in a simply-connected manifold \mathcal{N} with bounded curvature. Assume M has mean curvature bounded away from zero. Then, one has:*

$$|B_\sigma^{\mathcal{N}}(R) \cap M|_M \leq c|B_\sigma^{\mathcal{N}}(R)|_{\mathcal{N}} \quad (30)$$

where c is a constant depending on the radius ρ of the embedded half-tube given in Theorem 3.1 and on the curvature of M and $|\cdot|_M$ (respectively $|\cdot|_{\mathcal{N}}$) denotes the volume in M (respectively in \mathcal{N}).

Proof. By Proposition 3.1 there exists ρ such that the half tube $T^+(\rho)$ is embedded. Hence, by Weyl's formula, the volume of $T^+(\rho) \cap B_\sigma^{\mathcal{N}}(R)$ satisfies

$$|B_\sigma(R)^{\mathcal{N}} \cap M|_M \leq c|T^+(\rho) \cap B_\sigma^{\mathcal{N}}(R)| \quad (31)$$

where c is a constant depending only on ρ , the curvature of M and \mathcal{N} (see for instance [24] or [40]). As $|T^+(\rho) \cap B_\sigma^{\mathcal{N}}(R)| \leq |B_\sigma^{\mathcal{N}}(R)|$, inequality (30) follow from inequality (31). \square

Then, we are able to establish a relation between the intrinsic and the extrinsic volume entropies.

Corollary 5.3. *Let M be a noncompact, complete, hypersurface of bounded curvature, properly embedded in a simply-connected manifold \mathcal{N} with bounded curvature. Assume M has mean curvature bounded away from zero. Then*

$$\mu_M \leq \mu_M^{\mathcal{N}} \leq \mu_{\mathcal{N}}$$

and hence

$$\lambda_0^{ess}(M) \leq \frac{\mu_{\mathcal{N}}^2}{4}$$

Proof. Corollary 5.2 gives the first inequality. For the second inequality, we use Corollary 5.1 since M has infinite volume (see [20]). \square

In particular, we have the following result.

Corollary 5.4. *Let M and \mathcal{N} be manifolds of bounded curvature such that M is a noncompact, complete, hypersurface properly embedded in \mathcal{N} . Assume \mathcal{N} is simply connected. If $\mu_{\mathcal{N}} = 0$, then $\mu_M = \mu_M^{\mathcal{N}} = h_M = \lambda_0^{ess}(M) = \lambda_0(M) = 0$.*

Remark 5.6. • *Let M be as in Corollary 5.4. Then 0 belongs to σ_{ess} but 0 is not an eigenvalue of Δ , because M , having infinite volume, does not carry any L^2 harmonic function.*

• *Lemma 5.1 guarantees that, if $Ric_{\mathcal{N}} \geq 0$, then $\mu_{\mathcal{N}} = 0$. More generally, $\mu_{\mathcal{N}} = 0$ when the manifold \mathcal{N} satisfies the doubling volume property since, as in the case of nonnegative Ricci curvature, the volume of balls grows polynomially.*

There is a strong relation between Caccioppoli's inequality and the entropy of the p -total curvature. The following general Theorem will be used to show such relation in Section 6.

Theorem 5.2. *Let $w \in L^1_{loc}(M; \mathbb{R}^+)$ and $W(R) := \int_{B_\sigma(R)} w$. Assume that the entropy of W is zero. Then we have the following results.*

(1) *If, for some positive constant C , w satisfies*

$$\int_M w\psi^2 \leq C \int_M w|\nabla\psi|^2, \quad \forall \psi \in W_0^{1,2}(M), \quad (32)$$

then $w \equiv 0$.

(2) *If, for some positive constant C , and some compact subset K , w satisfies*

$$\int_{M \setminus K} w\psi^2 \leq C \int_{M \setminus K} w|\nabla\psi|^2, \quad \forall \psi \in W_0^{1,2}(M \setminus K), \quad (33)$$

then $\int_M w < \infty$.

Proof. (1) Fix a point σ in M and for any $x \in M$, let $r(x) = d(x, \sigma)$ be the distance between x and σ . Let $\alpha > 0$ and let $\psi \in W_0^{1,2}(M)$ be the radial function such that $\psi(x) = e^{-\alpha r(x)} - e^{-\alpha R}$ on $B_\sigma(R)$ and $\psi \equiv 0$ on $M \setminus B_\sigma(R)$.

Applying inequality (32) to ψ we obtain

$$\int_{B_\sigma(R)} w (e^{-\alpha r} - e^{-\alpha R})^2 \leq C\alpha^2 \int_{B_\sigma(R)} w e^{-2\alpha r}. \quad (34)$$

First, we prove that the right hand side of (34) is bounded independently of R . By the co-area formula (see for instance Formula 3.8 in [35] or [46])

$$\int_{B_\sigma(R)} w e^{-2\alpha r} = \int_0^R \left(\int_{S_r} w d\sigma_r \right) e^{-2\alpha r} dr \quad (35)$$

where $d\sigma_r$ is the volume element of $S_r = \partial B_\sigma(r)$. Then, integrating by parts, equality (35), one has

$$\int_{B_\sigma(R)} w e^{-2\alpha r} = W(R)e^{-2\alpha R} + 2\alpha \int_0^R W(r)e^{-2\alpha r} dr \quad (36)$$

As the entropy of W is zero, we deduce that the first term of the right hand side of (36) tends to zero for $R \rightarrow \infty$. On the other hand, as $W e^{-\alpha r}$ is bounded, the second term of the right hand side of (36) converges for $R \rightarrow \infty$. Hence the right hand side of inequality (34) is bounded independently of R and the left hand side is bounded as well.

Therefore, for any compact $K \subset M$, there exists R so that

$$\int_K w (e^{-\alpha r} - e^{-\alpha R})^2 \leq C \alpha^2 \int_M w e^{-2\alpha r}. \quad (37)$$

Letting $R \rightarrow \infty$ in (37) gives, for any compact $K \subset M$

$$\int_K w e^{-2\alpha r} \leq C \alpha^2 \int_M w e^{-2\alpha r}. \quad (38)$$

Hence

$$\int_M w e^{-2\alpha r} \leq C \alpha^2 \int_M w e^{-2\alpha r}. \quad (39)$$

In order to conclude the proof of (1), we choose α such that $C \alpha^2 < 1$.

(2) Fix a point σ in M and for any $x \in M$, let $r(x) = d(x, \sigma)$ be the distance between x and σ . Let $0 < R_0 < R_1 < R_2 < R_3$, $\alpha > 0$ and let $\phi \in W_0^{1,2}(M)$ be the radial function such that $\phi \equiv 0$ on $B_\sigma(R_0)$, ϕ is linear on $B_\sigma(R_1) \setminus B_\sigma(R_0)$, $\phi \equiv 0$ on $B_\sigma(R_2) \setminus B_\sigma(R_1)$, $\phi(r) = \frac{e^{-\alpha(r-R_2)} - e^{-\alpha(R_3-R_2)}}{1 - e^{-\alpha(R_3-R_2)}}$ on $B_\sigma(R_3) \setminus B_\sigma(R_2)$ and $\phi \equiv 0$ on $M \setminus B_\sigma(R_3)$.

Replacing ϕ in (33) yields

$$\int_{B_\sigma(R_1) \setminus B_\sigma(R_0)} w (\phi^2 - C |\nabla \phi|^2) + \int_{B_\sigma(R_2) \setminus B_\sigma(R_1)} w + \int_{B_\sigma(R_3) \setminus B_\sigma(R_2)} w \phi^2 \leq C \int_{B_\sigma(R_3) \setminus B_\sigma(R_2)} w |\nabla \phi|^2 \quad (40)$$

As the third term in the left hand side of (40) is positive, we can remove it and obtain

$$\int_{B_\sigma(R_1) \setminus B_\sigma(R_0)} w (\phi^2 - C |\nabla \phi|^2) + \int_{B_\sigma(R_2) \setminus B_\sigma(R_1)} w \leq C \int_{B_\sigma(R_3) \setminus B_\sigma(R_2)} w |\nabla \phi|^2 \quad (41)$$

We claim that the right hand term of (41) is bounded independently of R_3 .

A straightforward computation shows that there exists a constant C_1 independent of R_3 , such that on $B_\sigma(R_3) \setminus B_\sigma(R_2)$

$$|\nabla \phi(r)|^2 = \frac{\alpha^2}{(1 - e^{-\alpha(R_3-R_2)})^2} e^{-2\alpha(r-R_2)} \leq C_1 e^{-2\alpha(r-R_2)} \quad (42)$$

Then, in order to prove our claim, one needs to bound $\int_{B_\sigma(R_3) \setminus B_\sigma(R_2)} w e^{-2\alpha(r-R_2)}$ independently of R_3 .

This goes exactly as in the proof of the boundedness of the right hand side of (34). For the sake of completeness we do it. By the co-area formula (see for instance Formula 3.8 in [35])

$$\int_{B_\sigma(R_3) \setminus B_\sigma(R_2)} w e^{-2\alpha(r-R_2)} = \int_{R_2}^{R_3} \left(\int_{S_r} W d\sigma_r \right) e^{-2\alpha(r-R_2)} dr \quad (43)$$

where $d\sigma_r$ is the volume element of $S_r = \partial B_\sigma(R)$. Then, integration by parts of equality (43) yields

$$\int_{B_\sigma(R_3) \setminus B_\sigma(R_2)} w e^{-2\alpha(r-R_2)} = W(R_3) e^{-2\alpha(R_3-R_2)} - W(R_2) + 2\alpha \int_{R_2}^{R_3} W(r) e^{-2\alpha(r-R_2)} dr \quad (44)$$

As the entropy of W is zero, we deduce that the first term of the right hand side of (44) tends to zero as $R_3 \rightarrow \infty$. On the other hand, as $W e^{-\alpha r}$ is bounded, the third term of the right hand side of (44) converges for $R_3 \rightarrow \infty$. Hence our claim is proved and the left hand term of inequality (41) is bounded independently of R_3 . We let $R_3 \rightarrow \infty$ in (41). As the first term is independent of R_2, R_3 , there exists a constant C_2 (independent of R_2) such that

$$\int_{B_\sigma(R_2) \setminus B_\sigma(R_0)} w \leq C_2 \quad (45)$$

By letting R_2 go to infinity on (45) we obtain the result. \square

Remark 5.7. We note that Theorem 5.2 can be viewed as a generalization to manifolds with density of the result of Brooks [6], where the volume entropy is zero.

6 Applications of Caccioppoli's inequalities

In this section, we will apply Caccioppoli's inequalities of Section 4 to obtain some results about entropies.

When M is a minimal hypersurface immersed in a manifold with constant curvature, R. Schoen, L. Simon, S.T. Yau obtained the following inequality (see the last inequality of the proof of Theorem 2 in [44])

$$\int_{B_\sigma(tR)} |A|^p \leq \frac{\beta}{(1-t)^p R^p} |B_\sigma(R)| \quad (46)$$

where β is a positive constant, $t \in (0, 1)$, $p \in \left(0, 4 + \sqrt{\frac{8}{n}}\right)$. This inequality implies a comparison between the entropy of \mathcal{T}_p and the volume entropy of M . In the same spirit, using Caccioppoli's inequality (10) in Theorem 4.1, we deduce a similar comparison result.

In this section we assume that the ambient manifold \mathcal{N} is an orientable Riemannian manifold with bounded sectional curvature.

Theorem 6.1. *Let M be a complete, noncompact stable hypersurface with constant mean curvature, immersed in \mathcal{N} . Assume $x \in [1, 1 + \sqrt{\frac{2}{n}})$, then one has:*

$$\mu_{\mathcal{T}_{2x+2}} \leq \mu_M.$$

Proof. Fix $t \in (0, 1)$ and choose a radial test function $f \equiv 1$ on $B_\sigma(tR)$, $f \equiv 0$ on $M \setminus B_\sigma(R)$ and linear on the annulus $B_\sigma(R) \setminus B_\sigma(tR)$. By a straightforward computation, inequality (10) with $K \subset B_\sigma(tR)$ applied to f yields

$$\beta_1 \int_{B_\sigma(tR)} \varphi^{2x+2} \leq |B_\sigma(R)| \left(\frac{\beta_2}{(1-t)^{2x+2} R^{2x+2}} + \beta_3 \right). \quad (47)$$

In (47), we take the logarithm, divide by R and take the limit of both sides for $R \rightarrow \infty$ and we obtain the result. \square

Remark 6.1. • *The Hölder inequality gives for any $p > 0$ and $q \geq p$:*

$$\frac{1}{p} \mu_{\mathcal{T}_p} \leq \left(\frac{1}{p} - \frac{1}{q} \right) \mu_M + \frac{1}{q} \mu_{\mathcal{T}_q} \quad (48)$$

• *Note that inequality (12) yields the following comparison between entropies of total curvature*

$$\mu_{\mathcal{T}_{2x+2}} \leq \mu_{\mathcal{T}_{2x}}.$$

This last inequality combined with inequality (48) gives an alternative proof of the inequality of Theorem 6.1

Another application of Caccioppoli's inequality yields:

Theorem 6.2. *Let M be a complete, noncompact, hypersurface immersed with constant mean curvature $H \neq 0$ in a manifold \mathcal{N} with constant curvature c . Assume M has finite index and dimension $n \leq 5$. Then provided either*

(1) $c = 0$, or $c = 1$, $x \in [1, x_2]$

or

(2) $c = -1$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 \geq g_n(x)$.

one has

$$\int_M \varphi^{2x} < \infty \text{ if and only if } \mu_{\mathcal{T}_{2x}} \equiv 0. \quad (49)$$

Proof. We first observe that x_2 , given in (14), is greater than one if and only if $n \leq 5$.

The first implication is clear by the definition of entropy. Vice-versa, as inequality (15) holds in M , we can apply (2) of Theorem 5.2 with $w = \varphi^{2x}$ in order to have the result. \square

A direct consequence of Theorem 6.2 is the following result in the spirit of a result of J. Tysk concerning Euclidean minimal hypersurfaces (see the main theorem of [47]).

Corollary 6.1. *Let M be a complete, noncompact, hypersurface immersed with constant mean curvature $H \neq 0$ in a manifold \mathcal{N} with constant curvature c . Assume M has finite index and dimension $n \leq 5$. Then provided either*

(1) $c = 0$ or $c = 1$

or

(2) $c = -1$ and $H^2 \geq g_n(1)$ ($g_n(x)$ is defined in (13))

one has

$$\mu_{\mathcal{T}_2} = 0 \text{ implies } \int_M \varphi^n < \infty$$

Proof. By Theorem 6.2, we have $\int_M \varphi^2 < \infty$. Using the reverse Holder inequality of Theorem 4.2 for $x = 1$, we obtain $\int_M \varphi^4 < \infty$. Using Holder inequality, we obtain $\int_M \varphi^3 < \infty$. Using the reverse Holder inequality of Theorem 4.2 for $x = \frac{3}{2}$, we obtain $\int_M \varphi^5 < \infty$. \square

Remark 6.2. • As $\mu_M \geq \mu_{\mathcal{T}_{2x}}$, one can replace the hypothesis $\mu_{\mathcal{T}_{2x}} \equiv 0$ in Theorem 6.2 and the hypothesis $\mu_{\mathcal{T}_2} \equiv 0$ in Corollary 6.1 by $\mu_M \equiv 0$.

• P. Bérard, M. do Carmo and W. Santos [4] proved that if M is a complete hypersurface in \mathbb{H}^{n+1} , with constant mean curvature $H^2 < 1$, such that $\int_M \varphi^n < \infty$, then M has finite index. Notice that the converse is false as it is shown by the examples of A. da Silveira [14].

7 Some answers to do Carmo's question

In this Section we apply our previous results in order to answer, at least in some cases, to the following do Carmo's question [15] : *is a noncompact, complete, stable, constant mean curvature hypersurface of \mathbb{R}^{n+1} , $n \geq 3$, necessarily minimal?*

It will be clear in the following in which cases we are able to give an answer.

Theorem 7.1. *There is no complete, noncompact, finite index hypersurface M immersed in a manifold \mathcal{N} , provided the mean curvature function H satisfies $nH^2 + Ric(\nu, \nu) \geq \delta$, where δ is a constant such that $\delta > \frac{\mu_M^2}{4}$.*

Proof of Theorem 7.1. Assume such M exists. As M has finite index, there exists a compact K in M such that $M \setminus K$ is stable. Therefore, for any $f \in C_0^\infty(M \setminus K)$, one has

$$0 \leq Q(f, f) = \int_{M \setminus K} |\nabla f|^2 - (|A|^2 + Ric(\nu, \nu))f^2. \quad (50)$$

As $|A|^2 + Ric(\nu, \nu) \geq nH^2 + Ric(\nu, \nu) \geq \delta$, (50) yields

$$0 \leq \int_{M \setminus K} |\nabla f|^2 - \delta \int_{M \setminus K} f^2, \quad (51)$$

that is $\lambda_0(M \setminus K) \geq \delta$. This contradicts inequality (22) of Brooks' Theorem. \square

Remark 7.1. *A result weaker than Theorem 7.1 was proved by M. do Carmo and D. Zhou [16].*

A straightforward application of Theorem 7.1 gives the following result.

Corollary 7.1. *If M is a constant mean curvature hypersurface of finite index immersed in a space of constant curvature c , then $n(H^2 + c) \leq \frac{\mu_M^2}{4}$. In particular, if $H^2 + c > 0$, the volume growth of M is exponential.*

As an application of Corollary 7.1 we obtain the following result.

Corollary 7.2. *There is no complete, noncompact, finite index, constant mean curvature hypersurface M immersed in a manifold \mathcal{N} , with $\mu_M \equiv 0$, provided either*

- (1) $\mathcal{N} = \mathbb{S}^{n+1}$ or
- (2) $\mathcal{N} = \mathbb{R}^{n+1}$ and $H \neq 0$, or
- (3) $\mathcal{N} = \mathbb{H}^{n+1}$, $H > 1$.

One can replace the entropy μ_M by the entropy of the ambient space $\mu_{\mathcal{N}}$ by imposing some additional geometric conditions on M and \mathcal{N} , that guarantee the embeddedness of the tube around M .

Theorem 7.2. *There is no complete, noncompact, finite index M that is properly embedded in a simply connected manifold \mathcal{N} , where M and \mathcal{N} have bounded curvature and provided $H \geq \delta_1 > 0$ and $nH^2 + \text{Ric}(\nu, \nu) \geq \delta_2 > \frac{\mu_{\mathcal{N}}^2}{4}$ for some positive δ_1, δ_2 .*

Proof. One applies Corollary 5.3 and Theorem 7.1 □

As an immediate consequence of Theorem 7.2 one has the following result.

Corollary 7.3. *If M is a properly embedded hypersurface of bounded curvature and constant mean curvature with finite index in a space form of curvature c :*

- *If $c \geq 0$, then $c = 0$ and M is minimal.*
- *If $c < 0$, then $H^2 \leq -c\left(\frac{n}{4} + 1\right)$*

In Theorem 7.1, when \mathcal{N} is either \mathbb{R}^{n+1} or \mathbb{H}^{n+1} , one can replace the volume entropy by some total curvature entropy, provided some restriction on the dimension and on the mean curvature of M are satisfied.

Theorem 7.3. *There is no complete noncompact stable hypersurface M with constant mean curvature H in a manifold \mathcal{N} , $n \leq 5$, with $\mu_{\mathcal{T}_{2x}} = 0$, provided either*

(1) $\mathcal{N} = \mathbb{R}^{n+1}$, $x \in [1, x_2)$, $H > 0$,

or

(2) $\mathcal{N} = \mathbb{H}^{n+1}$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 > g_n(x)$ ($g_n(x)$ is defined in (13)).

Proof. First notice that, by Theorem 6.2, the hypothesis $\mu_{\mathcal{T}_{2x}} = 0$ implies $\int_M \varphi^{2x} < \infty$. One applies Corollary 6.3 of [28] in order to obtain that M is totally umbilic. In case (1), it follows that M is contained either in a sphere or in a plane. As M is complete noncompact, M is contained in a plane, then $H = 0$. Contradiction. In case (2), it follows that M is contained either in a sphere, or in a horosphere, or in an equidistant sphere. The inequality $H^2 > g_n(x) \geq 1$ yields that M can be only contained in a sphere. As M is complete and noncompact, this is a contradiction. □

Remark 7.2. • *It is worthwhile to note that the condition $\int_M \varphi^{2x} < \infty$ in Theorem 7.3 is equivalent to the apparently weaker condition $\mu_{\mathcal{T}_{2x}} = 0$ (see Theorem 6.2)*

- *In Theorem 7.3, under the same conditions, if \mathcal{N} has constant sectional curvature $c \leq 0$ but not necessarily simply connected, one prove similarly that M is totally umbilical.*

8 Appendix

We develop here, the missing parts of the principal arguments of the proof of Theorem 3.1. In particular we will prove estimate (2) and equality (3).

The main reason for this Appendix is the lack of references for the computations of the mean curvature equation in general Riemannian manifolds, and for the uniform estimates of its coefficients. The mean curvature equation has been extensively studied in Euclidian spaces, constant

curvature spaces and more recently for the particular case of constant mean curvature surfaces in homogenous spaces (see for instance [43]). A similar mean curvature equation was obtained in Fermi coordinates for minimal surfaces in [13]. We first, give a proof of the cheesebox argument, then we compute the mean curvature equation and deduce the desired estimates of its coefficients. This will allow us to fill the gaps in the proof of Theorem 3.1.

8.1 The cheesebox argument

In this paragraph we recall the cheesebox argument and show how it implies estimate (2). We use the same notations as in the proof of Theorem 3.1.

Recall that the section P is defined as the graph of the function ϕ defined at the beginning of the proof of the second step of Theorem 3.1.

We first study the case where the ambient space \mathcal{N} is \mathbb{R}^{n+1} .

Let p be a point in M and q be a point above p in the section P (see Figure 3). Consider the

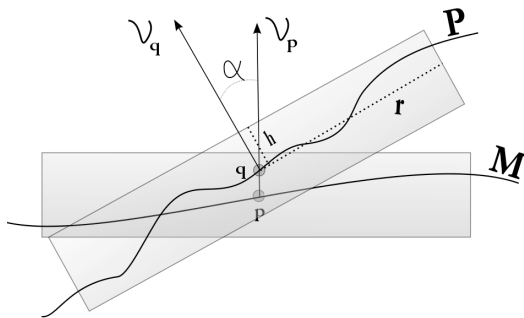


Figure 3: Intersection of the cheeseboxes at $p \in M$ and at $q \in P$ with $\mathbb{R}\nu_p \oplus \mathbb{R}\nu_q$

following boxes of \mathcal{N} :

$$C_p(\rho, h) := \exp_p(D(\rho) \times]0, h]) \text{ and } C_q(\rho, h) := \exp_q(D(\rho) \times]0, h]).$$

We choose ρ as in Theorem 3.1 so that M and $C_p(\rho, h)$ (respectively P and $C_q(\rho, h)$) intersect only on the boundary side $\partial D \times [0, h]$ of $C_p(\rho, h)$ (resp. on the boundary side $\partial D \times [0, h]$ of $C_q(\rho, h)$). The intersection of $C_p(\rho, h) \cup C_q(\rho, h)$ with the plane through p generated by the normal vectors $\nu(p)$ and $\nu(q)$, is the union of two Euclidean rectangles (see Figure 3).

For a sufficiently small ρ , we may choose the height of the box to be proportional to the square of its radius: $h = c\rho^2$ for some constant c depending on the C^1 -norm of the second fundamental form A of M . In fact, in a neighborhood of any $x_0 \in M$, M is the graph of the height function h defined on a ball $B_{x_0}(\rho)$ of the tangent space at x_0 of sufficiently small radius ρ , and such that for any $x \in B_{x_0}(\rho)$, $|h(x)| \leq C|x|^2$ where C is a uniform constant.

Indeed, let us clarify the relation between the second fundamental form of the aforementioned graph defined by the height function $h(x_1, \dots, x_n)$ and the Hessian of h . Notice that $h(0) = 0$ and $\nabla h(0) = 0$.

Using the Einstein's convention on indices, we have $|A|^2 = g^{ik}g^{jl}A_{il}A_{kj} = A_i^j A_j^i$, and

$$A_i^j = \frac{h_{ik}}{W} g^{kj}, \quad \text{with } W := \sqrt{1+f}, \quad f := |\nabla h|^2 \text{ and } g^{ij} := \delta^{ij} - \frac{h_i h_j}{W^3}$$

Therefore,

$$A_i^j = \frac{h_{ij}}{W} - \frac{h_{ki}h_k h_j}{W^3}$$

and computing $|A_i^j h_j|$, we get :

$$|A_i^j h_j| = \left| \frac{f_i}{2W} \left(1 - \frac{f}{W^2} \right) \right|$$

Since A is bounded, there exists a positive constant c such that $|A_i^j| \leq \frac{c}{2\sqrt{n}}$, and applying Cauchy-Schwarz's inequality, we derive:

$$2|A_i^j h_j| = \left| \frac{f_i}{(1+f)^{3/2}} \right| \leq c\sqrt{f}$$

Standard comparison between the solutions of the previous differential inequality and the corresponding differential equality, together with the initial condition $\nabla h(0) = 0$, shows that the condition $\sum |x_i|^2 \leq \rho$ implies $|\nabla h| \leq \frac{c\rho}{\sqrt{4-(c\rho)^2}}$.

Thus, h is C^1 -uniformly bounded on the ball $B_{x_0}(\rho)$ and W is also uniformly bounded. Finally, $h_{ij} \leq C$ on $B_{x_0}(\rho)$ (where C is a constant depending on c and ρ). In conclusion, since $h(0) = 0$ and $\nabla h(0) = 0$, we have $|h(x)| \leq C|x|^2$ on $B_{x_0}(\rho)$.

We may also suppose that the function ϕ defining the section P satisfies $\phi \leq \frac{h}{2}$ (in Figure 2, $\phi(p) := d_{\mathbb{R}^{n+1}}(p, q)$). Let α be the angle defined by $\tan \alpha = |\nabla \phi|(p)$ (Figure 2 represents a limit case for which M and P necessarily intersects for any q' such that $d_{\mathbb{R}^{n+1}}(p, q') \leq d_{\mathbb{R}^{n+1}}(p, q)$ or any $\nu(q')$ such that $\langle \nu(p), \nu(q') \rangle \leq \cos \alpha$).

For a given ρ , if $d_{\mathbb{R}^{n+1}}(p, q)$ is small enough then $|\nabla \phi| \leq 1$ unless M and P intersects. Thus $\alpha \leq \alpha_0 = \frac{\pi}{4}$. Then elementary plane geometry gives

$$\rho \sin \alpha \leq \rho \sin \alpha_0 \leq h(1 + \cos \alpha_0) + \phi(p) \cos \alpha_0$$

thus

$$\rho \sin \alpha \leq (h + \phi) \cos \alpha_0 + h \leq 3h$$

hence

$$\frac{|\nabla \phi|(p)}{\sqrt{1 + |\nabla \phi|^2(p)}} \leq 3c\rho.$$

which yields $|\nabla \phi|(p) \leq 6c\rho$.

Since, by hypothesis, $\phi(p) \leq \frac{c}{2}\rho^2$ and since the inequality holds for any point $p \in \Omega \subset M$, we obtain

$$\|\phi\|_1 := \left(\sup_{p \in \Omega} |\phi| + \sup_{p \in \Omega} |\nabla \phi| \right) \leq O(\rho). \quad (52)$$

Consider the general case, where the ambient space \mathcal{N} is not necessarily Euclidean. Since \mathcal{N} has bounded curvature, there exists a radius ρ_0 depending on the curvature of \mathcal{N} , such that for each point $p \in \mathcal{N}$, there is a harmonic coordinate chart ψ_p^{-1} , such that $\psi_p : U \left(:= B_0^{\mathbb{R}^{n+1}(\rho_0)} \right) \subset \mathbb{R}^{n+1} \rightarrow V := \psi_p(U) \subset \mathcal{N}$, and the pulled-back metric $g_{\mathcal{N}}$ is $C^{1,\alpha}$ -regular, $C^{1,\alpha}$ -close to the Euclidean one. The diffeomorphism ψ_p is C^1 -uniformly bounded in these coordinates. The previous result concerning Euclidean cheeseboxes applies to $\psi_p^{-1}(M) \cap V$ and $\psi_p^{-1}(P) \cap V$ to prove the C^1 -uniformly boundedness of $\phi \circ \psi_p$ with respect to $p \in M$. Finally since ψ_p is C^1 -uniformly bounded with respect to p , so is ϕ . In conclusion there exists a radius ρ_0 , depending on the curvature of \mathcal{N} and M , such that for each point $p \in M$, there a cheesebox of M around p of radius ρ_0 and height $c\rho_0^2$ in harmonic coordinate charts. For details about the theory of harmonic coordinates see for instance the survey [26] and the references therein.

8.2 Cmc equation of a section of the normal bundle of M

Notations are the same as in the previous paragraph and Section 3. Our purpose is to compute the mean curvature H_P of the section P in a neighborhood of $q \in P$, in terms of local coordinates around $p \in M$. More precisely we will show how to obtain the expansion (3) of H_P in the proof of Theorem 3.1.

Let ψ_p be a parametrization of a neighborhood V_p of p in \mathcal{N} as given in previous paragraph: $(\psi_p : B_0^{\mathbb{R}^{n+1}}(R) \subset \mathbb{R}^n \times \mathbb{R} \rightarrow V_p \subset \mathcal{N})$ with $\psi(0) = p$ and $\psi(B_0^{\mathbb{R}^{n+1}}(R) \cap \mathbb{R}^n \times \{0\}) = M \cap V_p$. The local section $P \cap V_p$, being in a cheesebox, is parametrized by a graph of a function $\phi : \mathbb{R}^n \times \{0\} \rightarrow \mathbb{R}$. Indeed $\psi_p^{-1}(P \cap V_p)$ and $\psi_p^{-1}(M \cap V_p)$ are C^1 -close in the pulled-back metric $g_{\mathcal{N}}$. For simplicity we identify the metric $g_{\mathcal{N}}$ of \mathcal{N} with its pulled-back $\psi^*(g_{\mathcal{N}})$ on the Euclidean ball $B_0^{\mathbb{R}^{n+1}}(R)$. We denote by $\{e_\alpha\}_{\alpha=1,\dots,n+1}$ the standard basis of \mathbb{R}^{n+1} and by $\{e_i\}_{i=1,\dots,n}$ the standard basis of $\mathbb{R}^n \times \{0\}$. With some abuse of notations, we identify $(x, \phi(x))$ with its image $\psi_p(x, \phi(x))$ and derivatives with respect to e_i of a function f will be denoted by $f_{,i}$.

8.2.1 Mean curvature equation of the section P

We first choose an adapted frame tangent to P such that the first n vectors $\{f_i(x, \phi(x)) := e_i + \phi_{,i}(x, \phi(x)) e_{n+1}\}_{i=1,\dots,n}$ are tangent to P at $(x, \phi(x))$ and the last vector is the unit normal field $\nu_P(x, \phi(x))$ to the graph of ϕ . In fact ν_P is a unit vector, solution of the system of linear equations given by: $\left\{ g_{\mathcal{N}}(\nu_P, F_i) = 0 \right\}_{i=1,\dots,n}$ and an easy computation yields :

$$\nu^\alpha := \frac{1}{W} \left(-g^{\alpha i} \phi_{,i} + g^{\alpha n+1} \right). \quad (53)$$

where $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta} := g_{\mathcal{N}}(e_\alpha, e_\beta)$, $\alpha, \beta = 1, \dots, n+1$, and $W^2 := g^{kl} \phi_{,k} \phi_{,l} - 2g^{n+1k} \phi_{,k} + g^{n+1n+1}$ (we use in all subsequent formulas the Einstein summation convention).

We denote by $\tilde{g}_{ij} := g_{\mathcal{N}}(f_i, f_j)$ the coefficients of the induced metric on $T_q P$. Therefore, replacing each f_i by its expression in terms of e_i , we obtain

$$\tilde{g}_{ij} = g_{ij} + g_{n+1(j\phi_{,i})} + g_{n+1n+1\phi_{,i}\phi_{,j}}$$

(where $g_{n+1(j\phi_{,i})} := g_{n+1j\phi_{,i}} + g_{n+1i\phi_{,j}}$).

Let us now compute the mean curvature equation for P . We have:

$$nH_P(q) = -\operatorname{div}(\nu_P)(q) = -\tilde{g}^{ij}g_{\mathcal{N}}(\nabla_{f_i}\nu_P, f_j)(q) = \tilde{g}^{ij}g_{\mathcal{N}}(\nu_P, \nabla_{f_i}f_j)(q) \quad (54)$$

Therefore, for $\alpha, \beta, \gamma = 1, \dots, n+1$ and $i, j = 1, \dots, n$ we obtain

$$(\nabla_{f_i}f_j)^\alpha = \Gamma_{ij}^\alpha + \Gamma_{n+1(j)\phi, i}^\alpha + \Gamma_{n+1n+1}^\alpha\phi, i\phi, j + \phi, ij\delta^{\alpha n+1}$$

We then compute $g_{\mathcal{N}}(\nu_P, \nabla_{f_i}f_j)$ and plug into equation (54). We obtain

$$nH_PW = \tilde{g}^{ij} \left(\phi, ij + (\Gamma_{n+1n+1}^k\phi, k + \Gamma_{n+1n+1}^{n+1})\phi, i\phi, j + \Gamma_{n+1(i)\phi, j}^k\phi, k + \Gamma_{n+1(i)\phi, j}^{n+1} + \Gamma_{ij}^k\phi, k + \Gamma_{ij}^{n+1} \right) \quad (55)$$

We use now harmonic charts as in the end of last paragraph. In these charts, the induced metric $g_{\mathcal{N}}$ of \mathcal{N} on $B_p^{\mathbb{R}^{n+1}}(\rho)$ is $C^{1,\alpha}$ -regular and $C^{1,\alpha}$ uniformly close to the Euclidean metric. Since ϕ is also C^1 uniformly bounded (see (52)), the coefficients of equation (55) are $C^{0,\alpha}$ uniformly bounded. Using Schauder estimates, we obtain uniform C^∞ bounds on ϕ .

Notice first that, when $\phi = 0$ the same equation (55) gives the mean curvature of the zero section:

$$nH_Mg^{n+1n+1} = g^{ij}\Gamma_{ij}^{n+1} \quad (56)$$

Replacing equation (56) in equation (55) we obtain the estimate

$$nH_P = nH_M + \tilde{g}^{ij}\phi, ij + O(\rho^\alpha) = nH_M + \Delta_M\phi + O(\rho^\alpha) \quad (57)$$

which gives estimate estimate (3) in the proof of theorem 3.1

References

- [1] L. BARBOSA, P. BÉRARD: *Eigenvalue and "twisted" eigenvalue problems, applications to cmc surfaces*, J. Math. Pure Appl. 79, 5 (2000) 427-450.
- [2] L. BARBOSA, M. DO CARMO: *Stability of hypersurfaces with constant mean curvature*, Math. Z., 185 (1984) 339-353.
- [3] L. BARBOSA, M. DO CARMO, J. ESCHENBURG: *Stability of hypersurfaces with constant mean curvature in a Riemannian manifold*, Math. Z., 197 (1988) 123-128.
- [4] P. BÉRARD, M. DO CARMO, W. SANTOS: *Complete hypersurfaces with constant mean curvature and finite total curvature*, Annals Glob. Anal. Geom. 16 (1998) 273-290.
- [5] P. BESSA, F. MONTENEGRO: *Eigenvalue estimates for submanifolds with locally bounded mean curvature*, Ann. Global Anal. Geom. 24, 3 (2003) 279-290.
- [6] R. BROOKS: *Exponential Growth and the Spectrum of the Laplacian* Proceedings of the American Mathematical Society, 82, 3 (1981) 473-477.
- [7] R. BROOKS: *On the spectrum of non-compact manifold with finite volume*, Math. Zeit. 187 (1984) 425-432.

- [8] G. CARRON: *Inégalités isopérimétriques et inégalités de Faber-Krahn*, Séminaire de géométrie spectrale et géométrie, tome 13 (1994-1995) 63-66.
- [9] J. CHEEGER: *A lower bound for the smallest eigenvalue of the Laplacian*, in Problems in analysis (ed. R.C. Gunning). A symposium in honor of Salomon Bochner (Princeton 1969), Princeton University Press (1970) 195-199.
- [10] L.-F. CHEUNG , P.-F. LEUNG: *Eigenvalue estimates for submanifolds with bounded mean curvature in hyperbolic space*, Math. Z. (3) 236 (2001) 525-530.
- [11] X. CHENG: *On Constant Mean Curvature Hypersurfaces with Finite Index*, Arch. Math. 86 (2006) 365-374.
- [12] J. CHOE, R. GULLIVER: *Isoperimetric inequalities on minimal submanifolds of space forms*, Manuscripta Math. 77 (2-3) (1992) 169-189.
- [13] T. H. COLDING, W. P. MINICOZZI II: *Minimal surfaces*, Courant lecture notes in math. 4, Courant institute of math. sciences, N.Y, 1999.
- [14] A. DA SILVEIRA: *Stability of Complete Noncompact Surfaces with Constant Mean Curvature*, Math. Ann. 277 (1987) 629-638.
- [15] M. DO CARMO: *Hypersurfaces of constant mean curvature*, Lecture Notes in Mathematics, Vol 1410, p. 133.
- [16] M. DO CARMO, D. ZHOU: *Eigenvalue estimate on complete noncompact Riemannian manifolds and applications*, Trans. Amer. Math. Soc. 351 (4) (1999) 1391-1401.
- [17] M. F. ELBERT, B. NELLI, H. ROSENBERG: *Stable constant mean curvature hypersurfaces*, Proc. Amer. Math. Soc. 135, 10 (2007) 3359-3366.
- [18] D. FISCHER-COLBRIE: *On complete minimal surfaces with finite Morse index in three manifolds*, Invent. Math. 82 (1985) 121-132.
- [19] D. FISCHER-COLBRIE, R. SCHOEN: *The structure of complete stable minimal surfaces in 3-manifolds of non negative scalar curvature*, Comm. Pure Appl. Math. 33 (1980) 199-211.
- [20] K. FRENSEL: *Stable complete surfaces with constant mean curvature*, Bull. of the Brazilian Math. Soc. Volume 27, Number 2 (1996) 129-144.
- [21] D. GILBARG, N.S. TRUDINGER: *Elliptic Partial Differential Equations of Second order*, Springer Verlag, 1983.
- [22] V. GIMENO: *On the fundamental tone of minimal submanifolds with controlled extrinsic curvature*, Potential Analysis (2013) DOI: 10.1007/s11118-013-9349-6.
- [23] V. GIMENO, V. PALMER: *Volume growth of submanifolds and the Cheeger isoperimetric constant*, PAMS (2013) <http://dx.doi.org/10.1090/S0002-9939-2013-11664-3>.
- [24] A. GRAY: *Tubes*, Addison-Wesley (1990).
- [25] E. HEBEY: *Sobolev Spaces on riemannian manifolds*, Lecture Notes in Maths 1635 (1996).

- [26] E. HEBEY, M. HERZLICH: *Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds*, Rendiconti di mat. 17 (1997) 569-605.
- [27] Y. HIGUCHI: *A remark on exponential growth and the spectrum of the Laplacian*, Kodai math. J. 24 (2001) 42-47.
- [28] S. ILIAS, B. NELLI, M. SORET: *Caccioppoli's inequality on constant mean curvature hypersurfaces in Riemannian manifolds*, , Annals Glob. Anal. Geom. 42,4 (2012) 443-471.
- [29] L. KARP: *Differential inequalities on complete Riemannian manifolds and applications*, Math. Ann. 272 (1985) 449-459.
- [30] L. LEFTON, D. WEI: *Numerical approximation of the first eigenpair of the p -Laplacian using finite elements and the penalty method*, Numer. Funct. Anal. Optim. 18 (1997) 389-399.
- [31] F. LOPEZ, A. ROS: *Complete minimal surfaces with index one and stable constant mean curvature surfaces*, Comment. Math. Helvetici 64 (1989) 34-43.
- [32] E. L. LIMA: *The Jordan-Brouwer Separation Theorem for Smooth Hypersurfaces* The American Mathematical Monthly, Vol. 95, No. 1 (Jan., 1988), pp. 39-42.
- [33] B.P. LIMA, J. F. MONTENEGRO, N. L. SANTOS: *Eigenvalues Estimates for the p -Laplace Operator on Manifolds*, Nonlinear Anal. 72 (2) (2010) 771-781.
- [34] A. M. MATEI: *First eigenvalue for the p -Laplace operator*, Nonlinear Anal. 39 (2000) 1051-1068.
- [35] F. MORGAN: *Geometric Measure Theory, a beginner's guide*, Academic Press (1988).
- [36] W. F. MOSS, J. PIEPENBRINK: *Positive solutions of elliptic equations*, Pacific J. Math., 75, 1 (1978) 219-226.
- [37] W. MEEKS III, J. PEREZ, A. ROS: *Stable constant mean curvature surfaces*, Handbook of Geometric Analysis, International Press of Boston (2008).
- [38] W. MEEKS III, G. TINAGLIA: *Existence of regular neighborhoods for H -surfaces*, arXiv:1002.4972v1 [math. DG].
- [39] M. NAKAI, L. SARIO: *Quasi-harmonic classification*, Proc. of the AMS 31, 1 (1972) 165-169.
- [40] B. NELLI, M. SORET: *Stably embedded minimal hypersurfaces*, Math. Z. 255, 3 (2007) 493-514.
- [41] H. OKUMURA: *Infimum of the exponential volume growth and the bottom of the essential spectrum of the Laplacian*, arXiv:0707.0185v3 [math.DG].
- [42] B. PALMER: PhD thesis, Stanford (1986).
- [43] H. ROSENBERG, R. SOUAM, E. TOUBIANA: *General curvature estimates for stable H -surfaces in 3-manifolds and applications*, J. Diff. Geom. 84, 3 (2010) 623-648.

- [44] R. SCHOEN, L. SIMON, S.T. YAU: *Curvature Estimates for Minimal Hypersurfaces*, Acta Math. 134 (1975) 275-288.
- [45] M. SORET: *Minimal surfaces with bounded curvature*, Comm. in Geom. and Analysis 9 (2001) 921-950.
- [46] R. SCHOEN, S. T. YAU: *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, Intern. Press, vol. 1 (1994).
- [47] J. TYSK: *Finiteness of index and total scalar curvature for minimal hypersurfaces*: Proc. of the AMS, 105, 2 (1989) 429-435.
- [48] S. T. YAU: *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Sc. E. N. S. 8, 4 (1975) 487-507.

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