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## APPROXIMATE CONTROLLABILITY CONDITIONS FOR SOME LINEAR 1D PARABOLIC SYSTEMS WITH SPACE-DEPENDENT COEFFICIENTS

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**ABSTRACT.** In this article we are interested in the controllability with one single control force of parabolic systems with space-dependent zero-order coupling terms. We particularly want to emphasize that, surprisingly enough for parabolic problems, the geometry of the control domain can have an important influence on the controllability properties of the system, depending on the structure of the coupling terms.

Our analysis is mainly based on a criterion given by Fattorini in [12] (and systematically used in [22] for instance), that reduces the problem to the study of a unique continuation property for elliptic systems. We provide several detailed examples of controllable and non-controllable systems. This work gives theoretical justifications of some numerical observations described in [9].

### 1. Introduction.

**1.1. Statement of the problem.** This paper deals with the controllability properties at time  $T > 0$  of the following class of 1D linear parabolic systems

$$\begin{cases} \partial_t y + \mathcal{L}y = A(x)y + 1_\omega Bv & \text{in } (0, T) \times \Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \quad (1)$$

Here, the domain is  $\Omega = (0, 1)$ ,  $y \in C^0([0, T], L^2(\Omega)^n)$  is the state,  $y_0 \in L^2(\Omega)^n$  is the initial data,  $A(x)$  is a  $n \times n$  real matrix with entries in  $L^\infty(\Omega)$ ,  $B$  is a constant vector in  $\mathbb{R}^n$  and  $v \in L^2((0, T) \times \Omega)$  is the (scalar-valued) control which is acting only on the control domain  $\omega$ , a non-empty open subset of  $\Omega$ . The diffusion operator  $\mathcal{L} = \mathcal{L} \text{Id}$  operates on vector-valued functions component-wise through the scalar elliptic operator  $\mathcal{L}$  defined by

$$\mathcal{L} = -\partial_x (\gamma(x) \partial_x \cdot) + \gamma_0(x) \cdot, \quad (2)$$

with domain  $\mathcal{D}(\mathcal{L}) = \{u \in H_0^1(\Omega), \mathcal{L}u \in L^2(\Omega)\}$  corresponding to homogeneous Dirichlet boundary condition. The coefficients of  $\mathcal{L}$  are supposed to satisfy the standard uniform ellipticity assumptions  $\gamma, \gamma_0 \in L^\infty(\Omega)$ , with  $\inf_\Omega \gamma > 0$ .

Since  $B$  is a non-trivial constant vector, and  $\mathcal{L} = \mathcal{L} \text{Id}$ , we see that a simple linear change of unknowns let us transform the system into the case where  $B =$

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$(1, 0, \dots, 0)^*$ , the first vector of the canonical basis of  $\mathbb{R}^n$  (in this work we denote by  $M^*$  the transpose of any matrix  $M$ ). This means that the direct action of the control  $v$  only concerns the first component of the system.

We are particularly interested in the study of system (1) under the following structural assumptions on the coupling terms:

1. Controllability of a  $2 \times 2$  cascade system

$$A(x) = \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix}. \quad (3)$$

2. Simultaneous controllability of two  $2 \times 2$  cascade systems

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ a_{31}(x) & 0 & 0 \end{pmatrix}. \quad (4)$$

3. Controllability of a  $3 \times 3$  cascade system

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix}. \quad (5)$$

The notion we deal with in this paper is the one of approximate controllability (which is weaker than null-controllability [16]), that can be stated as follows: For every  $\epsilon > 0$  and  $y_0, y_T \in L^2(\Omega)^n$ , find a control  $v \in L^2(0, T; L^2(\Omega))$  such that the solution  $y$  of (1) satisfies

$$\|y(T) - y_T\|_{L^2(\Omega)^n} \leq \epsilon.$$

**Remark 1.1.** Since the semigroup generated by the operator  $-\mathcal{L} + A(x)$  is analytic, this notion is in fact equivalent to the null-approximate controllability property, that is the one when the target state is  $y_T = 0$ . Moreover, analyticity also implies that the approximate controllability property does not depend on the control time  $T > 0$ .

**1.2. Known results and main achievements of the paper.** The class of systems presented above can be considered as "toy models" to understand how the structure of the coupling terms can influence the behavior of a system controlled with a few number of controls. In the case where  $A(x) = A$  is constant, it is shown in [4] that (1) is null-controllable if and only if the Kalman rank condition between matrices  $A$  and  $B$  holds. This result is thus independent of the control domain  $\omega$  and of the operator  $\mathcal{L}$  (and is actually true in any space dimension).

The situation is more complex for systems with space-dependent coupling coefficients in which case there exist only few controllability results [15, 17, 23, 2, 1, 20, 22, 8]. Most of them are still partial and deal with systems of 2 equations. In [15], the null-controllability was established for  $n \times n$  systems with some structural assumption on the coupling and under the crucial hypothesis that the control domain  $\omega$  intersects the support  $\mathcal{O}$  of the coupling terms. The structural assumption was removed in [8] and [20], however with some other technical hypothesis, still in the case  $\omega \cap \mathcal{O} \neq \emptyset$ . On the other hand, approximate controllability in the case where the coupling term only acts away from the control domain, that is  $\omega \cap \mathcal{O} = \emptyset$ , was

proved for a cascade system with non-negative coupling terms in [17]. In the same framework, the null-controllability was then obtained in the one-dimensional case in [23], and then in any dimension in [1] under a geometric condition on the control and the coupling domains, though. These restrictions come from the geometric control condition (GCC) for the wave-type systems that are used in these works to deduce results for parabolic systems through the transmutation method.

We will see in this paper that the geometry of the control domain  $\omega$  will play an important role in the study of those systems, even though the GCC is automatically satisfied in 1D; for instance we shall provide examples of systems which are controllable for some choices of  $\omega$  but not controllable for other choices. This is not usual in the parabolic framework.

We will also give some examples of one-parameter families of coupling matrices  $(x \mapsto A_d(x))_d$  such that, for suitable  $\omega$ ,  $\mathcal{L}$  and  $B$ , we have

$$(1) \text{ is approximately controllable} \iff d \notin \mathbb{Q},$$

showing that the influence of the coupling terms on the controllability property of parabolic systems can be quite complex. Formally similar results are for example given in [13] in the case of a boundary control problem and for a parameter  $d$  which is related to the ratio between the diffusion coefficients of the different components in the system.

Let us also underline that the results in [17], [23] and [1] require some sign conditions for the coupling terms. To the authors knowledge there is no available result in the literature in the case  $\omega \cap \mathcal{O} = \emptyset$  without such a sign assumption. However, it is worth mentioning that the proof of sufficient controllability conditions given in [17] still holds without this sign assumption, see Section 3.3. This is another achievement of the present paper to provide necessary and sufficient conditions in the general case, that is without *a priori* assumptions on the sign of the coupling terms.

Last but not least, we also investigate the case of some  $n \times n$  systems with  $n > 2$  that do not enter the framework of [15] and [20].

The paper is organized as follows. In Section 2 we characterize the unique continuation property for scalar non-homogeneous elliptic problems (Theorem 2.2). Together with the Fattorini theorem, this result is the key-point underlying all the analysis proposed in this work. This is illustrated in the next three sections. More precisely, Section 3 is dedicated to the controllability of  $2 \times 2$  cascade systems (cases (3) and (4)), Section 4 is concerned with  $3 \times 3$  cascade systems (case (5)). Finally in Section 5 we give some examples and counter-examples of simultaneous controllability for an uncoupled  $2 \times 2$  system ( $A \equiv 0$ ) with different diffusions on each equation, that is when the operator  $\mathcal{L}$  is not anymore of the form  $\mathcal{L} = \mathcal{L} \text{Id}$  (but still diagonal). A short conclusion is given in Section 6.

## 2. Unique continuation criterion for triangular systems.

**2.1. Some useful spectral properties.** Before starting the analysis, let us recap the main properties of some elliptic operators that will be useful to us.

- The operator  $(\mathcal{D}(\mathcal{L}), \mathcal{L})$  is selfadjoint in  $L^2(\Omega)$ , with compact resolvent and thus admits a countable sequence  $\{\lambda_k\}_{k \geq 1}$  of (simple) eigenvalues. We denote by  $(\phi_k)_k$  the hilbertian basis of  $L^2(\Omega)$  made of associated normalized eigenfunctions.

We recall that the non-homogeneous equation  $\mathcal{L}u - \lambda_k u = f$  has a solution if and only if the orthogonality condition  $\int_{\Omega} f \phi_k dx = 0$  holds. In that case, the set of solutions of this problem is a dimension 1 affine space directed along  $\phi_k$ .

- We will frequently use the fact that, for any  $u \in \mathcal{D}(\mathcal{L})$ , we have  $u, \gamma \partial_x u \in \mathcal{C}^0(\bar{\Omega})$ . Moreover, in order to simplify a little the notation, we shall write  $v'$  (resp.  $v''$ ) instead of  $\partial_x v$  (resp.  $\partial_x^2 v$ ) for functions  $v$  depending only on the 1D variable  $x$ .

Observe that for any  $k \geq 1$ , we have  $(\gamma \phi'_k)(0) \neq 0$ . Assume that it is not the case, then we can write

$$\begin{aligned}\phi_k(x) &= \int_0^x \frac{1}{\gamma(s)} (\gamma \phi'_k)(s) ds, \quad \forall x \in [0, 1], \\ (\gamma \phi'_k)(x) &= \int_0^x (\gamma_0(s) - \lambda_k) \phi_k(s) ds, \quad \forall x \in [0, 1].\end{aligned}$$

It follows that

$$|\phi_k(x)| + |(\gamma \phi'_k)(x)| \leq M \int_0^x |\phi_k(s)| + |(\gamma \phi'_k)(s)| ds, \quad \forall x \in [0, 1],$$

with  $M = ((\inf_{\Omega} \gamma)^{-1} + \|\gamma_0\|_{\infty} + \lambda_k)$ , which gives that  $\phi_k \equiv 0$  by the Gronwall inequality. This is a contradiction.

- For any  $k \geq 1$ , we choose  $\tilde{\phi}_k$  to be any solution of the ordinary differential equation  $\mathcal{L}\tilde{\phi}_k - \lambda_k \tilde{\phi}_k = 0$  which satisfies  $\tilde{\phi}_k(0) \neq 0$ . Observe that  $\phi_k$  and  $\tilde{\phi}_k$  are linearly independent, and that  $\tilde{\phi}_k \notin \mathcal{D}(\mathcal{L})$  since it does not satisfy the Dirichlet boundary condition. In the case  $\mathcal{L} = -\partial_x^2$ , one can choose for instance  $\tilde{\phi}_k(x) = \cos(k\pi x)$ . Obviously, one can check that all the results given in this paper do not depend on the particular choice of  $\tilde{\phi}_k$  satisfying the above properties.
- The spectral properties of the vectorial operator  $\mathcal{L}$  are easily deduced from the ones of  $\mathcal{L}$ . In the sequel of this paper, the following operator will play a very important role

$$\mathcal{A} = \mathcal{L} - A(x)^*.$$

By a perturbation argument (see for instance the Keldysh theorem, [19]) it can be proved that

$$\begin{cases} \mathcal{A} \text{ has a compact resolvent,} \\ \text{The system of root vectors of } \mathcal{A}^* \text{ is complete in } L^2(\Omega)^n. \end{cases} \quad (6)$$

- In all the cases considered here (3)-(5), we observe that for any  $x \in \Omega$ ,  $A(x)$  is strictly lower triangular. Thus, the eigenvalues of the operator  $\mathcal{A}$  are simply the  $\{\lambda_k\}_{k \geq 1}$ . Indeed, assume that  $u$  is an eigenfunction of  $\mathcal{A}$  associated with an eigenvalue  $s \in \mathbb{C}$  and let  $i \geq 1$  be the higher index for which  $u_i$  is not identically zero. Writing the  $i$ th component of the equation  $\mathcal{A}u = su$ , leads to

$$su_i = \mathcal{L}u_i - \sum_{j>i} a_{ji}(x)u_j = \mathcal{L}u_i,$$

so that  $s$  is an eigenvalue of  $\mathcal{L}$  and finally  $s = \lambda_k$  for some  $k \geq 1$ .

Moreover, we observe that the first component  $u_1$  of  $u$  solves an equation of the following form

$$\mathcal{L}u_1 - \lambda_k u_1 = F \text{ in } (0, T) \times \Omega, \quad (7)$$

where  $F$  can be computed as a function of the other components of  $u$  and the entries in  $A(x)$  as we shall see below.

**2.2. Approximate controllability criteria.** With the notation introduced above, the adjoint system of (1) is

$$\begin{cases} -\partial_t q + \mathcal{A}q = 0 & \text{in } (0, T) \times \Omega, \\ q(T) = q_F & \text{in } \Omega, \end{cases} \quad (8)$$

and it is well known (see for instance [11, Theorem 2.43]) that the approximate controllability at time  $T > 0$  of (1) is equivalent to the unique continuation property for the adjoint parabolic system: there is no non-trivial solutions of (8) such that  $B^*q = 0$  on  $(0, T) \times \omega$ . Following Remark 1.1, this unique continuation property does not depend on  $T > 0$ .

However, Fattorini proved in [12, Corollary 3.3] that, as soon as the properties (6) are satisfied, this parabolic unique continuation property is actually equivalent to an elliptic unique continuation property which is much easier to handle. More precisely, we thus have the following controllability criterion for our class of systems.

**Theorem 2.1.** *System (1) is approximately controllable, if and only if for any  $s \in \mathbb{C}$  and any  $u \in \mathcal{D}(\mathcal{L})$  we have*

$$\left. \begin{array}{l} \mathcal{A}u = su \quad \text{in } \Omega \\ B^*u = 0 \quad \text{in } \omega \end{array} \right\} \implies u = 0. \quad (9)$$

In the theory of ordinary differential systems, this controllability condition is also known as the Hautus test. The characterization given by Fattorini has been recently developed and used in [10] and [22] for the study of some other parabolic systems.

Note that, for the particular systems studied in the present paper (excepted in Section 5),  $B^*u$  is nothing but the first component of  $u$ . Thus, the study of the approximate controllability of all the systems considered in Sections 3 and 4 reduces to the following question: does it exist an eigenfunction of  $\mathcal{A}$  whose first component is identically zero on the control domain  $\omega$ ?

We have seen in the previous section that the first component of any eigenfunction of  $\mathcal{A}$  solves a non-homogeneous problem like (7). That's the reason why the starting point of our analysis consists in studying necessary and sufficient conditions on the source term  $F$  ensuring that (7) does not have any solution  $u_1$  which identically vanishes on the control domain  $\omega$ . This is the main goal of the next section.

**2.3. Unique continuation for a 1D non-homogeneous scalar problem.** We establish necessary and sufficient conditions for a non-homogeneous scalar problem to have a solution which vanishes identically on a given subset of the domain. As we will see below, this is the main tool for analyzing the elliptic unique continuation property for eigenfunctions of  $\mathcal{A}$ .

We denote by  $\mathcal{C}(\overline{\Omega \setminus \omega})$  the set of all connected components of  $\overline{\Omega \setminus \omega}$ , and for every  $C \in \mathcal{C}(\overline{\Omega \setminus \omega})$  and  $f \in L^1(\Omega)$ , we define the vector  $M_k(f, C) \in \mathbb{R}^2$  by

$$M_k(f, C) = \begin{cases} \begin{pmatrix} \int_C f \phi_k dx \\ 0 \end{pmatrix} & \text{if } C \cap \partial\Omega \neq \emptyset, \\ \begin{pmatrix} \int_C f \phi_k dx \\ \int_C f \tilde{\phi}_k dx \end{pmatrix} & \text{if } C \cap \partial\Omega = \emptyset. \end{cases} \quad (10)$$

Then, for any  $f \in L^1(\Omega)$  we define the following family of vectors of  $\mathbb{R}^2$

$$\mathcal{M}_k(f, \omega) = (M_k(f, C))_{C \in \mathcal{C}(\overline{\Omega \setminus \omega})} \in (\mathbb{R}^2)^{\mathcal{C}(\overline{\Omega \setminus \omega})}.$$

**Theorem 2.2.** *Let  $F \in L^2(\Omega)$  and  $\omega$  be a non-empty open subset of  $\Omega$ . Let  $k \geq 1$  be fixed. There exists a solution  $u \in \mathcal{D}(\mathcal{L})$  to the following problem*

$$\begin{cases} \mathcal{L}u - \lambda_k u = F & \text{in } \Omega, \\ u = 0 & \text{in } \omega, \end{cases} \quad (11)$$

if and only if

$$\begin{cases} F = 0 & \text{in } \omega, \\ \mathcal{M}_k(F, \omega) = 0. \end{cases} \quad (12)$$

*Proof.* Let us perform a preliminary computation. Let  $[\alpha, \beta] \subset [0, 1]$  and  $u \in \mathcal{D}(\mathcal{L})$  be a solution of  $\mathcal{L}u - \lambda_k u = F$ .

Let  $v \in L^2(\Omega)$  be any distribution solution of the ordinary differential equation  $\mathcal{L}v - \lambda_k v = 0$ . We multiply by  $v$  the equation satisfied by  $u$  and we perform two integrations by parts to get

$$\begin{aligned} \int_{\alpha}^{\beta} Fv dx &= - [(\gamma u')(\beta)v(\beta) - u(\beta)(\gamma v')(\beta)] \\ &\quad + [(\gamma u')(\alpha)v(\alpha) - u(\alpha)(\gamma v')(\alpha)]. \end{aligned} \quad (13)$$

This formula will be used in the sequel with  $v = \phi_k$  and  $v = \tilde{\phi}_k$  to compute  $\mathcal{M}_k(F, \omega)$ . We can now turn to the proof of the claimed equivalence.

$\Rightarrow$  Assume that there exists a  $u$  satisfying (11).

- Since  $u = 0$  in  $\omega$ , it is clear from the equation that  $F = 0$  on  $\omega$ . Moreover, by continuity,  $u$  and  $\gamma u'$  are identically 0 on  $\bar{\omega}$ .
- Let  $C = [\alpha, \beta]$  be a connected component of  $\overline{\Omega \setminus \omega}$ . Observe that  $\alpha$  (resp.  $\beta$ ) necessarily belongs either to  $\bar{\omega}$  or to  $\partial\Omega$ , and that

$$\begin{cases} \alpha \in \partial\Omega \implies u(\alpha) = 0 \text{ and } \phi_k(\alpha) = 0, \\ \alpha \in \bar{\omega} \implies u(\alpha) = 0 \text{ and } \gamma u'(\alpha) = 0. \end{cases}$$

Therefore, in both cases, we have  $u(\alpha) = 0$  and  $\phi_k(\alpha)(\gamma u')(\alpha) = 0$ , the same being true for when one changes  $\alpha$  into  $\beta$ .

It follows from (13) with  $v = \phi_k$  that

$$\int_C F \phi_k dx = 0.$$

- Assume additionally that the connected component  $C$  is such that  $C \cap \partial\Omega = \emptyset$ . As we have seen above, in that case we have  $u(\alpha) = u(\beta) = (\gamma u')(\alpha) = (\gamma u')(\beta) = 0$ .

Therefore, (13) with  $v = \tilde{\phi}_k$  immediately gives that

$$\int_C F \tilde{\phi}_k dx = 0.$$

- Finally, we have proved that  $M_k(F, C) = 0$  in any case, which is exactly the second equation of (12).
- $\Leftrightarrow$  Since  $\mathcal{M}_k(F, \omega) = 0$ , we can sum all the integrals corresponding to the various connected components to obtain that  $\int_{\overline{\Omega} \setminus \omega} F \phi_k dx = 0$ . Using that  $F = 0$  on  $\omega$ , we conclude that  $\int_{\Omega} F \phi_k dx = 0$ . This orthogonality condition implies the existence of at least one solution  $u_0 \in \mathcal{D}(\mathcal{L})$  of the non-homogeneous equation

$$\mathcal{L}u_0 - \lambda_k u_0 = F, \text{ in } \Omega.$$

Actually, any solution of this problem has the form  $u = u_0 + \mu \phi_k$ ,  $\mu \in \mathbb{R}$ . We will show that we can find a  $\mu$  such that this function  $u$  vanishes identically on  $\omega$ .

- We first show that one can choose  $\mu$  in such a way that there exists a point  $x_0 \in \overline{\omega}$  satisfying

$$u(x_0) = (\gamma u')(x_0) = 0. \quad (14)$$

- \* Assume first that  $\overline{\omega} \cap \partial\Omega \neq \emptyset$  and for instance that  $0 \in \overline{\omega}$ . Thanks to the Dirichlet boundary condition we have  $u(0) = 0$  and we just need to impose  $(\gamma u')(0) = 0$ , that is  $(\gamma u'_0)(0) + \mu(\gamma \phi'_k)(0) = 0$ . This determines  $\mu$  in a unique way since  $(\gamma \phi'_k)(0) \neq 0$  and gives  $x_0 = 0$ .
- \* In the case where  $\overline{\omega} \cap \partial\Omega = \emptyset$ , we denote by  $[0, \beta]$  the connected component of  $\overline{\Omega} \setminus \omega$  containing 0. By assumption

$$\int_0^\beta F \phi_k dx = 0. \quad (15)$$

Since  $F = 0$  in  $\omega$ , we can replace  $\beta$  in this formula by  $\beta + \delta$  with  $\delta > 0$  small enough such that  $] \beta, \beta + \delta ] \subset \omega$  and  $\phi_k(\beta + \delta) \neq 0$  (the zeros of the eigenfunction  $\phi_k$  are isolated).

We can then fix the parameter  $\mu$  in such a way that

$$u(\beta + \delta) = u_0(\beta + \delta) + \mu \phi_k(\beta + \delta) = 0.$$

It follows from (13) with  $v = \phi_k$ , (15) (with the upper bound  $\beta + \delta$  instead of  $\beta$ , and  $\alpha = 0$ ), and from the boundary condition satisfied by  $u$  and  $\phi_k$  at 0, that

$$0 = (\gamma u')(\beta + \delta) \phi_k(\beta + \delta) - u(\beta + \delta) (\gamma \phi'_k)(\beta + \delta).$$

Since  $u$  vanishes at  $\beta + \delta$ , but  $\phi_k$  does not, we deduce that

$$(\gamma u')(\beta + \delta) = 0.$$

Therefore  $u$  and  $(\gamma u')$  vanish at the same point  $x_0 = \beta + \delta$  in  $\omega$ .

- The parameter  $\mu$  is now fixed and we know that there is a  $x_0$  such that (14) holds.

We want to show that  $u = 0$  on  $\omega$ . By contradiction, we assume that there is a  $x_1 \in \omega$ , such that  $u(x_1) \neq 0$ . Without loss of generality we



assume for instance that  $x_0 < x_1$ . Observe that  $[x_0, x_1] \cap (\overline{\Omega \setminus \omega})$  is a (possibly empty) union of connected components of  $\overline{\Omega \setminus \omega}$  and that none of them touches the boundary of  $\Omega$ . Since  $F = 0$  in  $\omega$ , and  $\mathcal{M}_k(F, \omega) = 0$ , we deduce that

$$0 = \int_{x_0}^{x_1} F \phi_k dx = \int_{x_0}^{x_1} F \tilde{\phi}_k dx.$$

Using (13) with  $v = \phi_k$  (resp. with  $v = \tilde{\phi}_k$ ) and (14), we get

$$\begin{cases} 0 = (\gamma u')(x_1) \phi_k(x_1) - u(x_1) (\gamma \phi_k')(x_1), \\ 0 = (\gamma u')(x_1) \tilde{\phi}_k(x_1) - u(x_1) (\gamma \tilde{\phi}_k')(x_1). \end{cases}$$

Since the Wronskian matrix

$$\begin{pmatrix} \phi_k(x_1) & -(\gamma \phi_k')(x_1) \\ \tilde{\phi}_k(x_1) & -(\gamma \tilde{\phi}_k')(x_1) \end{pmatrix},$$

is invertible (recall that  $\phi_k$  and  $\tilde{\phi}_k$  are two independent solutions of the second order differential equation  $\mathcal{L}v - \lambda_k v = 0$ ) we deduce that  $u(x_1) = (\gamma u')(x_1) = 0$  which is a contradiction.  $\square$

**3. Simultaneous controllability of several  $2 \times 2$  cascade systems.** In this section we are interested in the controllability of system (1) when the matrix  $A(x)$  is of the following form

$$A(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & 0 & \cdots & 0 \end{pmatrix}. \quad (16)$$

In this system, the distributed control  $v$  only acts on the first component  $y_1$  and this component serves itself as a simultaneous control for the other components through the coupling terms  $a_{21}, \dots, a_{n1}$ .

**3.1. Reduction.** The goal of this discussion is to show that for the study of the approximate controllability of System (1) with  $A(x)$  given by (16), we can always assume, up to a suitable change of variable, that all the supports of the coupling functions  $a_{i1}(x)$ ,  $i = 2, \dots, n$  do not intersect the control domain  $\omega$ , that is

$$a_{i1} \mathbf{1}_\omega = 0, \quad \forall i \in \{2, \dots, n\}. \quad (17)$$

Indeed, we assume that (17) does not hold (if not the reduction is unnecessary) and we observe that we can always reorder the unknowns  $y_k$  and the entries  $a_{k1}$ , for  $2 \leq k \leq n$ , in such a way that for some  $p \in \{2, \dots, n\}$

$$\begin{cases} \text{Span}(a_{21} \mathbf{1}_\omega, \dots, a_{n1} \mathbf{1}_\omega) = \text{Span}(a_{p1} \mathbf{1}_\omega, \dots, a_{n1} \mathbf{1}_\omega), \\ a_{p1} \mathbf{1}_\omega, \dots, a_{n1} \mathbf{1}_\omega \text{ are linearly independent.} \end{cases} \quad (18)$$

By using (18), we can write

$$a_{i1}1_\omega = \sum_{j=p}^n \alpha_{ij} a_{j1} 1_\omega, \quad \forall i \in \{2, \dots, p-1\},$$

for some  $\alpha_{ij} \in \mathbb{R}$ . We perform now the (reversible) change of unknowns  $y \rightarrow \tilde{y}$  defined by

$$\begin{cases} \tilde{y}_i = y_i - \sum_{j=p}^n \alpha_{ij} y_j, & \forall i \in \{2, \dots, p-1\}, \\ \tilde{y}_i = y_i, & \forall i \in \{1\} \cup \{p, \dots, n\}. \end{cases}$$

It is easily verified that  $\tilde{y}$  solves a system of the same form as (1)-(16), with a new coupling matrix, still referred to as  $A(x)$ , which satisfies

$$\begin{cases} a_{i1} = 0, \text{ on } \omega, & \forall i \in \{2, \dots, p-1\}, \\ a_{p1}1_\omega, \dots, a_{n1}1_\omega, & \text{are linearly independent.} \end{cases} \quad (19)$$

Finally, since the change of variable is invertible, we observe that the controllability of the original system for  $y$  is equivalent to the one of the new system for  $\tilde{y}$ .

Therefore, from now on we shall assume that (19) holds and we introduce the following reduced system of size  $p-1$

$$\begin{cases} \partial_t \hat{y} + \mathcal{L} \hat{y} = \hat{A}(x) \hat{y} + 1_\omega B v & \text{in } (0, T) \times \Omega, \\ \hat{y}(0) = \hat{y}_0 & \text{in } \Omega, \end{cases} \quad (20)$$

where  $\hat{A}(x)$  is the  $(p-1) \times (p-1)$  matrix defined by

$$\hat{A}(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21}(x) & 0 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1}(x) & 0 & \cdots & 0 \end{pmatrix}.$$

**Proposition 3.1.** *Assume that (19) holds, then the following statements are equivalent.*

1. System (1) is approximately controllable for any initial data  $y_0 \in L^2(\Omega)^n$ .
2. System (20) is approximately controllable for any initial data  $\hat{y}_0 \in L^2(\Omega)^{p-1}$ .

*Proof.*

1. $\Rightarrow$ 2. This is obvious since (20) is a subsystem of (1).

2. $\Rightarrow$ 1. Assume that (1) is not approximately controllable. The criterion given in Theorem 2.1 implies that (9) is not true. Therefore, there exists a non-trivial  $u \in \mathcal{D}(\mathcal{L})$  which satisfies, for some  $k \geq 1$ ,

$$\begin{cases} \mathcal{L}u - A(x)^*u = \lambda_k u & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

Observe that, from the particular structure of  $A(x)^*$ ,  $u = (u_1, \dots, u_n)^*$  has necessarily the following form

$$u = \begin{pmatrix} u_1 \\ \delta_2 \phi_k \\ \vdots \\ \delta_n \phi_k \end{pmatrix},$$

with  $\delta_i \in \mathbb{R}$  for  $i = 2, \dots, n$  and that  $u_1$  solves

$$\mathcal{L}u_1 - \lambda_k u_1 = \left( \sum_{i=2}^n \delta_i a_{i1} \right) \phi_k.$$

Since  $u_1$  vanishes on  $\omega$  as well as  $a_{i1}$  for  $i = 2, \dots, p-1$  (from Assumption (19)), we deduce that

$$\left( \sum_{i=p}^n \delta_i a_{i1} \right) \phi_k = 0, \quad \text{almost everywhere in } \omega.$$

Since  $\phi_k \neq 0$  almost everywhere (its zeros are isolated), it follows that

$$\sum_{i=p}^n \delta_i a_{i1} 1_\omega = 0.$$

By (19), the functions  $a_{i1} 1_\omega$ ,  $i = p, \dots, n$  are linearly independent so that  $\delta_i = 0$  for any  $i = p, \dots, n$ .

Coming back to the equation satisfied by  $u_1$ , we get

$$\mathcal{L}u_1 - \lambda_k u_1 = \left( \sum_{i=2}^{p-1} \delta_i a_{i1} \right) \phi_k = \left( \sum_{i=2}^{p-1} a_{i1} u_i \right).$$

It follows that the reduced vector  $\hat{u}(x) = (u_1(x), \dots, u_{p-1}(x))^* \in \mathbb{R}^{p-1}$  is a non-trivial eigenfunction of the reduced adjoint system

$$\mathcal{L}\hat{u} - \hat{A}(x)^* \hat{u} = \lambda_k \hat{u},$$

that satisfies

$$\hat{u}_1 = u_1 = 0, \quad \text{in } \omega.$$

From Theorem 2.1, this is in contradiction with the approximate controllability of the reduced system (20). □

**3.2. Necessary and sufficient approximate controllability conditions.** The main result of this section is the following.

**Theorem 3.2.** *Consider the matrix  $A(x)$  as defined in (16) and assume that (17) holds. Then, System (1) is approximately controllable if and only if*

$$\forall k \geq 1, \quad \text{rank} \{ \mathcal{M}_k(a_{21} \phi_k, \omega), \dots, \mathcal{M}_k(a_{n1} \phi_k, \omega) \} = n - 1.$$

**Remark 3.1.** In this formula the rank condition is understood in the (possibly infinite dimensional) vector space  $(\mathbb{R}^2)^{c(\Omega \setminus \omega)}$ .

In the usual case where  $\overline{\Omega \setminus \omega}$  has a finite number of connected components, this condition can be more classically written in a matrix formulation.

**Remark 3.2.** The first conclusion that the rank condition above let us draw is that there is a minimal number of connected components of  $\overline{\Omega \setminus \omega}$  that are required to have a chance to control the system. Recall that the goal is to be able to control all the  $n$  components of the solution with only one control  $v$ .

More precisely, we see that it is necessary (but not at all sufficient) to have  $2 \operatorname{card} \mathcal{C}(\overline{\Omega \setminus \omega}) \geq n - 1$  for the approximate controllability to be possible. Observe that, if the system is not controllable, it is of course useless to split the control domain  $\omega$  into smaller parts: this will actually increase the number of connected components of  $\overline{\Omega \setminus \omega}$  but without adding non-trivial terms in the rank condition, because of (17).

Looking more attentively at the rank condition we see that, for instance, one can not hope to control a  $3 \times 3$  system (resp. a  $4 \times 4$  system) of this form if  $\omega$  is an interval that touches the boundary (resp. that does not touch the boundary). A more detailed description of such examples is given in Section 3.4.

*Proof.* We use the criterion given in Theorem 2.1 and we study whether or not (9) holds. As we have already seen in Section 2.2, the only non-trivial case is the one where  $s = \lambda_k$  for some  $k \geq 1$ , in which case a solution  $u$  of  $\mathcal{L}u - A(x)^*u = \lambda_k u$  can be written

$$u = \begin{pmatrix} u_1 \\ \delta_2 \phi_k \\ \vdots \\ \delta_n \phi_k \end{pmatrix},$$

with  $\delta_i \in \mathbb{R}$ ,  $i = 2, \dots, n$  and  $u_1 \in \mathcal{D}(\mathcal{L})$  satisfying

$$\mathcal{L}u_1 - \lambda_k u_1 = \left( \sum_{i=2}^n \delta_i a_{i1} \right) \phi_k.$$

From Theorem 2.2, and since by assumption all the  $a_{i1}$  vanish on  $\omega$ , such a solution  $u$  exists and satisfies  $u_1 = 0$  in  $\omega$ , if and only if

$$\mathcal{M}_k \left( \sum_{i=2}^n \delta_i a_{i1} \phi_k, \omega \right) = 0. \quad (21)$$

On the other hand, note that  $u = 0$  if and only if  $\delta_2 = \dots = \delta_n = 0$  and  $u_1 = 0$  on  $\omega$ . This follows from the unique continuation for a single parabolic equation (see for instance [21], [14] and [3], depending on the regularity required for the diffusion coefficient  $\gamma$ ).

In summary, (9) holds if and only if (21) implies  $\delta_2 = \dots = \delta_n = 0$ . Clearly,

$$\mathcal{M}_k \left( \sum_{i=2}^n \delta_i a_{i1} \phi_k, \omega \right) = \sum_{i=2}^n \delta_i \mathcal{M}_k(a_{i1} \phi_k, \omega),$$

and thus the approximate controllability is equivalent to the linear independence of the vectors  $(\mathcal{M}_k(a_{i1} \phi_k, \omega))_{2 \leq i \leq n}$ , for any  $k \geq 1$ , which gives exactly the claim.  $\square$

**3.3. Application to a single  $2 \times 2$  cascade system.** Let us study the following simplest example of system concerned by the previous analysis

$$\begin{cases} \partial_t y_1 + \mathcal{L}y_1 = 1_\omega v & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 + \mathcal{L}y_2 = a_{21}(x)y_1 & \text{in } (0, T) \times \Omega, \end{cases} \quad (22)$$

which corresponds to the case (3). Depending on the assumptions on the coupling term  $a_{21}$  different results can be obtained. A first result in this direction is the following.

**Theorem 3.3.** *Let us denote the support of  $a_{21}$  by  $\mathcal{O}_2$ .*

1. *If  $\mathcal{O}_2 \cap \omega \neq \emptyset$ , then System (22) is approximately controllable.*
2. *Assume now that  $\mathcal{O}_2 \cap \omega = \emptyset$ .*
  - (a) *If the coupling coefficient  $a_{21}$  satisfies*

$$\int_0^1 a_{21}(\phi_k)^2 dx \neq 0, \quad \forall k \geq 1, \quad (23)$$

*then System (22) is approximately controllable.*

- (b) *If System (22) is approximately controllable and  $\mathcal{O}_2$  is entirely included in a connected component of  $\Omega \setminus \omega$  that touches the boundary  $\partial\Omega$ , then (23) holds.*

**Remark 3.3.**

- In the first situation it can be proved under a slightly stronger assumption on the coupling coefficient that System (22) is even null-controllable in this case (see for instance [15]), but the proof is much longer and technical.
- With (23) we recover the (sufficient) condition of [17, Theorem 1.5]. It is easy to see that this condition is fulfilled if  $a_{21}$  has a sign on  $\Omega$ : for instance  $a_{21} \not\equiv 0$  and  $a_{21} \geq 0$  almost everywhere on  $\Omega$ . Actually, under this sign assumption, the null-controllability of this system is known (see [23, Theorem 5]).
- The geometric configuration required in the last point (2b) holds in particular if  $\mathcal{O}_2$  and  $\omega$  are two disjoint intervals. As it will be illustrated in the examples below, condition (23) is however not necessary in general.

*Proof.*

1. If  $a_{21}$  is not identically zero on  $\omega$ , we deduce from Proposition 3.1 (with  $p = n = 2$ ) that the approximate controllability of (22) is equivalent to the one of the scalar parabolic equation

$$\partial_t \hat{y} + \mathcal{L}\hat{y} = 1_\omega v$$

with Dirichlet boundary condition. This kind of scalar heat equation is known to be approximately controllable (see the references given in the proof of Theorem 3.2) and thus, we obtain that (22) is also approximately controllable.

2. Assume now that  $\mathcal{O}_2 \cap \omega = \emptyset$ . In this case, (17) holds and the rank condition in Theorem 3.2 simply reduces to the property

$$\mathcal{M}_k(a_{21}\phi_k, \omega) \neq 0, \quad \forall k \geq 1. \quad (24)$$

- (a) In particular, if we assume that (23) holds, then, for any  $k \geq 1$ , there exists at least one connected component  $C$  of  $\Omega \setminus \omega$  such that

$$\int_C a_{21}(\phi_k)^2 dx \neq 0.$$

This shows that the first component of  $M_k(a_{21}\phi_k, C)$  is not zero and thus Condition (24) holds and System (22) is approximately controllable.

- (b) Let  $C$  be the connected component of  $\overline{\Omega \setminus \omega}$  that contains  $\mathcal{O}_2$ . Since by assumption  $C$  touches the boundary of  $\Omega$ , we have  $\mathcal{M}_k(a_{21}\phi_k, \omega) \neq 0$  if and only if  $\int_C a_{21}(\phi_k)^2 dx \neq 0$ . On the other hand, since  $\mathcal{O}_2 \subset C$ , we have  $\int_0^1 a_{21}(\phi_k)^2 dx = \int_C a_{21}(\phi_k)^2 dx$  and the claim is proved.  $\square$

We will now investigate some examples (not necessarily under the assumptions of the previous theorem though).

3.3.1. *Example 1 : Influence of the geometry of the control domain.* In the first example we consider a coupling coefficient  $a_{21}$  that vanishes in  $\omega$  and does not have a constant sign in  $\overline{\Omega \setminus \omega}$ . We will provide in particular some controllable systems for which (23) fails. Up to our knowledge our analysis is the first available result in this framework.

We will study two slightly different situations depending on the geometry of the control domain  $\omega$ , as shown in Figure 1.

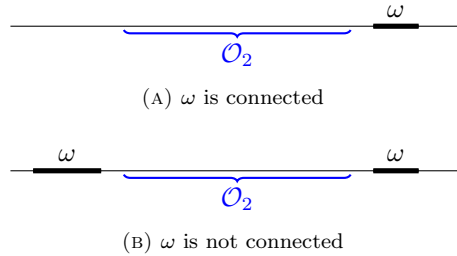


FIGURE 1. Two geometries for the study of the  $2 \times 2$  system (22)

- For some  $\alpha \in \mathbb{R}$ , we consider (see Figure 1a)

$$\omega \subset \left(\frac{3}{4}, 1\right), \quad a_{21}(x) = (x - \alpha)1_{\mathcal{O}_2}(x), \quad \mathcal{O}_2 = \left(\frac{1}{4}, \frac{3}{4}\right).$$

In this case, we are in the framework of Theorem 3.3 (case 2b) and, as a result, the approximate controllability holds if and only if (23) holds. If, for any  $k \geq 1$ , we set

$$\alpha_k = \frac{\int_{\mathcal{O}_2} x \phi_k^2(x) dx}{\int_{\mathcal{O}_2} \phi_k^2(x) dx},$$

then we obtain that

$$(22) \text{ is approximately controllable} \iff \alpha \notin \{\alpha_k\}_{k \geq 1}.$$

As an illustration, in the case  $\mathcal{L} = -\partial_x^2$ , we have  $\phi_k(x) = \sin(k\pi x)$  and a direct computation shows that  $\alpha_k = 1/2$  for any  $k \geq 1$ . Therefore, the system is approximately controllable if and only if  $\alpha \neq 1/2$ .

To our knowledge, no (positive or negative) null-controllability result is available for this system. However, the numerical results given in [9] in a similar case seem to suggest that it is possible that null-controllability does not hold in general when approximate controllability holds.

- With the same choice of  $a_{21}$  and  $\mathcal{O}_2$ , we consider now the case where

$$\omega \cap \mathcal{O}_2 = \emptyset, \omega \cap \left(\frac{3}{4}, 1\right) \neq \emptyset, \text{ and } \omega \cap \left(0, \frac{1}{4}\right) \neq \emptyset,$$

as shown in Figure 1b.

For  $\alpha \notin \{\alpha_k\}_{k \geq 1}$ , the controllability result obtained above immediately imply the approximate controllability of the system in this new framework.

However, for  $\alpha \in \{\alpha_k\}_{k \geq 1}$  it may happen that the system is approximately controllable with this new choice of the control domain  $\omega$  despite it is not approximately controllable for the previous choice of  $\omega$ . Indeed, we observe that the only connected component  $C$  of  $\Omega \setminus \omega$  that plays a role in the problem does not touch the boundary  $\partial\Omega$  anymore. Therefore, in the rank condition given in Theorem 3.2, the second components in  $\mathcal{M}_k(a_{21}\phi_k, \omega)$  are no more trivial (see (10)) and we have,

(22) is approximately controllable

$$\iff \int_{\mathcal{O}_2} (x - \alpha)\phi_k \tilde{\phi}_k dx \neq 0, \forall k \geq 1, \text{ s.t. } \alpha_k = \alpha.$$

This new condition is not explicit in general, but for instance in the case  $\mathcal{L} = -\partial_x^2$  we have  $\phi_k(x) = \sin(k\pi x)$  and  $\tilde{\phi}_k(x) = \cos(k\pi x)$  and we can check that (recall that the only interesting value here is  $\alpha = 1/2$  since  $\alpha_k = 1/2$  for any  $k \geq 1$ )

$$\begin{aligned} \int_C a_{21}(x)\phi_k(x)\tilde{\phi}_k(x) dx &= \int_{1/4}^{3/4} (x - 1/2) \sin(k\pi x) \cos(k\pi x) dx \\ &= \begin{cases} \frac{-1}{8k\pi} (-1)^{k/2} & \text{if } k \text{ is even,} \\ \frac{-1}{4k^2\pi^2} (-1)^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

As a consequence, those integrals are never equal to zero and the approximate controllability of the system is proved in this case for any value of  $\alpha$ .

It is worth mentioning that for this example the null-controllability of the system remains an open problem (it seems that there is no result available in this direction as soon as the coupling function  $a_{21}$  does not have a constant sign).

**3.3.2. Example 2 : Analysis of the set of controllable initial data.** Let us go back to the geometry of Figure 1a, in the particular case  $\mathcal{L} = -\partial_x^2$  and  $\alpha = \frac{1}{2}$ . We have seen above that this particular value of  $\alpha$  is the only one for which System (22) is not approximately controllable. This precisely means that there is at least one initial data  $y_0$  that can not be steered as close to zero as we would like to.

Actually, we can obtain a more precise result in that case since we have seen that, when  $\alpha = \frac{1}{2}$ , the integrals  $\int_C (x - \frac{1}{2})\phi_k^2 dx$  vanish for every  $k \geq 1$  (and not only for one value of  $k$ ). More precisely, we will identify a set of an infinite number of necessary conditions that should be satisfied by the initial data  $y_0$  in order for the system to be approximately controllable from  $y_0$ .

Using that  $\mathcal{M}_k(a_{21}\phi_k, \omega) = 0$  for any  $k \geq 1$ , so that from Theorem 2.2, we deduce the existence (and uniqueness) of a function denoted by  $\psi_k$  which satisfies

$$\begin{cases} -\psi_k'' - \lambda_k \psi_k = a_{21}\phi_k & \text{in } \Omega, \\ \psi_k = 0 & \text{in } \omega. \end{cases} \quad (25)$$

**Proposition 3.4.** *Let  $y_0 = (y_{0,1}, y_{0,2})^* \in L^2(\Omega)^2$ . If System (22) is approximately controllable from the initial data  $y_0$ , we have*

$$\langle y_{0,1}, \psi_k \rangle_{L^2(\Omega)} + \langle y_{0,2}, \phi_k \rangle_{L^2(\Omega)} = 0, \quad \forall k \geq 1. \quad (26)$$

*Proof.* We introduce the set  $Q_F$  of the non-observable adjoint states defined as follows

$$Q_F = \{q_F \in L^2(\Omega)^2, \text{ s.t. the solution of (8) satisfies } 1_\omega B^* q(t) = 0, \forall t\}.$$

In the present case, we recall that  $B^* q = q_1$  is the first component of  $q$ .

It is proved in [9, Proposition 1.17] that our system is approximately controllable at time  $T$  from a given initial data  $y_0$  if and only if

$$\langle y_0, q(0) \rangle_{L^2(\Omega)} = 0, \quad \text{for any solution } q \text{ of (8) with } q_F \in Q_F. \quad (27)$$

By construction of  $\psi_k$ , the vector  $q_F = (\psi_k, \phi_k)^*$  belongs to  $Q_F$  for any  $k \geq 1$ , and the associated solution of the adjoint problem (8) is nothing but

$$q(t) = e^{-\lambda_k(T-t)} q_F.$$

It follows that, if the system is controllable from  $y_0$ , we necessarily should have

$$\begin{aligned} 0 &= \langle y_0, q(0) \rangle_{L^2(\Omega)} = e^{-\lambda_k T} \langle y_0, q_F \rangle_{L^2(\Omega)} \\ &= e^{-\lambda_k T} \left( \langle y_{0,1}, \psi_k \rangle_{L^2(\Omega)} + \langle y_{0,2}, \phi_k \rangle_{L^2(\Omega)} \right), \end{aligned}$$

for any  $k \geq 1$ , and the proof is complete.  $\square$

**Remark 3.4.** It follows from this proposition that the set of initial data for which System (22) is approximately controllable is a closed subspace of  $L^2(\Omega)^2$  of infinite codimension.

However, we observe that this set is not trivial. Indeed, let us consider an initial data of the form  $y_0 = (y_{0,1}, y_{0,2})^*$  with  $y_{0,1}$  supported in  $\omega$  then we have

$$(22) \text{ is approximately controllable from } y_0 \iff y_{0,2} = 0.$$

$\Rightarrow$  Since for any  $k \geq 1$ ,  $\psi_k = 0$  in  $\omega$  and  $y_{0,1}$  is supported in  $\omega$ , the first term in (26) automatically vanishes. Therefore, we have  $\langle y_{0,2}, \phi_k \rangle_{L^2(\Omega)} = 0$  for any  $k \geq 1$  which leads to  $y_{0,2} = 0$ .

$\Leftarrow$  We use the characterization (27) which reduces to  $\langle y_{0,1}, q_1(0) \rangle_{L^2(\Omega)} = 0$  for any  $q_F \in Q_F$ , since  $y_{0,2} = 0$ . But this new condition is automatically satisfied because  $y_{0,1}$  is supported in  $\omega$  and  $q_1(0) = 0$  in  $\omega$  by definition of  $Q_F$ .

**3.3.3. Example 3 : Influence of the coupling terms.** Let us give another example where the controllability conditions are slightly more complex. Our aim is to emphasize that the notion of approximate controllability is very sensitive to the coupling terms in the system; in some sense we can say that it is not a stable notion with respect to the coefficients of the equation under study.



The situation we consider is the following (see Figure 2)

$$\begin{cases} \mathcal{L} = -\partial_x^2, & \text{so that } \phi_k(x) = \sin(k\pi x), \forall k \geq 1, \\ \omega \subset (\beta, 1), & a_{21} = 1_{\mathcal{O}_2} - 1_{\mathcal{O}'_2}, \\ \text{with } \mathcal{O}_2 = [\alpha, \alpha + L], \mathcal{O}'_2 = [\alpha + d, \alpha + d + L], \end{cases} \quad (28)$$

for some fixed  $\beta \in (0, 1)$  and  $L, d, \alpha \geq 0$  such that  $\alpha + L + d \leq \beta$ . Therefore, the coupling term  $a_{21}$  takes the values 1 and  $-1$  on two intervals of the same length and its support does not touch the control domain  $\omega$ .

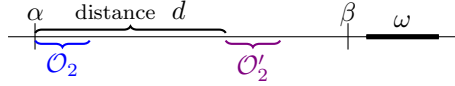


FIGURE 2. The geometry for example (28)

There is again one single connected component of  $\overline{\Omega \setminus \omega}$ , that we denote by  $C$ , that plays a role in the controllability, and this latter one touches the boundary  $\partial\Omega$ . Thus, we are in the framework of Theorem 3.3 (case 2b). Let us compute

$$\begin{aligned} \int_C a_{21}(x) \sin^2(k\pi x) dx &= \int_{\alpha}^{\alpha+L} \sin^2(k\pi x) dx - \int_{\alpha+d}^{\alpha+d+L} \sin^2(k\pi x) dx \\ &= \frac{-1}{k\pi} \sin(k\pi L) \sin(k\pi d) \sin(k\pi(2\alpha + d + L)). \end{aligned}$$

As a conclusion, System (22) is approximately controllable if and only if

$$L \notin \mathbb{Q}, \quad d \notin \mathbb{Q}, \quad 2\alpha + d + L \notin \mathbb{Q}.$$

Fix for instance  $L, \alpha \geq 0$  such that  $L \notin \mathbb{Q}$ ,  $2\alpha + L \in \mathbb{Q}$  and  $\alpha + L < \beta$ . Then, for any  $d \in [0, \beta - \alpha - L]$ , we have

$$\text{System (22) is approximately controllable} \iff d \notin \mathbb{Q}.$$

**3.4. Application to the simultaneous controllability of two  $2 \times 2$  cascade systems.** In this section we study the controllability properties of the following  $3 \times 3$  one-dimensional system,

$$\begin{cases} \partial_t y_1 + \mathcal{L}y_1 = 1_{\omega} v & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 + \mathcal{L}y_2 = a_{21}(x)y_1 & \text{in } (0, T) \times \Omega, \\ \partial_t y_3 + \mathcal{L}y_3 = a_{31}(x)y_1 & \text{in } (0, T) \times \Omega, \end{cases} \quad (29)$$

which corresponds to the case (4). Observe that there is no direct interaction between  $y_2$  and  $y_3$  so that the problem can be understood as follows: find a single control  $v \in L^2(0, T; L^2(\Omega))$  which simultaneously drives near zero at time  $T$  the solutions of the two  $2 \times 2$  subsystems for  $(y_1, y_2)$  in the one hand and for  $(y_1, y_3)$  in the other hand.

We recall that we can always assume that the coupling terms  $a_{21}$  and  $a_{31}$  identically vanish on  $\omega$ , see section 3.1. Let us denote by  $\mathcal{O}_2$  and  $\mathcal{O}_3$  the supports of  $a_{21}$  and  $a_{31}$ , respectively.

We will illustrate the controllability properties of the system in various situations depending on the geometric configuration of the coupling domains  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  and of the control domain  $\omega$ .

3.4.1. *Example 1 : the control domain  $\omega$  is connected.* We assume first that  $\omega$  is connected. In such case there is at most two connected components in  $\overline{\Omega \setminus \omega}$ , say  $C_1$  and  $C_2$ , and they necessarily touch the boundary. Theorem 3.2 then states that system (29) is approximately controllable if and only if

$$\text{rank} \begin{pmatrix} M_k(a_{21}\phi_k, C_1) & M_k(a_{31}\phi_k, C_1) \\ M_k(a_{21}\phi_k, C_2) & M_k(a_{31}\phi_k, C_2) \end{pmatrix} = 2, \quad \forall k \geq 1. \quad (30)$$

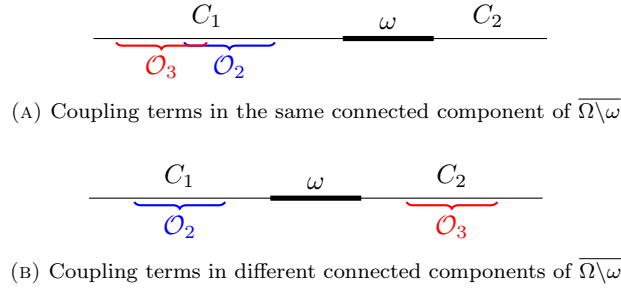


FIGURE 3. Various geometric situations for the  $3 \times 3$  system (29)

- First case:  $\mathcal{O}_2$  and  $\mathcal{O}_3$  are included in the same connected component of  $\overline{\Omega \setminus \omega}$ , see Figure 3a. We see that system (29) can not be approximately controllable (whether the supports of  $a_{21}$  and  $a_{31}$  intersect each other or not). Indeed, (30) cannot be true because

$$\text{rank} \begin{pmatrix} M_k(a_{21}\phi_k, C_1) & M_k(a_{31}\phi_k, C_1) \\ 0 & 0 \end{pmatrix} \leq 1, \quad \forall k \geq 1,$$

since  $C_1$  touches the boundary  $\partial\Omega$  and thus there is only one row in this  $4 \times 2$  matrix which can be non-trivial, see (10).

- Second case:  $\mathcal{O}_2$  and  $\mathcal{O}_3$  are included in two different connected components of  $\overline{\Omega \setminus \omega}$ , see Figure 3b.

Here, we have

$$\text{rank} \begin{pmatrix} M_k(a_{21}\phi_k, C_1) & 0 \\ 0 & M_k(a_{31}\phi_k, C_2) \end{pmatrix} = 2 \iff \begin{cases} M_k(a_{21}\phi_k, C_1) \neq 0, \\ M_k(a_{31}\phi_k, C_2) \neq 0. \end{cases}$$

Thus, the approximate controllability of system (29) in this case is equivalent to the approximate controllability of the two  $2 \times 2$  systems

$$\begin{cases} \partial_t \hat{y}_1 + \mathcal{L}\hat{y}_1 = 1_\omega \hat{v} & \text{in } (0, T) \times \Omega, \\ \partial_t \hat{y}_2 + \mathcal{L}\hat{y}_2 = a_{21}(x)\hat{y}_1 & \text{in } (0, T) \times \Omega, \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{y}_1 + \mathcal{L}\tilde{y}_1 = 1_\omega \tilde{v} & \text{in } (0, T) \times \Omega, \\ \partial_t \tilde{y}_3 + \mathcal{L}\tilde{y}_3 = a_{31}(x)\tilde{y}_1 & \text{in } (0, T) \times \Omega, \end{cases}$$

Of course, it is not required here that the controls  $\hat{v}$  and  $\tilde{v}$  are the same.

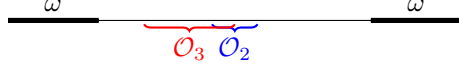


FIGURE 4. The case of a non-connected control domain

Actually, by a direct argument we can even prove that the null-controllability of system (29) is equivalent to the null-controllability of these  $2 \times 2$  systems. Indeed, let  $\omega = (a, b) \subset\subset \Omega = (0, 1)$  and take  $\mathcal{L} = -\partial_x^2$  for simplicity. Let  $\alpha, \beta \in C^\infty(\overline{\Omega})$  be smooth cut-off functions satisfying

$$\begin{cases} \alpha = 1 \text{ in } (0, a), \\ \alpha = 0 \text{ in } (b, 1), \end{cases} \quad \begin{cases} \beta = 0 \text{ in } (0, a), \\ \beta = 1 \text{ in } (b, 1). \end{cases}$$

If  $\hat{v}$  and  $\tilde{v}$  are null-controls for the  $2 \times 2$  systems above, we define the control  $v$  by

$$v = \alpha \hat{v} + \beta \tilde{v} + (\partial_x^2 \alpha) \hat{y}_1 + 2(\partial_x \alpha)(\partial_x \hat{y}_1) + (\partial_x^2 \beta) \tilde{y}_1 + 2(\partial_x \beta)(\partial_x \tilde{y}_1),$$

It is clear that  $v$  belongs to  $L^2(\Omega)$  and is supported in  $\omega$ . On the other hand we can check that  $y_1 = \alpha \hat{y}_1 + \beta \tilde{y}_1$ ,  $y_2 = \hat{y}_2$  and  $y_3 = \tilde{y}_3$  so that  $y_1(T) = y_2(T) = y_3(T) = 0$ .

**3.4.2. Example 2 : the control domain  $\omega$  is not connected.** We choose here  $\omega = (0, \alpha) \cup (\beta, 1)$  with, for instance,  $\alpha < 1/2 < \beta$ . In that case,  $\overline{\Omega \setminus \omega}$  has also one single connected component  $C$  but  $C$  does not touch the boundary of  $\Omega$ . In order to make the computations explicit, we set  $\mathcal{L} = -\partial_x^2$ .

We take  $a_{21} = 1_{\mathcal{O}_2}$ ,  $a_{31} = 1_{\mathcal{O}_3}$  where  $\mathcal{O}_2 = ]1/2 - \delta_2, 1/2 + \delta_2[$  is an interval centered at  $1/2$  and  $\mathcal{O}_3$  is another interval  $\mathcal{O}_3 = ]\alpha_3 - \delta_3, \alpha_3 + \delta_3[$ . They are chosen in such a way that  $\mathcal{O}_2 \cap \omega = \mathcal{O}_3 \cap \omega = \emptyset$ , see Figure 4. The controllability rank condition given by Theorem 3.2 then writes

$$\text{rank} \begin{pmatrix} \int_{\mathcal{O}_2} \sin(k\pi x)^2 dx & \int_{\mathcal{O}_3} \sin(k\pi x)^2 dx \\ \int_{\mathcal{O}_2} \sin(k\pi x) \cos(k\pi x) dx & \int_{\mathcal{O}_3} \sin(k\pi x) \cos(k\pi x) dx \end{pmatrix} = 2, \quad \forall k \geq 1.$$

Using the symmetry of  $\mathcal{O}_2$  with respect to  $1/2$ , we see that

$$\int_{\mathcal{O}_2} \sin(k\pi x) \cos(k\pi x) dx = 0.$$

Since  $\int_{\mathcal{O}_2} \sin(k\pi x)^2 dx > 0$ , it follows that the system is controllable if and only if  $\int_{\mathcal{O}_3} \sin(k\pi x) \cos(k\pi x) dx \neq 0$  for any  $k \geq 1$ . A straightforward computation shows that

$$\int_{\mathcal{O}_3} \sin(k\pi x) \cos(k\pi x) dx = \frac{\sin(2k\pi\delta_3) \sin(2k\pi\alpha_3)}{2k\pi},$$

so that we conclude that

$$\text{The system is approximately controllable} \iff \alpha_3 \notin \mathbb{Q} \text{ and } \delta_3 \notin \mathbb{Q}.$$

3.4.3. *Summary.* Let us draw a kind of summary of the previous discussion when  $a_{21} = 1_{\mathcal{O}_2}$  and  $a_{31} = 1_{\mathcal{O}_3}$  are the characteristic functions of intervals that do not intersect  $\omega$ :

- In the situation of Figure 3a, System (29) is never approximately controllable.
- In the situation of Figure 3b, System (29) is always approximately controllable.
- In the situation of Figure 4, the approximate controllability of System (29), depends on the precise size and position of the intervals  $\mathcal{O}_2$  and  $\mathcal{O}_3$ .

4. **Controllability of a  $3 \times 3$  cascade system.** In this section, we are interested in the controllability properties of the following system

$$\begin{cases} \partial_t y_1 + \mathcal{L}y_1 = 1_\omega v & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 + \mathcal{L}y_2 = a_{21}(x)y_1 & \text{in } (0, T) \times \Omega, \\ \partial_t y_3 + \mathcal{L}y_3 = a_{32}(x)y_2 & \text{in } (0, T) \times \Omega, \end{cases} \quad (31)$$

which corresponds to the case (5). This system has a cascade structure since the control  $v$  only acts on the first component of the solution which itself has an influence on the second component  $y_2$  through the coupling term  $a_{21}y_1$ , and finally  $y_2$  also acts on the third component through another coupling term  $a_{32}y_2$ .

For simplicity, we assume all along this section that there exists a non-empty open set  $\mathcal{O}_2 \subset \Omega$  such that

$$a_{21} \neq 0 \text{ a.e. in } \mathcal{O}_2, \quad a_{21} = 0 \text{ a.e. in } \Omega \setminus \mathcal{O}_2. \quad (32)$$

This is a (weak) regularity assumption which holds for instance if  $a_{21}$  is piecewise continuous and not identically zero.

**Remark 4.1.** A first necessary condition for the approximate controllability of System (31) is the approximate controllability of the subsystem (22), which has been studied in Section 3.3.

4.1. **Necessary and sufficient approximate controllability conditions.** The following result gives additional necessary and sufficient conditions that allow a quite simple analysis of the approximate controllability of System (31). Under particular assumptions on the coupling coefficients, we see that the study of the controllability for the  $3 \times 3$  system (31) reduces to the study of the controllability of some  $2 \times 2$  systems. This should be connected with Theorem 3.3 for  $2 \times 2$  systems.

**Theorem 4.1.** *Assume that the  $2 \times 2$  subsystem (22) is approximately controllable and let  $a_{21}$  satisfy (32).*

1. *Assume that  $\mathcal{O}_2 \cap \omega \neq \emptyset$ .*
  - (a) *If the  $2 \times 2$  system*

$$\begin{cases} \partial_t y_2 + \mathcal{L}y_2 = 1_{\mathcal{O}_2 \cap \omega} \hat{v} & \text{in } (0, T) \times \Omega, \\ \partial_t y_3 + \mathcal{L}y_3 = a_{32}(x)y_2 & \text{in } (0, T) \times \Omega, \end{cases} \quad (33)$$

*is approximately controllable, then System (31) is itself approximately controllable.*

- (b) *If System (31) is approximately controllable and  $\mathcal{O}_2 \subset \omega$ , then System (33) is approximately controllable.*
2. *Assume now that  $\mathcal{O}_2 \cap \omega = \emptyset$ .*

(a) If the coupling coefficient  $a_{32}$  satisfies

$$\int_0^1 a_{32}(\phi_k)^2 dx \neq 0, \quad \forall k \geq 1, \quad (34)$$

then, System (31) is approximately controllable.

(b) If System (31) is approximately controllable and  $\mathcal{O}_2$  is entirely included in a connected component of  $\Omega \setminus \omega$  that touches the boundary  $\partial\Omega$ , then (34) holds.

The proof of Theorem 4.1 relies on the following characterization.

**Proposition 4.2.** *Assume that the  $2 \times 2$  subsystem (22) is approximately controllable and let  $a_{21}$  satisfy (32). Then, System (31) is **not** approximately controllable if and only if there exists  $k \geq 1$  and  $v \in \mathcal{D}(\mathcal{L})$  such that*

$$\begin{cases} \mathcal{L}v - \lambda_k v = a_{32}(x)\phi_k & \text{in } \Omega, \\ v = 0 & \text{in } \mathcal{O}_2 \cap \omega, \\ \mathcal{M}_k(a_{21}v, \omega) = 0. \end{cases} \quad (35)$$

*Proof.* From Theorem 2.1, System (31) is **not** approximately controllable if and only if there exists  $k \geq 1$  and  $u = (u_1, u_2, u_3)^* \in \mathcal{D}(\mathcal{L})$  with  $u \neq 0$  such that

$$\begin{cases} \mathcal{L}u_1 - \lambda_k u_1 = a_{21}u_2 & \text{in } \Omega, \\ \mathcal{L}u_2 - \lambda_k u_2 = a_{32}u_3 & \text{in } \Omega, \\ \mathcal{L}u_3 - \lambda_k u_3 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

Clearly,  $u_3 = \delta\phi_k$  for some  $\delta \in \mathbb{R}$ . Moreover, we have  $\delta \neq 0$ . Indeed, if we assume that  $\delta = 0$  then  $(u_1, u_2)$  is not trivial and satisfies

$$\begin{cases} \mathcal{L}u_1 - \lambda_k u_1 = a_{21}u_2 & \text{in } \Omega, \\ \mathcal{L}u_2 - \lambda_k u_2 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega, \end{cases}$$

and this is a contradiction with the approximate controllability of the subsystem (22), by Theorem 2.1.

Thus, under this assumption, System (31) is **not** approximately controllable if and only if there exists  $k \geq 1$ ,  $\delta \neq 0$  and  $u_1, u_2 \in \mathcal{D}(\mathcal{L})$  such that

$$\begin{cases} \mathcal{L}u_1 - \lambda_k u_1 = a_{21}u_2 & \text{in } \Omega, \\ \mathcal{L}u_2 - \lambda_k u_2 = \delta a_{32}\phi_k & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

Using Theorem 2.2 this is equivalent to the existence of  $k \geq 1$ ,  $\delta \neq 0$  and  $u_2 \in \mathcal{D}(\mathcal{L})$  such that

$$\begin{cases} \mathcal{L}u_2 - \lambda_k u_2 = \delta a_{32}\phi_k & \text{in } \Omega, \\ a_{21}u_2 = 0 & \text{in } \omega, \\ \mathcal{M}_k(a_{21}u_2, \omega) = 0. \end{cases}$$

Finally, by definition of  $\mathcal{O}_2$  (see (32)), we have  $a_{21}u_2 = 0$  almost everywhere in  $\omega$  if and only if  $u_2 = 0$  almost everywhere in  $\mathcal{O}_2 \cap \omega$ . This proves the proposition with  $v = u_2/\delta$ .  $\square$

We turn out to the proof of Theorem 4.1.

*Proof of Theorem 4.1.* We use the characterization of Proposition 4.2.

1. Assume first that  $\mathcal{O}_2 \cap \omega \neq \emptyset$ . Note that this condition automatically implies the approximate controllability of the  $2 \times 2$  subsystem (22) by Theorem 3.3.
  - (a) Looking at the first two equations of (35) and using Theorem 2.1 with  $u = (v, \phi_k)^*$ , it is not difficult to see that, if System (31) is not approximately controllable, then the  $2 \times 2$  system (33) is not approximately controllable either.
  - (b) When  $\mathcal{O}_2 \subset \omega$ , the third condition  $\mathcal{M}_k(a_{21}v, \omega) = 0$  of (35) is always fulfilled since, in the one hand  $a_{21} = 0$  almost everywhere in  $\Omega \setminus \mathcal{O}_2$  (by (32)) and in the other hand  $v = 0$  almost everywhere in  $\mathcal{O}_2 \cap \omega = \mathcal{O}_2$ . It follows from Theorem 2.1 that System (31) is approximately controllable if and only if so is the  $2 \times 2$  system (33).
2. Assume now that  $\mathcal{O}_2 \cap \omega = \emptyset$ . Observe, in particular, that the second equation of (35) is now empty.
  - (a) The orthogonality condition  $\int_0^1 a_{32} \phi_k^2 dx = 0$  is necessary for the existence of a solution to the first equation of (35). Thus, System (31) is approximately controllable if this latter one fails.
  - (b) We assume that, for some  $k \geq 1$ , we have

$$\int_0^1 a_{32}(\phi_k)^2 dx = 0. \quad (36)$$

We are going to show that there is a solution of (35). By Proposition 4.2 this will prove that System (31) is not approximately controllable.

From (36), we deduce that the first equation in (35) admits an infinite number of solutions of the form  $v = v_0 + \alpha \phi_k$ ,  $\alpha \in \mathbb{R}$ , where  $v_0 \in \mathcal{D}(\mathcal{L})$  is the unique solution of this equation that satisfies  $\langle v_0, \phi_k \rangle_{L^2(\Omega)} = 0$ . Let  $C$  be the connected component of  $\overline{\Omega \setminus \omega}$  that contains  $\mathcal{O}_2$ . Since by assumption  $C$  touches the boundary of  $\Omega$ , and by (10), we have  $\mathcal{M}_k(a_{21}v, \omega) = 0$  if and only if  $\int_C a_{21}v \phi_k dx = 0$ . It remains to prove that we can choose  $\alpha$  such that

$$0 = \int_C a_{21}v \phi_k dx = \int_C a_{21}v_0 \phi_k dx + \alpha \int_C a_{21}(\phi_k)^2 dx.$$

In particular, it is enough to prove that  $\int_C a_{21}(\phi_k)^2 dx \neq 0$ . By assumption the  $2 \times 2$  subsystem (22) is approximately controllable, and  $a_{21} = 0$  almost everywhere in  $\omega \subset \Omega \setminus \mathcal{O}_2$ . Thus,  $\mathcal{M}_k(a_{21}\phi_k, \omega) \neq 0$  (by Theorem 3.2), so that  $\int_C a_{21}(\phi_k)^2 dx \neq 0$  (same reasoning as above), and the claim is proved.  $\square$

**4.2. Applications.** Let us consider some basic examples of applications of this result.

- Assumption (34) is for example fulfilled if  $a_{32}$  has a constant sign on  $\Omega$  and is not identically zero. Combining this result with the discussion in Section 3.3, we deduce that our  $3 \times 3$  system is approximately controllable if  $a_{32}$  and  $a_{21}$  both have constant signs (not necessarily the same sign) on  $\Omega$ , and are non-identically zero. This situation is illustrated numerically in [9, Sect 4.4.2].
- However, observe that the sign condition for  $a_{32}$  is not necessary for the previous corollaries to apply. For instance, as we have seen in the item #1 of

Section 3.3, (34) also holds for any  $k \geq 1$  in the case where  $\mathcal{L} = -\partial_x^2$  and  $a_{32} = (x - \alpha)1_{(1/4, 3/4)}(x)$  for  $\alpha \neq 1/2$ .

- Finally, consider the case where  $\mathcal{L} = -\partial_x^2$ ,  $\omega = (1/2, 1)$ ,  $a_{21}(x) = 1_{(0, 1/2)}(x)$  and  $a_{32}(x) = x - 1/2$ . Here the coupling domain  $\mathcal{O}_2 = (0, 1/2)$  and  $\omega$  are two disjoint intervals. Since a straightforward computation shows that (34) fails for any  $k \geq 1$ , we can apply Theorem 4.1 and see that the system is not approximately controllable. This result is also numerically illustrated in [9, Section 4.4.2].

Observe that the subsystem satisfied by  $(y_1, y_2)$  is approximately controllable. The lack of controllability is thus a consequence of the structure of the coupling term  $a_{32}$  between the second and third components. It is worth precising that  $a_{32}$  is however supported almost-everywhere in this example.

**5. Simultaneous controllability of uncoupled systems.** In this section we still study systems of the general form (1) but in a slightly different framework compared to the previous sections.

Since we are mainly going to deal with examples, we restrict ourselves to the case  $n = 2$  for simplicity. We assume here that  $B = (b_1, b_2)^*$  is any vector in  $\mathbb{R}^2$ , that the coupling terms satisfy  $A(x) = 0$  for any  $x \in \Omega$  and that the (diagonal) operator  $\mathcal{L}$  is given by

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix}, \quad (37)$$

where  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are two possibly different elliptic operators. Hence, the system we are interested in writes

$$\begin{cases} \partial_t y + \mathcal{L}y = 1_\omega Bv & \text{in } (0, T) \times \Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \quad (38)$$

We assume that  $b_1 \neq 0$  and  $b_2 \neq 0$ , because if it is not the case, the controllability of (38) clearly fails. Observe that the controllability also fails if  $\mathcal{L}_1 = \mathcal{L}_2$  since in such case the linear combination  $z = b_2 y_1 - b_1 y_2$  solves the scalar equation  $\partial_t z + \mathcal{L}z = 0$  which does not depend anymore on the control  $v$ .

**5.1. Controllability conditions.** In the case where the operators  $\mathcal{L}_i$  are different but multiples of the same operator  $\mathcal{L}$  the following null-controllability result was proved in [5, Remark 1.1].

**Theorem 5.1.** *Let  $\mathcal{L}$  be an elliptic operator as defined in the introduction (2) and  $\omega$  a non-empty open subset in  $\Omega$ . For  $i = 1, 2$ , we set  $\mathcal{L}_i = d_i \mathcal{L}$  for some  $d_i > 0$ , we define  $\mathcal{L}$  by (37) and we suppose given  $B = (b_1, b_2)^*$  with  $b_1 \neq 0$  and  $b_2 \neq 0$ . Then,*

$$(38) \text{ is null-controllable} \iff d_1 \neq d_2.$$

We are interested here in studying some examples where the operators  $\mathcal{L}_i$  are different but not proportional to a given elliptic operator; this appears to be a more intricate problem. The strategy is still based on the unique continuation criterion given by Fattorini and is therefore restricted to the approximate controllability property.

We will assume that  $\mathcal{L}_1 = -\partial_x^2$  and that  $\mathcal{L}_2 = -\partial_x(\gamma(x)\partial_x \cdot)$  for some  $\gamma \in L^\infty(\Omega)$  and  $\inf_\Omega \gamma > 0$ .

In this framework, Theorem 2.1 says that the system is approximately controllable if and only if, for any  $s \in \mathbb{C}$  we have

$$\left. \begin{array}{l} \mathcal{L}_1 u_1 = s u_1 \quad \text{in } \Omega \\ \mathcal{L}_2 u_2 = s u_2 \quad \text{in } \Omega \\ b_1 u_1 + b_2 u_2 = 0 \quad \text{in } \omega \end{array} \right\} \implies u_1 = u_2 = 0, \quad \forall u_1 \in \mathcal{D}(\mathcal{L}_1), u_2 \in \mathcal{D}(\mathcal{L}_2).$$

However, since  $b_i \neq 0$ , this condition is equivalent to

$$\left. \begin{array}{l} \mathcal{L}_1 u_1 = s u_1 \quad \text{in } \Omega \\ \mathcal{L}_2 u_2 = s u_2 \quad \text{in } \Omega \\ u_1 = u_2 \quad \text{in } \omega \end{array} \right\} \implies u_1 = u_2 = 0, \quad \forall u_1 \in \mathcal{D}(\mathcal{L}_1), u_2 \in \mathcal{D}(\mathcal{L}_2).$$

Of course, if  $\mathcal{L}_1$  and  $\mathcal{L}_2$  have no common eigenvalues then this condition is automatically satisfied and the system is approximately controllable. If  $\mathcal{L}_1$  and  $\mathcal{L}_2$  have a common eigenvalue, we have to analyze if the corresponding eigenfunctions can coincide on the control domain  $\omega$ .

**5.2. Examples.** Let us look more precisely at two different examples.

5.2.1. *Example 1: the diffusion coefficients coincide in the control domain.* More precisely, we set  $\omega = (0, 1/2)$  and we assume that  $\gamma$  is piecewise constant

$$\gamma(x) = \begin{cases} 1, & \text{for } x \in (0, 1/2) = \omega, \\ \gamma_2, & \text{for } x \in (1/2, 1), \end{cases}$$

with  $\gamma_2 > 0$ .

Let  $\lambda_k = k^2 \pi^2$ ,  $k \geq 1$  be an eigenvalue of  $\mathcal{L}_1 = -\partial_x^2$  and  $u_1(x) = \sin(k\pi x)$  be the associated eigenfunction. An eigenfunction  $u_2$  of  $\mathcal{L}_2$  for the same eigenvalue and which coincides with  $u_1$  in  $\omega$  has necessarily the following form

$$u_2(x) = \begin{cases} \sin(k\pi x), & \text{for } x \in (0, 1/2) \\ \delta \sin\left(\frac{k\pi}{\sqrt{\gamma_2}}(x-1)\right), & \text{for } x \in (1/2, 1), \end{cases}$$

and,  $\delta$  should be determined in order to satisfy the following transmission conditions at  $x = 1/2$

$$\begin{cases} -\delta \sin\left(\frac{k\pi}{2\sqrt{\gamma_2}}\right) = \sin\left(\frac{k\pi}{2}\right), \\ \frac{\delta}{\sqrt{\gamma_2}} \cos\left(\frac{k\pi}{2\sqrt{\gamma_2}}\right) = \cos\left(\frac{k\pi}{2}\right). \end{cases}$$

- If  $k = 2p$  is even, the existence of a  $\delta$  satisfying those equations is equivalent to

$$\sqrt{\gamma_2} = \frac{p}{q}, \quad \text{for some } q \in \mathbb{N}^*.$$

- If  $k = 2p + 1$  is odd, the existence of  $\delta$  is equivalent to

$$\sqrt{\gamma_2} = \frac{2p+1}{2q+1}, \quad \text{for some } q \in \mathbb{N}^*.$$

The conclusion of this study is that, the system is approximately controllable if and only if  $\sqrt{\gamma_2} \notin \mathbb{Q}$ .



5.2.2. *Example 2: the diffusion coefficients do not coincide in the control domain.*

The non-controllability situations that we underlined in Example 1 seem to be the consequence of the fact that the diffusion coefficients of the two operators  $\mathcal{L}_1$  and  $\mathcal{L}_2$  coincide in the control domain  $\omega$ . However, we want to show here that we can construct an example of a non-controllable system of the same kind even if the diffusion coefficients are completely different for the two operators.

We first choose  $0 < \alpha < 1/4$ , and the control domain  $\omega = (0, \alpha)$ . We set

$$\beta = \frac{\sin(2\pi\alpha)}{\sin(\pi\alpha)}, \quad \bar{\beta} = \frac{\sin(\pi\alpha) \cos(2\pi\alpha)}{2 \sin(2\pi\alpha)} - \cos(\pi\alpha).$$

We consider now the following definition of the diffusion coefficient that defines the operator  $\mathcal{L}_2$

$$\gamma(x) = \begin{cases} 1 + \frac{\bar{\beta}}{\cos(\pi x)}, & \text{for } x \in (0, \alpha) = \omega, \\ \frac{1}{4}, & \text{for } x \in (\alpha, 1). \end{cases}$$

Observe that, even if  $\bar{\beta} < 0$ , we still have  $\inf \gamma > 0$ . A straightforward computation shows that the function  $u_2$  defined by

$$u_2(x) = \begin{cases} \sin(\pi x), & \text{for } x \in (0, \alpha) = \omega, \\ \beta \sin(2\pi x), & \text{for } x \in (\alpha, 1), \end{cases}$$

is an eigenfunction of  $\mathcal{L}_2$  associated with the eigenvalue  $\pi^2$  which obviously coincides with  $\sin(\pi x)$  on  $\omega$ .

As a consequence, with this particular choice of the diffusion coefficient, the parabolic system under study is not approximately controllable.

**6. Conclusion and perspectives.** In this paper, we have given some easily checkable necessary and sufficient conditions for the approximate controllability of some 1D coupled parabolic systems with space-dependent coefficients. These conditions have been illustrated on many simple examples to show that the controllability issue for those systems can be an intricate problem depending on the geometry of the control domain and of the characteristics of the coupling terms in the system.

In particular, we explicitly described some one-parameter families of systems that are approximately controllable if and only if the parameter is not a rational number. Observe that the study of the null-controllability of such systems is completely open up to now. Actually, non-standard behaviors (in the parabolic framework) may be expected for the values of the parameters that give the approximate controllability. It is for instance possible that those systems are approximately controllable but not null-controllable, or that the null-controllability only holds for a large enough control time  $T$ . These kind of behaviors have been recently established in [18, 7] in the framework of the boundary control of parabolic systems. See also a recent review on this topic in [6].

Another point that we should explore is the link between the distributed control problem, that we studied in this paper, and the boundary control problem of parabolic systems. Even if it is known since [13] that there is no equivalence between these properties for systems with a few number of controls (in contrast with scalar equations), it seems that there exist however some relations between the two notions when the coupling domain does not meet the control domain. For instance, Theorem 3.3 should be connected with [22, Theorem 3.2].

Finally, we observe that some of our examples can be extended to simple Cartesian geometries but the study of the general multi-dimensional systems is far from

being straightforward and is still widely open. The main difficulty in higher dimensions is the lack of a result as simple as Theorem 2.2 to characterize the unique continuation property for non-homogeneous elliptic problems.

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