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Weak and strong singular solutions of semilinear fractional elliptic equations

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Abstract

If $p \in (0, \frac{N}{N-2\alpha})$, $\alpha \in (0, 1)$, $k > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain containing 0 and δ_0 is the Dirac measure at 0, we prove that the weak solution of $(E)_k$ $(-\Delta)^\alpha u + u^p = k\delta_0$ in Ω which vanishes in Ω^c is a weak singular solution of $(E)_\infty$ $(-\Delta)^\alpha u + u^p = 0$ in $\Omega \setminus \{0\}$ with the same outer data. Furthermore, we study the limit of weak solutions of $(E)_k$ when $k \rightarrow \infty$. For $p \in (0, 1 + \frac{2\alpha}{N}]$, the limit is infinity in Ω . For $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, the limit is a strong singular solution of $(E)_\infty$.

Contents

1	Introduction	1
2	Preliminaries	5
3	Regularity	7
4	The limit of weak solutions	9
4.1	The case of $p \in (0, 1 + \frac{2\alpha}{N}]$	9
4.2	The case of $p \in (1 + \frac{2\alpha}{N}, p_\alpha^*)$	15
4.3	Proof of Theorem 1.2 (ii) and (iii)	19

Key words: Fractional Laplacian, Dirac measure, Isolated singularity, Weak solution, Weakly singular solution, Strongly singular solution.

MSC2010: 35R11, 35J75, 35R06

1 Introduction

Let Ω be an open bounded C^2 domain of \mathbb{R}^N ($N \geq 2$) containing 0, $\alpha \in (0, 1)$ and δ_0 denote the Dirac measure at 0. In this paper, we study the properties

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of the weak solution the problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \tag{1.1}$$

where $k > 0$ and $p \in (0, \frac{N}{N-2\alpha})$. The operator $(-\Delta)^\alpha$ is the fractional Laplacian defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(x),$$

where for $\epsilon > 0$,

$$(-\Delta)_\epsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

In 1980, Benilan and Brezis (see [2, 1]) proved that when for $1 < q < N/(N - 2)$, the following equation

$$\begin{aligned} -\Delta u + u^q &= k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

admits a unique solution u_k , while no solution exists when $q \geq N/(N - 2)$. Soon after, Brezis and Véron [3] proved that the problem

$$\begin{aligned} -\Delta u + u^q &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

admits only the zero solution when $q \geq N/(N-2)$. When $1 < q < N/(N-2)$, Véron obtained in [16] the description of the all the possible singular behaviour of positive solutions (1.3). In particular he proved that this behaviour is always isotropic (when $(N + 1)/(N - 1) \leq q < N/(N - 2)$ the assumption of positivity is unnecessary) and that two types of singular behaviour occur:

- (i) either $u(x) \sim c_N k |x|^{2-N}$ when $x \rightarrow 0$ and k can take any positive value; u is said to have a *weak singularity* at 0, and actually $u = u_k$.
- (ii) or $u(x) \sim c_{N,q} |x|^{-\frac{2}{q-1}}$ when $x \rightarrow 0$ and u has a *strong singularity* at 0, and $u = u_\infty := \lim_{k \rightarrow \infty} u_k$.

For another interesting contributions to singularity problems see Chen, Matano and Véron [9], Vazquez and Véron [13, 14] and Véron [17].

In a recent work, Chen and Véron [7] considered the fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (1.4)$$

where $1 + \frac{2\alpha}{N} < p < p_\alpha^* := \frac{N}{N-2\alpha}$. They proved that (1.4) admits a strong singular solution u_s such that

$$\lim_{x \rightarrow 0} u_s(x) |x|^{\frac{2\alpha}{p-1}} = c_0, \quad (1.5)$$

for some $c_0 > 0$. Moreover, the strong singular solution u_s of (1.4) is unique in the class of solutions satisfying

$$0 < \liminf_{x \rightarrow 0} u(x) |x|^{\frac{2\alpha}{p-1}} \leq \limsup_{x \rightarrow 0} u(x) |x|^{\frac{2\alpha}{p-1}} < \infty. \quad (1.6)$$

Here in the fractional framework, we call u as a weakly singular solution if $\limsup_{x \rightarrow 0} |u(x)| |x|^{N-2\alpha} < \infty$ and u as a strongly singular solution if $\lim_{x \rightarrow 0} |u(x)| |x|^{N-2\alpha} = \infty$.

More recently, Chen and Véron in [8] studied the existence and uniqueness of weak solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (1.7)$$

under the hypotheses that g is subcritical integrability and ν is a Radon measure. Here the weak solution is given as following.

Definition 1.1 *We say that u is a weak solution of (1.7), if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho^\alpha dx)$, where $\rho(x) := \text{dist}(x, \Omega^c)$, and*

$$\int_{\Omega} [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_{\Omega} \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \quad (1.8)$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

- (i) $\text{supp}(\xi) \subset \bar{\Omega}$,
- (ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \leq c_1$ for some $c_1 > 0$,
- (iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)_\epsilon^\alpha \xi| \leq \varphi$ a.e. in Ω , for all $\epsilon \in (0, \epsilon_0]$.

For problem (1.7), we consider the simplest case $g(s) = s^p$ with $0 < p < p_\alpha^*$ and $\nu = k\delta_0$. According to Theorem 1.1 in [8], problem (1.1) admits a unique weak solution u_k , moreover,

$$\mathbb{G}_\alpha[k\delta_0] - \mathbb{G}_\alpha[(\mathbb{G}_\alpha[k\delta_0])^p] \leq u_k \leq \mathbb{G}_\alpha[k\delta_0] \quad \text{in } \Omega, \quad (1.9)$$

where $\mathbb{G}_\alpha[\cdot]$ is the Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_{\Omega} G_\alpha(x, y) d\nu(y), \quad \forall \nu \in \mathfrak{M}(\Omega, \rho^\alpha), \quad (1.10)$$

with G_α is the Green kernel of $(-\Delta)^\alpha$ in Ω and $\mathfrak{M}(\Omega, \rho^\alpha)$ denotes the space of Radon measures in Ω such that $\int_{\Omega} \rho^\alpha d|\rho| < \infty$. By (1.9),

$$\lim_{x \rightarrow 0} u_k(x) |x|^{N-2\alpha} = c_{\alpha, N} k. \quad (1.11)$$

for some $c_{\alpha, N} > 0$. Moreover, from Theorem 1.1 in [8], we have that

$$u_k(x) \leq u_{k+1}(x), \quad x \in \Omega, \quad (1.12)$$

then the limit of $u_k(x)$ exists or $+\infty$, denoted by u_∞ , that is

$$u_\infty(x) = \lim_{k \rightarrow \infty} u_k(x), \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (1.13)$$

Throughout of this paper, we denote u_k is the weak solution of (1.1) and u_∞ is defined by (1.13).

Motivated by these results and in view of the nonlocal character of the fractional Laplacian we are interested in considering the connection between the solutions of (1.1) and the ones of (1.4). Being more precise, our main theorem states as follows.

Theorem 1.1 *Assume that $1 + \frac{2\alpha}{N} \geq \frac{2\alpha}{N-2\alpha}$ and $p \in (0, p_\alpha^*)$. Then u_k is a classical solution of (1.4). Furthermore,*

$$(i) \text{ if } p \in (0, 1 + \frac{2\alpha}{N}), \quad u_\infty(x) = \infty, \quad \forall x \in \Omega; \quad (1.14)$$

$$(ii) \text{ if } p \in (1 + \frac{2\alpha}{N}, p_\alpha^*), \quad u_\infty = u_s,$$

where u_s is the solution of (1.4) satisfying (1.5).

Moreover, if $1 + \frac{2\alpha}{N} = \frac{2\alpha}{N-2\alpha}$, (1.14) holds for $p = 1 + \frac{2\alpha}{N}$.

The result of part (i) indicates that even if the absorption is superlinear, the diffusion dominates and there is no strongly singular solution to problem (1.4). On the contrary, part (ii) points out that the absorption dominates the diffusion; the limit function u_s is the least strongly singular solution of (1.4). Comparing Theorem 1.1 with the results for Laplacian case, part (i) with $p \in (0, 1]$ and (ii) are similar as the Laplacian case, but part (i) with $p \in (1, 1 + \frac{2\alpha}{N}]$ is totally different from the one in the case $\alpha = 1$. This difference comes from the fact that the fractional Laplacian is a nonlocal operator, which requires the solution to belong to $L^1(\Omega)$.

At end, we consider the case where $1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$. It occurs when $N = 2$ and $\frac{\sqrt{5}-1}{2} < \alpha < 1$ or $N = 3$ and $\frac{3(\sqrt{5}-1)}{4} < \alpha < 1$. In this situation, we have the following results.

Theorem 1.2 Assume that $1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$ and $p \in (0, p_\alpha^*)$. Then u_k is a classical solution of (1.4). Furthermore,

(i) if $p \in (0, \frac{N}{2\alpha})$, then

$$u_\infty(x) = \infty, \quad \forall x \in \Omega;$$

(ii) if $p \in (1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, then u_∞ is a classical solution of (1.4) and there exist $\rho_0 > 0$ and $c_2 > 0$ such that

$$c_2|x|^{-\frac{(N-2\alpha)p}{p-1}} \leq u_\infty \leq u_s, \quad \forall x \in B_{\rho_0} \setminus \{0\}; \quad (1.15)$$

(iii) if $p = \frac{2\alpha}{N-2\alpha}$, then u_∞ is a classical solution of (1.4) and there exist $\rho_0 > 0$ and $c_3 > 0$ such that

$$c_3 \frac{|x|^{-\frac{(N-2\alpha)p}{p-1}}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}} \leq u_\infty \leq u_s, \quad \forall x \in B_{\rho_0} \setminus \{0\}; \quad (1.16)$$

(iv) if $p \in (\frac{2\alpha}{N-2\alpha}, p_\alpha^*)$, then

$$u_\infty = u_s.$$

In Theorem 1.2 parts (ii) and (iii), we do not obtain that $u_\infty = u_s$, since (1.15) and (1.16) do not provide sharp estimate on u_∞ in order it to belong to the uniqueness class characterized by ((1.6)).

The paper is organized as follows. In Section 2, we present some preliminaries such as some estimate for Green kernel and comparison principle. In Section 3, we prove that the weak solution of (1.1) is a classical solution of (1.4). Section 4 is devoted to analyze the limit of weakly singular solutions as $k \rightarrow \infty$ in our theorems.

2 Preliminaries

The purpose of this section is to recall some known results. We denote by $B_r(x)$ the ball centered at x with radius r and $B_r := B_r(0)$.

Lemma 2.1 Assume that $0 < p < p_\alpha^*$, then there exists $c_4 > 1$ such that

(i) if $p \in (0, \frac{2\alpha}{N-2\alpha})$,

$$\frac{1}{c_4} \leq \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \leq c_4 \quad \text{in } B_r \setminus \{0\};$$

(ii) if $p = \frac{2\alpha}{N-2\alpha}$,

$$-\frac{1}{c_5} \ln|x| \leq \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \leq -c_5 \ln|x| \quad \text{in } B_r \setminus \{0\};$$

(iii) if $p \in (\frac{2\alpha}{N-2\alpha}, p_\alpha^*)$,

$$\frac{1}{c_6}|x|^{2\alpha-(N-2\alpha)p} \leq \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \leq c_6|x|^{2\alpha-(N-2\alpha)p} \quad \text{in } B_r \setminus \{0\},$$

where $r = \frac{1}{4} \min\{1, \text{dist}(0, \partial\Omega)\}$ and \mathbb{G}_α is defined by (1.9).

Proof. The proof refers to Lemma 5.3 in [6]. □

Theorem 2.1 *Assume that O is an open and bounded domain of \mathbb{R}^N and u_1, u_2 are continuous in \bar{O} and satisfy*

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in } O.$$

Moreover, we assume that $u_1 \geq u_2$ in O^c . Then

$$u_1 > u_2 \quad \text{in } O$$

or

$$u_1 \equiv u_2 \quad \text{a.e. in } \mathbb{R}^N.$$

Proof. The proof refers to Theorem 2.2 in [5] (see also Theorem 5.2 in [4]). □

The following stability result is proved in [5, Th 2.2]

Theorem 2.2 *Suppose that \mathcal{O} is an open, bounded and C^2 domain and $h : \mathbb{R} \rightarrow \mathbb{R}$. Assume that $(u_n)_n, n \in \mathbb{N}$ is a sequence of functions, uniformly bounded in $L^1(\mathcal{O}^c, \frac{dy}{1+|y|^{N+2\alpha}})$ and f_n, f are continuous in \mathcal{O} and that there holds*

$$(-\Delta)^\alpha u_n + h(u_n) \geq f_n \quad (\text{resp } (-\Delta)^\alpha u_n + h(u_n) \leq f_n) \quad \text{in } \mathcal{O}$$

in the viscosity sense and

- (i) $u_n \rightarrow u$ locally uniformly in \mathcal{O} ,
- (ii) $u_n \rightarrow u$ in $L^1(\mathbb{R}^N, \frac{dy}{1+|y|^{N+2\alpha}})$,
- (iii) $f_n \rightarrow f$ locally uniformly in \mathcal{O} .

Then

$$(-\Delta)^\alpha u + h(u) \geq f \quad (\text{resp } (-\Delta)^\alpha u + h(u) \leq f) \quad \text{in } \mathcal{O}$$

in the viscosity sense.

3 Regularity

In this section, we prove that a weak solution of (1.1) is a classical solution of (1.4). To this end, we introduce some auxiliary lemma.

Lemma 3.1 *Assume that $w \in C^{2\alpha+\epsilon}(\bar{B}_1)$ with $\epsilon > 0$ satisfies*

$$(-\Delta)^\alpha w = h \quad \text{in } B_1,$$

where $h \in C^1(\bar{B}_1)$. Then for $\beta \in (0, 2\alpha)$, there exists $c_7 > 0$ such that

$$\|w\|_{C^\beta(\bar{B}_{1/4})} \leq c_7(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|(1 + |\cdot|)^{-N-2\alpha}w\|_{L^1(\mathbb{R}^N)}). \quad (3.1)$$

Proof. Let $\eta : \mathbb{R}^N \rightarrow [0, 1]$ be a C^∞ function such that

$$\eta = 1 \quad \text{in } B_{\frac{3}{4}} \quad \text{and} \quad \eta = 0 \quad \text{in } B_1^c.$$

We denote $v = w\eta$, then $v \in C^{2\alpha+\epsilon}(\mathbb{R}^N)$ and for $x \in B_{\frac{1}{2}}$, $\epsilon \in (0, \frac{1}{4})$,

$$\begin{aligned} (-\Delta)_\epsilon^\alpha v(x) &= - \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{v(x+y) - v(x)}{|y|^{N+2\alpha}} dy \\ &= (-\Delta)_\epsilon^\alpha w(x) + \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{[1 - \eta(x+y)]w(x+y)}{|y|^{N+2\alpha}} dy. \end{aligned}$$

Together with the fact of $\eta(x+y) = 1$ for $y \in B_\epsilon$, we have

$$\int_{\mathbb{R}^N \setminus B_\epsilon} \frac{[1 - \eta(x+y)]w(x+y)}{|y|^{N+2\alpha}} dy = \int_{\mathbb{R}^N} \frac{[1 - \eta(x+y)]w(x+y)}{|y|^{N+2\alpha}} dy =: h_1(x),$$

thus,

$$(-\Delta)^\alpha v = h + h_1 \quad \text{in } B_{\frac{1}{2}}.$$

For $x \in B_{\frac{1}{2}}$ and $z \in \mathbb{R}^N \setminus B_{\frac{3}{4}}$, we have

$$|z - x| \geq |z| - |x| \geq |z| - \frac{1}{2} \geq \frac{1}{16}(1 + |z|)$$

and then

$$\begin{aligned} |h_1(x)| &= \left| \int_{\mathbb{R}^N} \frac{[1 - \eta(z)]w(z)}{|z - x|^{N+2\alpha}} dz \right| \leq \int_{\mathbb{R}^N \setminus B_{\frac{3}{4}}} \frac{|w(z)|}{|z - x|^{N+2\alpha}} dz \\ &\leq 16^{N+2\alpha} \int_{\mathbb{R}^N} \frac{|w(z)|}{(1 + |z|)^{N+2\alpha}} dz \\ &= 16^{N+2\alpha} \|(1 + |\cdot|)^{-N-2\alpha}w\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

By [12, Proposition 2.1.9], for $\beta \in (0, 2\alpha)$, there exists $c_8 > 0$ such that

$$\begin{aligned} \|v\|_{C^\beta(\bar{B}_{1/4})} &\leq c_8(\|v\|_{L^\infty(\mathbb{R}^N)} + \|h + h_1\|_{L^\infty(B_{1/2})}) \\ &\leq c_8(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|h_1\|_{L^\infty(B_{1/2})}) \\ &\leq c_9(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|(1 + |\cdot|)^{-N-2\alpha}w\|_{L^1(\mathbb{R}^N)}), \end{aligned}$$

where $c_9 = 16^{N+2\alpha}c_8$. Combining with $w = v$ in $B_{\frac{3}{4}}$, we obtain (3.1). \square

Theorem 3.1 *Let $\alpha \in (0,1)$ and $0 < p < p_\alpha^*$, then the weak solution of (1.1) is a classical solution of (1.4).*

Proof. Let u_k be the weak solution of (1.1). According to [8, Theorem 1.1], we have

$$0 \leq u_k = \mathbb{G}_\alpha[k\delta_0] - \mathbb{G}_\alpha[u_k^p] \leq \mathbb{G}_\alpha[k\delta_0]. \quad (3.2)$$

We observe that $\mathbb{G}_\alpha[k\delta_0] = k\mathbb{G}_\alpha[\delta_0] = kG_\alpha(\cdot, 0)$ is $C_{loc}^2(\Omega \setminus \{0\})$. Denote by O an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ with $r > 0$. Then $\mathbb{G}_\alpha[k\delta_0]$ is uniformly bounded in $\Omega \setminus B_{r/2}$, so is u_k^p by (3.2).

Let $\{g_n\}$ be a sequence nonnegative functions in $C_0^\infty(\mathbb{R}^N)$ such that $g_n \rightarrow \delta_0$ in the weak sense of measures and let w_n be the solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= kg_n & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (3.3)$$

From [8], we obtain that

$$u_k = \lim_{n \rightarrow \infty} w_n \quad \text{a.e. in } \Omega. \quad (3.4)$$

We observe that $0 \leq w_n = \mathbb{G}_\alpha[kg_n] - \mathbb{G}_\alpha[w_n^p] \leq k\mathbb{G}_\alpha[g_n]$, $\mathbb{G}_\alpha[g_n]$ converges to $\mathbb{G}_\alpha[\delta_0]$ uniformly in any compact set of $\Omega \setminus \{0\}$ and $\mathbb{G}_\alpha[g_n]$ converges to $\mathbb{G}_\alpha[\delta_0]$ in $L^1(\Omega)$; then there exists $c_{10} > 0$ independent of n such that

$$\|w_n\|_{L^\infty(\Omega \setminus B_{r/2})} \leq c_{10}k \quad \text{and} \quad \|w_n\|_{L^1(\Omega)} \leq c_{10}k.$$

By [11, Corollary 2.4] and Lemma 3.1, for some $\epsilon > 0$ and $\beta \in (0, 2\alpha)$, there exist constants $c_{11}, c_{12}, c_{13} > 0$ independent of n and k , such that

$$\begin{aligned} \|w_n\|_{C^{2\alpha+\epsilon}(O)} &\leq c_{11}(\|w_n\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})}^p + \|kg_n\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} + \|w_n\|_{C^\beta(\Omega \setminus B_{\frac{3r}{4}})}) \\ &\leq c_{12}(\|w_n\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})}^p + \|w_n\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} + \|kg_n\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} + \|w_n\|_{L^1(\Omega)}) \\ &\leq c_{13}(k + k^p). \end{aligned}$$

Therefore, together with (3.4) and Arzela-Ascoli Theorem, we have that $u_k \in C^{2\alpha+\frac{\epsilon}{2}}(O)$, which implies that u_k is $C^{2\alpha+\frac{\epsilon}{2}}$ locally in $\Omega \setminus \{0\}$. Therefore, $w_n \rightarrow u_k$ and $g_n \rightarrow 0$ uniformly in any compact subset of $\Omega \setminus \{0\}$ as $n \rightarrow \infty$. We conclude that u_k is a classical solution of (1.4) by Theorem 2.2. \square

Corollary 3.1 *Let u_k be the weak solution of (1.1) and O be an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ with $r > 0$. Then for some $\epsilon > 0$, there exists $c_{14} > 0$ independent of k such that*

$$\|u_k\|_{C^{2\alpha+\epsilon}(O)} \leq c_{14}(\|u_k\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})}^p + \|u_k\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} + \|u_k\|_{L^1(\Omega)}). \quad (3.5)$$

Proof. By Theorem 3.1, u_k is a solution of (1.4). By [11, Corollary 2.4] and Lemma 3.1, for some $\epsilon > 0$ and $\beta \in (0, 2\alpha)$, there exist $c_{15}, c_{16} > 0$, independent of k , such that

$$\begin{aligned} \|u_k\|_{C^{2\alpha+\epsilon}(O)} &\leq c_{15}(\|u_k\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})}^p + \|u_k\|_{C^\beta(\Omega \setminus B_{\frac{3r}{4}})}) \\ &\leq c_{16}\|u_k\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})}^p + \|u_k\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} + \|u_k\|_{L^1(\Omega)}. \end{aligned}$$

□

Theorem 3.2 *Assume that the weak solution u_k of (1.1) satisfy*

$$\|u_k\| \leq c_{17} \quad (3.6)$$

for some $c_{17} > 0$ independent of k and that for $r \in (0, \text{dist}(0, \partial\Omega))$, there exists $c_{18} > 0$ independent of k such that

$$\|u_k\|_{L^\infty(\Omega \setminus B_{\frac{r}{2}})} \leq c_{18}. \quad (3.7)$$

Then u_∞ is a classical solution of (1.4).

Proof. Let O be an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ for $0 < r < \text{dist}(0, \partial\Omega)$. By (3.5), (3.6) and (3.7), for some $\epsilon > 0$ there exists $c_{19} > 0$ independent of k such that

$$\|u_k\|_{C^{2\alpha+\epsilon}(O)} \leq c_{19}.$$

Together with (1.13) and Arzela-Ascoli Theorem, we have that $u_\infty \in C^{2\alpha+\frac{\epsilon}{2}}(O)$, which implies that u_∞ is $C^{2\alpha+\frac{\epsilon}{2}}$ locally in $\Omega \setminus \{0\}$. Therefore, $w_n \rightarrow u_k$ and $g_n \rightarrow 0$ uniformly in any compact set of $\Omega \setminus \{0\}$ as $n \rightarrow \infty$. We apply Theorem 2.2 and conclude that u_∞ is a classical solution of (1.4). □

4 The limit of weak solutions

We recall that u_k denotes the weak solution of (1.1) and $d = \min\{1, \text{dist}(0, \partial\Omega)\}$.

4.1 The case of $p \in (0, 1 + \frac{2\alpha}{N}]$

Proposition 4.1 *Let $p \in (0, 1]$, then $\lim_{k \rightarrow \infty} u_k(x) = \infty$ for $x \in \Omega$.*

Proof. We observe that $\mathbb{G}_\alpha[\delta_0], \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] > 0$ in Ω . By (1.9), for $p \in (0, 1)$ and $x \in \Omega$ we have

$$\begin{aligned} u_k &\geq k\mathbb{G}_\alpha[\delta_0] - k^p\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \\ &\rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

For $p = 1$, it is obvious that $u_k = ku_1$ and $u_1 > 0$ in Ω , then

$$\lim_{k \rightarrow \infty} u_k = \infty \quad \text{in } \Omega.$$

The proof is complete. \square

Now we consider the case of $p \in (1, 1 + \frac{2\alpha}{N}]$. Let $\{r_k\} \subset (0, \frac{d}{2}]$ be a strictly decreasing sequence of numbers satisfying $\lim_{k \rightarrow \infty} r_k = 0$. Denote by $\{z_k\}$ a sequence of functions as

$$z_k(x) = \begin{cases} -d^{-N}, & x \in B_{r_k}, \\ |x|^{-N} - d^{-N}, & x \in B_{r_k}^c. \end{cases} \quad (4.1)$$

Lemma 4.1 *Let $\{\rho_k\}$ be a strictly decreasing sequence of numbers such that $\frac{r_k}{\rho_k} < \frac{1}{2}$ and $\lim_{k \rightarrow \infty} \frac{r_k}{\rho_k} = 0$. Then*

$$(-\Delta)^\alpha z_k(x) \leq -c_{1,k} |x|^{-N-2\alpha}, \quad x \in B_{\rho_k}^c$$

where $c_{1,k} = -c_{20} \log(\frac{r_k}{\rho_k})$ with $c_{20} > 0$ independent of k .

Proof. For any $x \in B_{\rho_k}^c$, we have

$$\begin{aligned} (-\Delta)^\alpha z_k(x) &= -\frac{1}{2} \int_{\mathbb{R}^N} \frac{z_k(x+y) + z_k(x-y) - 2z_k(x)}{|x|^{N+2\alpha}} dy \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \frac{|x+y|^{-N} \chi_{B_{r_k}^c(-x)}(y) + |x-y|^{-N} \chi_{B_{r_k}^c(x)}(y) - 2|x|^{-N}}{|y|^{N+2\alpha}} dy \\ &= -\frac{1}{2} |x|^{-N-2\alpha} \int_{\mathbb{R}^N} \frac{\delta(x, z, r_k)}{|z|^{N+2\alpha}} dz, \end{aligned}$$

where $\delta(x, z, r_k) = |z + e_x|^{-N} \chi_{B_{\frac{r_k}{|x|}}(-e_x)}(z) + |z - e_x|^{-N} \chi_{B_{\frac{r_k}{|x|}}(e_x)}(z) - 2$ and $e_x = \frac{x}{|x|}$.

We observe that $\frac{r_k}{|x|} \leq \frac{r_k}{\rho_k} < \frac{1}{2}$ and $|z \pm e_x| \geq 1 - |z| \geq \frac{1}{2}$ for $z \in B_{\frac{1}{2}}$. Then there exists $c_{21} > 0$ such that

$$|\delta(x, z, r_k)| = ||z + e_x|^{-N} + |z - e_x|^{-N} - 2| \leq c_{21} |z|^2.$$

Thus,

$$\begin{aligned} \left| \int_{B_{\frac{1}{2}}(0)} \frac{\delta(x, z, r_k)}{|z|^{N+2\alpha}} dz \right| &\leq \int_{B_{\frac{1}{2}}(0)} \frac{|\delta(x, z, r_k)|}{|z|^{N+2\alpha}} dz \\ &\leq c_{21} \int_{B_{\frac{1}{2}}(0)} |z|^{2-N-2\alpha} dz \leq c_{22}, \end{aligned}$$

where $c_{22} > 0$ is independent of k .

For $z \in B_{\frac{1}{2}}(-e_x)$, we have

$$\begin{aligned} \int_{B_{\frac{1}{2}}(-e_x)} \frac{\delta(x, z, r_k)}{|z|^{N+2\alpha}} dz &\geq \int_{B_{\frac{1}{2}}^c(-e_x)} \frac{|z + e_x|^{-N} \chi_{B_{\frac{r_k}{|x|}}(-e_x)}(z) - 2}{|z|^{N+2\alpha}} dz \\ &\geq c_{23} \int_{B_{\frac{1}{2}}(0) \setminus B_{\frac{r_k}{|x|}}(0)} (|z|^{-N} - 2) dz \\ &\geq -c_{24} \log\left(\frac{r_k}{|x|}\right) \geq -c_{24} \log\left(\frac{r_k}{\rho_k}\right), \end{aligned}$$

where $c_{23}, c_{24} > 0$ are independent of k .

For $z \in B_{\frac{1}{2}}(e_x)$, we have

$$\int_{B_{\frac{1}{2}}(e_x)} \frac{\delta(x, z, r_k)}{|z|^{N+2\alpha}} dz = \int_{B_{\frac{1}{2}}(-e_x)} \frac{\delta(x, z, r_k)}{|z|^{N+2\alpha}} dz.$$

For $z \in O := \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(-e_x) \cup B_{\frac{1}{2}}(e_x))$, we have

$$\left| \int_O \frac{\delta(x, z, r_k)}{|z|^{N+2\alpha}} dz \right| \leq c_{25} \int_{B_{\frac{1}{2}}^c(0)} \frac{|z|^{-N} + 1}{|z|^{N+2\alpha}} dz \leq c_{26},$$

where $c_{25}, c_{26} > 0$ are independent of k . Therefore, there exists $c_{20} > 0$ independent of k such that

$$(-\Delta)^\alpha z_k(x) |x|^{N+2\alpha} \leq c_{20} \log\left(\frac{r_k}{\rho_k}\right) := c_{1,k},$$

which ends the proof. \square

Proposition 4.2 *Assume that*

$$\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}, \quad \max\left\{1, \frac{2\alpha}{N-2\alpha}\right\} < p < 1 + \frac{2\alpha}{N} \quad (4.2)$$

and z_k is defined by (4.1) with $r_k = k^{-\frac{p-1}{N-(N-2\alpha)p}} (\log k)^{-2}$. Then there exists $k_0 > 0$ such that for any $k \geq k_0$

$$u_k \geq c_{2,k}^{\frac{1}{p-1}} z_k \quad \text{in } B_d, \quad (4.3)$$

where $c_{2,k} = \ln \ln k$.

Proof. For $p \in (\max\{1, \frac{2\alpha}{N-2\alpha}\}, 1 + \frac{2\alpha}{N})$, it follows by (1.9) and Lemma 2.1(iii) that there exist $\rho_0 \in (0, d)$ and $c_{27}, c_{28} > 0$ independent of k such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$,

$$\begin{aligned} u_k(x) &\geq k \mathbb{G}_\alpha[\delta_0](x) - k^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) \\ &\geq c_{27} k |x|^{-N+2\alpha} - c_{28} k^p |x|^{-(N-2\alpha)p+2\alpha} \\ &= c_{27} k |x|^{-N+2\alpha} \left(1 - \frac{c_{28}}{c_{27}} k^{p-1} |x|^{N-(N-2\alpha)p}\right). \end{aligned}$$

We choose

$$\rho_k = k^{-\frac{p-1}{N-(N-2\alpha)p}} (\log k)^{-1}. \quad (4.4)$$

There exists $k_1 > 1$ such that for $k \geq k_1$

$$\begin{aligned} u_k(x) &\geq c_{27} k |x|^{-N+2\alpha} \left(1 - \frac{c_{28}}{c_{27}} k^{p-1} \rho_k^{N-(N-2\alpha)p}\right) \\ &\geq \frac{c_{27}}{2} k |x|^{-N+2\alpha}, \quad x \in \bar{B}_{\rho_k} \setminus \{0\}. \end{aligned} \quad (4.5)$$

Since $p < 1 + \frac{2\alpha}{N}$, $1 - \frac{2\alpha(p-1)}{N-(N-2\alpha)p} > 0$ and there exists $k_0 \geq k_1$ such that

$$\frac{c_{27}}{2} k r_k^{2\alpha} \geq (\ln \ln k)^{\frac{1}{p-1}}, \quad (4.6)$$

for $k \geq k_0$. This implies

$$\frac{c_{27}}{2} k |x|^{2\alpha} \geq (\ln \ln k)^{\frac{1}{p-1}}, \quad x \in \bar{B}_{\rho_k} \setminus B_{r_k}.$$

Together with (4.1) and (4.5), we have

$$u_k(x) \geq (\ln \ln k)^{\frac{1}{p-1}} z_k(x), \quad x \in \bar{B}_{\rho_k} \setminus B_{r_k},$$

for $k \geq k_0$. It is obvious that for $x \in B_{r_k}$ or $x \in B_d^c$,

$$(\ln \ln k)^{\frac{1}{p-1}} z_k(x) \leq 0 \leq u_k(x).$$

Set $c_{2,k} = \ln \ln k$, by Lemma 4.1, then for $x \in B_d \setminus B_{\rho_k}$,

$$(-\Delta)^\alpha c_{2,k}^{\frac{1}{p-1}} z_k(x) + c_{2,k}^{\frac{p}{p-1}} z_k(x)^p \leq c_{2,k}^{\frac{p}{p-1}} |x|^{-N-2\alpha} (-1 + |x|^{N+2\alpha-Np}) \leq 0,$$

since $N + 2\alpha - Np \geq 0$ and $d \leq 1$. By Theorem 2.1, we derive

$$c_{2,k}^{\frac{1}{p-1}} z_k(x) \leq u_k(x), \quad x \in \bar{B}_d,$$

which ends the proof. \square

Proposition 4.3 *Assume that*

$$1 < \frac{2\alpha}{N-2\alpha} \leq 1 + \frac{2\alpha}{N}, \quad p = \frac{2\alpha}{N-2\alpha} \quad (4.7)$$

and z_k is defined by (4.1) with $r_k = k^{-\frac{2\alpha}{N-(N-2\alpha)}} (\log k)^{-3}$ and $k > 2$. Then there exists $k_0 > 2$ such that (4.3) holds for $k \geq k_0$.

Proof. By (1.9) and Lemma 2.1(ii), there exist $\rho_0 \in (0, d)$ and $c_{30}, c_{31} > 0$ independent of k such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$

$$\begin{aligned} u_k(x) &\geq c_{30}k|x|^{-N+2\alpha} + c_{31}k^p \log|x| \\ &= c_{30}k|x|^{-N+2\alpha} \left(1 + \frac{c_{31}}{c_{30}}k^{p-1}|x|^{N-2\alpha} \log|x|\right), \end{aligned}$$

Then we choose $\rho_k = k^{-\frac{2\alpha}{N(N-2\alpha)}}(\log k)^{-2}$, and then there exists $k_1 > 1$ such that for $k \geq k_1$, we have $1 + \frac{c_{31}}{c_{30}}k^{p-1}\rho_k^{N-2\alpha} \log(\rho_k) \geq \frac{1}{2}$ and

$$u_k(x) \geq \frac{c_{30}}{2}k|x|^{-N+2\alpha}, \quad x \in \bar{B}_{\rho_k} \setminus \{0\}. \quad (4.8)$$

Since $\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$, there holds $1 - \frac{4\alpha^2}{N(N-2\alpha)} > 0$ and there exists $k_0 \geq k_1$ such that

$$\frac{c_{30}}{2}kr_k^{2\alpha} = \frac{c_{30}}{2}k^{1-\frac{4\alpha^2}{N(N-2\alpha)}}(\log k)^{-6\alpha} \geq (\ln \ln k)^{\frac{1}{p-1}}$$

for $k \geq k_0$. The remaining of the proof is the same as in Proposition 4.2. \square

In the sequel, the description of the limit behavior depends which of the following three cases holds:

$$\frac{2\alpha}{N-2\alpha} = 1 + \frac{2\alpha}{N} = \frac{N}{2\alpha}; \quad (4.9)$$

$$\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N} < \frac{N}{2\alpha}; \quad (4.10)$$

$$\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N} > \frac{N}{2\alpha}. \quad (4.11)$$

Proposition 4.4 *Assume that*

$$1 < \frac{2\alpha}{N-2\alpha} \leq 1 + \frac{2\alpha}{N}, \quad 1 < p < \frac{2\alpha}{N-2\alpha}, \quad (4.12)$$

or

$$1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}, \quad 1 < p < \frac{N}{2\alpha}, \quad (4.13)$$

and z_k is defined by (4.1) with $r_k = k^{-\frac{p-1}{N-2\alpha}}(\log k)^{-1}$. Then there exists $k_0 > 2$ such that (4.3) holds for $k \geq k_0$.

Proof. By (1.9) and Lemma 2.1(i), there exist $\rho_0 \in (0, d)$ and $c_{33}, c_{34} > 0$ independent of k such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$

$$\begin{aligned} u_k(x) &\geq c_{33}k|x|^{-N+2\alpha} - c_{34}k^p \\ &= c_{33}k|x|^{-N+2\alpha} \left(1 - \frac{c_{34}}{c_{33}}k^{p-1}|x|^{N-2\alpha}\right), \end{aligned}$$

Then we choose $\rho_k = k^{-\frac{p-1}{N-2\alpha}}$, and then there exists $k_1 > 1$ such that for $k \geq k_1$, $1 - \frac{c_{34}}{c_{33}} k^{p-1} \rho_k^{N-2\alpha} \geq \frac{1}{2}$ and

$$u_k(x) \geq \frac{c_{33}}{2} k |x|^{-N+2\alpha}, \quad x \in \bar{B}_{\rho_k} \setminus \{0\}. \quad (4.14)$$

By our setting (4.12), (4.13), together with relations (4.9)(4.10), (4.11), we have $p < \frac{N}{2\alpha}$ and then $1 - (p-1)\frac{2\alpha}{N-2\alpha} > 0$. Therefore there exists $k_0 \geq k_1$ such that

$$\frac{c_{33}}{2} k r_k^{2\alpha} = \frac{c_{33}}{2} k^{1-(p-1)\frac{2\alpha}{N-2\alpha}} (\log k)^{-2\alpha} \geq (\log \log k)^{\frac{1}{p-1}} = c_{2,k}^{\frac{1}{p-1}}$$

for $k \geq k_0$. The remaining of the proof is similar to the one of Proposition 4.2. \square

Proof of Theorem 1.1 part (i) with $p \in (1, 1 + \frac{2\alpha}{N}]$ and Theorem 1.2 part (i) with $p \in (1, 1 + \frac{2\alpha}{N}]$. We divided the proof into two steps.

Step 1. To prove $u_\infty = \infty$ in B_d . Indeed, we observe that for $\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$, Propositions 4.2, 4.3, 4.4 cover the case $p \in (\max\{1, \frac{2\alpha}{N-2\alpha}\}, 1 + \frac{2\alpha}{N})$, the case $1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$ along with $p = \frac{2\alpha}{N-2\alpha}$ and the case $1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$ along with $p \in (1, \frac{2\alpha}{N-2\alpha})$ respectively. For $\frac{2\alpha}{N-2\alpha} = 1 + \frac{2\alpha}{N}$, Proposition 4.3, 4.4 cover the case $p = \frac{2\alpha}{N-2\alpha}$ and the case $p \in (1, \frac{2\alpha}{N-2\alpha})$ respectively. So it covers $p \in (1, 1 + \frac{2\alpha}{N}]$ in Theorem 1.1 part (i). When $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$, Proposition 4.4 covers $p \in (1, \frac{N}{2\alpha})$ in Theorem 1.2 part (i). Therefore, we have

$$u_\infty \geq c_{2,k}^{\frac{1}{p-1}} z_k \quad \text{in } B_d$$

and for any $x \in B_d \setminus \{0\}$,

$$\lim_{k \rightarrow \infty} c_{2,k}^{\frac{1}{p-1}} z_k(x) = \infty.$$

Then

$$u_\infty = \infty \quad \text{in } B_d.$$

Step 2. To prove $u_\infty = \infty$ in Ω . By the fact of $u_\infty = \infty$ in B_d and $u_{k+1} \geq u_k$ in Ω , then for any $n > 1$ there exists $k_n > 0$ such that $u_{k_n} \geq n$ in B_d . For any $x_0 \in \Omega \setminus B_d$, there exists $\rho > 0$ such that $\bar{B}_\rho(x_0) \subset \Omega \cap B_{d/2}^c$. We denote by w_n the solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 & \text{in } B_\rho(x_0), \\ u &= 0 & \text{in } B_\rho^c(x_0) \setminus B_{d/2}, \\ u &= n & \text{in } B_{d/2} \end{aligned} \quad (4.15)$$

Then by Theorem 2.1, we have

$$u_{k_n} \geq w_n. \quad (4.16)$$

Let η_1 be the solution of

$$\begin{aligned} (-\Delta)^\alpha u &= 1 & \text{in } B_\rho(x_0), \\ u &= 0 & \text{in } B_\rho^c(x_0) \end{aligned}$$

and $v_n = w_n - n\chi_{B_{d/2}}$, then $v_n = w_n$ in $B_\rho(x_0)$ and

$$\begin{aligned} (-\Delta)^\alpha v_n(x) + v_n^p(x) &= (-\Delta)^\alpha w_n(x) - n(-\Delta)^\alpha \chi_{B_{d/2}}(x) + w_n^p(x) \\ &= n \int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}}, \quad x \in B_\rho(x_0), \end{aligned}$$

that is, v_n is a solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= n \int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}} & \text{in } B_\rho(x_0), \\ u &= 0 & \text{in } B_\rho^c(x_0). \end{aligned} \quad (4.17)$$

It is clear that $\frac{1}{c_{35}} \leq \int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}} \leq c_{35}$, $x \in B_\rho(x_0)$ for some $c_{35} > 1$ and $(\frac{n}{2C \max \eta_1})^{\frac{1}{p}} \eta_1$ is sub solution of (4.17). Then using Theorem 2.1, we obtain that

$$v_n \geq \left(\frac{n}{2C \max \eta_1}\right)^{\frac{1}{p}} \eta_1, \quad x \in B_\rho(x_0),$$

which implies that

$$w_n \geq \left(\frac{n}{2C \max \eta_1}\right)^{\frac{1}{p}} \eta_1, \quad x \in B_\rho(x_0).$$

Then

$$\lim_{n \rightarrow \infty} w_n(x_0) \rightarrow \infty.$$

Since x_0 is arbitrary and together with (4.16), it implies that $u_\infty = \infty$ in Ω . The proof is complete. \square

4.2 The case of $p \in (1 + \frac{2\alpha}{N}, p_\alpha^*)$

Proposition 4.5 *Let $\alpha \in (0, 1)$ and $r_0 = \text{dist}(0, \partial\Omega)$. Then*

(i) *if $\max\{1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha}\} < p < p_\alpha^*$, there exist $R_0 \in (0, r_0)$ and $c_{36} > 0$ such that*

$$u_\infty(x) \geq c_{36}|x|^{-\frac{2\alpha}{p-1}}, \quad x \in B_{R_0} \setminus \{0\}; \quad (4.18)$$

(ii) *if $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$ and $p = \frac{2\alpha}{N-2\alpha}$, there exist $R_0 \in (0, r_0)$ and $c_{37} > 0$ such that*

$$u_\infty(x) \geq \frac{c_{37}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}} |x|^{-\frac{p(N-2\alpha)}{p-1}}, \quad x \in B_{R_0} \setminus \{0\}. \quad (4.19)$$

(iii) if $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$ and $p \in (1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, there exist $R_0 \in (0, r_0)$ and $c_{38} > 0$ such that

$$u_\infty(x) \geq c_{38}|x|^{-\frac{p(N-2\alpha)}{p-1}}, \quad x \in B_{R_0} \setminus \{0\}. \quad (4.20)$$

Proof. (i) Using (1.9) and Lemma 2.1(i) with $\max\{1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha}\} < p < p_\alpha^*$, then there exist $\rho_0 \in (0, r_0)$ and $c_{39}, c_{40} > 0$ such that

$$u_k(x) \geq c_{39}k|x|^{-N+2\alpha} - c_{40}k^p|x|^{-(N-2\alpha)p+2\alpha}, \quad x \in B_{\rho_0} \setminus \{0\}. \quad (4.21)$$

Let us define

$$\rho_k = (2^{(N-2\alpha)p-2\alpha-1} \frac{c_{40}}{c_{39}} k^{p-1})^{\frac{1}{(N-2\alpha)(p-1)-2\alpha}}. \quad (4.22)$$

Since $(N-2\alpha)(p-1) - 2\alpha < 0$, we have that

$$\lim_{k \rightarrow \infty} \rho_k = 0$$

and then there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Then for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, we have

$$\begin{aligned} c_{40}k^p|x|^{-(N-2\alpha)p+2\alpha} &\leq c_{40}k^p\left(\frac{\rho_k}{2}\right)^{-(N-2\alpha)p+2\alpha} \\ &= \frac{c_{39}}{2}k\rho_k^{-N+2\alpha} \\ &\leq \frac{c_{39}}{2}k|x|^{-N+2\alpha} \end{aligned}$$

and

$$\begin{aligned} k &= (2^{(N-2\alpha)p-2\alpha-1} \frac{c_{40}}{c_{39}})^{-\frac{1}{p-1}} \rho_k^{N-2\alpha-\frac{2\alpha}{p-1}} \\ &\geq c_{41}|x|^{N-2\alpha-\frac{2\alpha}{p-1}}, \end{aligned}$$

where $c_{41} = (2^{(N-2\alpha)p-2\alpha-1} \frac{c_{40}}{c_{39}})^{-\frac{1}{p-1}} 2^{(N-2\alpha)(p-1)-2\alpha-1}$. Combining with (4.18), we obtain

$$\begin{aligned} u_k(x) &= c_{39}k|x|^{-N+2\alpha} - c_{40}k^p|x|^{-(N-2\alpha)p+2\alpha} \\ &\geq \frac{c_{39}}{2}k|x|^{-N+2\alpha} \\ &\geq c_{42}|x|^{-\frac{2\alpha}{p-1}}, \end{aligned} \quad (4.23)$$

for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, where $c_{42} = \frac{1}{2}c_{39}c_{41}$ is independent of k . By (4.22), we can choose a sequence $\{k_n\} \subset [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2}\rho_{k_n},$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$ and then by (4.23),

$$u_{k_n}(x) \geq c_{42}|x|^{-\frac{2\alpha}{p-1}}.$$

Together with $u_{k+1} > u_k$, we derive

$$u_\infty(x) \geq c_{42}|x|^{-\frac{2\alpha}{p-1}}, \quad x \in B_{\rho_{k_0}} \setminus \{0\}.$$

(ii) By (1.9) and Lemma 2.1(ii) with $p = \frac{2\alpha}{N-2\alpha}$, there exist $\rho_0 \in (0, r_0)$ and $c_{43}, c_{44} > 0$ such that

$$u_k(x) \geq c_{43}k|x|^{-N+2\alpha} - c_{44}k^p|\log(|x|)|, \quad x \in B_{\rho_0} \setminus \{0\}. \quad (4.24)$$

Denote by (ρ_k) a sequence number in $(0, 1)$ such that

$$c_{44}k^{p-1}|\log(\frac{\rho_k}{2})| = \frac{c_{43}}{2}\rho_k^{-N+2\alpha}, \quad (4.25)$$

then

$$\lim_{k \rightarrow \infty} \rho_k = 0$$

and there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Then for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ and $k \geq k_0$,

$$c_{43}k^p|\log(|x|)| \leq c_{44}k^p|\log(\frac{\rho_k}{2})| = \frac{c_{43}}{2}k\rho_k^{-N+2\alpha} \leq \frac{c_{43}}{2}k|x|^{-N+2\alpha}.$$

By (4.25), for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, we have that

$$k = \left(\frac{c_{44}}{2c_{43}}\right)^{-\frac{1}{p-1}} \left(\frac{\rho_k^{-N+2\alpha}}{1 + |\log(\rho_k)|}\right)^{\frac{1}{p-1}} \geq c_{45} \frac{|x|^{-\frac{N-2\alpha}{p-1}}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}},$$

where $c_{45} = 2^{-\frac{N-2\alpha}{p-1}} \left(\frac{c_{44}}{2c_{43}}\right)^{-\frac{1}{p-1}}$. Therefore, for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ we have

$$\begin{aligned} u_k(x) &\geq c_{43}k|x|^{-N+2\alpha} - c_{44}k^p|\log(|x|)| \\ &\geq \frac{c_{43}}{2}k|x|^{-N+2\alpha} \geq c_{46} \frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}}, \end{aligned} \quad (4.26)$$

where $c_{46} = \frac{1}{2}c_{43}c_{45}$ independent of k .

By (4.25), we can choose a sequence $k_n \in [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2}\rho_{k_n},$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$ and then by (4.26),

$$u_{k_n}(x) \geq c_{46} \frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}}.$$

Together with $u_{k+1} > u_k$, we have

$$u_\infty(x) \geq c_{46} \frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}}, \quad x \in B_{\rho_{k_0}} \setminus \{0\}.$$

(iii) By (1.9) and Lemma 2.1 (iii) with $p \in (1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, then there exist $\rho_0 \in (0, r_0)$ and $c_{47}, c_{48} > 0$ such that

$$u_k(x) \geq c_{47}k|x|^{-N+2\alpha} - c_{48}k^p, \quad x \in B_{\rho_0} \setminus \{0\}. \quad (4.27)$$

Let

$$\rho_k = \left(\frac{c_{48}}{2c_{47}}k^{p-1}\right)^{-\frac{1}{N-2\alpha}}, \quad (4.28)$$

then

$$\lim_{k \rightarrow \infty} \rho_k = 0$$

and there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Then for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ and $k \geq k_0$,

$$c_{48}k^p = \frac{c_{47}}{2}k\rho_k^{-N+2\alpha} \leq \frac{c_{47}}{2}k|x|^{-N+2\alpha}.$$

By (4.28), for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, we have that

$$k = \left(\frac{c_{48}}{2c_{47}}\right)^{-\frac{1}{p-1}} \rho_k^{-\frac{N-2\alpha}{p-1}} \geq c_{49}|x|^{-\frac{N-2\alpha}{p-1}},$$

where $c_{49} = 2^{-\frac{N-2\alpha}{p-1}} \left(\frac{c_{48}}{2c_{47}}\right)^{-\frac{1}{p-1}}$. Therefore, for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ we have

$$\begin{aligned} u_k(x) &\geq c_{47}k|x|^{-N+2\alpha} - c_{48}k^p \geq \frac{c_{47}}{2}k|x|^{-N+2\alpha} \\ &\geq c_{50}|x|^{-\frac{p}{p-1}(N-2\alpha)}, \end{aligned} \quad (4.29)$$

where $c_{50} = \frac{1}{2}c_{47}c_{49}$ independent of k .

By (4.28), we can choose a sequence $k_n \in [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2}\rho_{k_n},$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$ and then by (4.29),

$$u_{k_n}(x) \geq c_{50}|x|^{-\frac{p(N-2\alpha)}{p-1}}.$$

Together with $u_{k+1} > u_k$, we have

$$u_\infty(x) \geq c_{50}|x|^{-\frac{p(N-2\alpha)}{p-1}}, \quad x \in B_{\rho_{k_0}} \setminus \{0\}.$$

We complete the proof. \square

Lemma 4.2 *Let $p \in (1 + \frac{2\alpha}{N}, p_\alpha^*)$ and u_s be a strongly singular solution of (1.4) satisfying (1.5). Then*

$$u_\infty \leq u_s \quad \text{in } \Omega \setminus \{0\}. \quad (4.30)$$

where u_∞ is defined by (1.13).

Proof. By (1.5) and (1.9), we have that

$$\lim_{x \rightarrow 0} u_s |x|^{\frac{2\alpha}{p-1}} = c_0 \quad \text{and} \quad \lim_{x \rightarrow 0} u_k |x|^{N-2\alpha} = c_k,$$

then there exists $r_1 > 0$ such that

$$u_k < u_s \quad \text{in } B_{r_1} \setminus \{0\}.$$

By Theorem 3.1, we have that u_k satisfies

$$(-\Delta)^\alpha u_k + u_k^p = 0 \quad \text{in } \Omega \setminus B_{r_1}(0),$$

so does u_s . By Theorem 2.1, we have $u_k \leq u_s$ in $\Omega \setminus \{0\}$. Combining with (1.13), then

$$u_\infty \leq u_s \quad \text{in } \Omega \setminus \{0\}.$$

The proof is complete. \square

Proof of Theorem 1.1 (ii) and Theorem 1.2 part (iv). By Lemma 4.2 and Theorem 3.2, we obtain that u_∞ is a classical solution of (1.4). Moreover, by Proposition 4.5 part (i) and Lemma 4.2, we have

$$\frac{1}{c_{51}} |x|^{-\frac{2\alpha}{p-1}} \leq u_\infty(x) \leq c_{51} |x|^{-\frac{2\alpha}{p-1}},$$

for some $c_{51} > 1$. Since u_s is unique in the sense of (1.6), then $u_\infty = u_s$ in $\mathbb{R}^N \setminus \{0\}$. The proof is complete. \square

4.3 Proof of Theorem 1.2 (ii) and (iii)

Proof of Theorem 1.2 parts (ii) and (iii). By Lemma 4.2 and Theorem 3.2 we obtain that u_∞ is a classical solution of (1.4) satisfying

$$u_\infty \leq u_s \quad \text{in } \Omega \setminus \{0\}.$$

Moreover, we obtain (1.16) and (1.15) by Proposition 4.5 part (ii) and (iii), respectively. The proof is complete. \square

References

- [1] Ph. Benilan and H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, *J. Evolution Eq.* 3, 673-770, (2003).
- [2] H. Brezis, Some variational problems of the Thomas-Fermi type. Variational inequalities and complementarity problems, *Proc. Internat. School, Erice, Wiley, Chichester*, 53-73 (1980).
- [3] H. Brezis and L. Véron, Removable singularities of some nonlinear elliptic equations, *Arch. Rational Mech. Anal.* 75, 1-6 (1980).
- [4] L. Caffarelli and L. Silvestre, Regularity theory for fully non-linear integrodifferential equations, *Comm. Pure Appl. Math.* 62, 597-638 (2009).
- [5] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, *submitted*.
- [6] H. Chen and L. Véron, Solutions of fractional equations involving sources and Radon measures, *preprint*.
- [7] H. Chen and L. Véron, Singular solutions of fractional elliptic equations with absorption, *arXiv:1302.1427v1*, [*math.AP*], 6 (Feb 2013).
- [8] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *arXiv:1305.0945v2*, [*math.AP*], 15 (May 2013).
- [9] X.Y. Chen, H. Matano and L. Véron, Anisotropic singularities of nonlinear elliptic equations, *J. Funct. Anal.* 83, 50-97 (1989).
- [10] Z. Chen, and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann.* 312, 465-501 (1998).
- [11] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *J. Math. Pures Appl.*, to appear.
- [12] L. Silvestre, Regularity of the obstacle problem for a fractional power of the laplace operator, *Comm. Pure Appl. Math.* 60, 67-112 (2007).
- [13] J. Vazquez and L. Véron, Singularities of elliptic equations with an exponential nonlinearity, *Math. Ann.* 269, 119-135 (1984).
- [14] J. Vazquez and L. Véron, Isolated singularities of some semilinear elliptic equations, *J. Diff. Equ.* 60(3), 301-321 (1985).
- [15] L. Véron, Weak and strong singularities of nonlinear elliptic equations, *Proc. Symp. Pure Math.* 45, 477-495 (1986).

- [16] L. Véron, Singular solutions of some nonlinear elliptic equations, *Non-linear Anal. T. M. & A.* 5, 225-242 (1981).
- [17] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, *Vol. I, 593-712, Handb. Differ. Equ., North-Holland, Amsterdam* (2004).