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One-Year Volatility of Reserve Risk in a Multivariate Framework

Yannick APPERT-RAULLIN, Laurent DEVINEAU,
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July 30, 2013

Abstract

The one-year prediction error (one-year MSEP) proposed by Merz and Wüthrich [23] has become a market-standard approach for the assessment of reserve volatilities for Solvency II purposes. However, this approach is declined in a univariate framework. Moreover, Braun [2] proposed a closed-formed expression of the prediction error of several run-off portfolios at the ultimate horizon by taking into account their dependency.

This article proposes an analytical expression of the one-year MSEP obtained by generalizing the modeling developed by Braun [2] to the one-year horizon with an approach similar to Merz and Wüthrich [23]. A full mathematical demonstration of the formula has been provided in this paper. A case study is presented to assess the dependency between commercial and motor liabilities businesses based on data coming from a major international insurer.

Keywords: multivariate reserving, correlation, run-off portfolio, prediction error, one-year multivariate reserve risk, claims development result, Solvency II

1 Introduction

The European insurance regulator seeks to support the stability of the financial system as well as the protection of the policyholders of each insurance or reinsurance company operating in the European Union. For that purpose, the European Directive 2009/138/EC, generally known as Solvency II, requires estimating the economic value of the aggregated risk assumed by each insurance or reinsurance company. To cover the latter, Solvency II establishes some capital requirements named as the Solvency Capital Requirement (SCR). This capital charge needs to be calculated by each company according either to the so-called Standard Formula proposed in the European Directive, or to an Internal Model provided the previous validation of the supervisory authorities. Within the Solvency II regulatory framework, the SCR is estimated through a modular structure of risks related to insurance activity, including underwriting, market, credit and operational risks. In this paper, we focus on non-life underwriting risks and more specifically on the Reserve risk. Reserve risk concerns the liabilities for insurance policies covering historical years, often simply referred as the risk in the claims reserve, i.e., the provision for outstanding claims. Within the Solvency II framework, the reserve risk corresponds to the risk that the current reserves are insufficient to cover the ultimate claims over one-year horizon.

A classical measure of uncertainty in claims reserving is called the ultimate uncertainty based on the ultimate claims evaluation. In Solvency II, it is now required to quantify one-year uncertainty. The starting point is that in one year the company will update its prediction of the ultimate claims amount. Using the Mack [12] framework, Merz and Wüthrich [23] developed an analytical expression of the one-year prediction error (MSEP) based on parameters calculated from the claims triangles. This MSEP is henceforth a market-standard approach to measure reserve volatilities in the Solvency II framework (used in QIS 5 Technical Specifications, articles SCR 10.48 – SCR 10.55).

The article of Merz and Wüthrich [23] did not address the issue of the correlation between portfolios. A segmentation into homogenous risk groups for the computation of insurance undertakings' technical provisions is required by the Directive (Art. 80). This homogeneity criteria is reflected by the involvement of several lines of business within a company when calculating the SCR. These lines of business are not necessarily assumed to be statistically independent. An assumption is established regarding the combination of the results inherent to several lines of business within a particular company. Indeed, significant correlations may be present as a result of both endogenous and exogenous causes. To capture the correlation effects, the SCR standard formula as defined in the Level 1 text for the non-life premium and reserve sub-module uses linear correlations matrix between different lines of business to estimate the combined standard deviation of premium and reserve risks. However, this solution does not allow insurance or reinsurance company to properly manage the interaction between lines of business. To tackle this issue, the present paper considers the whole prediction error of the one-year claims reserving problem for several correlated run-off lines of business.

The quantification of the ultimate uncertainties in the claims reserves for a multivariate situation has been addressed in several actuarial papers *e.g.* Holmberg [10], Brehm [3], Kirschner et al. [11], Taylor and McGuire [22], Braun [2]. In the latter paper, the standalone reserves are estimated by the generalization of the well-known univariate chain-ladder model of Mack and a correlation is introduced in the aggregated prediction error of the total claims reserves for two triangles. As an extension of the Mack's model to the one-year view, the market-standard approach Merz and Wüthrich [23] provides an analytical expression of the reserve risk only in a univariate world.

Within the context of Solvency II, this paper aims at developing a one-year multivariate approach, based on the assumptions stated in Braun [2] and in Braun, Mack, Quarg [13], by providing a closed-form expression of the one-year prediction error (MSEP) on the correlated run-off triangles. Then, the implicit linear correlation is deduced from the analytical expression for informational purposes. The multivariate approach tackled in the present paper has the advantage that by understanding the MSEP of one run-off portfolio we are able to learn about the behavior of the other run-off subportfolios. Thus, our work seeks to contribute to the discussion around the methods underlying the SCR calculation and more globally to support insurance or reinsurance risk manager to analyse the prediction error of the reserves uncertainty on aggregated run-off portfolios in a Solvency II framework.

The rest of the paper is structured as follows. Section 2 provides a brief overview of currently applied methodologies. Section 3 describes the theoretical background and assumptions underlying our modeling. In section 4, we introduce the one-year multivariate model. In a first subsection, we remind some key concepts of Solvency II such as the Claims Development Result (CDR) and the purpose of the

subsequent sections is to describe our one-year multivariate modeling. Finally, in Section 5, we present our data and the results of our case study. In this last section we discuss our findings and present our conclusions.

2 Issues and overview of the methodologies

As introduced previously, two main issues have been raised: the necessity to introduce a one-year horizon, regarding the Solvency II framework, and the possibility to take into account some potential correlations between lines of business. These two issues have been identified and have to be considered in order to compute properly the MSEP of the claims reserves (over one or several lines of business).

In the literature, some papers focus on the assessment of a one-year prediction error inherent to the reserve risk. Merz and Wüthrich [23] proposed closed-form expressions for the one-year prediction error as an adaptation of the Mack approach [12]. Alternatively, a transposition of the GLM models to a one-year horizon was introduced by Diers [7] and Ohlsson and Lauzenings [15]: unlike the model developed by Wüthrich *et al.* [23], these ones allow establishing distributions of the variables of interest thanks to a simulatory approach.

These models are suitable in a univariate framework. They do not allow taking into account a potential correlation between lines of business. Indeed, a correlation might be supposed but is, the most often, integrated *a posteriori*: the independency assumption is required *a priori* to treat the triangles separately and then the dependency between independent distributions is added *a posteriori* through various methodologies (*e.g.* copulas).

Nevertheless, several lines of business include insurance companies' portfolios and there is a need for measuring the different interactions (mutual impact on the development patterns, common dependency regarding inflation, etc.). As a consequence, in order to assess the aggregated claims' reserves, it seems to be necessary to develop multivariate reserving models. Some methodologies are available in the reserving literature.

A first approach has been initiated by Ajne [1] who worked on an aggregated triangle, as a sum of two or three triangles, for the determination of the Best Estimate of reserves. The author noticed that working on an aggregated triangle provides improper results as soon as this aggregated triangle might lack homogeneity and may not satisfy the Chain-Ladder initial assumptions. Necessary and sufficient conditions have been identified but have appeared to be very restrictive. Then, the correlation has to be integrated while considering standalone triangles, linked by specific structures of dependence.

Several approaches might be considered while dealing with this issue, two main methodologies have been identified namely: the development of closed-form expressions to assess a multi-line of business reserves' volatility or approaches based on a simulatory framework which allow providing the whole distribution.

On the one hand, Kirschner *et al.* [11] and Taylor and McGuire [22] have proposed a modeling allowing taking into account the dependency between the lines of business thanks to an implicit correlation inferred by bootstrap simulations on standardized residuals obtained in a Generalized Linear Model framework. Alternatively, De Jong [5] focused on the Hertig model and proposed an *a priori* correlation between the log-link ratios depending on the three directions of the triangle (accident, development and calendar years) via a simulatory approach.

Serra and Gillet [8] provided a multivariate simulatory approach which aims at introducing a correlation between the credibility factors via a copula. They propose a generalization of the credibility theory in a two-dimensional framework: the main purpose is to specify a line by line dependency between triangles (*i.e.* for a particular accident year, a model providing a dependency between the credibility factors of the triangles). An approach based on the Bayesian credibility has also been developed by Shi *et al.* [20].

Some of these models rely on a copula-based approach. It is also the case for Brehm [3] who focused on the model developed by Zehnirith [30] to assess a common dependency to calendar distortions (mainly due to inflation) correlating the diagonal components of the triangles. This paper, then, proposed to resort to a Gaussian copula to aggregate multiple distribution of reserves (corresponding to different lines of business). The use of such a process involving a Gaussian copula was also presented by De Jong [6]. Shi and Frees [21] considered a flexible copula regression model to aggregate multiple loss triangles. Then, Shi [19] examined a Tweedie's compound Poisson distribution to model the incremental payment

in each triangle along with elliptical copulas for multivariate loss reserving.

On the other hand, closed-form expressions have been determined in a multivariate framework. For instance, Holmberg [10] focused on the several existing correlations *i.e.* between accident years, within accident years and between different lines of business by using the variance of the link ratios in order to assess the reserves' MSE. Later, Pröhl and Schmidt [16] and Schmidt [17] extended the model of Schnaus [18] to consider the dependence structure between several portfolios by constructing multivariate development factors. At the same time, Hess *et al.* [9] and Schmidt [17] provided a generalization of the additive model (or incremental loss-ratio method) which allowed dealing with portfolios comprised of correlated sub-portfolios. They managed to determine a Gauss-Markov estimator of incremental claim volumes to predict. Merz and Wüthrich [25] and [26] formulate two models which aim at providing a MSE relying on the structure developed by these former studies (Hess *et al.* [9] and Schmidt [17]). The first model consists in a stochastic modeling using the provided Gauss-Markov predictor and derives a MSE of the ultimate claims. As for the second methodology, its purpose is to design a model mixing a multivariate Chain-Ladder approach and a multivariate additive loss-reserving method (one or the other model being used for particular sub-portfolios). This latter study allows dealing simultaneously with different sub-portfolios (*e.g.* associated to different lines of business) but also provides an estimation of the conditional MSE of the total portfolio's ultimate claims.

A milestone of the work on dependency between triangles has been the generalization of Mack's model in a multivariate framework by Braun [2]: the stand-alone reserves are estimated thanks to the Chain-Ladder approach and a correlation is introduced in the aggregated MSE of the sum of the triangles' reserves. It is also worth mentioning the introduction of an iterative approach of the development factors in the Braun's methodology in Merz *et al.* [28].

These multivariate models provide some answers to the dependency problem between lines of business. However, they do not appear to be fully compliant with the Solvency II framework since they provide an expression of the prediction error of the estimators of the ultimate claims whereas in such a context, a one-year perspective is required.

We remind that two issues have been identified: the assessment of a one-year volatility and the possibility to work in a multivariate framework involving several lines of business and potential correlations. Hence, we need to assess the one-year prediction error by taking into account the dependency between the two run-off triangles.

Regarding these conclusions, we exhibit, in this paper, a model proposing closed-form expressions for the one-year prediction error of the Claims Development Result (defined as the difference between the estimation of the ultimate claims amount observed at the end of year I and the re-estimation of the ultimate claims amount observed at the end of the year $I + 1$) in a multivariate framework. This model relies on the Braun [2] one, which corresponds to the bivariate version of the Mack [12] model, the latter being adapted in a univariate context to a one-year horizon by Merz and Wüthrich [23] and Wüthrich *et al.* [29].

The methodology developed in the present paper, which provides an analytical expression of the one-year prediction error within a multivariate framework, is compliant with the standpoint of the Solvency II directive.

3 Theoretical background and assumptions

3.1 Notations and useful definitions

Hereinafter, the indexes $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$ will respectively refer to accident and development years. Besides, we are going to suppose that the triangles are such that $I = J$. Let us consider two lines of business with $(C_{i,j})_{\substack{1 \leq i, j \leq I \\ 1 \leq i+j-1 \leq I}}$ and $(D_{i,j})_{\substack{1 \leq i, j \leq I \\ 1 \leq i+j-1 \leq I}}$ their cumulative claims triangles.

The two cumulative claims triangles have the following format (the C -triangle is presented here):

Accident year i	Development year j						
	1	2	\dots	$I-i+1$	\dots	$I-1$	I
1	$C_{1,1}$	$C_{1,2}$	\dots	$C_{1,I-i+1}$	\dots	$C_{1,I-1}$	$C_{1,I}$
2	$C_{2,1}$	$C_{2,2}$	\dots	$C_{2,I-i+1}$	\dots	$C_{2,I-1}$	
\dots	\dots	\dots	\dots	\dots	\dots		
i	$C_{i,1}$	$C_{i,2}$	\dots	$C_{i,I-i+1}$			
\dots	\dots	\dots	\dots				
$I-1$	$C_{I-1,1}$	$C_{I-1,2}$					
I	$C_{I,1}$						

Let D_k be the available information at year I , *i.e.*:

$$\forall k \in \{1, \dots, I\}, D_k = \{C_{i,j}, i+j \leq k+1\} \cup \{D_{i,j}, i+j \leq k+1\}.$$

3.2 Model assumptions

We follow the assumption set up in Merz and Wüthrich [14]:

let us consider the cumulative payments $C_{i,j}$ and $D_{i,j}$ for accident year $i \in \{1, \dots, I\}$, there exist constants $f_j > 0$, $g_j > 0$, $\sigma_j \geq 0$ and $\tau_j \geq 0$ for $j \in \{1, \dots, I-1\}$ such that the following time series models are defined:

$$\begin{cases} \forall i \in \{1, \dots, I\}, \forall j \in \{1, \dots, I-1\}, & C_{i,j+1} = f_j \times C_{i,j} + \sigma_j \times \sqrt{C_{i,j}} \times \varepsilon_{i,j+1}^C \\ \forall i \in \{1, \dots, I\}, \forall j \in \{1, \dots, I-1\}, & D_{i,j+1} = g_j \times D_{i,j} + \tau_j \times \sqrt{D_{i,j}} \times \varepsilon_{i,j+1}^D \end{cases} \quad (H_0)$$

where $\varepsilon_{i,j}^C$ (resp. $\varepsilon_{i,j}^D$) are independent random variables in the C -triangle (resp. D -triangle) with the moments of the two first orders respectively equal to 0 and 1 for both triangles.

In the sequel, we deduce several properties, which are corollary of the main assumption (H_0) and question the dependency link that will be introduced when dealing with a multivariate framework. Since the hypothesis (H_0) is more restrictive than the assumptions underlying Mack's model [12], we deduce the following properties:

Property 1: The cumulative payments for two different accident years $i \neq j$ ($C_{i,1}, \dots, C_{i,I}$) and ($C_{j,1}, \dots, C_{j,I}$) are independent. Similarly, ($D_{i,1}, \dots, D_{i,I}$) and ($D_{j,1}, \dots, D_{j,I}$) are independent for $i \neq j$.

Property 2: There exist constants $f_j > 0$ and $g_j > 0$ such that:

$$\begin{aligned} \forall j \in \{2, \dots, I\}, \forall i \in \{1, \dots, I\}, \mathbb{E}(C_{i,j}|C_{i,j-1}) &= f_{j-1} \times C_{i,j-1}, \\ \forall j \in \{2, \dots, I\}, \forall i \in \{1, \dots, I\}, \mathbb{E}(D_{i,j}|D_{i,j-1}) &= g_{j-1} \times D_{i,j-1}. \end{aligned}$$

Property 3: There exist constants $\sigma_j \geq 0$ and $\tau_j \geq 0$ such that:

$$\begin{aligned} \forall j \in \{2, \dots, I\}, \forall i \in \{1, \dots, I\}, \mathbb{V}(C_{i,j}|C_{i,j-1}) &= \sigma_{j-1}^2 \times C_{i,j-1}, \\ \forall j \in \{2, \dots, I\}, \forall i \in \{1, \dots, I\}, \mathbb{V}(D_{i,j}|D_{i,j-1}) &= \tau_{j-1}^2 \times D_{i,j-1}. \end{aligned}$$

For the triangles $(C_{i,j})_{\substack{1 \leq i, j \leq I \\ 1 \leq i+j-1 \leq I}}$ and $(D_{i,j})_{\substack{1 \leq i, j \leq I \\ 1 \leq i+j-1 \leq I}}$, we introduce the notations of the individual development factors of the triangles ($C_{i,j}$) and ($D_{i,j}$):

$$\forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, F_{i,j} = \frac{C_{i,j+1}}{C_{i,j}} \quad \text{and} \quad G_{i,j} = \frac{D_{i,j+1}}{D_{i,j}}. \quad (1)$$

The moments of the two first orders of the individual development factors (for the C -triangle) are then given by:

$$\forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, \mathbb{E}(F_{i,j}|C_{i,j}) = f_j \quad \text{and} \quad \mathbb{V}(F_{i,j}|C_{i,j}) = \frac{\sigma_j^2}{C_{i,j}}. \quad (2)$$

Knowing D_I , aforementioned, we will denote S_j^I and T_j^I :

$$\forall j \in \{1, \dots, I-1\}, S_j^I = \sum_{i=1}^{I-j} C_{i,j} \quad \text{and} \quad T_j^I = \sum_{i=1}^{I-j} D_{i,j}.$$

and the estimators of the Chain-Ladder development factors at year I are given by the following formulas:

$$\left\{ \begin{array}{ll} \forall j \in \{1, \dots, I-1\}, \hat{f}_j^I = \frac{\sum_{i=1}^{I-j} C_{i,j+1}}{S_j^I} & \text{and } \forall j \in \{1, \dots, I-1\}, \hat{g}_j^I = \frac{\sum_{i=1}^{I-j} D_{i,j+1}}{T_j^I}, \\ \forall j \in \{1, \dots, I-1\}, \hat{f}_j^{I+1} = \frac{\sum_{i=1}^{I-j+1} C_{i,j+1}}{S_j^{I+1}} & \text{and } \forall j \in \{1, \dots, I-1\}, \hat{g}_j^{I+1} = \frac{\sum_{i=1}^{I-j+1} D_{i,j+1}}{T_j^{I+1}}. \end{array} \right. \quad (3)$$

We infer the following estimators.

Unbiased estimators of $\mathbb{E}(C_{i,j}|D_I)$ and $\mathbb{E}(D_{i,j}|D_I)$:

$$\forall i \in \{2, \dots, I\}, \forall j \in \{2, \dots, I\}, i+j > I+1, \hat{\mathbb{E}}(C_{i,j}|D_I) = \hat{C}_{i,j}^I = C_{i,I-i+1} \times \hat{f}_{I-i+1}^I \times \dots \times \hat{f}_{j-2}^I \times \hat{f}_{j-1}^I. \quad (4)$$

$$\forall i \in \{2, \dots, I\}, \forall j \in \{2, \dots, I\}, i+j > I+1, \hat{\mathbb{E}}(D_{i,j}|D_I) = \hat{D}_{i,j}^I = D_{i,I-i+1} \times \hat{g}_{I-i+1}^I \times \dots \times \hat{g}_{j-2}^I \times \hat{g}_{j-1}^I. \quad (5)$$

Unbiased estimators of $\mathbb{E}(C_{i,j}|D_{I+1})$ and $\mathbb{E}(D_{i,j}|D_{I+1})$:

$$\forall i \in \{2, \dots, I\}, \forall j \in \{2, \dots, I\}, \hat{\mathbb{E}}(C_{i,j}|D_{I+1}) = \hat{C}_{i,j}^{I+1} = C_{i,I-i+2} \times \hat{f}_{I-i+2}^{I+1} \times \dots \times \hat{f}_{j-2}^{I+1} \times \hat{f}_{j-1}^{I+1}. \quad (6)$$

$$\forall i \in \{2, \dots, I\}, \forall j \in \{2, \dots, I\}, \hat{\mathbb{E}}(D_{i,j}|D_{I+1}) = \hat{D}_{i,j}^{I+1} = D_{i,I-i+2} \times \hat{g}_{I-i+2}^{I+1} \times \dots \times \hat{g}_{j-2}^{I+1} \times \hat{g}_{j-1}^{I+1}. \quad (7)$$

Unbiased estimators of $\{\sigma_j^2, j = 1, \dots, I-1\}$ and $\{\tau_j^2, j = 1, \dots, I-1\}$:

$$\left\{ \begin{array}{ll} \forall j \in \{1, \dots, I-2\}, (\hat{\sigma}_j)^2 = \frac{1}{I-j-1} \times \sum_{i=1}^{I-j} C_{i,j} \times \left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j^I \right)^2 \\ \text{and } (\hat{\sigma}_{I-1})^2 = \min \left(\frac{\hat{\sigma}_{I-2}^4}{\hat{\sigma}_{I-3}^2}; \hat{\sigma}_{I-3}^2, \hat{\sigma}_{I-2}^2 \right). \end{array} \right. \quad (8)$$

$$\left\{ \begin{array}{ll} \forall j \in \{1, \dots, I-2\}, (\hat{\tau}_j)^2 = \frac{1}{I-j-1} \times \sum_{i=1}^{I-j} D_{i,j} \times \left(\frac{D_{i,j+1}}{D_{i,j}} - \hat{g}_j^I \right)^2 \\ \text{and } (\hat{\tau}_{I-1})^2 = \min \left(\frac{\hat{\tau}_{I-2}^4}{\hat{\tau}_{I-3}^2}; \hat{\tau}_{I-3}^2, \hat{\tau}_{I-2}^2 \right). \end{array} \right. \quad (9)$$

In order to generalize the properties, deduced above, of Mack's model [12] to a multivariate framework, we need to take into account dependency between the triangles $(C_{i,j})$ and $(D_{i,j})$. It will be modelled via a correlation structure between the random processes $(\varepsilon_{i,j}^C)_{i,j}$ and $(\varepsilon_{i,j}^D)_{i,j}$, which introduces dependency between the triangles $(C_{i,j})_{\substack{1 \leq i,j \leq I \\ 1 \leq i+j-1 \leq I}}$ and $(D_{i,j})_{\substack{1 \leq i,j \leq I \\ 1 \leq i+j-1 \leq I}}$.

In his paper [2], Braun has introduced the following correlation assumptions:

$$\left\{ \begin{array}{ll} \forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, \exists \rho_j, \text{cov}(F_{i,j}, G_{i,j}|C_{i,j}, D_{i,j}) = \frac{\rho_j}{\sqrt{C_{i,j} \times D_{i,j}}}, \\ \forall h, j, k, l \in \{1, \dots, I-1\}, \text{ with } (h, j) \neq (k, l), \text{cov}(F_{h,j}, G_{k,l}|C_{h,j}, D_{k,l}) = 0. \end{array} \right. \quad (10)$$

This is equivalent to the linear correlation between the individual development factors $F_{i,j}$ and $G_{i,j}$, given by the following formula:

$$\forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, \frac{\text{cov}(F_{i,j}, G_{i,j}|C_{i,j}, D_{i,j})}{\sqrt{\mathbb{V}(F_{i,j}|C_{i,j}) \times \mathbb{V}(G_{i,j}|D_{i,j})}} = \frac{\rho_j}{\sigma_j \tau_j}. \quad (11)$$

It is worth to mention that, in spite of the correlation assumption introduced by Braun [2], the standalone assumptions deduced above remain valid since the Braun model [2] is a generalization of Mack's model [12] in a multivariate framework.

Computation of the coefficients $\{\rho_k, k = 1, \dots, I-1\}$:

In analogy to the estimators of σ_j^2 and τ_j^2 , Braun (cf equation (25) in [2]) introduced the following estimators for the coefficients $\{\rho_k, k = 1, \dots, I-1\}$:

$$\forall k \in \{1, \dots, I-2\}, \hat{\rho}_k = \frac{1}{I-k-2 + (\hat{w}_k)^2} \times \sum_{i=1}^{I-k} \sqrt{C_{i,k} \times D_{i,k}} \left(F_{i,k} - \hat{f}_k^I \right) \times (G_{i,k} - \hat{g}_k^I) \quad (12)$$

with:

$$\forall k \in \{1, \dots, I-1\}, (\hat{w}_k)^2 = \frac{\left(\sum_{i=1}^{I-k} \sqrt{C_{i,k} \times D_{i,k}} \right)^2}{S_k^I \times T_k^I}.$$

Limit condition for $\frac{\hat{\rho}_{I-1}}{\hat{\sigma}_{I-1} \hat{\tau}_{I-1}}$:

We propose the following limit condition, similar to the ones applied to the variances, for the computation of the correlation coefficient $\frac{\hat{\rho}_{I-1}}{\hat{\sigma}_{I-1} \hat{\tau}_{I-1}}$ and we propose to set that:

$$\frac{\hat{\rho}_{I-1}}{\hat{\sigma}_{I-1} \hat{\tau}_{I-1}} = \max \left(\left| \frac{\hat{\rho}_{I-2}}{\hat{\sigma}_{I-2} \hat{\tau}_{I-2}} \right|; \left| \frac{\hat{\rho}_{I-3}}{\hat{\sigma}_{I-3} \hat{\tau}_{I-3}} \right| \right). \quad (13)$$

For the oldest maturities, since the development patterns become generally more stable, the correlation between run-off portfolios tends to stabilize. The correlation for the last development year has been set as a maximum between the absolute values of the ones observed at years $I-2$ and $I-3$ to keep a more prudent approach.

We propose to generalize the Braun modelling [2] by introducing some dependency between the white noise structures of the triangles ($C_{i,j}$) and ($D_{i,j}$), respectively denoted as $\varepsilon_{i,j}^C$ and $\varepsilon_{i,j}^D$.

As a preliminary, we remind that $\varepsilon_{i,j}^C$ (resp. $\varepsilon_{i,j}^D$) are mutually independent in the C-triangle (resp. D-triangle).

By applying the assumption (H_0), the following equality is straightforward:

$$\forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, \frac{\text{cov}(F_{i,j}, G_{i,j} | C_{i,j}, D_{i,j})}{\sqrt{\mathbb{V}(F_{i,j} | C_{i,j}) \mathbb{V}(G_{i,j} | D_{i,j})}} = \text{cov}(\varepsilon_{i,j+1}^C, \varepsilon_{i,j+1}^D | C_{i,j}, D_{i,j}). \quad (14)$$

We deduce from the white noise structure of $\varepsilon_{i,j}^C$ and $\varepsilon_{i,j}^D$ ($\varepsilon_{i,j}^C$ and $\varepsilon_{i,j}^D$ are mutually independent) the following:

$$\forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, \frac{\text{cov}(F_{i,j}, G_{i,j} | C_{i,j}, D_{i,j})}{\sqrt{\mathbb{V}(F_{i,j} | C_{i,j}) \mathbb{V}(G_{i,j} | D_{i,j})}} = \text{cov}(\varepsilon_{i,j+1}^C, \varepsilon_{i,j+1}^D). \quad (15)$$

As a consequence:

$$\forall i \in \{1, \dots, I-1\}, \forall j \in \{1, \dots, I-i\}, \frac{\text{cov}(F_{i,j}, G_{i,j} | C_{i,j}, D_{i,j})}{\sqrt{\mathbb{V}(F_{i,j} | C_{i,j}) \mathbb{V}(G_{i,j} | D_{i,j})}} = \frac{\rho_j}{\sigma_j \tau_j} \iff \text{cov}(\varepsilon_{i,j+1}^C, \varepsilon_{i,j+1}^D) = \frac{\rho_j}{\sigma_j \tau_j}.$$

$\forall i, j, k, l, (\varepsilon_{i,j}^C) \perp (\varepsilon_{k,l}^D)_{(k,l) \neq (i,j)}$, *i.e.* the residuals are supposed to be uncorrelated.

Although more restrictive, the whole set of assumptions we suppose allows replicating the ultimate MSEF proposed by Braun [2]. The methodology proposed by Braun [2] is a generalization of Mack's model [12] to a multivariate framework for the ultimate horizon. Since the Merz and Wüthrich methodology [23] is an adaptation of the Mack methodology [12] to the Solvency II view, we propose, in the following of this paper, to extend the Braun methodology to the one-year horizon with the model assumptions set in this paragraph.

4 One-year multivariate modeling

In this section, we exhibit closed-form expressions for the computation of the Claims Development Result's prediction error. Without loss of generality, the results are going to be presented for a pair of lines of business.

In the Solvency II framework, the time-horizon to be considered is one year. Therefore, we aim at measuring the volatility inherent to the revision of the Best Estimate between years I and $I + 1$.

We define the CDR (*Claims Development Result*) as the difference between the estimation of the ultimate claims amount of the accident year i observed at the end of the year I , the re-estimation of the ultimate claims amount of the accident year i observed at the end of the year $I + 1$.

There are two types of CDR (see Merz and Wüthrich [14]) : the true CDR and the observed CDR. If the former is not observable at the end of year I due to the fact that the true Chain-Ladder factors remain unknown, the latter can be assessed. Indeed, the computation of the observed CDR is based on the expected ultimate claims amount (Chain-Ladder methodology) and it represents the position observed in the income statement at time $I + 1$. Typically, it corresponds to the registered result in the current year P&L statement at the end of the exercise. For Solvency II prospects, it is necessary to measure the quality of the prediction based on the observable CDR. Therefore, the true CDR by accident year i is defined by the following expressions:

$$\begin{cases} CDR_1^C(I+1) = CDR_1^D(I+1) = 0, \\ \forall i \in \{2, \dots, I\}, CDR_i^C(I+1) = \mathbb{E}(C_{i,I}|D_I) - \mathbb{E}(C_{i,I}|D_{I+1}), \\ \forall i \in \{2, \dots, I\}, CDR_i^D(I+1) = \mathbb{E}(D_{i,I}|D_I) - \mathbb{E}(D_{i,I}|D_{I+1}) \end{cases} \quad (16)$$

where D_I and D_{I+1} denote respectively the available information at year I and at year $I + 1$.

The observed CDR of a given accident year i estimated at the end of year $I + 1$ is given by the following formula:

$$\begin{cases} \widehat{CDR}_1^C(I+1) = 0 \quad \text{and} \quad \widehat{CDR}_1^D(I+1) = 0, \\ \forall i \in \{2, \dots, I\}, \widehat{CDR}_i^C(I+1) = \widehat{C}_{i,I}^I - \widehat{C}_{i,I}^{I+1} \quad \text{and} \quad \widehat{CDR}_i^D(I+1) = \widehat{D}_{i,I}^I - \widehat{D}_{i,I}^{I+1}. \end{cases} \quad (17)$$

4.1 Determination of a closed-form expression for the prediction error of the CDR

The one-year prediction uncertainty is assessed through the MSEP within a multivariate framework.

Hereinafter, the purpose is to determine a closed-form expression for the prediction error of the sum of the CDR. The term $\widehat{CDR}_i^C(I+1)$ (resp. $\widehat{CDR}_i^D(I+1)$) is the observable CDR of the $(C_{i,j})_{1 \leq i,j \leq I}$ triangle (resp. $(D_{i,j})_{1 \leq i,j \leq I}$).

We recall that $D_I = \{C_{i,j}, 1 \leq i+j \leq I+1\} \cup \{D_{i,j}, 1 \leq i+j \leq I+1\}$. The prediction error of the observable aggregated CDR, in comparison with its predicted value 0, for the accident year i is defined, in Solvency II purpose, by:

$$\forall i \in \{1, \dots, I\}, MSEP_{\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1)|D_I}(0) = \mathbb{E} \left[\left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right)^2 | D_I \right]. \quad (18)$$

The prediction error can be decomposed into a conditional process error and a conditional estimation error as follows, for a single accident year i :

$$\begin{aligned} & \forall i \in \{1, \dots, I\}, \mathbb{E} \left[\left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right)^2 | D_I \right] \\ &= \underbrace{\mathbb{V} \left[\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) | D_I \right]}_{\text{Process error}} + \underbrace{\left[\mathbb{E} \left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) | D_I \right) \right]^2}_{\text{Estimation error}}. \end{aligned} \quad (19)$$

In order to determine the prediction error of the observable CDR in comparison with 0, for all accident years, let us define the aggregated CDR as the sum of the CDR for each line of business. Thus, the

prediction error of the aggregated CDR for all the accidents years is then determined as follows:

$$\begin{aligned}
MSEP_{\sum_{i=1}^I (\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1)) | D_I}(0) &= \mathbb{E} \left[\left(\sum_{i=1}^I \left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right) \right)^2 | D_I \right] \\
&= \underbrace{\mathbb{V} \left[\sum_{i=1}^I \left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right) | D_I \right]}_{\text{Process error}} + \underbrace{\left[\mathbb{E} \left[\sum_{i=1}^I \left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right) | D_I \right] \right]^2}_{\text{Estimation error}}.
\end{aligned}$$

4.1.1 Process error

In this section, the detailed expression inherent to the computation of the process error is exhibited:

$$\begin{aligned}
&\mathbb{V} \left[\sum_{i=1}^I \left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right) | D_I \right] \\
&= \underbrace{\sum_{i=1}^I \mathbb{V} \left[\widehat{CDR}_i^C(I+1) | D_I \right] + 2 \sum_{I \geq i > j \geq 1} \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^C(I+1) | D_I \right]}_{C\text{-triangle process error}} \\
&+ \underbrace{\sum_{i=1}^I \mathbb{V} \left[\widehat{CDR}_i^D(I+1) | D_I \right] + 2 \sum_{I \geq i > j \geq 1} \text{cov} \left[\widehat{CDR}_i^D(I+1), \widehat{CDR}_j^D(I+1) | D_I \right]}_{D\text{-triangle process error}} \quad (20) \\
&+ 2 \times \sum_{i=1}^I \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) \right] \\
&+ 2 \times \sum_{I \geq i > j \geq 1} \left\{ \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) \right] + \text{cov} \left[\widehat{CDR}_j^C(I+1), \widehat{CDR}_i^D(I+1) \right] \right\}.
\end{aligned}$$

In this expression, we can identify some terms, which also appear in the standalone process error in closed-form expressions developed in Wüthrich *et al.* [29]. We deduce, then, the following (for the C -triangle) expressions:

$$\widehat{\mathbb{V}} \left[\widehat{CDR}_i^C(I+1) | D_I \right] = \widehat{\Gamma}_{i,I}^I \quad \text{and} \quad \widehat{\text{cov}} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^C(I+1) | D_I \right] = \widehat{\Upsilon}_{i,j}^I \quad \text{with } i > j.$$

Notice that:

$$\widehat{\Gamma}_{i,I}^I = \left(\widehat{C}_{i,I} \right)^2 \left\{ \left(1 + \frac{(\widehat{\sigma}_{I-i+1}^I)^2}{(\widehat{f}_{I-i+1}^I)^2 C_{i,I-i+1}} \right) \prod_{l=I-i+2}^{I-1} \left(1 + \frac{(\widehat{\sigma}_l^I)^2 C_{I-l+1,l}}{(\widehat{f}_l^I)^2 (S_l^{I+1})^2} \right) - 1 \right\}$$

and

$$\forall i > j, \widehat{\Upsilon}_{i,j}^I = \widehat{C}_{i,I} \widehat{C}_{j,I} \left\{ \left(1 + \frac{(\widehat{\sigma}_{I-j+1}^I)^2}{(\widehat{f}_{I-j+1}^I)^2 S_{I-j+1}^{I+1}} \right) \prod_{l=I-j+2}^{I-1} \left(1 + \frac{(\widehat{\sigma}_l^I)^2 C_{I-l+1,l}}{(\widehat{f}_l^I)^2 (S_l^{I+1})^2} \right) - 1 \right\}.$$

For further details, see Wüthrich *et al.* [29] equations (3.16) to (3.18). Notice that in Wüthrich *et al.* [29], the indexes start from 0, contrary to the present paper starting from 1.

Nevertheless, in addition to those terms, we have cross-elements, which imply both triangles. Indeed, the covariance terms have to be computed and involve both the C and the D -triangles. In the sequel, we will focus on these terms in section 4.2.

4.1.2 Estimation error

In this section, the detailed expression inherent to the computation of the estimation error is exhibited:

$$\begin{aligned}
& \left(\mathbb{E} \left[\sum_{i=1}^I \left(\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1) \right) | D_I \right] \right)^2 \\
&= \underbrace{\sum_{i=1}^I \left(\mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \right)^2 + 2 \times \sum_{I \geq i > j \geq 1} \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^C(I+1) | D_I \right]}_{C\text{-triangle estimation error}} \\
&+ \underbrace{\sum_{i=1}^I \left(\mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \right)^2 + 2 \times \sum_{I \geq i > j \geq 1} \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right]}_{D\text{-triangle estimation error}} \\
&+ 2 \times \sum_{i=1}^I \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \\
&+ 2 \times \sum_{I \geq i > j \geq 1} \left\{ \begin{array}{l} \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \\ + \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^C(I+1) | D_I \right] \end{array} \right\}.
\end{aligned} \tag{21}$$

In this expression, we can see that some terms have already been computed in closed formulas in Wüthrich *et al.* [29] as soon as the standalone estimation errors are identified. For the estimation of these terms, Wüthrich used Approach 3 of Buchwalder *et al.* [4], for the C -triangle they correspond to (we suppose $i > j$):

$$\begin{cases} \widehat{\mathbb{E}}_{D_I} \left[\left(\mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \right)^2 \right] = \left(\widehat{C}_{i,I}^I \right)^2 \widehat{\Delta}_{i,I}^I \\ \widehat{\mathbb{E}}_{D_I} \left(\mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^C(I+1) | D_I \right] \right) = \widehat{C}_{i,I}^I \widehat{C}_{j,I}^I \widehat{\Lambda}_{j,I}^I. \end{cases}$$

Notice that:

$$\widehat{\Delta}_{i,I}^I = \frac{(\widehat{\sigma}_{I-i+1}^I)^2}{\left(\widehat{f}_{I-i+1}^I \right)^2 S_{I-i+1}^I} + \sum_{j=I-i+2}^{I-1} \left\{ \frac{(\widehat{\sigma}_j^I)^2}{\left(\widehat{f}_j^I \right)^2 S_j^I} \left(\frac{C_{I-j+1,j}}{S_j^{I+1}} \right)^2 \right\}$$

and

$$\widehat{\Lambda}_j^I = \frac{C_{j,I-j+1} (\widehat{\sigma}_{I-j+1}^I)^2}{S_{I-j+1}^{I+1} \left(\widehat{f}_{I-j+1}^I \right)^2 S_{I-j+1}^I} + \sum_{l=I-j+2}^{I-1} \left(\frac{C_{I-l+1,l}}{S_l^{I+1}} \right)^2 \frac{(\widehat{\sigma}_l^I)^2}{\left(\widehat{f}_l^I \right)^2 S_l^I}.$$

For further details, see Wüthrich *et al.* [29] equations (3.13), (5.14), (5.15) and (5.27). Notice that in Wüthrich *et al.* [29], the indexes start from 0, contrary to the present paper starting from 1.

Nevertheless, in addition to those terms, we have cross-elements that have to be computed. We will focus on these terms in section 4.3.

4.1.3 Final expression

By summing the exhibited expressions, estimation and process errors, we are able to determine the prediction error for all accident years. The terms

$$MSEP_{\sum_{i=1}^I \widehat{CDR}_i^C(I+1) | D_I} (0)$$

and

$$MSEP_{\sum_{i=1}^I \widehat{CDR}_i^D(I+1) | D_I} (0)$$

correspond to the standalone mean square errors of prediction computed thanks to the Wüthrich *et al.* model [29].

$$\begin{aligned}
& MSEP_{\sum_{i=1}^I (\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1)) | D_I}^{(0)} \\
&= MSEP_{\sum_{i=1}^I \widehat{CDR}_i^C(I+1) | D_I}^{(0)} + MSEP_{\sum_{i=1}^I \widehat{CDR}_i^D(I+1) | D_I}^{(0)} \\
&+ 2 \times \sum_{i=1}^I \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right] \\
&+ 2 \times \sum_{I \geq i > j \geq 1} \left\{ \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I \right] + \text{cov} \left[\widehat{CDR}_i^D(I+1), \widehat{CDR}_j^C(I+1) | D_I \right] \right\} \\
&+ 2 \times \sum_{i=1}^I \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \\
&+ 2 \times \sum_{I \geq i > j \geq 1} \left\{ \begin{array}{l} \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \\ + \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^C(I+1) | D_I \right] \end{array} \right\}.
\end{aligned} \tag{22}$$

As a consequence, four new elements have to be determined in addition to the standalone Wüthrich MSEP as for the computation of the consolidated MSEP:

Expressions inherent to the process error: $i \neq j$

$$\text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right] \text{ and } \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I \right]$$

Expressions inherent to the estimation error: $i \neq j$

$$\mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \text{ and } \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right]$$

4.2 Determination of closed-form expressions for the process error term in the prediction error

First of all, we are going to focus on the covariance-terms inherent to the process error and required to compute the prediction error in a multivariate framework.

4.2.1 Single accident year

Lemma 4.1. Conditional process error for a Single Accident Year *For a given accident year $i \in \{2, \dots, I\}$, the covariance term for a single accident year can be estimated as follows:*

$$\begin{aligned}
\widehat{\text{cov}} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right] &\approx \widehat{C}_{i,I}^I \times \widehat{D}_{i,I}^I \times \left\{ \left[\frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \times \widehat{g}_{I-i+1}^I} \times \frac{1}{\sqrt{C_{i,I-i+1} \times D_{i,I-i+1}}} \right] \right. \\
&\quad \left. + \sum_{k=I-i+2}^{I-1} \frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{\widehat{f}_k^I \times \widehat{g}_k^I} \right\}.
\end{aligned} \tag{23}$$

Proof. Notice that intermediate computations are provided in Appendices (B.2).

$$\begin{aligned}
\forall i \in \{2, \dots, I\}, \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right] &= \text{cov} \left[\widehat{C}_{i,I}^I - \widehat{C}_{i,I}^{I+1}, \widehat{D}_{i,I}^I - \widehat{D}_{i,I}^{I+1} | D_I \right] \\
&= \text{cov} \left[\widehat{C}_{i,I}^{I+1}, \widehat{D}_{i,I}^{I+1} | D_I \right] \quad \text{due to } D_I\text{-measurability of the random variables } C_{i,I}^I \text{ and } D_{i,I}^I \\
&= \underbrace{\mathbb{E} \left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{i,I}^{I+1} | D_I \right]}_A - \underbrace{\mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right]}_B.
\end{aligned}$$

A-term: thanks to the computed estimators, the following expression can be assessed (see equations (29) and (30) in Appendices (B.2)):

$$\begin{aligned}\widehat{\mathbb{E}}\left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{i,I}^{I+1} | D_I\right] &= \widehat{\mathbb{E}}\left[C_{i,I-i+2} \times D_{i,I-i+2} | D_I\right] \times \prod_{k=I-i+2}^{I-1} \widehat{\mathbb{E}}\left[\widehat{f}_k^{I+1} \times \widehat{g}_k^{I+1} | D_I\right] \\ &= \widehat{C}_{i,I}^I \times \widehat{D}_{i,I}^I \times \left[1 + \frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \times \widehat{g}_{I-i+1}^I} \times \frac{1}{\sqrt{C_{i,I-i+1} \times D_{i,I-i+1}}}\right] \\ &\quad \times \prod_{k=I-i+2}^{I-1} \left[\frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{\widehat{f}_k^I \times \widehat{g}_k^I} + 1\right].\end{aligned}$$

B-term: thanks to the computed estimators, the following expression can be assessed:

$$\widehat{\mathbb{E}}\left[\widehat{C}_{i,I}^{I+1} | D_I\right] \times \widehat{\mathbb{E}}\left[\widehat{D}_{i,I}^{I+1} | D_I\right] = \widehat{C}_{i,I}^I \times \widehat{D}_{i,I}^I.$$

As a consequence, the closed-form expression can be expressed as follows, by using estimators and the first order approximation $\forall L \in \mathbf{N}^*, \prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$ (Taylor approximation), (see equations (31) and (32) in Appendices (B.2)):

$$\begin{aligned}\widehat{\text{cov}}\left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I\right] &\approx \widehat{C}_{i,I}^I \times \widehat{D}_{i,I}^I \times \left\{ \left[\frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \times \widehat{g}_{I-i+1}^I} \times \frac{1}{\sqrt{C_{i,I-i+1} \times D_{i,I-i+1}}} \right] \right. \\ &\quad \left. + \sum_{k=I-i+2}^{I-1} \frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{\widehat{f}_k^I \times \widehat{g}_k^I} \right\}.\end{aligned}$$

□

4.2.2 All accident years

Lemma 4.2. Conditional process error for all accident years

Let us consider two accident years $i \in \{2, \dots, I\}, j \in \{1, \dots, I-1\}$, such that $i > j$. The covariance term, when considering all accident years, can be estimated as follows:

$$\begin{aligned}\widehat{\text{cov}}\left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I\right] &\approx \widehat{C}_{i,I}^I \times \widehat{D}_{j,I}^I \times \left[\frac{\widehat{\rho}_{I-j+1}}{\widehat{f}_{I-j+1}^I \times \widehat{g}_{I-j+1}^I} \times \sqrt{\frac{C_{j,I-j+1}}{D_{j,I-j+1}}} \times \frac{1}{S_{I-j+1}^{I+1}} \right. \\ &\quad \left. + \sum_{k=I-j+2}^{I-1} \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{S_k^{I+1} \times T_k^{I+1}} \times \frac{\widehat{\rho}_k}{\widehat{f}_k^I \times \widehat{g}_k^I} \right].\end{aligned}\tag{24}$$

Proof. Notice that intermediate computations are provided in Appendices (B.3).

$\forall i > j$,

$$\begin{aligned}\widehat{\text{cov}}\left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I\right] &= \widehat{\text{cov}}\left[\widehat{C}_{i,I}^I - \widehat{C}_{i,I}^{I+1}, \widehat{D}_{j,I}^I - \widehat{D}_{j,I}^{I+1} | D_I\right] \\ &= \widehat{\text{cov}}\left[\widehat{C}_{i,I}^{I+1}, \widehat{D}_{j,I}^{I+1} | D_I\right] \quad \text{due to } D_I \text{ measurability of the random variables } C_{i,I}^I \text{ and } D_{j,I}^I \\ &= \underbrace{\mathbb{E}\left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{j,I}^{I+1} | D_I\right]}_A - \underbrace{\mathbb{E}\left[\widehat{C}_{i,I}^{I+1} | D_I\right] \mathbb{E}\left[\widehat{D}_{j,I}^{I+1} | D_I\right]}_B.\end{aligned}$$

A-term: thanks to the computed estimators, the following expression can be assessed (see equations (32), (33) and (B.3) in Appendices (B.3)):

$\forall i > j$,

$$\widehat{\mathbb{E}}\left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{j,I}^{I+1} | D_I\right] = \left\{ \begin{aligned} &\widehat{f}_{I-i+1}^I \times C_{i,I-i+1} \times \prod_{k=I-i+2}^{I-j} \widehat{f}_k^I \times \frac{\widehat{\rho}_{I-j+1} \times \sqrt{C_{j,I-j+1} \times D_{j,I-j+1}}}{S_{I-j+1}^{I+1}} \\ &\quad + \widehat{f}_{I-j+1}^I \times \widehat{g}_{I-j+1}^I \times D_{j,I-j+1} \\ &\times \prod_{k=I-j+2}^{I-1} \left(\frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \widehat{\rho}_k \times \sqrt{C_{I-k+1,k} \times D_{I-k+1,k}} + \widehat{f}_k^I \times \widehat{g}_k^I \right). \end{aligned} \right.$$

B-term: thanks to the computed estimators, the following expression can be assessed:
 $\forall i > j$,

$$\widehat{\mathbb{E}} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \widehat{\mathbb{E}} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] = \left(\widehat{f}_{I-i+1}^I \times C_{i,I-i+1} \times \prod_{k=I-i+2}^{I-j} \widehat{f}_k^I \right) \times \left(\widehat{g}_{I-j+1}^I \times D_{j,I-j+1} \times \prod_{k=I-j+2}^{I-1} \widehat{g}_k^I \right).$$

As a consequence, the closed-form expression can be expressed as follows, by using estimators and the first order approximation $\forall L \in \mathbf{N}^*$, $\prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$ (see equations (35) and (36) in Appendices (B.3)):

$\forall i > j$,

$$\begin{aligned} \widehat{\text{cov}} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I \right] &\approx \widehat{C}_{i,I} \times \widehat{D}_{j,I} \times \left[\frac{\widehat{\rho}_{I-j+1}}{\widehat{f}_{I-j+1}^I \times \widehat{g}_{I-j+1}^I} \times \sqrt{\frac{C_{j,I-j+1}}{D_{j,I-j+1}}} \times \frac{1}{S_{I-j+1}^{I+1}} \right. \\ &\quad \left. + \sum_{k=I-j+2}^{I-1} \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{S_k^{I+1} \times T_k^{I+1}} \times \frac{\widehat{\rho}_k}{\widehat{f}_k^I \times \widehat{g}_k^I} \right]. \end{aligned}$$

□

Remark: the estimation of the symmetric expression $\widehat{\text{cov}} \left[\widehat{CDR}_j^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right]$, with $i > j$, is straightforward.

4.3 Determination of closed-form expressions for the estimation error term in the prediction error

In this section, we focus on the expressions inherent to the estimation error and required to compute the prediction error in a multivariate framework, *i.e.*:

$$\mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \text{ and } \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \text{ for } i \neq j.$$

As observed by Wüthrich *et al.* [29], these terms cannot be assessed explicitly since the true chain-ladder factors f_j are not known. To overcome this issue, we follow the approach developed in this study, *i.e.* use a conditional resampling methodology which corresponds to Approach 3 in Buchwalder *et al.* [4].

As a consequence the two expressions are going to be estimated subsequently by:

$$\begin{aligned} &\widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \right\}, \\ &\widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\} \text{ with } i \neq j. \end{aligned}$$

4.3.1 Single accident year

Lemma 4.3. Conditional estimation error for a single accident year For a given accident year $i \in \{2, \dots, I\}$, the following approximation has been exhibited:

$$\begin{aligned} \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \right\} &\approx \widehat{C}_{i,I} \widehat{D}_{i,I} \left\{ \sum_{k=I-i+1}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \right. \\ &\quad \left. - \sum_{k=I-i+2}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \left(\frac{T_k^I}{T_k^{I+1}} + \frac{S_k^I}{S_k^{I+1}} - \frac{S_k^I T_k^I}{S_k^{I+1} T_k^{I+1}} \right) \right\}. \end{aligned} \quad (25)$$

Proof. Notice that intermediate computations are provided in Appendices (C.2).

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned} & \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \right\} \\ &= \underbrace{\mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{i,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\}}_B \\ &+ \underbrace{\mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\}}_A. \end{aligned}$$

A-term: the following expression can be assessed (see equation (36) in Appendices (C.2)):

$$\begin{aligned} & \forall i \in \{2, \dots, I\}, \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\} \\ &= C_{i,I-i+1} f_{I-i+1} D_{i,I-i+1} g_{I-i+1} \prod_{k=I-i+2}^{I-1} \left\{ \frac{1}{S_k^{I+1} T_k^{I+1}} \left(\rho_k \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right\} \end{aligned}$$

B-term: the following expression can be assessed (see equation (37) in Appendices (C.2)): $\forall i \in \{2, \dots, I\}$,

$$\begin{aligned} & \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{i,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\} \\ &= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \left[\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + f_k g_k \right] \\ &- C_{i,I-i+1} D_{i,I-i+1} g_{I-i+1} f_{I-i+1} \left\{ \prod_{k=I-i+2}^{I-1} \left[\frac{T_k^I}{T_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \right. \\ &\quad \left. + \prod_{k=I-i+2}^{I-1} \left[\frac{S_k^I}{S_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \right\}. \end{aligned}$$

As a consequence, the closed-form expression can be expressed as follows, by using estimators and the first order approximation $\forall L \in \mathbf{N}^*$, $\prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$ (see equations (38) and (C.2) in Appendices (C.2)):

$$\begin{aligned} \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \right\} &\approx \widehat{C}_{i,I} \widehat{D}_{i,I} \left\{ \sum_{k=I-i+1}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \right. \\ &\quad \left. - \sum_{k=I-i+2}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \left(\frac{T_k^I}{T_k^{I+1}} + \frac{S_k^I}{S_k^{I+1}} - \frac{S_k^I T_k^I}{S_k^{I+1} T_k^{I+1}} \right) \right\}. \end{aligned}$$

□

4.3.2 All accident years

Lemma 4.4. Conditional estimation error for all accident years *Let us consider two accident years $i \in \{2, \dots, I\}$ and $j \in \{1, \dots, I-1\}$. For all accident years, the following approximation has been exhibited:*

$$\begin{aligned} & \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\} \\ &\approx \widehat{C}_{i,I} \widehat{D}_{j,I} \left\{ \frac{\widehat{\rho}_{I-j+1}}{S_{I-j+1}^I T_{I-j+1}^I} \left(\frac{\sum_{i=1}^{j-1} \sqrt{C_{i,I-j+1} D_{i,I-j+1}}}{\widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \right) \left(1 - \frac{S_{I-j+1}^I}{S_{I-j+1}^{I+1}} \right) \right. \\ &\quad \left. + \sum_{k=I-j+2}^{I-1} \left(\rho_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \frac{C_{I-k+1,k} D_{I-k+1,k}}{S_k^I T_k^I S_k^{I+1} T_k^{I+1}} \right) \right\}. \end{aligned} \tag{26}$$

Proof. Notice that intermediate computations are provided in Appendices (C.3).

$$\begin{aligned}
& \forall i > j, \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\} \\
&= \underbrace{\mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{j,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\}}_B \\
&+ \underbrace{\mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\}}_A.
\end{aligned}$$

A-term: the following expression can be assessed (see equation (39) in Appendices (C.3)):

$$\begin{aligned}
\mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} &= f_{I-i+1} C_{i,I-i+1} g_{I-j+1} D_{j,I-j+1} \\
&\times \prod_{k=I-i+2}^{I-j+1} f_k \prod_{k=I-j+2}^{I-1} \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \left(\rho_k \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right]
\end{aligned}$$

B-term: the following expression can be assessed (see equation (40) in Appendices (C.3)):

$$\begin{aligned}
& \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{j,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\} \\
&= C_{i,I-i+1} D_{j,I-j+1} \prod_{k=I-i+1}^{I-j} f_k \prod_{k=I-j+1}^{I-1} \left[\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + f_k g_k \right] \\
&- C_{i,I-i+1} g_{I-j+1} D_{j,I-j+1} \prod_{k=I-i+1}^{I-j+1} f_k \prod_{k=I-j+2}^{I-1} \left[\frac{T_k^I}{T_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \\
&- D_{j,I-j+1} f_{I-i+1} C_{i,I-i+1} \prod_{k=I-i+2}^{I-j} f_k \prod_{k=I-j+1}^{I-1} \left[\frac{S_k^I}{S_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right].
\end{aligned}$$

As a consequence, the closed-form expression can be expressed as follows, by using estimators and the first order approximation $\forall L \in \mathbf{N}^*, \prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$ (see equations (41) and (42) in Appendices (C.3)):

$$\begin{aligned}
& \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\} \\
&\approx \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \left\{ \frac{\widehat{\rho}_{I-j+1}}{S_{I-j+1}^I T_{I-j+1}^I} \left(\frac{\sum_{i=1}^{j-1} \sqrt{C_{i,I-j+1} D_{i,I-j+1}}}{\widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \right) \left(1 - \frac{S_{I-j+1}^I}{S_{I-j+1}^{I+1}} \right) \right. \\
&+ \left. \sum_{k=I-j+2}^{I-1} \left(\rho_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \frac{C_{I-k+1,k} D_{I-k+1,k}}{S_k^I T_k^I S_k^{I+1} T_k^{I+1}} \right) \right\}.
\end{aligned}$$

□

Remark: the computation of the symmetric expression:

$$\mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\},$$

with $i > j$, is straightforward.

As a synthesis of part 4, we remind the key equations which lead to the one-year MSEP in the multi-

variate framework. The one-year MSEP is defined as follows:

$$\begin{aligned}
MSEP_{\sum_{i=1}^I (\widehat{CDR}_i^C(I+1) + \widehat{CDR}_i^D(I+1))|D_I}(0) &= MSEP_{\sum_{i=1}^I \widehat{CDR}_i^C(I+1)|D_I}(0) + MSEP_{\sum_{i=1}^I \widehat{CDR}_i^D(I+1)|D_I}(0) \\
&+ 2 \times \sum_{i=1}^I \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1)|D_I \right] \\
&+ 2 \times \sum_{I \geq i > j \geq 1} \left\{ \begin{array}{l} \text{cov}[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1)|D_I] \\ + \text{cov}[\widehat{CDR}_i^D(I+1), \widehat{CDR}_j^C(I+1)|D_I] \end{array} \right\} \\
&+ 2 \times \sum_{i=1}^I \mathbb{E} \left[\widehat{CDR}_i^C(I+1)|D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1)|D_I \right] \\
&+ 2 \times \sum_{I \geq i > j \geq 1} \left\{ \begin{array}{l} \mathbb{E}[\widehat{CDR}_i^C(I+1)|D_I] \mathbb{E}[\widehat{CDR}_j^D(I+1)|D_I] \\ + \mathbb{E}[\widehat{CDR}_i^D(I+1)|D_I] \mathbb{E}[\widehat{CDR}_j^C(I+1)|D_I] \end{array} \right\}.
\end{aligned}$$

The terms $MSEP_{\sum_{i=1}^I \widehat{CDR}_i^C(I+1)|D_I}(0)$ and $MSEP_{\sum_{i=1}^I \widehat{CDR}_i^D(I+1)|D_I}(0)$ correspond to the standalone mean square errors of prediction computed thanks to the Wüthrich *et al.* model [29].

$$\begin{aligned}
\sum_{i=1}^I \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1)|D_I \right] &= \sum_{i=1}^I \widehat{C}_{i,I}^I \times \widehat{D}_{i,I}^I \times \left\{ \left[\frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \times \widehat{g}_{I-i+1}^I} \times \frac{1}{\sqrt{C_{i,I-i+1} \times D_{i,I-i+1}}} \right] \right. \\
&\quad \left. + \sum_{k=I-i+2}^{I-1} \frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{\widehat{f}_k^I \times \widehat{g}_k^I} \right\} ..
\end{aligned}$$

$$\begin{aligned}
&\sum_{I \geq i > j \geq 1} \left\{ \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1)|D_I \right] + \text{cov} \left[\widehat{CDR}_i^D(I+1), \widehat{CDR}_j^C(I+1)|D_I \right] \right\} \\
&= \sum_{i=1}^I \widehat{C}_{i,I} \times \widehat{D}_{j,I} \times \left[\frac{\widehat{\rho}_{I-j+1}}{\widehat{f}_{I-j+1}^I \times \widehat{g}_{I-j+1}^I} \times \sqrt{\frac{C_{j,I-j+1}}{D_{j,I-j+1}}} \times \frac{1}{S_{I-j+1}^{I+1}} \right. \\
&\quad \left. + \sum_{k=I-j+2}^{I-1} \frac{\sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}}{S_k^{I+1} \times T_k^{I+1}} \times \frac{\widehat{\rho}_k}{\widehat{f}_k^I \times \widehat{g}_k^I} \right] \\
&\quad + \sum_{i=1}^I \widehat{D}_{i,I} \times \widehat{C}_{j,I} \times \left[\frac{\widehat{\rho}_{I-j+1}}{\widehat{f}_{I-j+1}^I \times \widehat{g}_{I-j+1}^I} \times \sqrt{\frac{D_{j,I-j+1}}{C_{j,I-j+1}}} \times \frac{1}{T_{I-j+1}^{I+1}} \right. \\
&\quad \left. + \sum_{k=I-j+2}^{I-1} \frac{\sqrt{D_{I-k+1,k} \times C_{I-k+1,k}}}{T_k^{I+1} \times S_k^{I+1}} \times \frac{\widehat{\rho}_k}{\widehat{f}_k^I \times \widehat{g}_k^I} \right].
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^I \mathbb{E} \left[\widehat{CDR}_i^C(I+1)|D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1)|D_I \right] &= \sum_{i=1}^I \widehat{C}_{i,I} \widehat{D}_{i,I} \left\{ \sum_{k=I-i+1}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \right. \\
&\quad \left. - \sum_{k=I-i+2}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \left(\frac{T_k^I}{T_k^{I+1}} + \frac{S_k^I}{S_k^{I+1}} - \frac{S_k^I T_k^I}{S_k^{I+1} T_k^{I+1}} \right) \right\}.
\end{aligned}$$

$$\begin{aligned}
& \sum_{I \geq i > j \geq 1} \left\{ \begin{array}{l} \mathbb{E}[\widehat{CDR}_i^C(I+1)|D_I] \mathbb{E}[\widehat{CDR}_j^D(I+1)|D_I] \\ + \mathbb{E}[\widehat{CDR}_i^D(I+1)|D_I] \mathbb{E}[\widehat{CDR}_j^C(I+1)|D_I] \end{array} \right\} \\
&= \sum_{I \geq i > j \geq 1} \widehat{C}_{i,I} \widehat{D}_{j,I} \left\{ \frac{\widehat{\rho}_{I-j+1}}{S_{I-j+1}^I T_{I-j+1}^I} \left(\frac{\sum_{i=1}^{j-1} \sqrt{C_{i,I-j+1} D_{i,I-j+1}}}{\widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \right) \left(1 - \frac{S_{I-j+1}^I}{S_{I-j+1}^{I+1}} \right) \right. \\
&+ \sum_{k=I-j+2}^{I-1} \left(\rho_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \frac{C_{I-k+1,k} D_{I-k+1,k}}{S_k^I T_k^I S_k^{I+1} T_k^{I+1}} \right) \left. \right\} \\
&+ \sum_{I \geq i > j \geq 1} \widehat{D}_{i,I} \widehat{C}_{j,I} \left\{ \frac{\widehat{\rho}_{I-j+1}}{T_{I-j+1}^I S_{I-j+1}^I} \left(\frac{\sum_{i=1}^{j-1} \sqrt{D_{i,I-j+1} C_{i,I-j+1}}}{\widehat{g}_{I-j+1}^I \widehat{f}_{I-j+1}^I} \right) \left(1 - \frac{T_{I-j+1}^I}{T_{I-j+1}^{I+1}} \right) \right. \\
&+ \sum_{k=I-j+2}^{I-1} \left(\rho_k \frac{\sum_{i=1}^{I-k} \sqrt{D_{i,k} C_{i,k}}}{\widehat{g}_k^I \widehat{f}_k^I} \frac{D_{I-k+1,k} C_{I-k+1,k}}{T_k^I S_k^I T_k^{I+1} S_k^{I+1}} \right) \left. \right\}.
\end{aligned}$$

5 Case study

To give a numerical example, we consider two run-off claims triangles of a European insurance company, which contain information of the motor third party liability (MTPL) and commercial third party liability business (CTPL). The motor third party liability business covers the motor and third party guarantees for motorised vehicles (motorcycles, cars). The commercial third party liability covers third party liabilities guarantees for products such as pharmaceuticals and environmental damages. These two lines of business show mutual dependency since certain events (*e.g.* bodily injury claims or change in case settlement policy of the company) may influence both run-off portfolios.

The cumulative claims incurred triangles observed for the MTPL and CTPL lines of business are given in Table 2 and Table 3 (page 19) respectively. We suppose that $I = J = 14$.

In Table 1, we assess the univariate CL estimates of the ultimate claims per accident year and then deduce the outstanding claims reserve per accident year by subtracting the ultimate claims with the observed cumulative payments for both portfolios.

We notice that for the first accident year, there are still reserves, which means that there are still some outstanding claims after 14 years of development. A solution to this issue could be to use a tail-factor on paid triangles instead on incurred triangles as mentioned in Wüthrich [27].

Moreover, these two lines of business produce a close mean outstanding claims reserve, but with different levels of volatility around their respective means. By having two lines of business with comparable size, we will be able to compare our results with those based on aggregated cumulative incurred triangles.

Table 4 (page 19) shows the estimates of the chain-ladder development factors \hat{f}_j^I and \hat{g}_j^I , the volatilities of the development factors $\hat{\sigma}_j^2$ and $\hat{\tau}_j^2$ of the respective claims triangles of motor liability and commer-

	Ultimate Estimates		Reserve Estimates	
	MTPL	CTPL	MTPL	CTPL
1	892 250	805 366	57 012	61 567
2	892 558	1 082 429	49 723	57 437
3	916 196	892 785	52 191	153 547
4	919 561	1 436 382	60 173	78 156
5	833 509	603 234	57 236	78 499
6	715 618	657 787	39 132	78 346
7	691 584	574 683	57 311	109 862
8	597 702	488 959	42 121	95 498
9	584 857	624 957	41 904	218 887
10	610 667	560 564	48 707	224 829
11	593 734	715 365	50 244	358 481
12	665 059	829 814	87 743	536 594
13	629 255	813 671	96 313	615 329
14	569 105	819 206	221 025	741 059
All Accident Years	10 111 655	10 905 200	960 834	3 408 090

Table 1: Univariate Chain-Ladder estimates of the ultimate claims and reserves per accident year and for aggregated accident years

cial liability businesses. We have also estimated the correlation factors between these two portfolios by development year noted $\frac{\hat{\rho}_j}{\hat{\sigma}_j \hat{\tau}_j}$ according to the formula (3.13.) in Wüthrich [24] divided by the corresponding volatilities of both portfolios.

All the coefficients $\frac{\hat{\rho}_j}{\hat{\sigma}_j \hat{\tau}_j}$ are positive except for the development year 6. It is worth mentioning that the estimators of the coefficients of correlation $\frac{\hat{\rho}_j}{\hat{\sigma}_j \hat{\tau}_j}$ may lead to values outside the interval $[-1; 1]$. Moreover, this exception should not be overstated since the estimates $\frac{\hat{\rho}_{11}}{\hat{\sigma}_{11} \hat{\tau}_{11}}, \frac{\hat{\rho}_{12}}{\hat{\sigma}_{12} \hat{\tau}_{12}}, \frac{\hat{\rho}_{13}}{\hat{\sigma}_{13} \hat{\tau}_{13}}$ are based on a smaller number of points (less than 4 points).

For both lines of business, it is assumed that the development coefficients reach 1 after 14 years of development.

The CTPL is a more volatile business than the MTPL one due to the underlying guarantees. Indeed, the CTPL is more exposed to market cycles with claims volumes at the first development year varying significantly for certain accident years.

Table 5 shows the prediction error according to Mack [12] with its two components process and estimation errors of the total ultimate claims amount for all aggregated accident years.

	$\sqrt{MSEP^{\text{Mack}}}$	Mack process error	Mack estimation error
MTPL	50 633	30 693	40 270
CTPL	287 618	204 427	202 321
Whole portfolio	326 358	214 537	245 934

Table 5: Mack prediction, process and estimation errors of the total ultimate claims amount

The whole portfolio mentioned in the third line corresponds to the aggregated triangle built by adding two by two each term of the individual triangles (MTPL and CTPL). The results are obtained by Mack's methodology on the whole portfolio.

The formula presented in Wüthrich [23] allows estimating the one-year prediction error (split into the process and the estimation errors). Table 6 summarizes the numerical results.

	$\sqrt{MSEP^{\text{Wüthrich}}}$	Wüthrich process error	Wüthrich estimation error
MTPL	34 705	19 251	28 876
CTPL	190 107	133 190	135 651
Whole portfolio	215 519	134 218	168 624

Table 6: One-year prediction, process and estimation errors

The results for the whole portfolio are obtained by applying the Wüthrich methodology [23] on the aggregated triangle (MTPL and CTPL).

The Wüthrich estimation error calculated in Table 6 are obtained by applying a proxy of the analytical expression, which has been obtained with a bootstrap-based approach (see Buchwalder *et al.* [4]). Since the estimation error has been calculated using a bootstrap-based approach, it is interesting to notice that the probability spaces used for the determination of the process and the estimation errors are different. Indeed, the process error is based on the available information at time I whereas the estimation error is based on the pseudo-universe obtained by resampling of the bootstrap procedure. Therefore, we have:

$$\forall i \in \{1, \dots, I\}, \mathbb{E} \left[\left(C \hat{D}R_i^C + C \hat{D}R_i^D \right)^2 | D_I \right] = \underbrace{\mathbb{V} \left[C \hat{D}R_i^C + C \hat{D}R_i^D | D_I \right]}_{\text{Process error}} + \underbrace{\left[\mathbb{E} \left(C \hat{D}R_i^C + C \hat{D}R_i^D | D_I^* \right) \right]^2}_{\text{Estimation error}}. \quad (27)$$

where D_I denotes the information available at calendar year I and D_I^* the pseudo-information generated by bootstrap for calendar year I .

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	834 611	923 569	943 699	936 804	932 637	927 111	922 944	927 817	920 572	915 508	911 078	904 014	895 944	892 250
2	858 474	955 785	946 468	950 824	945 156	930 778	930 042	924 274	919 936	912 177	907 319	900 203	896 253	
3	830 316	935 959	960 512	967 358	969 739	961 614	950 089	945 055	936 860	928 636	928 666	926 160		
4	845 315	965 770	981 802	984 162	973 942	972 347	962 975	954 452	946 994	941 234	935 242			
5	787 572	878 920	888 915	896 325	889 604	886 389	875 703	871 071	862 843	851 233				
6	692 643	765 518	764 502	767 610	763 254	755 667	750 365	742 703	737 007					
7	655 898	741 571	742 328	740 093	733 828	729 162	724 620	717 762						
8	591 871	640 821	643 165	638 470	633 228	627 796	623 752							
9	570 246	666 402	627 613	628 243	623 673	614 909								
10	622 216	666 402	657 536	654 398	647 200									
11	568 618	638 340	634 628	633 276										
12	607 842	706 250	708 836											
13	578 502	667 146												
14	539 380													

Table 2: Observed cumulative incurred data in the MTPL line of business

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	587 191	891 617	807 303	860 106	843 256	827 731	807 276	806 538	810 362	813 412	813 080	812 143	808 640	805 366
2	790 405	1 059 629	1 086 339	1 085 717	1 077 136	1 066 249	1 065 045	1 078 606	1 096 027	1 090 848	1 076 231	1 073 815	1 086 829	
3	530 802	774 194	884 255	852 404	908 956	881 827	880 711	874 587	885 103	882 540	865 962	891 916		
4	687 419	1 366 725	1 435 400	1 463 897	1 553 233	1 510 406	1 541 317	1 496 476	1 527 887	1 524 715	1 423 310			
5	474 872	585 629	573 094	636 121	638 504	642 020	646 794	648 179	616 355	616 760				
6	399 151	570 497	651 594	712 736	730 269	715 261	698 880	690 195	673 554					
7	340 700	519 797	577 136	607 645	604 778	603 979	591 685	586 915						
8	445 240	625 730	593 153	598 443	590 624	525 914	503 423							
9	459 560	614 959	611 054	589 067	595 457	647 099								
10	480 263	697 246	736 954	743 668										
11	511 344	766 512	852 561											
12	564 255													
13														
14	538 173													

Table 3: Observed cumulative incurred data in the CTPL line of business

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
f_j^I	1.119	1.005	1.001	0.994	0.992	0.993	0.995	0.992	0.992	0.996	0.994	0.993	0.996	1.000
g_j^I	1.481	1.056	1.012	1.022	0.975	0.994	0.992	1.003	0.998	0.969	1.008	1.005	0.996	1.000
$\hat{\sigma}_j$	20.43	11.55	5.04	3.61	4.14	3.67	4.94	1.54	2.96	2.74	2.83	3.05	2.83	
$\hat{\tau}_j$	143.55	47.82	41.50	31.36	26.18	19.08	14.52	24.54	3.12	32.42	17.31	11.18	7.22	
$\frac{\rho_j}{\sigma_j \tau_j}$	33%	22%	43%	18%	6%	-39%	36%	52%	11%	38%	100%	100%	100%	

Table 4: Estimated Values of the parameters

For the sake of simplicity, the estimation error obtained from the pseudo-data converges towards the one knowing only the available information. Therefore, we consider the following equality:

$$\forall i \in \{1, \dots, I\}, \mathbb{E} \left[\left(C \hat{D} R_i^C + C \hat{D} R_i^D \right) | D_i^* \right] = \mathbb{E} \left[\left(C \hat{D} R_i^C + C \hat{D} R_i^D \right) | D_i \right]. \quad (28)$$

Table 7 shows a comparative table of the ultimate prediction errors with Mack [12] and Braun [2].

	Mack Ultimate	Reserves	$\sqrt{MSEP}^{\text{Mack}}$	$\sqrt{MSEP}^{\text{Braun-Ultimate}}$
MTPL	10 111 655	960 834	50 633	-
CTPL	10 756 715	3 259 604	287 618	-
Whole portfolio	20 868 369	4 220 439	326 358	308 747

Table 7: Comparison between the prediction errors obtained with Mack [12] and Braun [2]

The reserves for the whole portfolio are obtained by summing the levels of reserve of MTPL and CTPL. The total reserves estimated from the aggregated triangle with the CL approach would be equal to 2 233 417, which is lower than the sum 4 220 439. Indeed, according to Ajne [1], the necessary and sufficient conditions to have the same reserves CL estimate with the aggregated triangle are not respected by the current triangles. However, this difference in the estimated values of reserves will not impact the value of the correlation between the two LoBs since we focus on the MSEPs' values.

The column $\sqrt{MSEP}^{\text{Mack}}$ shows the prediction error obtained by applying Mack's methodology [12] on the aggregated triangle of the MTPL and CTPL business. The column $\sqrt{MSEP}^{\text{Braun-Ultimate}}$ gives us the prediction error of the ultimate claims amounts for the run-off triangles by taking into account their correlation according to Braun's multivariate methodology [2]. In our case study, his prediction error is lower than the prediction error obtained on the aggregated triangle.

Table 8 shows the conditional standard error of prediction for the one-year run-off uncertainty for the aggregated claims over all accident years within the Wüthrich framework [24] and the framework developed in this paper respectively. To be comparable with our approach, the Wüthrich [23] formulas have been applied on the whole run-off portfolio (MTPL+CTPL).

	Mack Ultimate	Reserves	$\sqrt{MSEP}^{\text{Wüthrich}}$	$\sqrt{MSEP}^{\text{1Y-Multivariate}}$
MTPL	10 111 655	960 834	34 705	-
CTPL	10 756 715	3 259 604	190 107	-
Whole portfolio	20 868 369	4 220 439	215 519	212 289

Table 8: Comparison between the one-year prediction errors obtained with Wüthrich [23] and our modeling

We see that the multivariate approach developed in this paper leads to a lower conditional standard error of prediction than the values obtained by applying the formulas in Wüthrich [23] on the aggregated triangle. The Wüthrich one-year prediction error remains, by construction, smaller than the ultimate Mack prediction error (see Table 7).

We then deduce the implied average linear correlation coefficient $C_{MTPL,CTPL}^m$ on the one-year risk between the MTPL and the CTPL lines of business from the Wüthrich MSEP (in this case, $m = \text{Wüthrich}$) and the MSEP developed in this paper (in this case, $m = \text{1Y-multivariate}$), *i.e.* we deduce the solution $C_{MTPL,CTPL}^m$ from the equation (notice that the standalone MSEPs are always estimated thanks to the Wüthrich methodology):

$$MSEP_{MTPL+CTPL}^m = MSEP_{MTPL} + MSEP_{CTPL} + 2 \times C_{MTPL,CTPL}^m \times \sqrt{MSEP_{MTPL} \times MSEP_{CTPL}}.$$

Thus:

$$C_{MTPL,CTPL}^m = \frac{MSEP_{MTPL+CTPL}^m - MSEP_{MTPL} - MSEP_{CTPL}}{2 \times \sqrt{MSEP_{MTPL} \times MSEP_{CTPL}}}.$$

	Implicit correlation between MTPL and CTPL
$C_{MTPL,CTPL}^{Wüthrich}$	69%
$C_{MTPL,CTPL}^{1Y-multivariate}$	59%

Table 9: Comparison of the correlation coefficients between MTPL and CTPL

The formula above provides only a proxy of the linear correlation between the MSEPs of MTPL and CTPL due to the aforementioned approximation of the estimation error.

Table 9¹ compares these two correlation coefficients between the MTPL and CTPL lines of business.

Finally, we obtain a lower implicit correlation between MTPL and CTPL based on the MSEP calculated with the approach developed in this paper, compared to the one determined with the Wüthrich formula on the whole portfolio.

The prediction error of Braun [2] relies on the assumption of a simultaneous correlation between the two claims triangles. However, this simultaneity may not be true for the insurance industry.

Therefore, the correlation between the two lines of business may be different from the simultaneous case. For instance, manufacturing defects of vehicles may impact first the MTPL business due to bodily claims injury and lawsuits caused by car accidents. Then, the CTPL line will be affected due to a massive recall of cars. For instance, if we consider that the manufacturing defects will affect the two lines with a year long delay, we obtain an implicit 1y-multivariate correlation equal to 23% by having applied the formula of the 1-year multivariate prediction error on modified triangles (removal of the claims corresponding to the first development year for the CTPL line (Table 10) and the last diagonal for the MTPL business (Table 11)).

¹For an informational purpose, we specify the QIS5 linear correlation parameter between the MTPL and CTPL lines of business, which is equal to 50%. However, this value cannot be compared to the values obtained in Table 9 for several reasons:

- the premium and reserve risks are aggregated within each LoB by determining the standard deviation for the two sub-risks under the assumption of a correlation coefficient of 50%,
- then, the overall standard deviation is calculated by aggregating the standard deviations of the individual LoBs with the QIS5 correlation matrix.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	801 617	897 303	860 106	843 256	827 731	807 276	806 538	810 362	813 412	813 080	812 143	808 640	805 366
2	1 050 629	1 086 339	1 081 717	1 077 126	1 066 549	1 065 045	1 078 606	1 086 027	1 089 848	1 076 291	1 073 815	1 086 829	
3	774 164	884 255	852 474	908 936	881 827	880 711	874 587	885 433	882 540	865 962	891 916		
4	1 366 725	1 433 400	1 463 897	1 533 233	1 510 406	1 541 317	1 496 476	1 327 887	1 524 715	1 425 310			
5	583 629	573 094	636 121	638 504	642 020	646 794	648 179	616 585	616 760				
6	370 497	651 594	712 736	730 269	713 261	698 880	680 195	673 554					
7	319 797	577 136	607 645	604 778	603 979	591 685	586 915						
8	349 248	593 153	598 443	590 624	525 914	503 423							
9	625 730	627 660	626 732	665 918	647 099								
10	614 959	611 054	589 067	595 457									
11	697 246	736 954	743 668										
12	766 512	852 561											
13	852 561												

Table 10: Claims triangle with the first column removed for the CTPL line of business

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	834 611	923 569	943 699	936 804	932 637	927 111	922 944	927 817	920 572	915 508	911 078	904 014	895 944
2	858 474	955 785	946 468	950 824	945 156	930 778	930 042	924 274	919 936	912 177	907 319	900 203	
3	830 316	935 959	960 512	967 358	969 739	961 614	950 089	945 055	936 860	928 636	928 666		
4	845 315	965 770	981 802	984 162	973 942	972 347	962 975	954 452	946 994	941 234			
5	787 572	878 920	888 915	896 325	889 604	886 389	875 703	871 071	862 843				
6	692 643	765 518	764 502	767 610	763 254	755 667	750 365	742 703					
7	655 898	741 571	742 328	740 093	733 802	729 162	724 620						
8	591 871	640 821	643 165	638 470	633 228	627 796							
9	570 246	631 108	627 613	628 243	623 673								
10	622 216	666 402	657 536	654 398									
11	568 618	638 340	634 628										
12	607 842	706 250											
13	578 502												

Table 11: Claims triangle with the last diagonal removed for the MTPL line of business

Conclusion

In this paper, we provide an analytical expression of the multivariate prediction error by generalizing the approach proposed by Braun [2] to the one-year horizon, which is Solvency II compliant. The present approach allows to incorporate a priori correlation between lines of business for regulatory and modeling purposes and to give information about the MSEF of a whole run-off portfolio, which helps the risk manager to learn about the behavior of the run-off subportfolios.

We have chosen to develop a convenient and straightforward modeling despite possible limitations especially due to imposed underlying hypotheses. For instance, the considered triangles have to be of the same size: this can be partially overcome when considering truncated triangles in order to get equal sizes. While doing so, some information is voluntarily neglected and thus the prediction error is not going to be well estimated. But we underline the fact that each risk manager should investigate the reasons why the capital charges in front of the reserve risk for different line of business are correlated before using mathematical model in order to understand the limit of the underlying assumptions.

Finally, the result obtained on our case study confirms the interest of this new methodology allowing deeper analysis to understand interaction between subportfolios. Further research is needed in order to generalize multivariate approaches in the Solvency II framework taking into account tail dependencies and interactions with other non-life risks, such as premium and catastrophic risks.

References

- [1] AJNE, B. Additivity of Chain-Ladder Projections. *ASTIN Bulletin* 24 (1994), 313–318.
- [2] BRAUN, C. The Prediction Error of the Chain-Ladder Method applied to Correlated Run-off Triangles. *ASTIN Bulletin* 34, 2 (2004), 399–423.
- [3] BREHM, P. Correlation and the Aggregation of Unpaid Loss Distributions. *Casualty Actuarial Society Forum* (2002), 1–23.
- [4] BUCHWALDER, M., BÜHLMANN, H., MERZ, M., AND WÜTHRICH, M. The Mean Square Error of Prediction in the Chain-Ladder Reserving Method. *ASTIN Bulletin* 36, 2 (2006), 521–542.
- [5] DE JONG, P. Forecasting Run-off Triangles. *North American Actuarial Journal* 10, 2 (2006), 28–38.
- [6] DE JONG, P. Modeling Dependence between Loss Triangles. *North American Actuarial Journal* 16, 1 (2011).
- [7] DIERS, D. Stochastic Re-Reserving in Multi-Year Internal Models. An Approach based on Simulations. *ASTIN Colloquium, Helsinki* (2008).
- [8] GILLET, A., AND SERRA, B. Effets de la dépendance entre différentes branches sur le calcul des provisions.
- [9] HESS, K., SCHMIDT, K., AND ZOCHER, M. Multivariate Loss Prediction in the Multivariate Additive Model. *Insurance: Mathematics and Economics* 39 (2006), 185–191.
- [10] HOLMBERG, R. Correlation and the Measurement of Loss Reserve Variability. *Casualty Actuarial Society Forum* (1994), 247–277.
- [11] KIRSCHNER, G., KERLEY, C., AND ISAACS, B. Two Approaches to Calculating Correlated Reserve Indications across Multiple Lines of Business. *Casualty Actuarial Society Forum* (2002), 211–246.
- [12] MACK, T. Distribution-free Calculation of the Standard Error of Chain-Ladder Reserve Estimates. *ASTIN Bulletin* 23, 2 (1993), 213–225.
- [13] MACK, T., QUARG, G., AND BRAUN, C. The Mean Square Error of Prediction in the Chain-Ladder Reserving Method - A Comment. *ASTIN Bulletin* 36, 2 (2006), 543–552.
- [14] MERZ, M., AND WÜTHRICH, M. Prediction Error of the Expected Claims Development Result. *Bulletin Swiss Assoc. Act. 1* (2007), 117–137.
- [15] OHLSSON, E., AND LAUZENINGKS, J. The One-Year Non-Life Insurance Risk. *ASTIN Colloquium in Manchester* (2008).
- [16] PRÖHL, C., AND SCHMIDT, K. Multivariate Chain-Ladder. *Dresdner Schriften zur Versicherungsmathematik* (2005).
- [17] SCHMIDT, K. Optimal and Additive Loss Reserving for Dependent Lines of Business. *Casualty Actuarial Society Forum* (2006), 319–351.
- [18] SCHMIDT, K., AND SCHNAUS, A. An Extension of Mack’s Model for the Chain-Ladder Method. *ASTIN Bulletin* 26, 2 (1996).
- [19] SHI, P. Multivariate Claims Reserving using Tweedie Model. *Working Paper* (2011b).
- [20] SHI, P., BASU, S., AND MEYERS, G. A Bayesian log-normal Model for multivariate Loss Reserving. *North American Actuarial Journal* 16, 1 (2009).
- [21] SHI, P., AND FREES, E. Dependent Loss Reserving using Copulas. *ASTIN Bulletin* 41, 2 (2011a).

- [22] TAYLOR, G., AND MCGUIRE, G. A Synchronous Bootstrap to Account for Dependencies between Lines of Business in the Estimation of Loss Reserve Prediction Error. *North American Actuarial Journal* 11, 3 (2007).
- [23] WÜTHRICH, M., AND MERZ, M. Modelling the Claims Development Result for Solvency Purposes. *Casualty Actuarial Society E-Forum* (2008a), 542–568.
- [24] WÜTHRICH, M., AND MERZ, M. Prediction Error of the Multivariate Chain-Ladder Reserving Method. *North American Actuarial Journal* 12, 2 (2008b), 175–197.
- [25] WÜTHRICH, M., AND MERZ, M. Prediction Error of the Multivariate Additive Loss Reserving Method for Dependent Lines of Business. *Variance* 3, 2 (2009a), 131–151.
- [26] WÜTHRICH, M., AND MERZ, M. Combining Chain-Ladder and Additive Loss Reserving Methods for Dependent Lines of Business. *Variance* 3, 2 (2009b), 270–291.
- [27] WÜTHRICH, M., AND MERZ, M. Estimation of Tail Development Factors in the paid-incurred Chain Reserving Method. *Preprint version, to be appeared in Variance* (2013).
- [28] WÜTHRICH, M., MERZ, M., AND HASHORVA, E. Dependence Modelling in Multivariate Claims Run-off. *Annals of Actuarial Science* 7, 1 (2013), 3–25.
- [29] WÜTHRICH, M., MERZ, M., AND LYSENKO, N. Uncertainty in the Claims Development Result in the Chain-Ladder Method. *Scandinavian Actuarial Journal* (2008), 1–22.
- [30] ZEHNWIRTH, B. Probabilistic Development Factor Models with Applications to Loss Reserve Variability Prediction Intervals and Risk based Capital. *Casualty Actuarial Society Forum* (1994), 447.

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Appendices

A Preliminary results

Elementary terms needed for the computation of the process and estimation errors are exhibited in this introductory section:

$$\mathbb{E} \left[\hat{f}_k^{I+1} | D_I \right] = \mathbb{E} \left[\frac{S_k^I}{S_k^{I+1}} \times \hat{f}_k^I + \frac{C_{I-k+1,k+1}}{S_k^{I+1}} | D_I \right] = \frac{S_k^I}{S_k^{I+1}} \times \hat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} \times f_k.$$

Similarly,

$$\mathbb{E} \left[\hat{g}_k^{I+1} | D_I \right] = \mathbb{E} \left[\frac{T_k^I}{T_k^{I+1}} \times \hat{g}_k^I + \frac{D_{I-k+1,k+1}}{T_k^{I+1}} | D_I \right] = \frac{T_k^I}{T_k^{I+1}} \times \hat{g}_k^I + \frac{D_{I-k+1,k}}{T_k^{I+1}} \times g_k.$$

By D_I -measurability and independence between development years,

$$\mathbb{E} \left[\hat{C}_{i,I}^{I+1} | D_I \right] = \mathbb{E} \left[C_{i,I-i+1} \times \prod_{k=I-i+1}^{I-1} \hat{f}_j^{I+1} | D_I \right] = \left[\hat{f}_{I-i+1}^I \times C_{i,I-i+1} \right] \times \prod_{k=I-i+2}^{I-1} \mathbb{E} \left[\hat{f}_k^{I+1} | D_I \right].$$

Similarly,

$$\mathbb{E} \left[\hat{D}_{i,I}^{I+1} | D_I \right] = \left[\hat{g}_{I-i+1}^I \times D_{i,I-i+1} \right] \times \prod_{k=I-i+2}^{I-1} \mathbb{E} \left[\hat{g}_k^{I+1} | D_I \right].$$

B Determination of closed-form expressions for the CDR's process error

B.1 Preliminary results

Elementary terms inherent to the computation of the process error are exhibited in this section.

Frequently used expressions

$$\frac{S_k^I}{S_k^{I+1}} + \frac{C_{I-k+1,k}}{S_k^{I+1}} = 1, \quad \frac{T_k^I}{T_k^{I+1}} + \frac{D_{I-k+1,k}}{T_k^{I+1}} = 1.$$

This covariance term can be directly assessed thanks to the assumption made for the computation of the covariance between individual development factors:

$$\hat{\mathbb{E}} \left[\hat{f}_k^{I+1} | D_I \right] = \hat{f}_k^I \quad \text{and} \quad \hat{\mathbb{E}} \left[\hat{g}_k^{I+1} | D_I \right] = \hat{g}_k^I.$$

Lemma B.1.

$$\begin{aligned} \text{cov} \left[\hat{f}_k^{I+1}, \hat{g}_k^{I+1} | D_I \right] &= \text{cov} \left[\frac{S_k^I}{S_k^{I+1}} \times \hat{f}_k^I + \frac{C_{I-k+1,k+1}}{S_k^{I+1}}, \frac{T_k^I}{T_k^{I+1}} \times \hat{g}_k^I + \frac{D_{I-k+1,k+1}}{T_k^{I+1}} | D_I \right] \\ &= \frac{1}{S_k^{I+1} \times T_k^{I+1}} \text{cov} \left[\sum_{i=1}^{I-k} C_{i,k+1} + C_{I-k+1,k+1}, \sum_{i=1}^{I-k} D_{i,k+1} + D_{I-k+1,k+1} | D_I \right] \\ &= \frac{1}{S_k^{I+1} \times T_k^{I+1}} \text{cov} [C_{I-k+1,k+1}, D_{I-k+1,k+1} | D_I] \quad \text{by } D_I\text{-measurability} \\ &= \frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \rho_k \times \sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}. \end{aligned}$$

$$\text{Estimator: } \widehat{\text{cov}} \left[\hat{f}_k^{I+1}, \hat{g}_k^{I+1} | D_I \right] = \frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \hat{\rho}_k \times \sqrt{C_{I-k+1,k} \times D_{I-k+1,k}}.$$

Lemma B.2. For different development years k and l , \hat{f}_k^{I+1} and \hat{f}_l^{I+1} are uncorrelated (see Theorem 2 in Mack [12]). For different accident years k and l , \hat{f}_k^{I+1} and \hat{g}_l^{I+1} are uncorrelated.

Proof. $k \neq l$

$$\begin{aligned}
\text{cov} \left[\hat{f}_k^{I+1}, \hat{g}_l^{I+1} | D_I \right] &= \text{cov} \left[\frac{S_k^I}{S_k^{I+1}} \times \hat{f}_k^I + \frac{C_{I-k+1,k+1}}{S_k^{I+1}}, \frac{T_l^I}{T_l^{I+1}} \times \hat{g}_l^I + \frac{D_{I-l+1,l+1}}{T_l^{I+1}} | D_I \right] \\
&= \frac{1}{S_k^{I+1} \times T_l^{I+1}} \text{cov} \left[\sum_{i=1}^{I-k} C_{i,k+1} + C_{I-k+1,k+1}, \sum_{i=1}^{I-l} D_{i,l+1} + D_{I-l+1,l+1} | D_I \right] \\
&= \frac{1}{S_k^{I+1} \times T_l^{I+1}} \left\{ \text{cov} \left[\sum_{i=1}^{I-k} C_{i,k+1}, \sum_{i=1}^{I-l} D_{i,l+1} | D_I \right] + \text{cov} \left[\sum_{i=1}^{I-k} C_{i,k+1}, D_{I-l+1,l+1} | D_I \right] \right. \\
&\quad \left. + \text{cov} \left[C_{I-k+1,k+1}, \sum_{i=1}^{I-l} D_{i,l+1} | D_I \right] + \text{cov} [C_{I-k+1,k+1}, D_{I-l+1,l+1} | D_I] \right\} \\
&= \frac{1}{S_k^{I+1} \times T_l^{I+1}} \times \text{cov} [C_{I-k+1,k+1}, D_{I-l+1,l+1} | D_I] \quad \text{by } D_I\text{-measurability} \\
&= 0
\end{aligned}$$

□

Besides:

$$\hat{\mathbb{E}} \left[\hat{C}_{i,I}^{I+1} | D_I \right] = \left(\hat{f}_{I-i+1}^I \times C_{i,I-i+1} \right) \times \prod_{k=I-i+2}^{I-1} \hat{f}_k^I = C_{i,I-i+1} \times \prod_{k=I-i+1}^{I-1} \hat{f}_k^I = \hat{C}_{i,I}^I$$

and

$$\hat{\mathbb{E}} \left[\hat{D}_{i,I}^{I+1} | D_I \right] = \hat{D}_{i,I}^I.$$

$$\mathbb{E} [C_{i,I-i+2} | D_I] = \mathbb{E} \left[f_{I-i+1} \times C_{i,I-i+1} + \sigma_{I-i+1} \times \sqrt{C_{i,I-i+1}} \times \varepsilon_{i,I-i+1}^C | D_I \right] \stackrel{H_0}{=} f_{I-i+1} \times C_{i,I-i+1}$$

and

$$\mathbb{E} [D_{i,I-i+2} | D_I] = g_{I-i+1} \times D_{i,I-i+1}.$$

Estimators: $\hat{\mathbb{E}} [C_{i,I-i+2} | D_I] = \hat{f}_{I-i+1} \times C_{i,I-i+1}$ and $\hat{\mathbb{E}} [D_{i,I-i+2} | D_I] = \hat{g}_{I-i+1} \times D_{i,I-i+1}$.

B.2 Conditional process error for a single accident year

Proof. 4.1

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}
\text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right] &= \text{cov} \left[\hat{C}_{i,I}^{I+1}, \hat{D}_{i,I}^{I+1} | D_I \right] \\
&= \mathbb{E} \left[\hat{C}_{i,I}^{I+1} \times \hat{D}_{i,I}^{I+1} | D_I \right] - \mathbb{E} \left[\hat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\hat{D}_{i,I}^{I+1} | D_I \right].
\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left[\hat{C}_{i,I}^{I+1} \times \hat{D}_{i,I}^{I+1} | D_I \right] &= \mathbb{E} \left[C_{i,I-i+2} \times \prod_{k=I-i+2}^{I-1} \hat{f}_k^{I+1} \times D_{i,I-i+2} \times \prod_{k=I-i+2}^{I-1} \hat{g}_k^{I+1} | D_I \right] \\
&= \mathbb{E} [C_{i,I-i+2} \times D_{i,I-i+2} | D_I] \times \mathbb{E} \left[\prod_{k=I-i+2}^{I-1} \hat{f}_k^{I+1} \times \prod_{k=I-i+2}^{I-1} \hat{g}_k^{I+1} | D_I \right] \\
&= \underbrace{\mathbb{E} [C_{i,I-i+2} \times D_{i,I-i+2} | D_I]}_{1^{st} \text{ term}} \times \prod_{k=I-i+2}^{I-1} \underbrace{\mathbb{E} [\hat{f}_k^{I+1} \hat{g}_k^{I+1} | D_I]}_{2^{nd} \text{ term}}.
\end{aligned}$$

By independency between accident years, the elements are, by definition, disjoint elements.

1st term: $\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}\mathbb{E}[C_{i,I-i+2} \times D_{i,I-i+2} | D_I] &= \text{cov}[C_{i,I-i+2}, D_{i,I-i+2} | D_I] + \mathbb{E}[C_{i,I-i+2} | D_I] \mathbb{E}[D_{i,I-i+2} | D_I] \\ &= \rho_{I-i+1} \times \sqrt{C_{i,I-i+1} \times D_{i,I-i+1}} + f_{I-i+1} \times C_{i,I-i+1} \times g_{I-i+1} \times D_{i,I-i+1}.\end{aligned}$$

In order to finalize the computation, one has to move to the estimators:

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}\widehat{\mathbb{E}}[C_{i,I-i+2} \times D_{i,I-i+2} | D_I] &= \widehat{\rho}_{I-i+1} \times \sqrt{C_{i,I-i+1} \times D_{i,I-i+1}} \\ &\quad + \left(\widehat{f}_{I-i+1}^I \times C_{i,I-i+1} \right) \times \left(\widehat{g}_{I-i+1}^I \times D_{i,I-i+1} \right).\end{aligned}\tag{29}$$

2nd term:

$\forall k \in \{1, \dots, I-1\}$,

$$\mathbb{E}[\widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I] = \text{cov}[\widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I] + \mathbb{E}[\widehat{f}_k^{I+1} | D_I] \times \mathbb{E}[\widehat{g}_k^{I+1} | D_I].$$

To compute these terms, we use the following estimators:

$\forall k \in \{1, \dots, I-1\}$,

$$\widehat{\mathbb{E}}[\widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I] = \frac{1}{S_k^{I+1} \times T_k^{I+1}} \times \widehat{\rho}_k \times \sqrt{C_{I-k+1,k} \times D_{I-k+1,k}} + \widehat{f}_k^I \times \widehat{g}_k^I.\tag{30}$$

Final expression: as a consequence, we are able to exhibit the whole calculation:

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}\widehat{\text{cov}}[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I] &= \widehat{\mathbb{E}}[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{i,I}^{I+1} | D_I] - \widehat{\mathbb{E}}[\widehat{C}_{i,I}^{I+1} | D_I] \widehat{\mathbb{E}}[\widehat{D}_{i,I}^{I+1} | D_I] \\ &= \left[\widehat{\rho}_{I-i+1} \times \sqrt{C_{i,I-i+1} \times D_{i,I-i+1}} + \left(\widehat{f}_{I-i+1}^I \times C_{i,I-i+1} \right) \times \left(\widehat{g}_{I-i+1}^I \times D_{i,I-i+1} \right) \right] \\ &\quad \times \prod_{k=I-i+2}^{I-1} \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \sqrt{C_{I-k+1,k} D_{I-k+1,k}} + \widehat{f}_k^I \widehat{g}_k^I \right] - \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \\ &= \prod_{k=I-i+1}^{I-1} \left[\widehat{f}_k^I \widehat{g}_k^I \right] \times \left[C_{i,I-i+1} D_{i,I-i+1} + \frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \widehat{g}_{I-i+1}^I} \times \sqrt{C_{i,I-i+1} D_{i,I-i+1}} \right] \\ &\quad \times \prod_{k=I-i+2}^{I-1} \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{\widehat{f}_k^I \widehat{g}_k^I} + 1 \right] - \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \\ &= \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \times \left\{ \left[1 + \frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \widehat{g}_{I-i+1}^I} \times \frac{1}{\sqrt{C_{i,I-i+1} D_{i,I-i+1}}} \right] \right. \\ &\quad \times \left. \prod_{k=I-i+2}^{I-1} \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{\widehat{f}_k^I \widehat{g}_k^I} + 1 \right] - 1 \right\}.\end{aligned}\tag{31}$$

By using the approximation $\forall L \in \mathbf{N}^*, \prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$, the following formula is proposed:

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}\widehat{\text{cov}}[\widehat{CDR}_i^C(I+1), \widehat{CDR}_i^D(I+1) | D_I] &\approx \widehat{C}_{i,I}^I \times \widehat{D}_{i,I}^I \times \left\{ \left[\frac{\widehat{\rho}_{I-i+1}}{\widehat{f}_{I-i+1}^I \widehat{g}_{I-i+1}^I} \times \frac{1}{\sqrt{C_{i,I-i+1} D_{i,I-i+1}}} \right] \right. \\ &\quad \left. + \sum_{k=I-i+2}^{I-1} \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right] \right\}.\end{aligned}$$

Remark: this approximation could be discussed since:

$$\frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{S_k^{I+1} T_k^{I+1}}$$

might turn to be non-negligeable in some cases (especially for high values of k). \square

B.3 Conditional process error for all accident years

Proof. 4.2 We have: $\forall i > j$,

$$\begin{aligned} \text{cov} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I \right] &= \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{j,I}^{I+1} | D_I \right] - \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \\ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{j,I}^{I+1} | D_I \right] &= \mathbb{E} \left[C_{i,I-i+2} \times \prod_{k=I-i+2}^{I-1} \widehat{f}_k^{I+1} \times D_{j,I-j+2} \times \prod_{k=I-j+2}^{I-1} \widehat{g}_k^{I+1} | D_I \right] \\ &= \mathbb{E} \left[C_{i,I-i+2} \times \prod_{k=I-i+2}^{I-j} \widehat{f}_k^{I+1} \times \widehat{f}_{I-j+1}^{I+1} \times D_{j,I-j+2} \times \prod_{k=I-j+2}^{I-1} \widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I \right]. \end{aligned}$$

By independency between accident years, the elements are disjoint:

$\forall i > j$,

$$\begin{aligned} \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{j,I}^{I+1} | D_I \right] & \quad (32) \\ &= \mathbb{E} [C_{i,I-i+2} | D_I] \times \mathbb{E} \left[\prod_{k=I-i+2}^{I-j} \widehat{f}_k^{I+1} | D_I \right] \times \mathbb{E} \left[\widehat{f}_{I-j+1}^{I+1} \times D_{j,I-j+2} | D_I \right] \times \mathbb{E} \left[\prod_{k=I-j+2}^{I-1} \widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I \right] \\ &= \mathbb{E} [C_{i,I-i+2} | D_I] \times \prod_{k=I-i+2}^{I-j} \mathbb{E} \left[\widehat{f}_k^{I+1} | D_I \right] \times \underbrace{\mathbb{E} \left[\widehat{f}_{I-j+1}^{I+1} \times D_{j,I-j+2} | D_I \right]}_{1^{st} term} \times \prod_{k=I-j+2}^{I-1} \underbrace{\mathbb{E} \left[\widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I \right]}_{2^{nd} term}. \end{aligned}$$

1st term:

$$\begin{aligned} \mathbb{E} \left[\widehat{f}_{I-j+1}^{I+1} \times D_{j,I-j+2} | D_I \right] &= \mathbb{E} \left[\frac{\sum_{k=1}^j C_{k,I-j+2}}{S_{I-j+1}^{I+1}} \times D_{j,I-j+2} | D_I \right] \\ &= \text{cov} \left[\frac{\sum_{k=1}^{j-1} C_{k,I-j+2} + C_{j,I-j+2}}{S_{I-j+1}^{I+1}}; D_{j,I-j+2} | D_I \right] + \mathbb{E} \left[\frac{\sum_{k=1}^{j-1} C_{k,I-j+2} + C_{j,I-j+2}}{S_{I-j+1}^{I+1}} | D_I \right] \\ &\quad \times \mathbb{E} [D_{j,I-j+2} | D_I] \\ &= \frac{1}{S_{I-j+1}^{I+1}} \text{cov} [C_{j,I-j+2}, D_{j,I-j+2} | D_I] + \left[\frac{S_{I-j+1}^I}{S_{I-j+1}^{I+1}} \times f_{I-j+1} + f_{I-j+1} \times \frac{C_{j,I-j+1}}{S_{I-j+1}^{I+1}} \right] \\ &\quad \times g_{I-j+1} \times D_{j,I-j+1}. \end{aligned}$$

By considering estimators, we obtain:

$$\widehat{\mathbb{E}} \left[\widehat{f}_{I-j+1}^{I+1} \times D_{j,I-j+2} | D_I \right] = \frac{\widehat{\rho}_{I-j+1} \times \sqrt{C_{j,I-j+1} D_{j,I-j+1}}}{S_{I-j+1}^{I+1}} + \widehat{f}_{I-j+1}^I \times \widehat{g}_{I-j+1}^I \times D_{j,I-j+1}. \quad (33)$$

2nd term:

$\forall k \in \{I-j+2, \dots, I\}$,

$$\mathbb{E} \left[\widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I \right] = \text{cov} \left[\widehat{f}_k^{I+1}, \widehat{g}_k^{I+1} | D_I \right] + \mathbb{E} \left[\widehat{f}_k^{I+1} | D_I \right] \mathbb{E} \left[\widehat{g}_k^{I+1} | D_I \right].$$

By considering estimators:

$$\forall k \in \{I-j+2, \dots, I\}, \widehat{\mathbb{E}} \left[\widehat{f}_k^{I+1} \widehat{g}_k^{I+1} | D_I \right] = \frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \sqrt{C_{I-k+1,k} D_{I-k+1,k}} + \widehat{f}_k^I \widehat{g}_k^I. \quad (34)$$

Final expression:

as a consequence, we are able to exhibit the whole calculation:

$\forall i > j$

$$\begin{aligned}
\widehat{\text{cov}} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I \right] &= \widehat{\mathbb{E}} \left[\widehat{C}_{i,I}^{I+1} \times \widehat{D}_{j,I}^{I+1} | D_I \right] - \widehat{\mathbb{E}} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \widehat{\mathbb{E}} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \\
&= \widehat{f}_{I-i+1}^I \times C_{i,I-i+1} \times \prod_{k=I-i+2}^{I-j} \widehat{f}_k^I \times \left(\frac{\widehat{\rho}_{I-j+1} \times \sqrt{C_{j,I-j+1} D_{j,I-j+1}}}{S_{I-j+1}^{I+1}} + \widehat{f}_{I-j+1}^I \right. \\
&\quad \times \left. \widehat{g}_{I-j+1}^I \times D_{j,I-j+1} \right) \times \prod_{k=I-j+2}^{I-1} \left(\frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \sqrt{C_{I-k+1,k} D_{I-k+1,k}} + \widehat{f}_k^I \widehat{g}_k^I \right) - \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \\
&= \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \left\{ \left(\frac{\widehat{\rho}_{I-j+1}}{S_{I-j+1}^{I+1} \widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \times \sqrt{\frac{C_{j,I-j+1}}{D_{j,I-j+1}}} + 1 \right) \right. \\
&\quad \times \left. \prod_{k=I-j+2}^{I-1} \left(\frac{1}{S_k^{I+1} T_k^{I+1}} \times \widehat{\rho}_k \times \frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{\widehat{f}_k^I \widehat{g}_k^I} + 1 \right) - 1 \right\}. \tag{35}
\end{aligned}$$

By using the approximation $\forall L \in \mathbf{N}^*, \prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$, the following formula is specified:
 $\forall i > j$,

$$\begin{aligned}
\widehat{\text{cov}} \left[\widehat{CDR}_i^C(I+1), \widehat{CDR}_j^D(I+1) | D_I \right] &\approx \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \times \left[\frac{\widehat{\rho}_{I-j+1}}{\widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \times \sqrt{\frac{C_{j,I-j+1}}{D_{j,I-j+1}}} \times \frac{1}{S_{I-j+1}^{I+1}} \right. \\
&\quad \left. + \sum_{k=I-j+2}^{I-1} \frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{S_k^{I+1} T_k^{I+1}} \times \frac{\widehat{\rho}_k}{\widehat{f}_k^I \widehat{g}_k^I} \right].
\end{aligned}$$

Remark: this approximation could be discutable since:

$$\frac{\sqrt{C_{I-k+1,k} D_{I-k+1,k}}}{S_k^{I+1} T_k^{I+1}}$$

might turn to be non-negligible in some cases (especially for high values of k). \square

Remark: the computation of the symmetric expression, $\widehat{\text{cov}} \left[\widehat{CDR}_j^C(I+1), \widehat{CDR}_i^D(I+1) | D_I \right]$ is straightforward.

C Determination of closed-form expressions for the CDR's estimation error

In order to compute the terms inherent to the CDR's estimation error, we use a conditional resampling methodology which corresponds to Approach 3 in Buchwalder *et al.* [4].

As a consequence the two expressions (with $i \neq j$):

$$\mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \text{ and } \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right]$$

are going to be estimated subsequently by:

$$\mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D(I+1) | D_I \right] \right\} \text{ and } \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\}.$$

C.1 Preliminary results

Notice that we will use the conditional resampling technique followed by Wüthrich *et al.* [29] corresponding to Approach 3 of Buchwalder *et al.* [4].

Frequently used expressions

$$\forall k \in \{1, \dots, I-1\}, \text{cov}_{D_I} [\hat{f}_k^I, \hat{g}_k^I] = \frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \text{ and } \mathbb{E}_{D_I} [\hat{f}_k^I] = f_k, \mathbb{E}_{D_I} [\hat{g}_k^I] = g_k.$$

$$\forall i \in \{2, \dots, I\},$$

$$\begin{aligned} \mathbb{E}_{D_I} [\hat{C}_{i,I}^I \hat{D}_{i,I}^I] &= \mathbb{E}_{D_I} \left[C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \hat{f}_k^I \hat{g}_k^I \right] \\ &= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \mathbb{E}_{D_I} [\hat{f}_k^I \hat{g}_k^I] \\ &= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + f_k g_k \right). \end{aligned}$$

C.2 Conditional estimation error for a single accident year

Proof. 4.3

$$\begin{aligned} \mathbb{E}_{D_I} \left\{ \mathbb{E} [\widehat{CDR}_i^C(I+1) | D_I] \mathbb{E} [\widehat{CDR}_i^D(I+1) | D_I] \right\} &= \underbrace{\mathbb{E}_{D_I} \left\{ \mathbb{E} [\hat{C}_{i,I}^{I+1} | D_I] \mathbb{E} [\hat{D}_{i,I}^{I+1} | D_I] \right\}}_A \\ &+ \underbrace{\mathbb{E}_{D_I} \left\{ \hat{C}_{i,I}^I \hat{D}_{i,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \hat{C}_{i,I}^I \mathbb{E} [\hat{D}_{i,I}^{I+1} | D_I] \right\} - \mathbb{E}_{D_I} \left\{ \hat{D}_{i,I}^I \mathbb{E} [\hat{C}_{i,I}^{I+1} | D_I] \right\}}_B. \end{aligned}$$

A-term:
 $\forall i \in \{2, \dots, I\},$

$$\begin{aligned} &\mathbb{E}_{D_I} \left\{ \mathbb{E} [\hat{C}_{i,I}^{I+1} | D_I] \mathbb{E} [\hat{D}_{i,I}^{I+1} | D_I] \right\} \\ &= \mathbb{E}_{D_I} \left\{ \hat{f}_{I-i+1}^I C_{i,I-i+1} \hat{g}_{I-i+1}^I D_{i,I-i+1} \prod_{k=I-i+2}^{I-1} \left(\frac{S_k^I}{S_k^{I+1}} \hat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \left(\frac{T_k^I}{T_k^{I+1}} \hat{g}_k^I + \frac{D_{I-k+1,k}}{T_k^{I+1}} g_k \right) \right\} \\ &= C_{i,I-i+1} f_{I-i+1} D_{i,I-i+1} g_{I-i+1} \prod_{k=I-i+2}^{I-1} \mathbb{E}_{D_I} \left\{ \left(\frac{S_k^I}{S_k^{I+1}} \hat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \left(\frac{T_k^I}{T_k^{I+1}} \hat{g}_k^I + \frac{D_{I-k+1,k}}{T_k^{I+1}} g_k \right) \right\} \\ &= C_{i,I-i+1} f_{I-i+1} D_{i,I-i+1} g_{I-i+1} \prod_{k=I-i+2}^{I-1} \left\{ \frac{1}{S_k^{I+1} T_k^{I+1}} \left(\rho_k \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right\}. \end{aligned} \quad (36)$$

B-term:

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}
& \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{i,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\} \\
&= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \mathbb{E}_{D_I} \left\{ \widehat{f}_k^I \widehat{g}_k^I \right\} - \mathbb{E}_{D_I} \left\{ C_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \widehat{f}_k^I \times g_{I-i+1} D_{i,I-i+1} \right. \\
&\quad \times \left. \prod_{k=I-i+2}^{I-1} \left(\frac{T_k^I}{T_k^{I+1}} \widehat{g}_k^I + \frac{D_{I-k+1,k}}{T_k^{I+1}} g_k \right) \right\} \\
&\quad - \mathbb{E}_{D_I} \left\{ D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \widehat{g}_k^I \times f_{I-i+1} C_{i,I-i+1} \prod_{k=I-i+2}^{I-1} \left(\frac{S_k^I}{S_k^{I+1}} \widehat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \right\} \\
&= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \left[\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + f_k g_k \right] - C_{i,I-i+1} D_{i,I-i+1} f_{I-i+1} g_{I-i+1} \\
&\quad \times \left\{ \prod_{k=I-i+2}^{I-1} \left[\frac{T_k^I}{T_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \right. \\
&\quad \left. + \prod_{k=I-i+2}^{I-1} \left[\frac{S_k^I}{S_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \right\}. \tag{37}
\end{aligned}$$

Final expression: by concatenating all the computations, the following formula is proposed:

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}
& \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C (I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D (I+1) | D_I \right] \right\} \\
&= \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{i,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{i,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\} \\
&\quad + \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{i,I}^{I+1} | D_I \right] \right\} \\
&= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} \left[\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + f_k g_k \right] - C_{i,I-i+1} D_{i,I-i+1} f_{I-i+1} g_{I-i+1} \\
&\quad \times \left\{ \prod_{k=I-i+2}^{I-1} \left[\frac{T_k^I}{T_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] + \prod_{k=I-i+2}^{I-1} \left[\frac{S_k^I}{S_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \right. \\
&\quad \left. - \prod_{k=I-i+2}^{I-1} \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \left(\rho_k \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + f_k g_k \right] \right\} \\
&= C_{i,I-i+1} D_{i,I-i+1} \prod_{k=I-i+1}^{I-1} f_k g_k \left\{ \prod_{k=I-i+1}^{I-1} \left(\frac{\rho_k}{S_k^I T_k^I f_k g_k} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + 1 \right) \right. \\
&\quad \left. - \prod_{k=I-i+2}^{I-1} \left[\frac{T_k^I}{T_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I f_k g_k} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + 1 \right] - \prod_{k=I-i+2}^{I-1} \left[\frac{S_k^I}{S_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I f_k g_k} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) + 1 \right] \right. \\
&\quad \left. + \prod_{k=I-i+2}^{I-1} \left(\frac{\rho_k}{S_k^{I+1} T_k^{I+1} f_k g_k} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} + 1 \right) \right\}. \tag{38}
\end{aligned}$$

By considering estimators and using the approximation $\forall L \in \mathbf{N}^*, \prod_{l=1}^L (1 + a_l) - 1 \approx \sum_{l=1}^L a_l$:

$\forall i \in \{2, \dots, I\}$,

$$\begin{aligned}
& \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C (I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_i^D (I+1) | D_I \right] \right\} \approx \widehat{C}_{i,I} \widehat{D}_{i,I} \left\{ \sum_{k=I-i+1}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \right. \\
&\quad \left. - \sum_{k=I-i+2}^{I-1} \left(\frac{\widehat{\rho}_k}{S_k^I T_k^I \widehat{f}_k^I \widehat{g}_k^I} \sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}} \right) \left(\frac{T_k^I}{T_k^{I+1}} + \frac{S_k^I}{S_k^{I+1}} - \frac{S_k^I T_k^I}{S_k^{I+1} T_k^{I+1}} \right) \right\}.
\end{aligned}$$

□

C.3 Conditional estimation error for all accident years

Proof. 4.4 Assumption: i and j are two accident years, let us suppose that $i > j$

$$\begin{aligned}
& \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C(I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D(I+1) | D_I \right] \right\} \\
&= \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^I - \widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^I - \widehat{D}_{j,I}^{I+1} | D_I \right] \right\} \\
&= \underbrace{\mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{j,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\}}_B \\
&\quad + \underbrace{\mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\}}_A
\end{aligned}$$

A-term: $\forall i > j$,

$$\begin{aligned}
& \mathbb{E}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} \\
&= \mathbb{E}_{D_I} \left\{ \mathbb{E} [C_{i,I-i+2} | D_I] \prod_{k=I-i+2}^{I-j+1} \mathbb{E} [\widehat{f}_k^{I+1} | D_I] \prod_{k=I-j+2}^{I-1} \mathbb{E} [\widehat{f}_k^{I+1} | D_I] \mathbb{E} [D_{j,I-j+2} | D_I] \mathbb{E} [\widehat{g}_k^{I+1} | D_I] \right\} \\
&= \mathbb{E}_{D_I} \left\{ f_{I-i+1} C_{i,I-i+1} D_{j,I-j+1} g_{I-j+1} \prod_{k=I-i+2}^{I-j+1} \left(\frac{S_k^I}{S_k^{I+1}} \widehat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \right. \\
&\quad \times \left. \prod_{k=I-j+2}^{I-1} \left(\frac{S_k^I}{S_k^{I+1}} \widehat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \left(\frac{T_k^I}{T_k^{I+1}} \widehat{g}_k^I + \frac{D_{I-k+1,k}}{T_k^{I+1}} g_k \right) \right\} \\
&= f_{I-i+1} C_{i,I-i+1} D_{j,I-j+1} g_{I-j+1} \prod_{k=I-i+2}^{I-j+1} f_k \prod_{k=I-j+2}^{I-1} f_k g_k \left[\frac{1}{S_k^{I+1} T_k^{I+1}} \left(\rho_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{f_k g_k} \right) + 1 \right].
\end{aligned} \tag{39}$$

By using estimators and a first order approximation:

$$\forall i > j, \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} \approx \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \left\{ 1 + \sum_{k=I-j+2}^{I-1} \frac{1}{S_k^{I+1} T_k^{I+1}} \left(\widehat{\rho}_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right) \right\}.$$

B-term: $\forall i > j$,

$$\begin{aligned}
& \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \right\} - \mathbb{E}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{j,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\} \\
&= \mathbb{E}_{D_I} \left\{ C_{i,I-i+1} D_{j,I-j+1} \prod_{k=I-i+1}^{I-j} \widehat{f}_k^I \prod_{k=I-j+1}^{I-1} \widehat{f}_k^I \widehat{g}_k^I \right\} \\
&\quad - \mathbb{E}_{D_I} \left\{ C_{i,I-i+1} g_{I-j+1} D_{j,I-j+1} \prod_{k=I-i+1}^{I-j+1} \widehat{f}_k^I \prod_{k=I-j+2}^{I-1} \widehat{f}_k^I \left(\frac{T_k^I}{T_k^{I+1}} \widehat{g}_k^I + \frac{D_{I-k+1,k}}{T_k^{I+1}} g_k \right) \right\} \\
&\quad - \mathbb{E}_{D_I} \left\{ D_{j,I-j+1} f_{I-i+1} C_{i,I-i+1} \prod_{k=I-j+1}^{I-1} \widehat{g}_k^I \prod_{k=I-i+2}^{I-j} \left(\frac{S_k^I}{S_k^{I+1}} \widehat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \right. \\
&\quad \times \left. \prod_{k=I-j+1}^{I-1} \left(\frac{S_k^I}{S_k^{I+1}} \widehat{f}_k^I + \frac{C_{I-k+1,k}}{S_k^{I+1}} f_k \right) \right\} \\
&= C_{i,I-i+1} D_{j,I-j+1} \prod_{k=I-i+1}^{I-1} f_k \prod_{k=I-j+1}^{I-1} g_k \left\{ \prod_{k=I-j+1}^{I-1} \left[\left(\frac{\rho_k}{S_k^I T_k^I} \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{f_k g_k} \right) + 1 \right] \right. \\
&\quad \left. - \prod_{k=I-j+2}^{I-1} \left[\frac{T_k^I}{T_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{f_k g_k} \right) + 1 \right] - \prod_{k=I-j+1}^{I-1} \left[\frac{S_k^I}{S_k^{I+1}} \left(\frac{\rho_k}{S_k^I T_k^I} \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{f_k g_k} \right) + 1 \right] \right\}.
\end{aligned} \tag{40}$$

By considering estimators and using a first order approximation: $\forall i > j$,

$$\begin{aligned} & \widehat{\mathbb{E}}_{D_I} \left\{ \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \right\} - \widehat{\mathbb{E}}_{D_I} \left\{ \widehat{C}_{i,I}^I \mathbb{E} \left[\widehat{D}_{j,I}^{I+1} | D_I \right] \right\} - \mathbb{E}_{D_I} \left\{ \widehat{D}_{j,I}^I \mathbb{E} \left[\widehat{C}_{i,I}^{I+1} | D_I \right] \right\} \\ & \approx \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \left\{ \sum_{k=I-j+1}^{I-1} \frac{1}{S_k^I T_k^I} \left(\widehat{\rho}_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right) - \sum_{k=I-j+2}^{I-1} \frac{1}{S_k^I T_k^{I+1}} \left(\widehat{\rho}_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right) \right. \\ & \quad \left. - \sum_{k=I-j+1}^{I-1} \frac{1}{S_k^{I+1} T_k^I} \left(\widehat{\rho}_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right) - 1 \right\}. \end{aligned}$$

Final expression: $\forall i > j$,

$$\begin{aligned} & \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C (I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D (I+1) | D_I \right] \right\} \\ & \approx \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \left\{ \frac{\widehat{\rho}_{I-j+1}}{S_{I-j+1}^I T_{I-j+1}^I} \left(\frac{\sum_{i=1}^{j-1} \sqrt{C_{i,I-j+1} D_{i,I-j+1}}}{\widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \right) \left(1 - \frac{S_{I-j+1}^I}{S_{I-j+1}^{I+1}} \right) \right. \\ & \quad \left. + \sum_{k=I-j+2}^{I-1} \left(\widehat{\rho}_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right) \left(\frac{1}{S_k^I T_k^I} + \frac{1}{S_k^{I+1} T_k^{I+1}} - \frac{1}{S_k^I T_k^{I+1}} - \frac{1}{S_k^{I+1} T_k^I} \right) \right\}. \end{aligned} \quad (41)$$

Besides, we recall that: $S_k^{I+1} = S_k^I + C_{I-k+1,k}$ and $T_k^{I+1} = T_k^I + D_{I-k+1,k}$. Then:

$$\forall i > j, \left(\frac{1}{S_k^I T_k^I} + \frac{1}{S_k^{I+1} T_k^{I+1}} - \frac{1}{S_k^I T_k^{I+1}} - \frac{1}{S_k^{I+1} T_k^I} \right) = \frac{C_{I-k+1,k} D_{I-k+1,k}}{S_k^I T_k^I S_k^{I+1} T_k^{I+1}}.$$

As a consequence, the final closed-form expression is the following one: $\forall i > j$,

$$\begin{aligned} & \widehat{\mathbb{E}}_{D_I} \left\{ \mathbb{E} \left[\widehat{CDR}_i^C (I+1) | D_I \right] \mathbb{E} \left[\widehat{CDR}_j^D (I+1) | D_I \right] \right\} \\ & \approx \widehat{C}_{i,I}^I \widehat{D}_{j,I}^I \left\{ \frac{\widehat{\rho}_{I-j+1}}{S_{I-j+1}^I T_{I-j+1}^I} \left(\frac{\sum_{i=1}^{j-1} \sqrt{C_{i,I-j+1} D_{i,I-j+1}}}{\widehat{f}_{I-j+1}^I \widehat{g}_{I-j+1}^I} \right) \left(1 - \frac{S_{I-j+1}^I}{S_{I-j+1}^{I+1}} \right) \right. \\ & \quad \left. + \sum_{k=I-j+2}^{I-1} \left(\widehat{\rho}_k \frac{\sum_{i=1}^{I-k} \sqrt{C_{i,k} D_{i,k}}}{\widehat{f}_k^I \widehat{g}_k^I} \right) \frac{C_{I-k+1,k} D_{I-k+1,k}}{S_k^I T_k^I S_k^{I+1} T_k^{I+1}} \right\}. \end{aligned} \quad (42)$$

□