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# A General Framework for Dynamic Programming Equation and Time Consistency

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## Abstract

Time consistency is an important property for a sequence of stochastic optimization problem, especially with regard to risk attitudes. It is known that a sequence of problem is time-consistent if a Dynamic Programming equation holds. In this paper we present a general framework for Dynamic Programming where general aggregators in time and uncertainties are considered instead of the expectation and sum. The key assumptions are monotonicity, commutation property, and decomposability of aggregators.

## 1 Introduction

Dynamic consistency has been studied in the field of economics (see Hammond, [11]) and more recently introduced in the context of risk measures (see Artzner et al., [2] and references therein for definitions and properties of coherent and consistent dynamic risk measures).

In discrete time dynamic optimal control problem, the property of time consistency is loosely stated as follows. The decision maker formulates an optimization problem at time  $t_0$  that yields a sequence of optimal decision rules for  $t_0$  and for the following time steps  $t_1, \dots, t_N = T$ . Then, at the next time step  $t_1$ , he formulates a new problem starting at  $t_1$  that yields a new sequence of optimal decision rules from time steps  $t_1$  to  $T$ . Suppose the process continues until time  $T$  is reached. The sequence of optimization problems is said to be time consistent if the optimal strategies obtained when solving the original problem at time  $t_0$  remain optimal for all subsequent problems. In other words, time consistency means that strategies obtained by solving the problem at the very first stage do not have to be questioned later on. It is well known that the existence of a Dynamic Programming Principle imply the time consistency of the sequence of problems (see [7]).

The traditional framework for Dynamic Programming consist in minimizing the expectation of the sum of costs. However there is other less known case where a Dynamic Programming equation holds, for example for the expectation of a product of costs or for the supremum over the realisations of a sum of costs. It appears that there is two aggregators, one that aggregates the costs over time, and one that aggregates the costs over the uncertainties. We aim at giving general conditions on those aggregators under which a Dynamic Programming equation holds for the selected information state. Indeed if the state is increased it is sometimes possible to find a Dynamic Programming equation, and thus a time-consistent sequence of problems.

In §2 we present three usual settings where a Dynamic Programming principle can be derived. In §3 we present the general setting we consider, and the general conditions that are sufficient for a Dynamic Programming principle. In §4 we present some class of problems satisfying our conditions and the Dynamic Programming Principle associated to them. Finally in §5 we present some tools that allow to construct new aggregators and the associated Dynamic Programming equation.

## 2 Examples of Dynamic Programming equations

In this section we present three settings in which a Dynamic Programming equation holds for discrete time systems. We stress out the structures that will be generalized in Section 3. Proof of the Dynamic Programming result exposed here are not presented as they fit into the framework developed in Section 3. Finally for the sake of clarity we will assume in this section that every mathematical expectation is well defined.

### 2.1 Expected and worst case scenario case for additive costs

#### Expected costs

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. For any  $t \in \llbracket 0, T \rrbracket$ , where  $\llbracket a, b \rrbracket$  denote the set of integers between  $a$  and  $b$ , we consider a state set  $\mathbb{X}_t$ , a control set  $\mathbb{U}_t$  and a noise set  $\mathbb{W}_t$ . Appropriate measurability assumptions are supposed to hold on this sets. Consider measurable functions  $f_t$  that maps  $\mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t$  into  $\mathbb{X}_{t+1}$ , a controlled stochastic dynamic system,

$$\forall t \in \llbracket 0, T-1 \rrbracket, \quad X_{t+1} = f_t(X_t, U_t, W_t), \quad (1)$$

where the state of the system is a stochastic process  $x_t$  that take values into the set  $\mathbb{X}_t$ , the control  $U_t$  is a random variable with values in  $\mathbb{U}_t$ , and  $W_t$  is a random variable with values in  $\mathbb{W}_t$ . Non-anticipativity constraint implies that the control  $U_t$  is measurable with respect to  $\sigma(X_0, \dots, X_t, W_0, \dots, W_t)$ . It is widely known that, under independance assumption over the noise, the control can be chosen as a function of  $X_t$ . We call  $s = (s_t)_{t \in \llbracket 0, T \rrbracket}$ , where for all  $t \in \llbracket 0, T \rrbracket$ ,  $s_t$  maps  $\mathbb{X}_t$  into  $\mathbb{U}_t$ , a *strategy*. We denote  $S$  the set of all strategies.

The problem considered reads as follows

$$\min_{s \in S} \mathbb{E} \left[ \sum_{t=0}^{T-1} J_t(X_t, U_t, W_t) + K(X_T) \right] \quad (2a)$$

$$\text{s.t. } X_{t+1} = f_t(X_t, U_t, W_t) \quad (2b)$$

$$U_t = s_t(X_t) \quad (2c)$$

In order to present the Dynamic Programming Principle associated to problem (2) we need to introduce the value functions  $V_t$  recursively defined by

$$\begin{cases} V_T(x) &= K(x) & \forall x \in \mathbb{X}_T \\ V_t(x) &= \min_{u \in \mathbb{U}_t} \mathbb{E} \left[ J_t(x, u, W_t) + V_{t+1} \circ f_t(x, u, W_t) \right] & \forall x \in \mathbb{X}_t, \forall t \in \llbracket 0, T-1 \rrbracket \end{cases} \quad (3)$$

The Dynamic Programming principle states that, if there exists a strategy  $s^*$  (with proper measurability assumptions that we do not discuss here (see [6])) such that, for each  $t \in \llbracket 0, T-1 \rrbracket$ , and each  $x \in \mathbb{X}_t$ , we have

$$s_t(x) \in \arg \min_{u \in \mathbb{U}_t} \mathbb{E} \left[ J_t(x, u, W_t) + V_{t+1} \circ f_t(x, u, W_t) \right], \quad (4)$$

then  $s^*$  is an optimal strategy for problem (2).

In problem (2) we make two choices to aggregate the time-step cost. The first one is to consider the sum over time of the instantaneous costs. And the second choice is to consider the expectation the costs. A usual alternative to the sum over time is the discounted sum, and we show in Section 2.2 another example where we naturally consider the product of the cost instead of the sum. The so-called fear-operator (as defined by Bernhard in [4]), and presented next, is another usual aggregator over the uncertainties.

#### Worst-case scenario

The fear-operator, or worst-case scenario operator, which take the supremum over the possible realisation of the cost also lead to a Dynamic Programming principle. It is often considered in the field of robust optimization (see [14] and [3]).

More precisely if  $W_t$  take values in  $\mathbb{W}_t$ , and we denote  $\mathbb{W} = \mathbb{W}_1 \times \dots \times \mathbb{W}_T$  then the problem considered is

$$\min_{u_t \in \mathcal{F}(\mathbb{X}_t; \mathbb{U}_t)} \sup_{w \in \mathbb{W}} \left[ \sum_{t=0}^{T-1} J_t(x_t, u_t, w_t) + K(x_T) \right] \quad (5a)$$

$$\text{s.t. } x_{t+1} = f_t(x_t, u_t, w_t) \quad (5b)$$

in [5], section 1.6, it is shown that we can write a Dynamic Programming equation for this problem

$$\begin{cases} V_T(x) = K(x) & \forall x \in \mathbb{X}_T \\ V_t(x) = \min_{u \in \mathbb{U}_t} \sup_{w_t \in \mathbb{W}_t} \left[ J_t(x, u, w_t) + V_{t+1}(f_t(x, u, w_t)) \right] & \forall x \in \mathbb{X}_t, \forall t \in \llbracket 0, T-1 \rrbracket \end{cases} \quad (6)$$

## 2.2 Expectation and multiplicative costs

An expected multiplicative cost appears in a financial context if we consider a final payoff  $K(X_{T+1})$  depending on the final state of our system, but discounted with a controlled rate  $r_t(X_t)$ . In this case the problem of maximizing the discounted expected product reads

$$\max \mathbb{E} \left( \prod_{t=1}^{T-1} \left( \frac{1}{1 + r_t(X_t)} \right) K(X_T) \right),$$

where  $r_t$  is the interest rate at time  $t$ .

We present another interesting setting where multiplicative cost appears. In control problems we consider a dynamic system and thrive to find a control such that the state  $x_t$  satisfy some constraints  $x_t \in \mathcal{X}_t \subset \mathbb{X}_t$ . In a deterministic setting the problem is either impossible (there is no strategy such that for all  $t \in \llbracket 0, T-1 \rrbracket$ ,  $x_t \in \mathcal{X}_t$ ) or have a solution depending on the starting point  $x_0$ . However in a stochastic setting satisfying the constraint for all time  $t \in \llbracket 0, T-1 \rrbracket$ ,  $x_t \in \mathcal{X}_t$  for all  $\omega$  (or  $\mathbb{P}$ -almost surely) can be extremely difficult, and will lead to problems without solution. Indeed if for example we add to a controlled dynamic an unknown gaussian error, then the resulting state can be anywhere in the state space, and thus a constraint  $x_t \in \mathcal{X}_t \subset \mathbb{X}_t$  where  $\mathcal{X}_t$  is, say, a compact set, can not be satisfied almost surely.

For such a control problem we propose alternatively to maximize the probability of satisfying the constraint (see [9]) :

$$\max_{s_t} \mathbb{P} \left( \{ \forall t \in \llbracket 0, T \rrbracket, X_t \in \mathcal{X}_t \} \right), \quad (7a)$$

$$\text{s.t. } X_{t+1} = f_t(X_t, s_t(X_t), W_t), \quad \forall t \in \llbracket 0, T-1 \rrbracket. \quad (7b)$$

This is the so called stochastic viability approach.

This problem can be written

$$\max_{s_t \in \mathcal{F}(\mathbb{X}_t, \mathbb{U}_t)} \mathbb{E} \left[ \prod_{t=0}^T \mathbb{1}_{\{X_t \in \mathcal{X}_t\}} \right], \quad (8a)$$

$$\text{s.t. } X_{t+1} = f_t(X_t, s_t(X_t), W_t), \quad \forall t \in \llbracket 0, T-1 \rrbracket. \quad (8b)$$

And it is shown in [8] that under independance of noises assumption the Dynamic Programming equation associated is given by

$$V_t(x) = \min_{u \in \mathbb{U}_t} \mathbb{E} \left[ \mathbb{1}_{\{x \in \mathcal{X}_t\}} \cdot V_{t+1}(f_t(x, u, W_t)) \right]. \quad (9)$$

More precisely if we can define a strategy  $s^*$  such that for all  $t \in \llbracket 0, T-1 \rrbracket$ , and all  $x \in \mathbb{X}_t$ ,

$$s_t^*(x) \in \arg \min_{u \in \mathbb{U}_t} \mathbb{E} \left[ \mathbb{1}_{\{x \in \mathcal{X}_t\}} \cdot V_{t+1}(f_t(x, u, W_t)) \right], \quad (10)$$

then this strategy is optimal for Problem (7).

### 3 A general Dynamic Programming equation

#### 3.1 Problem statement

##### Dynamic system and state

In a first place we define a controlled  $T$ -step dynamical system, with  $T \geq 2$ . We consider a sequence of set of states  $(\mathbb{X}_0, \dots, \mathbb{X}_T)$  and define  $\mathbb{X} = \mathbb{X}_0 \times \dots \times \mathbb{X}_T$ ; a sequence of set of uncertainties  $(\mathbb{W}_0, \dots, \mathbb{W}_{T-1})$ , and define  $\mathbb{W} = \mathbb{W}_0 \times \dots \times \mathbb{W}_{T-1}$ ; and a sequence of set of controls  $(\mathbb{U}_0, \dots, \mathbb{U}_{T-1})$  and define  $\mathbb{U} = \mathbb{U}_0 \times \dots \times \mathbb{U}_{T-1}$ . We consider  $T$  evolution functions  $f_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \rightarrow \mathbb{X}_{t+1}$ , for  $t \in \llbracket 0, T-1 \rrbracket$ . We consider the constraint multifunctions  $U_t : \mathbb{X}_t \rightrightarrows \mathbb{U}_t$ , for  $t \in \llbracket 0, T-1 \rrbracket$ .

A strategy  $s = (s_t)_{t \in \llbracket 0, T-1 \rrbracket}$  is a sequence of functions such that for all  $t \in \llbracket 0, T-1 \rrbracket$ ,  $s_t$  maps  $\mathbb{X}_t$  into  $\mathbb{U}_t$ . We denote  $S$  the set of all strategies, and  $S^{ad} \subset S$  the set of all admissible strategy. An admissible strategy  $s$  satisfy in particular :

$$\forall t \in \llbracket 0, T-1 \rrbracket, \quad \forall x \in \mathbb{X}_t, \quad s_t(x) \in U_t(x).$$

Other constraints, such as measurability constraint can be required for a strategy to be admissible.

Given  $w \in \mathbb{W}$ , and a strategy  $s \in S$ , a trajectory follow the equation

$$\forall t \in \llbracket 0, T-1 \rrbracket, \quad x_{t+1} = f_t(x_t, s_t(x_t), w_t),$$

where  $u_t \in \mathbb{U}_t$  and  $w_t \in \mathbb{W}_t$ .

##### Time preferences

Consider that we have  $T$  time-step cost function  $J_t : \mathbb{X}_t \times \mathbb{U}_t \times \mathbb{W}_t \mapsto \bar{\mathbb{R}}$ , (where we denote by  $\bar{\mathbb{R}}$  the set  $\mathbb{R} \cup \{+\infty\}$ ) and a final cost function  $K : \mathbb{X}_T \rightarrow \bar{\mathbb{R}}$ . This means that we have  $T+1$  values that we want to minimize. However to be able to compare two sets of values we aggregate them through a global time-aggregator  $\Psi : \bar{\mathbb{R}}^{T+1} \rightarrow \bar{\mathbb{R}}$ .

**Example 1.** For example in Section 2.1 the global time aggregator is simply given by  $\Psi\{c_0, \dots, c_T\} = \sum_{i=0}^T c_i$ , and in Section 2.2 the time aggregator is  $\Psi\{c_0, \dots, c_T\} = \prod_{i=0}^T c_i$ .

Thus the cost to minimize reads

$$\Psi\left\{J_0(x_0, u_0, W_0), \dots, J_{T-1}(x_{T-1}, u_{T-1}, W_{T-1}), K(x_T)\right\}.$$

##### Risk attitudes

As we did with time we now define an aggregation with respect to uncertainties. We define a global noise aggregator  $\mathbb{G} : \mathcal{F}(\mathbb{W}, \bar{\mathbb{R}}) \rightarrow \bar{\mathbb{R}}$ .

Consequently the objective function of our problem reads

$$\mathbb{G}\left[w \mapsto \Psi\left\{J_0(x_0, u_0, w_0), \dots, J_{T-1}(x_{T-1}, u_{T-1}, w_{T-1}), K(x_T)\right\}\right].$$

**Example 2.** For example in the first part of Section 2.1 the noise aggregator is the extended expectation with respect to the law image of  $\mathbb{P}$  by  $W$ . Where the extended expectation operator is defined as the usual expectation if the operand is measurable and integrable and  $+\infty$  otherwise. The extended expectation with respect to the probability  $\mathbb{P}$  will be denoted  $\bar{\mathbb{E}}_{\mathbb{P}}$ . In the second part the aggregator is the maximum over  $\mathbb{W}$ .

##### Optimization problem statement

Thus the problem we want to address is

$$\min_{s \in S^{\text{ad}}} \mathbb{G} \left[ w \mapsto \Psi \left\{ J_0(x_0, u_0, w_0), \dots, J_{T-1}(x_{T-1}, u_{T-1}, w_{T-1}), K(x_T) \right\} \right], \quad (11a)$$

$$x_{t+1} = f_t(x_t, u_t, w_t) \quad t \in \llbracket 0, T-1 \rrbracket, \quad (11b)$$

$$u_t = s_t(x_t) \quad t \in \llbracket 0, T-1 \rrbracket, \quad (11c)$$

$$u_t \in U_t(x_t) \quad t \in \llbracket 0, T-1 \rrbracket. \quad (11d)$$

### 3.2 Ingredients for a Dynamic Programming Principle

We give some definitions that represent two key assumptions for our theorem : the global aggregators can be written as a composition of time step aggregators, and there is a commutation property between those aggregators.

**Definition 1.** We consider  $\Psi_t : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}}$ , for  $t \in \llbracket 0, T-1 \rrbracket$  a sequence of time-step time aggregators. Their composition  $\Psi = \bigcirc_{\tau=0}^t \Psi_\tau = \Psi_0 \circ \dots \circ \Psi_t : \bar{\mathbb{R}}^t \rightarrow \bar{\mathbb{R}}$  is recursively defined by, for all  $c \in \bar{\mathbb{R}}^{t+1}$ ,

$$\left( \bigcirc_{\tau=t'}^t \Psi_\tau \right) \{c\} := \Psi_{t'} \left\{ c_{t'}, \left( \bigcirc_{\tau=t'+1}^t \Psi_\tau \right) \{c_t, \dots, c_{t+1}\} \right\}, \quad (12)$$

for all  $t' < t$ .

**Definition 2.** We consider  $\mathbb{G}_t : \mathcal{F}(\mathbb{W}_t; \bar{\mathbb{R}}) \rightarrow \bar{\mathbb{R}}$ , for  $t \in \llbracket 0, T-1 \rrbracket$  a sequence of time-step noise aggregators. Their composition  $\mathbb{G} = \square_{\tau=0}^t \mathbb{G}_\tau = \mathbb{G}_0 \square \dots \square \mathbb{G}_t : \mathcal{F}(\mathbb{W}; \bar{\mathbb{R}}) \rightarrow \bar{\mathbb{R}}$  is recursively defined, for all functions  $A \in \mathcal{F}(\mathbb{W}_{t'} \times \dots \times \mathbb{W}_t; \bar{\mathbb{R}})$ , by

$$\left( \square_{\tau=t'}^t \mathbb{G}_\tau \right) [A] := \mathbb{G}_{t'} \left[ w_{t'} \mapsto \left( \square_{\tau=t'+1}^t \mathbb{G}_\tau \right) \left[ (w_{t'+1}, \dots, w_t) \mapsto A(w_{t'}, w_{t'+1}, \dots, w_t) \right] \right]. \quad (13)$$

for all  $t' < t$ .

**Example 3.** In order to shed some light on those notations we define

$$\Psi\{c_1, c_2\} = c_1 + c_2, \quad \Psi'\{c_1, c_2\} = c_1 c_2,$$

then we have

$$\begin{aligned} \Psi \circ \Psi\{c_1, c_2, c_3\} &= c_1 + c_2 + c_3, \\ \Psi' \circ \Psi'\{c_1, c_2, c_3\} &= c_1 c_2 c_3, \\ \Psi \circ \Psi'\{c_1, c_2, c_3\} &= c_1 + c_2 c_3, \\ \Psi' \circ \Psi\{c_1, c_2, c_3\} &= c_1 (c_2 + c_3). \end{aligned}$$

**Definition 3.** A sequence of time aggregators  $(\Psi_0, \dots, \Psi_t)$  is said to commute with a sequence of noise aggregators  $(\mathbb{G}_0, \dots, \mathbb{G}_t)$  if for all  $t' \in \llbracket 0, t \rrbracket$ , all  $C_{t'} \in \mathcal{F}(\mathbb{W}_0 \times \dots \times \mathbb{W}_{t'}; \bar{\mathbb{R}})$ , and all  $C_{t+1} \in \mathcal{F}(\mathbb{W}_0 \times \dots \times \mathbb{W}_t; \bar{\mathbb{R}})$  we have

$$\left( \square_{\tau=0}^t \mathbb{G}_\tau \right) \circ \left( \bigcirc_{\tau=0}^t \Psi_\tau \right) (C_1, \dots, C_{t+1}) = \mathbb{G}_1 \left[ \Psi_1 \left( C_1, \dots, \mathbb{G}_t [\Psi_t \{C_t, C_{t+1}\}] \right) \right].$$

Or, if we extend slightly the definition of time-aggregators,

$$\left( \square_{\tau=0}^t \mathbb{G}_\tau \right) \circ \left( \bigcirc_{\tau=0}^t \Psi_\tau \right) = \bigcirc_{\tau=0}^t (\mathbb{G}_\tau \circ \Psi_\tau).$$

**Definition 4.** A noise aggregator  $\mathbb{G}$  is said to strongly commute with a time aggregator  $\Psi$  if for any  $A \in \mathcal{F}(\mathbb{W}; \bar{\mathbb{R}})$ , and any  $c \in \bar{\mathbb{R}}$ ,

$$\mathbb{G}[w \mapsto \Psi\{c, A(w)\}] = \Psi\{c, \mathbb{G}[w \mapsto A(w)]\}$$

**Example 4.** If  $\mathbb{W}$  is a probability space  $(\mathbb{W}, \mathcal{F}, \mathbb{P})$  and  $\Psi(c_1, c_2) = c_1 + c_2$ , then the extended expectation  $\bar{\mathbb{E}}_{\mathbb{P}}$  is strongly commuting with  $\Psi$ .

**Proposition 1.** If for any  $t \in \llbracket 0, T-2 \rrbracket$ ,  $\mathbb{G}_{t+1}$  strongly commute with  $\Psi_t$ , then the sequence  $(\mathbb{G}_0, \dots, \mathbb{G}_{T-1})$  commutes with the sequence  $(\Psi_0, \dots, \Psi_{T-1})$ .

*Proof.* We prove by backward induction the property

$$(H_k) \quad (\mathbb{G}_k, \dots, \mathbb{G}_{T-1}) \text{ commutes with } (\Psi_k, \dots, \Psi_{T-1}).$$

For  $k = T-2$  the property holds because for any  $C_{T-2} \in \mathcal{F}(\mathbb{W}_{T-2}; \bar{\mathbb{R}})$ , and any  $C_{T-1}$  and  $C_T$  in  $\mathcal{F}(\mathbb{W}_{T-2} \times \mathbb{W}_{T-1}; \bar{\mathbb{R}})$  we have

$$\begin{aligned} \mathbb{G}_{T-1}[\Psi_{T-2} \odot \Psi_{T-1}\{C_{T-2}, C_{T-1}, C_T\}] &= \mathbb{G}_{T-1}[\Psi_{T-2}\{C_{T-2}, \Psi_{T-1}\{C_{T-1}, C_T\}\}], \\ &= \Psi_{T-2}\left\{C_{T-2}, \mathbb{G}_{T-1}[\Psi_{T-1}\{C_{T-1}, C_T\}]\right\}. \end{aligned}$$

Aggregating with  $\mathbb{G}_{T-2}$  gives  $H_{T-2}$ . Assuming that  $(H_k)$  holds true, we show  $(H_{k-1})$  in a similar fashion by defining

$$C'_{T-k} := \mathbb{G}_{T-k} \left[ \Psi_{T-k} \left\{ C_{T-k}, \dots, \mathbb{G}_{T-1} [\Psi_{T-1} \{ C_{T-1}, C_T \}] \right\} \right],$$

we have

$$\left( \bigoplus_{t=T-k-1}^{T-1} \mathbb{G}_t \right) \left[ \bigodot_{t=T-k-1}^{T-1} \Psi_t \{ C_{T-k-1}, C_{T-k}, \dots, C_T \} \right] = \mathbb{G}_{T-k-1} \left[ \Psi_{T-k-1} \{ C_{T-k-1}, C'_{T-k} \} \right]$$

□

We can note that there is no commutation assumption concerning  $\mathbb{G}_1$  or  $\Psi_T$ .

### 3.3 Dynamic Programming principle

We now give the main theorem.

**Theorem 2.** Assume that  $(\mathbb{G}_1, \dots, \mathbb{G}_T)$  commute with  $(\Psi_1, \dots, \Psi_T)$ . Moreover assume that, for all  $t \in \llbracket 1, T-1 \rrbracket$ , all  $C_1 \in \bar{\mathbb{R}}$ ,  $C \mapsto \Psi_t\{C_1, C\}$  is non decreasing and  $\mathbb{G}_t$  is monotonous.

We define the value functions

$$V_T := K, \tag{14a}$$

$$V_t(x) := \inf_{u \in \mathbb{U}_t(x)} \mathbb{G}_t \left[ \Psi_t \left\{ J_t(x, u, \cdot), V_{t+1}(f_t(x, u, \cdot)) \right\} \right] \quad \forall t \in \llbracket 0, T-1 \rrbracket, \quad \forall x \in \mathbb{X}_t. \tag{14b}$$

If there exists an admissible strategy  $s^*$  such that for all  $t \in \llbracket 0, T-1 \rrbracket$ ,

$$s_t^*(x) \in \arg \min_{u \in \mathbb{U}_t(x)} \mathbb{G}_t \left[ \Psi_t \left\{ J_t(x, u, \cdot), V_{t+1}(f_t(x, u, \cdot)) \right\} \right]$$

then  $s^*$  is an optimal strategy for Problem (11), and its value is  $V_0(x_0)$ .

From now on we consider the Problem (11) with  $T = 2$  for simplicity of notation. The proof can easily be extended to the  $T$ -step case. We assume that  $\Psi = \Psi_0 \odot \Psi_1$  and  $\mathbb{G} = \mathbb{G}_0 \odot \mathbb{G}_1$  where  $(\Psi_0, \Psi_1)$  commutes with  $(\mathbb{G}_0, \mathbb{G}_1)$ .

Before presenting the proof itself we need some notations and a technical lemma. We denote, for any strategy  $s \in S$ ,

$$\begin{aligned} x_1^s(w_0) &:= f_0(x_0, s_0(x_0), w_0) \\ x_2^s(w_0, w_1) &:= f_1(x_1^s, s_1(x_1^s), w_1) \\ J^s(x_0)(w_0, w_1) &:= (\Psi_0 \odot \Psi_1) \left\{ J_0(x_0, s_0(x_0), w_0), J_1(x_1^s, s_1(x_1^s), w_1), K(x_2^s)) \right\} \\ J_0^{s_0}(x_0, C)(w_0) &:= \Psi_0 \left\{ J_0(x_0, s_0(x_0), w_0), C \right\} \\ J_1^{s_1}(x_1, K)(w_0, w_1) &:= \Psi_1 \left\{ J_1(x_1, s_1(x_1), w_1), K(x_2^s) \right\} \end{aligned}$$

Thus Problem (11) reads reads

$$\min_{s \in S^{\text{ad}}} \mathbb{G} \left[ (w_0, w_1) \mapsto J^s(x_0)(w_0, w_1) \right].$$

**Lemma 1.** *For any strategy  $s$  we have*

$$(\mathbb{G}_0 \odot \mathbb{G}_1) [J^s(x_0)] = \mathbb{G}_0 \left[ J_0^{s_0} \left( x_0, \mathbb{G}_1 [J_1^{s_1}(x_1^s, K)] \right) \right].$$

*Proof.* We have

$$\begin{aligned} J^s(x_0)(w_0, w_1) &= (\Psi_0 \odot \Psi_1) \left\{ J_0(x_0, s_0(x_0), w_0), J_1(x_1^s, s_1(x_1^s), w_1), K(x_2^s)) \right\} \quad \text{by definition of } J^s \\ &= \Psi_0 \left\{ J_0(x_0, s_0(x_0), w_0), \Psi_1 \left\{ J_1(x_1^s, s_1(x_1^s), w_1), K(x_2^s) \right\} \right\} \end{aligned}$$

Then we have

$$\begin{aligned} (\mathbb{G}_0 \odot \mathbb{G}_1) [J^s(x_0)] &= (\mathbb{G}_0 \odot \mathbb{G}_1) \left[ \Psi_0 \left\{ J_0(x_0, s_0(x_0), \cdot), \Psi_1 \left\{ J_1(x_1^s, s_1(x_1^s), \cdot), K(x_2^s) \right\} \right\} \right], \\ &= \mathbb{G}_1 \left[ \Psi_1 \left\{ J_1(x_1, s_1(x_1), \cdot), \mathbb{G}_2 \left[ \Psi_2 \left\{ J_2(x_2^s, s_2(x_2^s), \cdot), K(x_3^s) \right\} \right] \right\} \right], \\ &\quad \text{By commutativity} \\ &= \mathbb{G}_0 \left[ J_0^{s_0} \left( x_0, \mathbb{G}_1 [J_1^{s_1}(x_1^s, K)] \right) \right]. \end{aligned}$$

□

And now we gives the proof of theorem 2.

*Proof.* We have

$$\mathbb{G}_1 [J_1^{s_1^*}(x_1, K)] = \mathbb{G}_1 \left[ \Psi_1 \left\{ J_1(x_1, s_1^*(x_1), \cdot), K \right\} \right] = V_1(x_1). \quad (15)$$

Moreover

$$\begin{aligned} \mathbb{G} [J^{s^*}(x_0)] &= \mathbb{G}_0 \left[ J_0^{s_0^*} \left( x_0, \mathbb{G}_1 [J_1^{s_1^*}(x_1^*, K)] \right) \right] && \text{by Lemma 1} \\ &= \mathbb{G}_0 \left[ J_0^{s_0^*} \left( x_0, \mathbb{G}_1 \left[ V_1 \circ f_0(x_0, s_0^*(x_0), \cdot) \right] \right) \right] && \text{by (15)} \\ &= V_0(x_0) && \text{by definition of } s_0^* \end{aligned}$$



Now we show that  $s^*$  is optimal. Considering an admissible strategy  $s$ , we have by definition of  $V_1$ ,

$$\forall x \in \mathbb{X}_1, \quad V_1(x) \leq \mathbb{G}_1 [J_1^{s_1}(x)].$$

Moreover note that  $C \mapsto J_0^{s_0}(x_0, C)$  is non-decreasing. Thus

$$\begin{aligned} \mathbb{G}[J^{s^*}(x_0)] &= \mathbb{G}_0 \left[ J_0^{s_0^*} \left( x_0, V_1 \circ f_0(x_0, s_0^*(x_0), \cdot) \right) \right] \\ &\leq \mathbb{G}_0 \left[ J_0^{s_0} \left( x_0, V_1 \circ f_0(x_0, s_0(x), \cdot) \right) \right] && \text{by monotonicity of } \mathbb{G}_1, \\ &\leq \mathbb{G}_0 \left[ J_0^{s_0} \left( x_0, \mathbb{G}_1 \left[ J_1^{s_1} \circ f_0(x_0, s_0(x), \cdot), K \right] \right) \right] && \text{by monotonicity of } \mathbb{G}_1 \text{ and } J_0^{s_0}(x_0, \cdot) \end{aligned}$$

□

## 4 Applications

In this Section we present some class of time aggregators and noise aggregators that are strongly commuting, and monotonous, and thus yield a Dynamic Programming principle.

In all the example we consider a Stochastic Dynamic System with the notations presented in Section 3.1.

### 4.1 Coherent risk measures

For  $t \in \llbracket 0, T-1 \rrbracket$ , let  $(\Omega_t, \mathcal{F}_t)$  be a measurable space. Consider a set of probabilities  $\mathcal{P}_t$  on  $(\Omega_t, \mathcal{F}_t)$ , and denote  $\mathcal{P} = \mathcal{P}_0 \otimes \cdots \otimes \mathcal{P}_{T-1}$ . Consider a sequence of random variables  $(W_t)_{t \in \llbracket 0, T-1 \rrbracket}$  such that each  $W_t$  is measurable with respect to  $\mathcal{F}_t$ .

Consider the following problem, with  $\beta_t$  non negative,

$$\min_{s \in S^{ad}} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[ \sum_{t=0}^{T-1} \left\{ \alpha_t(J_t(x_t, s_t(x_t), W_t)) \prod_{t' < t} \beta_{t'}(J_{t'}(x_{t'}, s_{t'}(x_{t'}), W_{t'})) \right\} \right] \quad (16a)$$

$$+ \prod_{t=0}^{T-1} \beta_t(J_t(x_t, s_t(x_t), W_t)) K(x_T) \Big]. \quad (16b)$$

$$s.t. \quad x_{t+1} = f_t(x_t, s_t(x_t), W_t) \quad (16c)$$

**Proposition 3.** *Problem (16) can be solved by using the following Dynamic Programming Principle*

$$V_T(x) = K(x), \quad (17a)$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t} \left\{ \mathbb{E}_{\mathbb{P}_t} \left[ \alpha_t(J_t(x, u, W_t)) \right. \right. \quad (17b)$$

$$\left. \left. + \beta_t(J_t(x, u, W_t)) \cdot V_{t+1} \circ f_t(x, u, W_t) \right] \right\}. \quad (17c)$$

*Proof.* Problem (16) fit the framework of Problem 11 where  $\Psi = \Psi_0 \odot \cdots \odot \Psi_{T-1}$ , and  $\mathbb{G} = \mathbb{G}_0 \odot \cdots \odot \mathbb{G}_{T-1}$  with

$$\Psi_t \{C_1, C_2\} = \alpha_t(C_1) + \beta_t(C_1)C_2, \quad (18)$$

$$\mathbb{G}_t[A] = \sup_{\mathbb{P}_t \in \mathcal{P}_t} \mathbb{E}_{\mathbb{P}_t}[A], \quad (19)$$

We show that if  $\beta_t$  is non-negative,  $\mathbb{G}_t$  strongly-commute with  $\Psi_{t+1}$ .

We have for any  $C_t \in \bar{\mathbb{R}}$ , and any function  $C_{t+1} \in \mathcal{F}(\Omega_{t+1}; \bar{\mathbb{R}})$ ,

$$\mathbb{G}_{t+1}[\Psi_t\{C_t, C_{t+1}\}] = \sup_{\mathbb{P}_{t+1} \in \mathcal{P}_{t+1}} \left\{ \bar{\mathbb{E}}_{\mathbb{P}_{t+1}}[\alpha(C_t) + \beta(C_t)C_{t+1}] \right\} \quad (20)$$

$$= \alpha_t(C_t) + \beta_t(C_t) \sup_{\mathbb{P}_{t+1} \in \mathcal{P}_{t+1}} \left\{ \bar{\mathbb{E}}_{\mathbb{P}_{t+1}}[C_{t+1}] \right\} \quad \text{as } \beta \text{ is non-negative} \quad (21)$$

$$= \alpha_t(C_t) + \beta_t(C_t)\mathbb{G}_{t+1}[C_{t+1}] \quad (22)$$

Moreover it is easy to check that  $\mathbb{G}_t$  is monotonous, and that  $C \mapsto \alpha(C_t) + \beta(C_t)C$  is non decreasing.

The Dynamic Programming equation directly derive from Theorem 2.  $\square$

We conclude by a few remarks on the class of aggregator studied here.

If  $\beta = 1$   $\Psi$  is the additive-costs case, and if  $\alpha = 0$   $\Psi$  represent the multiplicative-costs case.

Let give three example of random aggregators constructed this way:

- if  $\mathcal{P}_t$  is the singleton  $\{\mathbb{P}_t\}$ , then  $\mathbb{G}_t$  is simply the extended expectation operator with respect to probability  $\mathbb{P}_t$ .
- if  $\mathcal{P}_t$  contains all the Dirac mass of each  $\omega \in \Omega_t$  then  $\mathbb{G}$  is the worst-case scenario aggregator.
- if  $\mathcal{P}_t$  is obtain from a closed convex set of densities then  $\mathbb{G}_t$  is a coherent risk measure as defined by Artzner et al. in [1].

## 4.2 Convex risk measures

In the same setting we consider the following problem

$$\min_{s \in S^{ad}} \sup_{\mathbb{P} \in \mathcal{P}} \bar{\mathbb{E}}_{\mathbb{P}} \left[ \sum_{t=0}^{T-1} \left\{ \alpha_t(J_t(x_t, s_t(x_t), W_t)) - g_t(\mathbb{P}_t) \right\} + K(x_T) \right]. \quad (23a)$$

$$s.t. \quad x_{t+1} = f_t(x_t, s_t(x_t), W_t) \quad (23b)$$

**Proposition 4.** *Problem (23) can be solved by using the following Dynamic Programming Principle*

$$V_T(x) = K(x), \quad (24a)$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{\mathbb{P}_t \in \mathcal{P}_t} \left\{ \bar{\mathbb{E}}_{\mathbb{P}_t} \left[ \alpha_t(J_t(x, u, W_t)) + V_{t+1} \circ f_t(x, u, W_t) \right] \right\} - g_t(\mathbb{P}_t). \quad (24b)$$

Note that if  $\mathcal{P}_t$  is a closed convex set of probability and  $f_t$  is convex and lower semi continuous then  $\mathbb{G}_t$  is a convex measure of risk as defined in [10].

*Proof.* Problem 23 fit the framework of Problem 11 where  $\Psi = \Psi_0 \odot \dots \odot \Psi_{T-1}$ , and  $\mathbb{G} = \mathbb{G}_0 \odot \dots \odot \mathbb{G}_{T-1}$  with

$$\Psi_t\{C_1, C_2\} = \alpha_t(C_1) + C_2, \quad (25)$$

$$\mathbb{G}_t[A] = \sup_{\mathbb{P}_t \in \mathcal{P}_t} \bar{\mathbb{E}}_{\mathbb{P}_t}[A] - g_t(\mathbb{P}_t), \quad (26)$$

We show that for all  $t \in \llbracket 0, T-1 \rrbracket$ ,  $\mathbb{G}_{t+1}$  strongly commute with  $\Psi_t$ .

Let  $C \in \bar{\mathbb{R}} \times \mathcal{F}(\Omega_t, \bar{\mathbb{R}})$ , we have

$$\mathbb{G}_{t+1}[\Psi_t\{C_t, C_{t+1}\}] = \sup_{\mathbb{P}_{t+1} \in \mathcal{P}_{t+1}} \left\{ \bar{\mathbb{E}}_{\mathbb{P}_{t+1}}[\alpha(C_t) + C_{t+1}] - g_{t+1}(\mathbb{P}_{t+1}) \right\}, \quad (27)$$

$$= \alpha_t(C_t) + \sup_{\mathbb{P}_{t+1} \in \mathcal{P}_{t+1}} \left\{ \bar{\mathbb{E}}_{\mathbb{P}_{t+1}}[C_{t+1}] - g_{t+1}(\mathbb{P}_{t+1}) \right\}, \quad (28)$$

$$= \alpha_t(C_t) + \mathbb{G}_{t+1}[C_{t+1}]. \quad (29)$$

Moreover  $\mathbb{G}_t$  is monotonous and  $C \mapsto \alpha(C_t) + C$  is non decreasing. We conclude with Theorem 2.  $\square$

### 4.3 Fear Operator

A special case of coherent risk measures are the worst case scenario operators, also called the fear operator. For each time  $t$  we consider  $\widetilde{\mathbb{W}}_t$  a subset of  $\mathbb{W}_t$ , and a function  $\Psi_t : \mathbb{R}^2 \rightarrow \mathbb{R}$  which is non-decreasing in its second variable. We denote  $\widetilde{\mathbb{W}} = \widetilde{\mathbb{W}}_1 \times \cdots \times \widetilde{\mathbb{W}}_T$  and consider the following problem

$$\min_{s \in S^{ad}} \sup_{w \in \widetilde{\mathbb{W}}} \Psi_0 \left\{ J_0(x_0, s_0(x_0), w_0), \Psi_1 \left\{ \cdots, \Psi_{T-1} \left\{ J_{T-1}(x_{T-1}, s_{T-1}(x_{T-1}), w_{T-1}), K(x_T(w_{T-1})) \right\} \right\} \right\} \quad (30a)$$

$$s.t. \quad x_{t+1} = f_t(x_t, s_t(x_t), W_t) \quad (30b)$$

**Proposition 5.** *Problem (30) can be solved by using the following Dynamic Programming Principle*

$$V_T(x) = K(x), \quad (31a)$$

$$V_t(x) = \min_{u \in U_t(x)} \sup_{w_t \in \widetilde{\mathbb{W}}_t} \left\{ \Psi_t \left\{ J_t(x, u, w_t), V_{t+1} \circ f_t(x, u, w_t) \right\} \right\}. \quad (31b)$$

*Proof.* This problem fit the framework of Problem (11) where  $\Psi = \Psi_0 \odot \cdots \odot \Psi_{T-1}$ , and  $\mathbb{G} = \mathbb{G}_0 \odot \cdots \odot \mathbb{G}_{T-1}$  with

$$\Psi_t \{C_1, C_2\} = \Phi_t \{C_1, C_2\}, \quad (32)$$

$$\mathbb{G}_t[A] = \sup_{w_t \in \widetilde{\mathbb{W}}_t} A(w_t), \quad (33)$$

Note that the composition  $\bigodot_{t=0}^{T-1} \mathbb{G}_t$  is simply the fear operator on the cartesian product  $\widetilde{\mathbb{W}} = \widetilde{\mathbb{W}}_0 \times \cdots \times \widetilde{\mathbb{W}}_{T-1}$ .

We show that if, for any  $C_1 \in \bar{\mathbb{R}}$ ,  $C_2 \mapsto \Psi_t(C_1, C_2)$  is non-decreasing then  $\mathbb{G}_{t+1}$  strongly commutes with  $\Psi_t$ .

Let  $C \in \bar{\mathbb{R}} \times \mathcal{F}(\mathbb{W}_2, \bar{\mathbb{R}})$ , we have

$$\mathbb{G}_{t+1} \left[ \Psi_t \{C_t, C_{t+1}\} \right] = \sup_{w \in \widetilde{\mathbb{W}}_{t+1}} \left[ \Psi_t \{C_t, C_{t+1}(w_{t+1})\} \right], \quad (34)$$

$$= \Psi_t \left\{ C_t, \sup_{w \in \widetilde{\mathbb{W}}_{t+1}} [C_{t+1}(w_{t+1})] \right\}, \quad (35)$$

$$= \Psi_t \{C_t, \mathbb{G}_{t+1}[C_{t+1}]\}. \quad (36)$$

Moreover  $\mathbb{G}_t$  is monotonous. Theorem 2 concludes the proof.  $\square$

A good example of monotonous time aggregator is  $\Psi_t \{C_1, C_2\} = \max \{C_1, C_2\}$  that is used in intertemporal comparison as in [8].

## 5 Combining commuting aggregators

### 5.1 Maximum and infimum

We consider time aggregator  $\Psi$  non-decreasing in its second argument and a family of monotonous uncertainties aggregators  $\mathbb{G}^i$  that strongly commute with  $\Psi$ .

**Proposition 6.** *Let  $\Psi$  be a time aggregator non-decreasing in its second argument. Suppose that  $(\mathbb{G}^i)_{i \in I}$  is a family of monotonous random aggregators strongly commuting with  $\Psi$ . Moreover if we have either one of the two following conditions*

*i) for all  $C_2 \in \mathcal{F}(\mathbb{W}_2, \bar{\mathbb{R}})$ ,  $\sup_{i \in I} \mathbb{G}^i[C_2]$  is attained (always true for  $I$  finite);*

*ii) for all  $C_1 \in \bar{\mathbb{R}}$   $\Psi \{C_1, \cdot\}$  is lower-semicontinuous (resp. upper-semicontinuous);*

then the aggregator  $\bar{\mathbb{G}} := \sup_{i \in I} \mathbb{G}^i$  (resp.  $\underline{\mathbb{G}} := \inf_i \mathbb{G}^i$ ) is monotonous and strongly commutes with  $\Psi$ .

*Proof.* Consider  $C \in \bar{\mathbb{R}} \times \mathcal{F}(\mathbb{W}_2, \bar{\mathbb{R}})$ , we have

$$\bar{\mathbb{G}}[\Psi\{C_1, C_2\}] = \sup_{i \in I} \mathbb{G}^i[\Psi\{C_1, C_2\}] \quad \text{by definition of } \bar{\mathbb{G}}, \quad (37)$$

$$= \sup_{i \in I} \Psi\{C_1, \mathbb{G}^i[C_2]\} \quad \text{by strong commutation,} \quad (38)$$

$$= \Psi\{C_1, \sup_{i \in I} \mathbb{G}^i[C_2]\} \quad \text{by monotonicity of } \Psi. \quad (39)$$

$$(40)$$

The last equality being obtained either if the supremum over  $i \in I$  of  $\mathbb{G}^i[C_2]$  is attained or by semicontinuity. Thus we have shown that

$$\bar{\mathbb{G}}[\Psi\{C_1, C_2\}] = \Psi\{C_1, \bar{\mathbb{G}}[C_2]\}.$$

The proof for the minimum is obtained by substituting sup by inf.  $\square$

Moreover if  $\Psi$  is affine in its second variable (i.e.  $\Psi\{c_1, c_2\} = \alpha(c_1) + \beta(c_1)c_2$ ) then any convex combinaison of infimum or maximum of monotonous aggregators strongly-commuting with  $\Psi$  is monotonous and strongly-commuting with  $\Psi$ . In addition if  $\Psi$  is linear in its second variable we can consider positive linear combinaison instead of convex combinaison.

### 5.1.1 Change of variables

We now focus on a way to derive a new pair of commuting agregators from another one via the use of a ‘‘utility’’ function (not assumed to be concave).

**Proposition 7.** *Let  $\nu$  be an increasing bijection from  $I \subset \bar{\mathbb{R}}$  onto  $J \subset \bar{\mathbb{R}}$ . Assume that we have an aggregator such that the image of  $\mathbb{G} \circ \nu$  is contained in  $I$ . We define  $\tilde{\mathbb{G}}_t := \nu^{-1} \circ \mathbb{G} \circ \nu$ .*

*In a similar fashion, we define  $\tilde{\Psi}_t$  given by, for all  $C \in \bar{\mathbb{R}}^2$ ,*

$$\tilde{\Psi}_t : (C_1, C_2) \mapsto \nu^{-1}(\Psi_t\{C_1, \nu(C_2)\}).$$

*If the sequence  $(\mathbb{G}_1, \dots, \mathbb{G}_t)$  commutes with  $(\Psi_1, \dots, \Psi_t)$ , then the sequence  $(\tilde{\mathbb{G}}_1, \dots, \tilde{\mathbb{G}}_t)$  commutes with  $(\tilde{\Psi}_1, \dots, \tilde{\Psi}_t)$ . Finally if  $\mathbb{G}_t$  is monotonous then so is  $\tilde{\mathbb{G}}_t$ , and if  $\Psi_t\{C, \cdot\}$  is monotonous so is  $\tilde{\Psi}_t\{C, \cdot\}$ .*

When we modify the aggregators with a ‘‘utility’’ function it is natural to also modify the time-stage cost, and consider  $\tilde{J}_t = \nu_t \circ J_t$ . Note that the result is mathematically weak however it gives the possibility of writting a sequence of consistent problems that are similar.

*Proof.* For simplicity the proof is written for  $T = 2$ , but can easily be extended to the general

case. For any  $C \in \mathcal{L}(\mathbb{W}_1; \bar{\mathbb{R}}) \times \mathcal{L}(\mathbb{W}_1 \times \mathbb{W}_2; \bar{\mathbb{R}})^2$ ,

$$\begin{aligned}
(\tilde{\mathbb{G}}_1 \odot \tilde{\mathbb{G}}_2) \left[ (\tilde{\Psi}_1 \odot \tilde{\Psi}_2)(C) \right] &= \tilde{\mathbb{G}}_1 \left[ \tilde{\mathbb{G}}_2 \left[ \tilde{\Psi}_1 \{ C_1, \tilde{\Psi}_2 \{ C_2, C_3 \} \} \right] \right], \\
&\text{by definition of compositions,} \\
&= \nu^{-1} \left( \mathbb{G}_1 \circ \nu \left[ \nu^{-1} \circ \mathbb{G}_2 \circ \nu \left[ \nu^{-1} \circ \Psi_1 \{ C_1, \nu \circ \nu^{-1} \circ \Psi_2 \{ C_2, \nu(C_3) \} \} \right] \right] \right), \\
&= \nu^{-1} \left( \mathbb{G}_1 \left[ \mathbb{G}_2 \left[ \Psi_1 \{ C_1, \Psi_2 \{ C_2, \nu(C_3) \} \} \right] \right] \right), \\
&= \nu^{-1} \left( (\mathbb{G}_1 \odot \mathbb{G}_2) \left[ (\Psi_1 \odot \Psi_2) \{ C_1, C_2, \nu(C_3) \} \right] \right), \\
&\text{by definition of compositions,} \\
&= \nu^{-1} \left( \mathbb{G}_1 \left[ \Psi_1 \left( C_1, \mathbb{G}_2 \left[ \Psi_2 \{ C_2, \nu(C_3) \} \right] \right) \right] \right), \\
&\text{by commutativity of } \mathbb{G} \text{ with } \Psi, \\
&= \nu^{-1} \left( \mathbb{G}_1 \circ \nu \circ \nu^{-1} \left[ \Psi_1 \left( C_1, \nu \circ \nu^{-1} \circ \mathbb{G}_2 \circ \nu \circ \nu^{-1} \left[ \Psi_2 \{ C_2, \nu(C_3) \} \right] \right) \right] \right), \\
&= \tilde{\mathbb{G}}_1 \left[ \tilde{\Psi}_1 \left( C_1, \tilde{\mathbb{G}}_2 \left[ \tilde{\Psi}_2 \{ C_2, C_3 \} \right] \right) \right].
\end{aligned}$$

Which conclude the proof of commutativity. Monotonicity is directly given by composition of non-decreasing functions.  $\square$

**Example 5.** We give the risk sensitive approach as an example of application. Risk-sensitive approach as expected utility goes back to Pratt ([15]) and Howard and Thomason ([12]). Even if this approach has some drawbacks (see for example [13]) it is still widely used and studied.

We consider the following problem

$$\inf_{s \in S^{ad}} -\frac{1}{\gamma} \ln \left( \bar{\mathbb{E}} \left[ e^{-\gamma \left( \sum_{t=1}^T J_t(x_t, s_t(x_t), W_t) + K(x_T) \right)} \right] \right). \quad (41)$$

Then Problem (41) can be solved by using the following Dynamic Programming Principle

$$\begin{aligned}
V_{T+1}(x) &= K(x), \\
V_t(x) &= \min_{u \in U_t(x)} -\frac{1}{\gamma} \ln \left( \bar{\mathbb{E}} \left[ e^{-\gamma \left( J_t(x_t, u, W_t) + V_{t+1} \circ f_t(x_t, u, W_t) \right)} \right] \right).
\end{aligned}$$

*Proof.* Defining

$$\begin{aligned}
\Psi_t \{ c_1, c_2 \} &= e^{-\gamma c_1} \times c_2, \\
\mathbb{G}_t &= \bar{\mathbb{E}}_{\mathbb{P}_t}, \\
\nu &: x \mapsto -e^{-\gamma x},
\end{aligned}$$

we see that Problem (41) is simply the transformation by  $\nu$  of the expected product of costs.  $\square$

## 5.2 Multi- $\mathbb{G}$ -linearity

We define a new concept of commutation between a  $\bar{\mathbb{R}}$ -operator of arity  $n$ , i.e a function  $\Psi : \bar{\mathbb{R}}^n \rightarrow \bar{\mathbb{R}}$  and an uncertainty aggregator: we say that  $\Psi$  is *multi- $\mathbb{G}_2$ -linear* if for all  $C \in \mathcal{F}(\mathbb{W}_2, \bar{\mathbb{R}})^n$  we have

$$\mathbb{G} \left[ \Psi \{ C_1, \dots, C_n \} \right] = \Psi \left\{ \mathbb{G} [C_1], \dots, \mathbb{G} [C_n] \right\}.$$

In particular if the uncertainty aggregator  $\mathbb{G}$  conserve the constants that is

$$\forall c \in \bar{\mathbb{R}}, \quad \mathbb{G}[c] = c$$

and  $\Psi_1$  is multi- $\mathbb{G}$ -linear then  $\mathbb{G}$  strongly commutes with  $\Psi_1$ .

**Proposition 8.** *If  $\Psi$  is a multi- $\mathbb{G}_2$ -linear  $\bar{\mathbb{R}}$ -operator of arity  $n$ , and for all  $i \in \llbracket 1, n \rrbracket$ , and  $\mathbb{G}$  strongly commutes with  $\psi^i$  then  $\mathbb{G}$  strongly commutes with  $\tilde{\Psi}$  defined as, for all  $C \in \mathcal{F}(\mathbb{W}, \bar{\mathbb{R}})^2$ ,*

$$\tilde{\Psi}\{C_1, C_2\} = \Psi\{\Psi^1\{C_1, C_2\}, \dots, \Psi^n\{C_1, C_2\}\}.$$

*Proof.* For  $C \in \mathcal{F}(\mathbb{W}_2, \bar{\mathbb{R}})^2$  we have

$$\begin{aligned} \mathbb{G}\left[\tilde{\Psi}\{C_1, C_2\}\right] &= \mathbb{G}\left[\Psi\{\Psi^1\{C_1, C_2\}, \dots, \Psi^n\{C_1, C_2\}\}\right] && \text{by defining } \tilde{\Psi}, \\ &= \Psi\left\{\mathbb{G}\left[\Psi^1\{C_1, C_2\}\right], \dots, \mathbb{G}\left[\Psi^n\{C_1, C_2\}\right]\right\} && \text{by multilinearity,} \\ &= \Psi\left\{\Psi^1\{C_1, \mathbb{G}_2[C_2]\}, \dots, \Psi^n\{C_1, \mathbb{G}[C_2]\}\right\} && \text{by strong commutation,} \\ &= \tilde{\Psi}\{C_1, \mathbb{G}_2[C_2]\}. \end{aligned}$$

□

## 6 Conclusion

In this paper we have described a general framework for a Dynamic Programming Principle. For simplicity reasons we have presented fixed uncertainty aggregators and not conditional ones. However the framework will be extended to conditional aggregators and thus allow us to make more precise links between this dynamic optimization concept of time-consistency and the time-consistency defined in the risk measure litterature.

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