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# Proper Hamiltonian Paths in Edge-Coloured Multigraphs

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## Abstract

A  $c$ -edge-coloured multigraph has each edge coloured with one of the  $c$  available colours where no two parallel edges have the same colour. A proper Hamiltonian path is a path that contains all the vertices of the multigraph such that no two adjacent edges have the same colour. In this work we establish sufficient conditions for a multigraph to have a proper Hamiltonian path, depending on several parameters such as the number of edges, the rainbow degree, etc.

*Keywords:* Multigraph, Proper Hamiltonian Path, Edge-Coloured Graph

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# 1 Introduction

The study of problems modeled by edge-coloured graphs have resulted in important developments recently. For instance, the research on long coloured cycles and paths for edge-coloured graphs has provided interesting results [3]. From a practical perspective, problems arising in molecular biology are often modeled using coloured graphs, i.e., graphs with coloured edges and/or vertices [9]. Given an edge-coloured graph, the original problems are equivalent to extract subgraphs coloured in a specified pattern. The most natural pattern in such a context is that of proper colourings, i.e., adjacent edges have different colours.

The proper Hamiltonian path and proper Hamiltonian cycle problems are  $NP$ -complete in the general case. It is polynomial to find a proper Hamiltonian path in  $c$ -edge-coloured complete graphs,  $c \geq 2$  [7]. It is also polynomial to find a proper Hamiltonian cycle in 2-edge-coloured complete graphs [4]. It is still open to determine the computational complexity for proper Hamiltonian cycles,  $c \geq 3$  [5]. Many other partial results for edge-coloured multigraphs can be found in the survey by Bang-Jensen and Gutin [2].

In this work we give sufficient conditions involving various parameters as the number of edges, rainbow degree, etc, in order to guarantee the existence of proper Hamiltonian paths in edge-coloured multigraphs where parallel edges with same colours are not allowed. All of our results are tight. We remark that this problem arises naturally since it is not reasonable to study such conditions just for simple edge-coloured graphs. Results involving only degree conditions can be found in [1].

Formally, let  $I_c = \{1, 2, \dots, c\}$  be a set of  $c \geq 2$  colours. Throughout this paper,  $G^c$  denotes a  $c$ -edge-coloured multigraph such that each edge is coloured with one colour in  $I_c$  and no two parallel edges joining the same pair of vertices have the same colour. Let  $n$  be the number of vertices and  $m$  be the number of edges of  $G^c$ . If  $H$  is a subgraph of  $G^c$ , then  $N_H^i(x)$  denotes the set of vertices of  $H$  adjacent to  $x$  with an edge of colour  $i$ . Whenever  $H$  is isomorphic to  $G^c$ , we write  $N^i(x)$  instead of  $N_{G^c}^i(x)$ . The *coloured  $i$ -degree* of a vertex  $x$ , denoted by  $d^i(x)$ , is the cardinality of  $N^i(x)$ . The *rainbow degree* of a vertex  $x$ , denoted by  $rd(x)$ , is the number of different colours on the edges incident to  $x$ . The *rainbow degree* of a multigraph  $G^c$ , denoted by  $rd(G^c)$ , is the minimum rainbow degree among its vertices. An edge with endpoints  $x$  and  $y$  is denoted by  $xy$ , and its colour by  $c(xy)$ . A *rainbow complete multigraph* is the one having all possible coloured edges between any pair of vertices (its number of edges is therefore  $c\binom{n}{2}$ ). The *complement* of a

multigraph  $G^c$  denoted by  $\overline{G^c}$ , is a multigraph with the same vertices as  $G^c$  and an edge  $vw \in E(\overline{G^c})$  if and only if  $vw \notin E(G^c)$ . A subgraph of  $G^c$  is said to be *properly edge-coloured* if any two adjacent edges in this subgraph differ in colour. A *Hamiltonian path (cycle)* is a path (cycle) containing all vertices of the multigraph. A path is said to be *compatible* with a given matching  $M$  if the edges of the path are alternatively in  $M$  and not in  $M$ . We assume that the first and the last edge of the path are in  $M$  otherwise we just remove one (or both) of them in order to have this property. All multigraphs are assumed to be connected.

This paper is organized as follows: In Section 2 we present some preliminary results that will be useful for the rest of the paper. In Section 3 we study proper Hamiltonian paths in 2-edge-coloured multigraphs. In Section 4 we study proper Hamiltonian paths in  $c$ -edge-coloured multigraphs, for  $c \geq 3$ . We remark that this division is because of proper Hamiltonian paths in 2-edge-coloured multigraphs are just alternating paths in two colours, therefore the results are different of those for  $c \geq 3$  colours. Finally, in Section 5 we present the conclusions.

## 2 Preliminary results

**Lemma 2.1** *Let  $G$  be a connected non-coloured simple graph on  $n$  vertices,  $n \geq 9$ . If  $m \geq \binom{n-2}{2} + 3$ , then  $G$  has a matching  $M$  of size  $|M| = \lfloor \frac{n}{2} \rfloor$ .*

**Proof.** By a theorem in [6], a 2-connected graph on  $n \geq 10$  vertices and  $m \geq \binom{n-2}{2} + 5$  edges has a Hamiltonian cycle. So if we add a new vertex  $v$  and we join it to all the vertices of  $G$  we have that  $G + \{v\}$  has  $m \geq \binom{n-1}{2} + 5$  edges. Therefore  $G + \{v\}$  has a Hamiltonian cycle, i.e.,  $G$  has a Hamiltonian path and this implies that there exists a matching  $M$  in  $G$  of size  $|M| = \lfloor \frac{n}{2} \rfloor$ .  $\square$

**Lemma 2.2** ([8]) *Let  $G$  be a simple non-coloured graph on  $n \geq 14$  vertices. If  $m \geq \binom{n-3}{2} + 4$  and for every vertex  $x$ ,  $d(x) \geq 1$ , then  $G$  has a matching  $M$  of size  $|M| \geq \lceil \frac{n-2}{2} \rceil$ .*

**Lemma 2.3** *Let  $G^c$  be a 2-edge-coloured multigraph on  $n \geq 14$  vertices. Suppose that for every vertex  $x$  in  $G^c$ ,  $rd(x) = 2$ . If  $m \geq \binom{n}{2} + \binom{n-3}{2} + 4$ , then  $G^c$  has two matchings  $M^r$  and  $M^b$  of colours red and blue respectively, such that  $|M^r| = \lfloor \frac{n}{2} \rfloor$  and  $|M^b| \geq \lceil \frac{n-2}{2} \rceil$ .*

**Proof.** Let us denote  $E^r$  and  $E^b$  as the set of edges coloured in red and blue, of sizes  $|E^r| = m^r$  and  $|E^b| = m^b$  respectively. Observe that, as for every

vertex  $x$  in  $G^c$ ,  $rd(x) = 2$ , we have that  $d^i(x) \geq 1$  for  $i \in \{r, b\}$ . Observe also that  $m^i \geq \binom{n-3}{2} + 4$  for  $i \in \{r, b\}$ , since this threshold is tight when the multigraph is complete on one of the colours.

First, if  $n$  is odd, by Lemma 2.2 there exist two matchings  $M^r$  and  $M^b$ , each one of size  $\frac{n-1}{2}$ , so the result follows straightforward. Second, if  $n$  is even, suppose without loss of generality that  $m^r \geq m^b$ . Then  $m^r \geq (\binom{n}{2} + \binom{n-3}{2} + 4)/2 > \binom{n-2}{2} + 3$ . It is sufficient to show that  $G^r$  has a matching of size  $\lfloor \frac{n}{2} \rfloor$  because  $G^b$  has one of size  $\lceil \frac{n-2}{2} \rceil$  by Lemma 2.2. Now, since  $\delta(G^r) \geq 1$ ,  $G^r$  is connected. Lemma 2.1 implies that  $G^r$  has a matching of size  $\lfloor \frac{n}{2} \rfloor$  as desired.  $\square$

**Lemma 2.4** *Let  $G^c$  be a connected  $c$ -edge-coloured multigraph,  $c \geq 2$ . Suppose that  $G^c$  contains a proper path  $P = x_1y_1x_2y_2 \dots x_p y_p$ ,  $p \geq 3$ , such that each edge  $x_i y_i$  is red. If  $G^c$  does not contain a proper cycle  $C$  such that  $V(C) = V(P)$  then there are at least  $(c-1)(2p-2)$  missing edges in  $G^c$ .*

**Proof.** Let blue be any other colour different from red. The blue edge  $x_1 y_p$  cannot be in  $G^c$  otherwise  $x_1 y_1 \dots x_p y_p x_1$  is a proper cycle. Suppose that the blue edges  $x_1 x_i$  are present in  $G^c$  for  $i = 2, \dots, p$ . Then the blue edges  $y_{i-1} y_p$  cannot be in  $G^c$  otherwise we have the proper cycle  $x_1 x_i \dots y_p y_{i-1} \dots x_1$  contradicting our hypothesis. Therefore for each edge  $y_{i-1} x_i$  in the path, either the blue edge  $x_1 x_i$  or the blue edge  $y_{i-1} y_p$  is missing. So there are  $p-1$  blue missing edges. Now suppose that the blue edges  $x_1 y_i$  are present in  $G^c$ , for  $i = 3, \dots, p-2$ . Then the blue edges  $x_{i+1} y_p$  cannot be in  $G^c$  at same time as the blue edges  $x_i y_{i+1}$ ,  $y_{i-1} x_{i+2}$  or  $y_{i-1} y_{i+1}$ ,  $x_i x_{i+2}$ , otherwise we have the proper cycles  $x_1 y_i x_i y_{i+1} x_{i+1} y_p \dots x_{i+2} y_{i-1} \dots x_1$  or  $x_1 y_i x_i x_{i+2} \dots y_p x_{i+1} y_{i+1} y_{i-1} \dots x_1$ . Then for each edge  $y_i x_{i+1}$  in the path, the minimum in this case corresponds to one missing edge among  $x_{i+1} y_p$ ,  $x_1 y_i$  for  $i = 2, \dots, p-1$ . Therefore there are  $p-3$  blue missing edges.

For the moment we have  $2p-3$  blue missing edges. To obtain the last missing edge suppose that the blue edge  $x_2 y_p$  is present in  $G^c$ . Then, it cannot be at the same time with the blue edges  $x_1 y_2$ ,  $y_1 x_3$  or  $x_1 x_3$ ,  $y_1 y_2$ , otherwise we obtain the proper cycles  $x_1 y_2 \dots x_2 y_p \dots x_3 y_1 x_1$  or  $x_1 x_3 \dots y_p x_2 \dots y_2 y_1 x_1$ . The minimum in this case corresponds to one missing edge  $x_2 y_p$ . We remark that the blue edges  $x_2 y_p$ ,  $y_1 y_2$  and  $y_1 x_3$  were not counted before. The edges  $x_1 x_3$  and  $x_1 y_2$  were supposed to exist, otherwise, to obtain the last missing edge we consider the symmetric case, i.e., using the blue edge  $y_{p-1} x_1$  (if it exists).

In total there are  $2p-2$  blue missing edges in  $G^c$ . As we have  $c-1$  colours different from red, that gives  $(c-1)(2p-2)$  missing edges as desired. Note that this number of missing edges is the same as in the simplest case, that is,

if all edges different from red  $x_1x_i$  and  $x_1y_i$ , for  $i = 2, \dots, p$  are not present in  $G^c$ .  $\square$

**Lemma 2.5** *Let  $G^c$  be a connected  $c$ -edge-coloured multigraph,  $c \geq 2$ . Let  $M$  be a matching of  $G^c$  in one colour, say red, of size  $|M| \geq \lceil \frac{n-2}{2} \rceil$ . Let  $P = x_1y_1x_2y_2 \dots x_py_p$ ,  $p \geq 3$ , be a longest proper path compatible with  $M$ . Then the following holds:*

- **(1)**  $n$  is even,  $|M| = \frac{n}{2}$  and  $2p < n$ .
  - **(1a)** If there is no proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(n - 2 + pn - 2p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 4)(c - 1)$ , for  $p = \frac{n-2}{2}$ .
  - **(1b)** If there is a proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(2pn - 4p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 4)(c - 1)$ , for  $p = \frac{n-2}{2}$ .
- **(2)**  $n$  is odd,  $|M| = \frac{n-1}{2}$  and  $2p < n - 1$ .
  - **(2a)** If there is no proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(n - 3 - p + pn - 2p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 6)(c - 1)$ , for  $p = \frac{n-3}{2}$ .
  - **(2b)** If there is a proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(2pn - 2p - 4p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 6)(c - 1)$ , for  $p = \frac{n-3}{2}$ .
- **(3)**  $n$  is even,  $|M| = \frac{n-2}{2}$  and  $2p < n - 2$ .
  - **(3a)** If there is no proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(n - 4 - 2p + pn - 2p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 8)(c - 1)$ , for  $p = \frac{n-4}{2}$ .
  - **(3b)** If there is a proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(2pn - 4p - 4p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 8)(c - 1)$ , for  $p = \frac{n-4}{2}$ .

**Proof.** Suppose first that  $n$  is even and  $2p < n$ . Since  $M$  has size  $\frac{n}{2}$  there are  $\frac{n-2p}{2}$  red edges outside  $P$ . Let us denote these edges by  $e_i$  for  $i = 1, \dots, \frac{n-2p}{2}$ . Suppose there is no proper  $C$  cycle such that  $V(C) = V(P)$ . Let blue denote any other colour different from red. By Lemma 2.4 there are  $(2p - 2)$  blue missing edges. As the path is maximum, we cannot extend  $P$  having an edge  $e_i$  neither at the beginning nor at the end of it. Then, there are no blue edges between the vertices  $x_1, y_p$  and the edges  $e_i$ . Thus, there are  $2(n - 2p)$  blue missing edges. Finally, as we cannot add any edge  $e_i$  in-between the path, there are at most two blue edges between the edges  $e_i$  and the edges  $y_ix_{i+1}$ ,

$i = 1, \dots, p - 1$ . So, there are  $\frac{(n-2p)(2p-2)}{2}$  blue missing edges. Adding up and simplifying all these numbers and having  $c - 1$  colours different from red, we arrive to  $(n - 2 + pn - 2p^2)(c - 1)$  missing edges in  $G^c$ . If we search the minimum value of this function we obtain  $(2n - 4)(c - 1)$  for  $p = \frac{n-2}{2}$  and case **(1a)** holds. Now if there is a proper cycle  $C$  such that  $V(C) = V(P)$  then there cannot exist any edge at all different from red between all vertices of  $C$  and the edges  $e_i$ . Therefore there are  $(2pn - 4p^2)(c - 1)$  missing edges. Again, by minimizing the function we obtain the same result as above and case **(1b)** holds.

Suppose now that  $n$  is odd,  $M = \frac{n-1}{2}$  and  $2p < n - 1$ , or  $n$  is even,  $M = \frac{n-2}{2}$  and  $2p < n - 2$ . In both cases the same arguments listed above apply just replacing  $n$  with  $n - 1$  or  $n - 2$  respectively, in the number of missing edges. This is because we have  $n - 1$  matched vertices and one non-matched vertex for the first case and  $n - 2$  matched vertices and two non-matched vertices for the second one.  $\square$

### 3 2-edge-coloured multigraphs

In this section we study the existence of proper Hamiltonian paths in 2-edge-coloured multigraphs. We present two main results. The first one involves just the number of edges. The second one involves the number of edges and the rainbow degree. Both results are tight.

**Theorem 3.1** *Let  $G^c$  be a 2-edge-coloured multigraph on  $n \geq 8$  vertices coloured with  $\{r, b\}$ . If  $m \geq \binom{n}{2} + \binom{n-2}{2} + 1$ , then  $G^c$  has a proper Hamiltonian path.*

For the extremal example,  $n \geq 8$ , consider a rainbow complete 2-edge-coloured multigraph on  $n - 2$  vertices,  $n$  odd. Add two new vertices  $x_1$  and  $x_2$ . Then add a red edge  $x_1x_2$  and all red edges between  $\{x_1, x_2\}$  and the complete graph. Although the resulting graph has  $\binom{n}{2} + \binom{n-2}{2}$  edges, it has no proper Hamiltonian path, since there is no blue matching of size  $(n - 1)/2$ .

**Proof.** The proof is by induction on  $n$ . For  $n = 8, 9$  by inspection the result can be shown. Suppose now that  $n \geq 10$ . Observe that  $|E(\overline{G^c})| \leq 2n - 4$ . By a theorem in [1], if every vertex  $v \in G^c$  has  $d^r(v) \geq \lceil \frac{n+1}{2} \rceil$  and  $d^b(v) \geq \lceil \frac{n+1}{2} \rceil$ , then  $G^c$  has a proper Hamiltonian path. Thus, we can assume that there exists a vertex  $v$  such that  $d^r(v) \leq \lceil \frac{n+1}{2} \rceil - 1$ , otherwise there is nothing to prove.

Suppose first that there exist two neighbours  $u, w$  of  $v$  such that  $c(vu) = r$

and  $c(vw) = b$ . We then construct a new multigraph  $G'^c$  by replacing the vertices  $v, u$  and  $w$  with a new vertex  $z$  such that  $N^r(z) = N_{G^c - \{v, u\}}^r(w)$  and  $N^b(z) = N_{G^c - \{v, w\}}^b(u)$ . We remark that  $N_{G^c - \{v, u\}}^r(w)$  and  $N_{G^c - \{v, w\}}^b(u)$  cannot both be empty at the same time otherwise  $|E(\overline{G^c})| \geq 3n - 5 - \lceil \frac{n+1}{2} \rceil > 2n - 4$ , which is a contradiction in the total number of edges. So by this we remove at most  $n - 1$  blue edges and  $\lceil \frac{n+1}{2} \rceil - 1$  red edges from  $v$ ,  $n - 3$  red edges from  $u$ ,  $n - 3$  blue edges from  $w$ , and one red and one blue between  $u$  and  $w$ . Therefore  $G'^c$  has at least  $(n - 2)(n - 3) - n + 4 - \lceil \frac{n+1}{2} \rceil \geq \binom{n-2}{2} + \binom{n-4}{2} + 1$  edges, i.e., the number of edges required to have a proper Hamiltonian path in a multigraph on  $n - 2$  vertices. By the inductive hypothesis,  $G'^c$  has a proper Hamiltonian path  $P$ . Finally, as we have chosen the appropriate edges to remove at  $u$  and  $w$  it is easy to extend  $P$  to a proper Hamiltonian path for  $G^c$ .

Suppose now that there does not exist two neighbours of  $v$  such that  $c(vu) = r$  and  $c(vw) = b$ . So we have two possible cases. In the first case  $v$  has one only neighbour  $w$  in both colours. It is easy to observe that  $G^c - \{v\}$  has  $(n - 1)(n - 2)$  edges, i.e., it is a rainbow complete multigraph. Therefore, we have every possible proper Hamiltonian path, in particular, one that starts at  $w$ . Then we easily extend the path to  $G^c$ . For the second case,  $v$  has neighbours in just one colour, say  $b$ . Observe that for every vertex  $w \neq v$ ,  $w$  has a red neighbour. Otherwise if there is a vertex  $w$  without red neighbours, we have  $|E(\overline{G^c})| \geq 2n - 3 > 2n - 4$ , which is a contradiction. Now suppose first that  $v$  has at most  $n - 2$  blue neighbours. Consider a neighbour  $w$  of  $v$  and remove all its blue incident edges. Then remove  $v$  from  $G^c$  and call this multigraph  $G'^c$ . In  $G'^c$ ,  $w$  is monochromatic in red and  $G'^c$  has at least  $\binom{n-1}{2} + \binom{n-3}{2} + 1$ , i.e., the number of edges required to have a proper Hamiltonian path in a multigraph on  $n - 1$  vertices. Then by the inductive hypothesis in  $G'^c$  we obtain a proper Hamiltonian path that starts at  $w$  since it was monochromatic. So we have a proper Hamiltonian path for  $G^c$ . Finally  $v$  has  $n - 1$  blue neighbours. If one neighbour  $w$  of  $v$  has at most  $n - 3$  blue neighbours, then we proceed exactly as before. Otherwise every vertex  $w$  has  $n - 2$  blue neighbours, i.e., the multigraph  $G^c - \{v\} = G'^c$  is complete in blue and has at least  $n^2 - 4n + 5$  edges. By the inductive hypothesis we have a proper Hamiltonian path in  $G'^c$ . Now if  $n - 1$  is odd, one of the endpoints of the path is red and we can trivially add  $v$  to the path. If  $n - 1$  is even, both endpoints have the same colour. If they are red we are done. Otherwise remove all the blue edges from  $G'^c$ . This new (red) graph has  $n - 1$  vertices and at least  $\binom{n-2}{2} + 1$  edges. Therefore by a theorem in [6], it has a Hamiltonian path  $P$ . Now since  $G'^c$  is complete in blue, we can use those blue edges

along with  $P$  to form a proper Hamiltonian  $P'$  for  $G^c$  that starts and ends with colour red. Finally, we can trivially add  $v$  to  $P'$ .

Since we covered all cases, the proof is finished.  $\square$

**Theorem 3.2** *Let  $G^c$  be a 2-edge-coloured multigraph on  $n \geq 14$  vertices coloured with  $\{r, b\}$ . Suppose that for every vertex  $x$  in  $G^c$ ,  $rd(x) = 2$ . If  $m \geq \binom{n}{2} + \binom{n-3}{2} + 4$ , then  $G^c$  has a proper Hamiltonian path.*

For the extremal example,  $n \geq 14$  odd, consider a complete blue graph, say  $A$ , on  $n - 3$  vertices. Add three new vertices  $v_1, v_2, v_3$  and join them to a same vertex  $v$  in  $A$  with blue edges. Finally, superpose the obtained graph with a complete red graph on  $n$  vertices. Although the resulting 2-edge-coloured multigraph has  $\binom{n}{2} + \binom{n-3}{2} + 3$  edges, it has no proper Hamiltonian path since one of the vertices  $v_1, v_2, v_3$  cannot belong to such a path.

**Proof.** Let us suppose that  $G^c$  does not have a proper Hamiltonian path. We will show that  $\overline{G^c}$  has more than  $3n - 10$  edges, i.e.,  $G^c$  has less than  $\binom{n}{2} + \binom{n-3}{2} + 4$  edges, contradicting the hypothesis of the theorem. We distinguish between two cases depending on the parity of  $n$ .

**Case A:**  $n$  is even. By Lemma 2.3  $G^c$  has two matchings  $M^r, M^b$  such that  $|M^r| = \frac{n}{2}$  and  $|M^b| \geq \frac{n-2}{2}$ . Take the longest proper paths  $P = x_1y_1x_2y_2 \dots x_py_p$  and  $P' = x'_1y'_1x'_2y'_2 \dots x'_{p'}y'_{p'}$  compatibles with  $M^r$  and  $M^b$ , respectively.

Notice now that if  $2p = n$  or  $2p' = n$  then we are finished. In addition, if  $2p < n - 2$  or  $2p' < n - 2$ , then by Lemma 2.5 there are at least  $2n - 4$  blue missing edges and  $2n - 8$  red ones. This gives a total of  $4n - 12 > 3n - 10$  missing edges, which is a contradiction. Consequently, in what follows we may suppose that  $2p = 2p' = n - 2$ .

Suppose first that there exists a proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ . Let  $e$  be the red edge of  $M^r - C$ . If there exists a blue edge  $e'$  between an endpoint of  $e$  and  $C$ , we can easily obtain a proper Hamiltonian path considering  $e, e'$  and a segment of  $C$  in the appropriate direction. Otherwise as the multigraph is connected, all edges between the endpoints of  $e$  and  $C$  are red. Now as  $rd(G^c) = 2$ , there must exist a blue edge  $e''$  parallel to  $e$  and therefore we can obtain a proper Hamiltonian path just as before but starting with  $e''$ .

Next suppose that there exists no proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ . By Lemma 2.5 there are at least  $2n - 4$  blue missing edges. Consider now the path  $P'$  and let  $v_1, w_1$  be the vertices in  $G^c - P'$ . It is clear that if there exists a blue edge joining  $v_1$  and  $w_1$ , then by symmetry on the colours there are at least  $2n - 4$  red missing edges. This gives a total of  $4n - 8$  missing

edges, a contradiction. Otherwise, assume that there is no blue edge between  $v_1$  and  $w_1$ . In this case we will count the red missing edges assuming that we cannot extend  $P'$  to a proper Hamiltonian path. If there exists no cycle  $C'$  in  $G^c$  such that  $V(C') = V(P')$  then by Lemma 2.4 there are  $2p' - 2 = n - 4$  red missing edges. By summing up we obtain  $3n - 8 > 3n - 10$  missing edges, which is a contradiction. Finally, assume that there exists a proper cycle  $C'$  in  $G^c$  such that  $V(C') = V(P')$ . Set  $C = c_1c_2 \dots c_{2p'}c_1$  where  $c(c_i c_{i+1}) = r$  for  $i = 1, 3, \dots, 2p' - 1$ . If there are three or more red edges between  $\{v_1, w_1\}$  and  $\{c_i, c_{i+1}\}$ , for some  $i = 1, 3, \dots, 2p' - 1$ , then either the edges  $v_1c_i$  and  $w_1c_{i+1}$ , or  $v_1c_{i+1}$  and  $w_1c_i$  are red. Suppose  $v_1c_i$  and  $w_1c_{i+1}$  are red. In this case, the path  $v_1c_i c_{i-1} \dots c_1 c_{2p'} \dots c_{i+1} w_1$  is a proper Hamiltonian one. Otherwise, if there are not three or more red edges between  $\{v_1, w_1\}$  and  $\{c_i, c_{i+1}\}$ , for all  $i = 1, 3, \dots, 2p' - 1$ , then there are  $2p' - 2 = n - 4$  red missing edges. If we sum up we obtain a total of  $3n - 8 > 3n - 10$  missing edges, which is a contradiction.

**Case B:**  $n$  is odd. By Lemma 2.3  $G^c$  has two matchings  $M^r, M^b$  such that  $|M^r| = |M^b| = \frac{n-1}{2}$ . As in **Case A**, we consider the longest proper paths  $P$  and  $P'$  compatibles with the matchings  $M^r$  and  $M^b$  respectively. Suppose first that  $2p < n - 1$  and  $2p' < n - 1$ . By Lemma 2.5 there are at least  $2n - 6$  blue missing edges and  $2n - 6$  red ones. We obtain a total of  $4n - 12 > 3n - 10$  missing edges, which is a contradiction.

Suppose next  $2p = 2p' = n - 1$  (the cases where  $2p < n - 1$  and  $2p' = n - 1$ , or  $2p = n - 1$  and  $2p' < n - 1$  are similar). In the rest of the proof because of symmetry, we will consider only the path  $P$  since the same arguments may be applied for  $P'$ . In this case we will count the blue missing edges assuming that we cannot extend  $P$  to a proper Hamiltonian path. Now let  $v$  be the unique vertex in  $G^c - P$ . It is clear that if there is a proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ , we can trivially obtain a proper Hamiltonian path since the multigraph is connected. Then, as there is no proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ , we have by Lemma 2.4 that there are  $2p - 2 = n - 3$  blue missing edges. If there exists a blue edge between  $x_1$  and  $x_i$ , for some  $i = 2, \dots, p$ , then the blue edge  $vy_{i-1}$  cannot exist, otherwise we would obtain the proper Hamiltonian path  $vy_{i-1} \dots x_1 x_i \dots y_p$ . We can complete the argument in a similar way if both edges  $y_p y_i$  and  $vx_{i+1}$ ,  $i = 1, \dots, p - 1$  exist in  $G^c$  and are on colour blue. Note that since there is no proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ , we cannot have the blue edges  $x_1 x_i$  and  $y_p y_{i-1}$ ,  $i = 2, \dots, p$  at the same time. Therefore there are  $p - 1 = \frac{n-3}{2}$  blue missing edges. If we make the sum and multiply it by two (since the same number of red missing edges is obtained with  $P'$ ), we conclude that there are  $3n - 9$  missing edges,

which is a contradiction.

Since we covered all the cases, the theorem is proved.  $\square$

## 4 $c$ -edge-coloured multigraphs, $c \geq 3$

In this section we study the existence of proper Hamiltonian paths in  $c$ -edge-coloured multigraphs, for  $c \geq 3$ . We present three main results that involve: (1) the total number of edges, (2) the total number of edges and the connectivity of the multigraph, (3) the total number of edges and the rainbow degree. All results are tight. First, we present a key result that reduces the case  $c \geq 4$  just to the case  $c = 3$ .

**Lemma 4.1** *Let  $G^c$  be a  $c$ -edge-coloured connected multigraph on  $n$  vertices,  $c \geq 4$  and  $m \geq c f(n) + 1$  edges. There exists one colour  $c_j$  such that if we colour its edges with another used colour and we delete parallel edges with the same colour, then the resulting  $(c - 1)$ -edge-coloured multigraph is connected and has  $m' \geq (c - 1) f(n) + 1$  edges, such that if  $G^{c-1}$  has a proper Hamiltonian path then  $G^c$  has one too. Moreover, if  $rd(G^c) = k$ , then  $rd(G^{c-1}) = k - 1$  for  $1 \leq k \leq c$ .*

**Proof.** Let  $c_i$  denote the colour  $i$ , for  $i = 1, \dots, c$ , in  $G^c$ , and denote by  $|c_i|$  the number of edges with colour  $i$ . Let  $c_j$  be the colour with the less number of edges. Colour the edges on colour  $c_j$  with another used colour, say  $c_l$ , and delete (if necessary) parallel edges with that colour. Call this multigraph  $G^{c-1}$ . By this, we delete at most  $|c_j|$  edges. It is clear that this multigraph is connected since we deleted just parallel edges. Also if  $G^{c-1}$  has a proper Hamiltonian path, then this path is also proper Hamiltonian in  $G^c$  but perhaps with some edges on colour  $c_j$  (in the case that they have been recoloured with  $c_l$ ). Observe also that if  $rd(G^c) = k$  then  $rd(G^{c-1}) = k - 1$  since only the colour  $c_j$  disappeared. We will show now that  $m' \geq (c - 1) f(n) + 1$ . We have two cases. First, if  $|c_j| > f(n)$ , then clearly  $m' \geq (c - 1) f(n) + 1$  edges since for all  $i$ ,  $|c_i| > f(n)$ . For the second case, we have that  $|c_j| \leq f(n)$ . Now,  $m = \sum_{i=1}^c |c_i| \geq c f(n) + 1$  and therefore  $\sum_{i=1, i \neq j}^c |c_i| \geq c f(n) - |c_j| + 1 = (c - 1) f(n) + f(n) - |c_j| + 1$ . This last expression is greater than or equal to  $(c - 1) f(n) + 1$  since  $f(n) - |c_j| \geq 0$ . Finally, we have that  $G^{c-1}$  has  $m' \geq (c - 1) f(n) + 1$  edges as desired.  $\square$

**Theorem 4.2** *Let  $G^c$  be a  $c$ -edge-coloured multigraph on  $n$  vertices,  $n \geq 2$  and  $c \geq 3$ . If  $m \geq c \binom{n-1}{2} + 1$ , then  $G^c$  has a proper Hamiltonian path.*

For the extremal graph consider a rainbow complete multigraph on  $n - 1$

vertices with  $c$  colours and add a new isolated vertex  $x$ . Although the resulting multigraph has  $c\binom{n-1}{2}$  edges, it contains no proper Hamiltonian path since it is not connected.

**Proof.** By Lemma 4.1 we can assume that  $c = 3$  and  $m \geq 3\binom{n-1}{2} + 1$ . Furthermore, cases  $n \leq 7$  can be checked by exhaustive methods. Assume  $n \geq 8$ . Since there exists one colour, say red, such that the number of red edges are at least  $\binom{n-1}{2} + 1$ , then by a theorem in [6] there is a Hamiltonian red path and therefore a perfect or almost perfect matching  $M^r$ . Take the longest proper path  $P = x_1y_1x_2y_2 \dots x_p y_p$  compatible with  $M^r$ . As  $m \geq 3\binom{n-1}{2} + 1$ ,  $|E(\overline{G^c})| \leq 3n - 4$ . Now we will distinguish between two cases depending on the parity of  $n$ .

**Case A:**  $n$  even. Clearly  $|M^r| = \frac{n}{2}$ . By contradiction suppose that  $2p < n$  otherwise we are finished. Now in both cases, if there exists a proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$  or if it does not, we have by Lemma 2.5 a contradiction with the number of edges of  $G^c$  for  $n \geq 8$ .

**Case B:**  $n$  odd. Clearly  $|M^r| = \frac{n-1}{2}$ . First if  $2p < n - 1$  we obtain a contradiction in both cases exactly as before by Lemma 2.5. Second if  $2p = n - 1$  and there is a proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$  we can obtain a proper Hamiltonian path by taking any edge between the unique vertex outside  $C$  and  $C$  and then following the cycle in the appropriate direction. Finally suppose that  $2p = n - 1$  and there is no proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$ . Let  $v$  be the unique vertex of  $G^c - P$ . We have the following cases depending on the degree and neighbours of  $v$ .

**Subcase B1:** *All edges incident to  $v$  have the same colour, say red.*

Suppose first that  $d(v) \leq n - 2$ . Consider the multigraph  $G^c - \{v\} = G'^c$  and delete from a neighbour  $w$  of  $v$  all the edges in two colours in order to have  $w$  monochromatic in any colour but red. Observe that we can always do this since it is impossible to have two monochromatic vertices. By this, we delete at most  $3n - 6$  edges. We can see that the multigraph  $G'^c$  on  $n - 1$  vertices has at least  $3\binom{n-2}{2} + 1$  edges. Then by the even case, we have a proper Hamiltonian path  $P'$  in  $G'^c$  and since  $w$  is monochromatic we easily add  $v$  to  $P'$ .

Suppose finally that  $d(v) = n - 1$ . Therefore the multigraph  $G^c - \{v\} = G'^c$  on  $n - 1$  vertices has at least  $3\binom{n-1}{2} + 1 - (n - 1) \geq 3\binom{n-2}{2} + 1$  edges and it has a proper Hamiltonian path  $P'$ . Now if the path either starts or ends with a colour different from red, we are done. If not, both of them are red. Now, if we can take any parallel edge either to the start or to the end of the path without losing the property of being properly coloured we can, once again,

easily add  $v$  to the path. Otherwise the degree in some colour of the first (or last) vertex of the path is at most  $n - 3$ . Thus we are in the same case as before.

**Subcase B2:** *There exist two distinct edges  $vx$  and  $vy$  such that  $c(vx) = c_1 \neq c_2 = c(vy)$ .*

Assume first that  $x = y$ . This case is similar to the previous one just taking the multigraph  $G^c - \{v\}$  and deleting from  $x$  the edges in the appropriate two colours in order to have it monochromatic.

Assume next that  $x \neq y$ . We will prove the result by induction. If we cannot attach  $v$  to the path  $P$  we have four missing edges in colours different from red between  $v$  and the vertices  $x_1, y_p$ . We also have at most two edges different from red between  $v$  and the endpoints of the edges  $y_i x_{i+1}$  of the path. Therefore there are  $n - 3$  missing edges. Adding up all this, we conclude that the degree of  $v$  in colours different from red is at most  $n + 1$ . Now we contract  $v, x, y$  to a new vertex  $v'$  such that  $N^{c_1}(v') = N_{G^c - \{v, x\}}^{c_1}(y)$ ,  $N^{c_2}(v') = N_{G^c - \{v, y\}}^{c_2}(x)$  and  $N^{c_3}(v') = N_{G^c - \{v, y\}}^{c_3}(x) \cap N_{G^c - \{v, x\}}^{c_3}(y)$ . By this we delete at most  $5n - 6$  edges. We can see that this new 3-edge-coloured multigraph  $G'^c$  on  $n - 2$  vertices has at least  $3\binom{n-1}{2} + 1 - (5n - 6) \geq 3\binom{n-3}{2} + 1$  edges, for  $n \geq 9$ . Thus, by the inductive hypothesis we have a proper Hamiltonian path  $P'$  in  $G'^c$ . Because of the way that we have chosen the edges to delete at  $x$  and  $y$ , it is easy to obtain from  $P'$  a proper Hamiltonian path for  $G^c$ .

Since we covered all the cases, the theorem is proved.  $\square$

Notice that in the above theorem there is no condition guaranteeing the connectivity of the underlying graph. The next result will add this condition.

**Theorem 4.3** *Let  $G^c$  be a connected  $c$ -edge-coloured multigraph on  $n$  vertices,  $n \geq 9$  and  $c \geq 3$ . If  $m \geq c\binom{n-2}{2} + n$ , then  $G^c$  has a proper Hamiltonian path.*

For the extremal example,  $n \geq 9$ , consider a rainbow complete multigraph on  $n - 2$  vertices with  $c$  colours and add two new vertices  $x$  and  $y$ . Now add the edge  $xy$  and all edges between  $y$  and the complete multigraph, all on the same colour. The resulting multigraph, although it has  $c\binom{n-2}{2} + n - 1$  edges, contains no proper Hamiltonian path, as  $x$  cannot belong to such a path.

**Proof.** By Lemma 4.1 it is enough to prove the theorem for  $c = 3$ . The proof is by induction on  $n$ . For  $n = 9, 10$  it can be shown by case analysis that the result holds. Now we have two cases depending on whether  $G^c$  contains a monochromatic vertex or not.

**Case A:** *There exists a monochromatic vertex  $x \in G^c$ .* Let  $y$  be a neighbour of  $x$ . If  $y$  is also monochromatic then the multigraph  $G^c - \{x, y\}$  has  $n - 2$

vertices and at least  $3\binom{n-2}{2} - n + 3$  edges, i.e., it is almost rainbow complete and therefore it has a proper Hamiltonian cycle. Then it is easy to add  $x$  and  $y$  to the cycle to obtain a proper Hamiltonian path in  $G^c$ . Otherwise we would have a contradiction in the total number of edges. Suppose then that  $y$  is not monochromatic and  $c(xy) = b$ . Let us contract  $x$  and  $y$  to a new vertex  $z$  such that  $N^b(z) = N^r(z) = \emptyset$  and  $N^g(z) = N^g(y)$  (or  $N^b(z) = N^g(z) = \emptyset$  and  $N^r(z) = N^r(y)$ ). Observe that if the resulting multigraph on  $n - 1$  vertices is connected (we will show this later) and we delete at most  $3n - 8$  edges then the induction hypothesis would hold. Therefore it would contain a proper Hamiltonian path with  $z$  as an endpoint ( $z$  is monochromatic).

Let us now consider  $d^b(x)$ . Observe that if  $d^b(x) \leq n - 4$ , we delete at most  $3n - 8$  edges from  $x$  and any selected neighbour  $y$  of  $x$  and we are done. Further, from a theorem in [1], if  $d^i(z) \geq \lceil \frac{n}{2} \rceil \forall z \in G^c - \{x\}, i \in \{r, g, b\}$ , then  $G^c - \{x\}$  has a proper Hamiltonian cycle. This would imply a proper Hamiltonian path in  $G$ . Thus, we may assume that there exists some vertex  $w \in G^c - \{x\}$  such that  $d^i(w) < \lceil \frac{n}{2} \rceil$  for some  $i \in \{r, g, b\}$ . We have the remaining cases:

**Subcase A1:**  $d^b(x) = n - 1$ . Observe that  $w \in N^b(x)$ . In this case, the contraction process deletes  $n - 1$  edges from  $x$ , and at most  $n + \frac{n}{2} - 3$  from  $w$ , which is much less than  $3n - 8$  for  $n > 10$ .

**Subcase A2:**  $d^b(x) = n - 2$ . If there is a vertex  $y$  adjacent to  $x$  such that  $d_{G^c - \{x\}}^{r,b}(y) \leq 2n - 6$  or  $d_{G^c - \{x\}}^{b,g}(y) \leq 2n - 6$  then we just take  $x$  and  $y$  for the contraction process. Otherwise for all  $y$  adjacent to  $x$  we have  $d_{G^c - \{x\}}^{r,b}(y) \geq 2n - 5$  and  $d_{G^c - \{x\}}^{b,g}(y) \geq 2n - 5$ . Therefore the multigraph  $G^c - \{x\}$  is almost rainbow complete and has a proper Hamiltonian cycle. Finally, we can trivially add  $x$  to the cycle to obtain a proper Hamiltonian path for  $G^c$ .

**Subcase A3:**  $d^b(x) = n - 3$ . This case is similar to the last one but finding a vertex  $y$  adjacent to  $x$  such that  $d_{G^c - \{x\}}^{r,b}(y) \leq 2n - 5$  or  $d_{G^c - \{x\}}^{b,g}(y) \leq 2n - 5$ . Otherwise the multigraph  $G^c - \{x\}$  is rainbow complete and we easily obtain a proper Hamiltonian path.

**Case B:** *There is no monochromatic vertex in  $G^c$ .*

First suppose that there exists a vertex  $x$  such that  $|N(x)| = 1$ . Pick  $x$  and its unique neighbour  $y$  for the contraction. We delete at most three edges at  $x$  and  $2n - 4$  at  $y$ , which guarantees the induction hypothesis and leaves  $y$  monochromatic.

In what follows we suppose that  $|N(x)| \geq 2$  for all  $x \in G^c$ . Consider a vertex  $x$  and  $y, z \in N(x)$ . Suppose that  $c(xy) = b$  and  $c(xz) = r \neq c(xy)$ . We

will contract  $x, y$  and  $z$  to a new vertex  $s$  as in Theorem 4.2. We can delete at most  $6n - 19$  edges, which is necessary for the induction hypothesis to hold. Since we delete at most  $3n - 6$  edges incident to  $y$  and  $z$  in  $G^c - \{x\}$ , we will show that either the degree of  $x$  is at most  $3n - 13$  or there exist  $x, y, z$  such that number of deleted edges is less than or equal to  $6n - 19$ .

Let  $E^i$  be the set of edges in colour  $i$  and suppose that the colour  $b$  maximizes  $|E^i|$ . The monochromatic subgraph in colour  $b$  has at least  $m^b \geq \binom{n-2}{2} + \lceil \frac{n}{3} \rceil$  edges. If this subgraph is connected, then by Lemma 2.1 there is a perfect matching for  $n$  even and an almost perfect matching for  $n$  odd. Otherwise there is a matching of size  $\frac{n-2}{2}$  for  $n$  even and  $\frac{n-1}{2}$  if  $n$  is odd. Let  $M^b$  be a maximum matching in  $E^b$ .

Suppose  $n$  is even (for  $n$  odd, similar arguments can be used). Let  $P = x_1y_1x_2y_2 \dots x_py_p$  be the longest proper path compatible with  $M^b$ . Let us suppose that there is no proper cycle  $C$  such that  $V(C) = V(P)$  (the case where a proper cycle exists is similar and even easier). By Lemma 2.5 we can check that  $|P| \geq n - 4$ . Consider the following cases:

- (i)  $|M^b| = \frac{n}{2}$ . If  $P$  is Hamiltonian we are done. Otherwise there is at least one edge in  $M^b - P$ . Clearly there are no edges in colour  $r$  and  $g$  from the endpoints of  $P$  to the edges in  $M^b - P$ . Further, for each edge in  $M^b - P$  there are at most four edges between their endpoints and the endpoints of the edges  $y_i x_{i+1}$ , otherwise we would obtain a longer path. Therefore taking the vertex  $x$  outside  $P$  with minimum degree, we have  $d^{r,g}(x) \leq n$ , for  $|P| = n - 4$ , and  $d^{r,g}(x) \leq n - 2$ , for  $|P| = n - 2$ . In both cases, if we consider  $d^b(x) = n - 1$  we get  $d(x) \leq 3n - 13$ .
- (ii)  $|M^b| = \frac{n-2}{2}$ . For  $|P| = n - 4$  the proof is similar as the previous case. Suppose then that  $|P| = n - 2$ . Let  $v, w$  be the unmatched vertices. We try to add these vertices to the path either at the endpoints or between the vertices  $y_i$  and  $x_{i+1}$ . Suppose first that we cannot add any of them. Then  $d^{r,g}(v) \leq n - 2$ . Summing this up with at most  $n - 2$  blue edges, we obtain  $d(v) \leq 2n - 4 < 3n - 13$  and we take  $v$  for the contraction. Suppose last that we can add  $v$  but we cannot add  $w$ . If  $v$  was added at one endpoint of the path we obtain that  $d(w) \leq 2n - 5 < 3n - 13$ . If  $v$  was added in-between the path we have  $d(w) \leq 2n - 6 < 3n - 13$ . In both cases we take  $w$  for the contraction.

Since we covered all cases, we can always find three appropriate vertices to make the contraction process. We now check the connectivity of the resulting multigraph after the contraction of two or three vertices.

**Connectivity:** Assume that we contract two vertices  $x, y$  to a vertex  $s$  and

the multigraph is disconnected. It can be easily shown that the graph has two components with one vertex and  $n - 2$  vertices respectively. Observe first that the isolated vertex cannot be  $s$  unless  $x$  and  $y$  are both monochromatic, but this case has been solved independently. Suppose now that a vertex  $z \neq s$  is the isolated vertex.

Suppose first the case where  $x$  is monochromatic and consider the original multigraph. There are at least  $2(n - 1)$  missing edges at  $x$  since it is monochromatic and  $3(n - 3)$  missing edges at  $z$  since  $z$  is isolated in the contracted multigraph. Further, there are two more missing edges between  $y$  and  $z$  since we have the choice of which colours to delete at  $y$ . This gives us a total of  $5n - 9 = |E(\overline{G^c})|$  missing edges. Therefore, if there is one more missing edge, we would have a contradiction with the total number of edges. Suppose now that there is another vertex  $y' \neq y$  adjacent to  $x$  and take  $x, y'$  for the contraction. If the contracted multigraph is connected we are done. Otherwise we have more new missing edges and therefore a contradiction. Finally if there is no other vertex than  $y$  adjacent to  $x$  we obtain more missing edges at  $x$  and again a contradiction.

Suppose last that  $x$  is not monochromatic. In this case  $x$  has  $y$  as its unique neighbour. So, there are  $3(n - 2)$  missing edges at  $x$  and  $3(n - 3)$  missing edges at  $z$  since  $z$  is isolated in the contracted multigraph. This gives us a total of  $6n - 15$  missing edges and this is greater than  $|E(\overline{G^c})| = 5n - 9$  which is a contradiction.

Assume now that we contract three vertices  $x, y, z$  where  $c(xy) = b$  and  $c(xz) = r$  to a vertex  $s$  and the multigraph is disconnected. This multigraph has exactly two components with one vertex  $u$  and  $n - 3$  vertices respectively.

Suppose first that  $u \neq s$ . In the original multigraph  $u$  must have at least two different neighbours in two different colours among the vertices  $x, y, z$ . Otherwise we would be in the case where either  $u$  is monochromatic or  $u$  has one unique neighbour. Let  $y'$  and  $z'$  be two neighbours of  $u$  among  $x, y, z$  such that  $c(uy') \neq c(uz')$ . Now we contract the vertices  $u, y', z'$  as before. Observe that at  $u$  we delete only six edges since  $u$  has only  $x, y, z$  as neighbours, and the red edge  $uy$ , the blue edge  $uz$  and at least one green edge among  $uy, uz$  are missing. At  $y'$  and  $z'$  we delete  $3(n - 3) + 3$  edges as usual. With this contraction we delete in total  $3n$  edges and therefore the contracted multigraph on  $n - 2$  vertices has at least  $3\binom{n-3}{2} + n - 9$  edges which guarantees not only the inductive hypothesis but also the connectivity for  $n \geq 10$ .

Suppose last that  $u = s$ . Then there are no red edges between  $y$  and the multigraph  $G^c - \{x, y, z\}$  and no blue edges between  $z$  and  $G^c - \{x, y, z\}$ . Now, since we are not in the previous cases,  $y$  has at least two different neighbours

$y'$  and  $z'$  such that  $c(yy') \neq c(yz')$ . Then we contract the vertices  $y, y', z'$ . In the contraction process we delete at most  $2(n-3)$  edges between  $y$  and  $G^c - \{x, y, z\}$  (since there are no red edges), six between  $y$  and the vertices  $x, z$ , and  $3(n-3) + 3$  at  $y'$  and  $z'$ . We obtain in total at most  $5n - 6$  deleted edges. Now, this new contracted multigraph has  $n - 2$  vertices and at least  $3\binom{n-3}{2} - n - 3$  edges. Clearly, if the multigraph is connected we are done. Otherwise, as in the situation before, it has two components with one vertex and  $n - 3$  vertices respectively. We can suppose that the contracted vertex is the isolated one. Observe now that the component on  $n - 3$  vertices has at least  $3\binom{n-3}{2} - n - 3$ , therefore it is almost rainbow complete and it has a proper Hamiltonian cycle. Finally, in the original multigraph it is easy to add the vertices  $y, y', z'$  to the proper cycle in order to obtain a proper Hamiltonian path.

Now as the connectivity is proved the theorem holds.  $\square$

**Theorem 4.4** *Let  $G^c$  be a  $c$ -edge-coloured multigraph on  $n$  vertices,  $n \geq 11$  and  $c \geq 3$ . Assume that for every vertex  $x$  of  $G^c$ ,  $rd(x) = c$ . If  $m \geq c\binom{n-2}{2} + 2c + 1$ , then  $G^c$  has a proper Hamiltonian path.*

For the extremal graphs,  $n \geq 11$ , consider a rainbow complete multigraph, say  $A$ , on  $n - 2$  vertices. Add two new vertices  $v_1, v_2$  and join them to a vertex  $v$  of  $A$  with all possible colours. The resulting  $c$ -edge-coloured multigraph has  $c\binom{n-2}{2} + 2c$  edges and clearly has no proper Hamiltonian path.

**Proof.** By Lemma 4.1 it is enough to prove the theorem for  $c = 3$ . As  $m \geq 3\binom{n-2}{2} + 7$  then  $|E(\overline{G^c})| \leq 6n - 16$ . The proof will be done either by construction of a proper Hamiltonian path or using Theorem 4.3. We will do this by contracting two or three vertices depending on if there exists a vertex  $x$  in  $G^c$  such that  $|N^{r,g,b}(x)| = 1$  or not.

If there exists a vertex  $x \in G^c$  such that  $|N(x)| = 1$  we proceed as usual and we contract  $x$  and its unique neighbour  $y$  to a new vertex  $s$  in order to leave  $s$  monochromatic. We can check that we delete at most  $2n - 1$  edges and therefore we can apply Theorem 4.3. Observe that the resulting multigraph on  $n - 1$  vertices is connected since as  $rd(x) = 3$ , we can choose at  $y$  which two colours to delete.

Suppose next that there is no vertex  $x \in G^c$  such that  $|N(x)| = 1$ . Let us suppose that there are three vertices  $x, y, z$  such that  $c(xy) = b$  and  $c(xz) = r$ . Then, we contract  $x, y, z$  to a new vertex  $s$  as in Theorem 4.2. In this case only  $5n - 12$  edges can be deleted to apply Theorem 4.3. Therefore, we need to find a vertex  $x$  such that  $d(x) \leq 2n - 6$ .

Suppose that without losing generality that  $|E^b| \geq |E^r| \geq |E^g|$ , then  $|E^b| \geq \binom{n-2}{2} + 3$ . Since the subgraph in colour  $b$  is connected because of the rainbow degree of the vertices, we have by Lemma 2.1 that there is a matching  $M^b$  such that  $|M^b| = \frac{n}{2}$  for  $n$  even and  $|M^b| = \frac{n-1}{2}$  for  $n$  odd. Let  $P = x_1y_1x_2y_2 \dots x_py_p$  be the longest proper path compatible with  $M^b$ .

Suppose now that  $n$  is even (the case  $n$  odd is similar, refer to [8] for more details). If there is a proper cycle  $C$  such that  $V(C) = V(P)$  the result easily follows. Assume therefore that there is no proper cycle  $C$  such that  $V(C) = V(P)$ . By Lemma 2.5 we can check that  $|P| \geq n - 4$ . Let  $x$  be the vertex in  $M^b - P$  with minimum degree and let  $y$  be its adjacent vertex in the matching. Clearly there cannot be edges in colours  $r$  and  $g$  between  $M^b - P$  and the endpoints of  $P$ . Additionally, there can be at most 4 edges in colours  $r$  and  $g$  from each edge in  $M^b - P$  and the edges  $y_{i-1}x_i$  in  $P$ . We can conclude that there are at most  $2p - 2$  edges in colour  $r$  and  $g$  between  $x$  and the vertices in  $P$ .

Suppose now that there is one parallel edge to the blue edge  $xy$  in colour  $r$  or  $g$ . Using the edges  $xx_i, yy_i$  or  $yx_i, xy_i$  in colour  $b$  we can add the edge  $xy$  in colour  $r$  or  $g$  into  $P$ . Since this would contradict our hypothesis we can conclude that there are two missing edges in colour  $b$  from  $xy$  to each edge  $x_iy_i$  in  $P$ . Now since there are  $p$  edges  $x_iy_i$  we conclude that the vertex  $x$  has  $d^b(x) \leq (n - p - 1)$ . In total we have  $d(x) \leq 3n - 3p - 5$ . For  $|P| = n - 2$ , this is less than or equal to  $2n - 6$  and we can make the contraction with it.

Now if  $|P| = n - 2$  and there are no parallel edges in  $r$  and  $g$  at  $xy$ , we have that  $d(x) \leq 2n - 5$  and  $d(y) \leq 2n - 5$ . We can now use one of these vertices unless both inequalities become equalities. In this case we try to replace the edges  $x_iy_i$  in  $P$  by  $x_ixy_i$  and  $x_jyy_j, i \neq j$ . Suppose that we have the edges  $xx_i$  in colours  $r$  and  $g$ ,  $xy_i$  in colour  $b$ ,  $yx_j$  in colours  $r$  and  $g$ , and  $yy_j$  in colour  $b$ . Then we have a proper Hamiltonian path  $P' = x_1y_1 \dots x_ixy_ix_{i+1} \dots x_jyy_j \dots x_py_p$ . Otherwise at least one of those edges is missing and therefore either  $d(x) \leq 2n - 6$  or  $d(y) \leq 2n - 6$  as desired.

If  $|P| = n - 4$  by Lemma 2.5 there are  $6n - 20$  missing edges in colours  $r$  and  $g$ . Now, if there is one parallel edge to the blue edge  $xy$  in colour  $r$  or  $g$ , similar to the situation before, there would be  $p = \frac{n-4}{2}$  blue missing edges from  $xy$  to each edge  $x_iy_i$  in  $P$ . Therefore since  $6n - 20 + \frac{n-4}{2} > 6n - 16$ , we obtain a contradiction in the total number of edges.

Finally if  $|P| = n - 4$  with no parallel edges in colours  $r$  and  $g$  to the blue edge  $xy$  and to the other edge of the matching outside the path. We obtain four more missing edges in colours  $r$  and  $g$  and therefore  $6n - 16$ . Observe now that if we can replace the edges  $xy$  and the other edge of the matching

outside  $P$ , say  $uv$ , in  $M^b$  by  $xu$  and  $yv$ , then there are four missing parallel edges in colours  $r$  and  $g$ , obtaining  $6n - 12$ . A contradiction. Otherwise if not, there are two more missing edges obtaining  $6n - 14$  edges, which is again a contradiction.

Since all cases were covered, there always exists a vertex  $x$  such that  $d(x) \leq 2n - 6$  and the contraction can be done. Therefore by Theorem 4.3 we obtain a proper Hamiltonian path in this new multigraph. Then, it is simple to extend this path to a proper Hamiltonian one in the initial multigraph because of the choice of the edges to delete at the contracted vertices.

**Connectivity:** Suppose we contract three vertices  $x, y, z$ , such that  $c(xy) = b$ ,  $c(xz) = r$  to a vertex  $s$  and the multigraph is disconnected. This multigraph has exactly two components with one vertex  $u$  and  $n - 3$  vertices respectively.

First, if  $u \neq s$ , we proceed exactly as in the equivalent case in Theorem 4.3 and we find three other vertices  $u, y', z'$  to contract to a vertex  $s'$ , just deleting  $3n$  edges. This new contracted multigraph on  $n - 2$  vertices has at least  $3\binom{n-3}{2} - 2$  edges. Then, if it is connected we are done, otherwise there is a component with one vertex  $u'$  and another one on  $n - 3$  vertices with at least  $3\binom{n-3}{2} - 2$  edges, i.e., almost rainbow complete. Therefore, we obtain a proper Hamiltonian cycle and then we can easily add either the isolated vertex  $u'$  (if  $u' \neq s'$ ) or the three  $u, y', z'$  (if  $u' = s'$ ) vertices to the cycle to obtain a proper Hamiltonian path for the initial multigraph.

Finally, if  $u = s$  we will consider two cases. Suppose first that  $x$  has at most one neighbour with parallel edges. Therefore,  $d(x) \leq n$ . Now in this case, the contraction process deletes  $4n - 6$  edges instead of  $5n - 12$ . We have then that the contracted multigraph on  $n - 2$  vertices is disconnected and the component on  $n - 3$  vertices has at least  $3\binom{n-3}{2} - n + 4$  edges. This component is almost rainbow complete and as before we can obtain a proper Hamiltonian cycle. This allows us to easily add  $x, y, z$  in order to obtain a proper Hamiltonian path in the initial multigraph. Suppose last that  $x$  has two different neighbors  $y'$  and  $z'$  with parallel edges. Consider the next two cases:

Assume first that the parallel edges are on the same two colours, that is,  $c(xy') = c(xz') = \{b, r\}$  (cases with other two colours are similar). Now, suppose that in both possible contractions the multigraph is disconnected and the contracted vertex is always the isolated one. We can observe that the initial multigraph has  $n + 3$  missing edges at  $x$  (since  $d(x) \leq 2n - 6$ ),  $n - 3$  green edges and  $4(n - 3)$  blue and red edges at  $y'$  and  $z'$  (since in both contractions the multigraph is disconnected). By this we obtain a total of  $6n - 12 > 6n - 16$  missing edges, which is a contradiction.

Thus, assume that the parallel edges are not on the same two colours, that is,  $c(xy') = \{b, r\}$  and  $c(xz') = \{b, g\}$  (cases with other combinations are similar). Now since we are not in the previous case we do not have either the green edge  $xy'$  or the red one  $xz'$ . Try any of the three possible contractions. Then, if the multigraph is disconnected and the contracted vertex is always the isolated one, we can observe that there can exist just the red edges between  $y'$  and  $G^c - \{x, y', z'\}$ , and the green edges between  $z'$  and  $G^c - \{x, y', z'\}$ . Now as  $rd(G^c) = 3$  there must exist the green edge  $y'z'$  and the red edge  $z'y'$ . Since we are not in the previous case, the blue edge  $y'z'$  is not present. We find us in the situation that  $c(xy') = \{b, r\}$ ,  $c(xz') = \{b, g\}$  and  $c(y'z') = \{r, g\}$ . Now, we have nine different contractions to try, three for each triplet  $xy'z'$ ,  $y'xz'$  and  $z'xy'$ . If in all of them we are in this same situation (the contracted multigraph is disconnected and the isolated vertex is the contracted one) we can conclude that in the original multigraph there can exist just the blue edges between  $x$  and  $G^c - \{x, y', z'\}$ , the red edges between  $y'$  and  $G^c - \{x, y', z'\}$ , and the green edges between  $z'$  and  $G^c - \{x, y', z'\}$ , therefore  $6(n-3)$  missing ones. Finally, adding the three missing edges  $xy'$ ,  $xz'$  and  $y'z'$  in green, red and blue respectively, we obtain  $6n-15$  missing edges which is a contradiction.

Now as the connectivity is proved the theorem holds.  $\square$

## 5 Summary and remarks

In this work, we studied the existence of proper Hamiltonian paths, in edge-coloured multigraphs depending on the number of edges, the rainbow degree and the connectivity. Here, the notable fact is that the proofs were sometimes long and tedious despite the lower bounds for the edges in the considered multigraphs were really high. It should be also interesting to study similar conditions guaranteeing other patterns than proper paths, as for instance proper cycles, proper trees, etc.

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