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# Proper Hamiltonian Paths in Edge-Colored Multigraphs

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## Abstract

A  $c$ -edge-colored multigraph has each edge colored with one of the  $c$  available colors and no two parallel edges have the same color. A proper hamiltonian path is a path containing all the vertices of the multigraph such that no two adjacent edges have the same color. In this work we establish sufficient conditions for a multigraph to have a proper hamiltonian path, depending on several parameters such as the number of edges, the rainbow degree, etc.

*Keywords:* Multigraph, Proper Hamiltonian Path, Edge-Colored Graph

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# 1 Introduction

The study of problems modeled by edge-colored graphs gave place to important developments over the last years. For instance, the research on long colored cycles and paths for edge-colored graphs has given interesting results. Refer to [3] for a survey on related results. From the point of view of applicability, problems arising in molecular biology are often modeled using colored graphs, i.e., graphs with colored edges and/or vertices [9]. Given such an edge-colored graph, original problems translate to extracting subgraphs colored in a specified pattern. The most natural pattern in such a context is that of a proper coloring, i.e., adjacent edges have different colors.

Clearly, the proper hamiltonian path and proper hamiltonian cycle problems are  $NP$ -complete in the general case. It is polynomial to find a proper hamiltonian path in  $c$ -edge-colored complete graphs,  $c \geq 2$  [8]. It is also polynomial to find a proper hamiltonian cycle in 2-edge-colored complete graphs [4]. It is still open to determine the computational complexity for proper hamiltonian cycles,  $c \geq 3$  [5]. Many other partial results for edge-colored multigraphs can be found in the survey by Bang-Jensen and Gutin [2].

In this work we give sufficient conditions involving various parameters as the number of edges, rainbow degree, etc, in order to guaranty the existence of proper hamiltonian paths in edge-colored multigraphs where parallel edges with same colors are not allowed. We remark that this problem arises naturally since it has no sense to study such conditions just for simple edge-colored graphs. Results involving only degree conditions can be found in [1].

Formally, let  $I_c = \{1, 2, \dots, c\}$  be a set of  $c \geq 2$  colors. Throughout this paper,  $G^c$  denotes a  $c$ -edge-colored multigraph such that, each edge is colored with one color in  $I_c$  and no two parallel edges joining the same pair of vertices have the same color. Let  $n$  be the number of vertices and  $m$  be the number of edges of  $G^c$ . If  $H$  is a subgraph of  $G^c$ , then  $N_H^i(x)$  denotes the set of vertices of  $H$  adjacent to  $x$  with an edge of color  $i$ . Whenever  $H$  is isomorphic to  $G^c$ , we write  $N^i(x)$  instead of  $N_{G^c}^i(x)$ . The *colored  $i$ -degree* of a vertex  $x$ , denoted by  $d^i(x)$ , is the cardinality of  $N^i(x)$ . The *rainbow degree* of a vertex  $x$ , denoted by  $rd(x)$ , is the number of different colors on the edges incident to  $x$ . The *rainbow degree* of a multigraph  $G^c$ , denoted by  $rd(G^c)$ , is the minimum rainbow degree among its vertices. An edge with endpoints  $x$  and  $y$  is denoted by  $xy$ , and its color by  $c(xy)$ . A *rainbow complete multigraph* is the one having all possible colored edges between any pair of vertices (its number of edges is therefore  $\frac{cn(n-1)}{2}$ ). The *complement* of a multigraph  $G^c$ , denoted by  $\overline{G^c}$ , is a multigraph with the same vertices as  $G^c$  and an edge  $vw \in E(\overline{G^c})$  if and only

if  $vw \notin E(\overline{G^c})$ . A subgraph of  $G^c$  is said to be *properly edge-colored*, if any two adjacent edges in this subgraph differ in color. A *hamiltonian path (cycle)* is a path (cycle) containing all vertices of the multigraph. A path (cycle) is said to be *compatible* with a given matching  $M$  if the edges of the path (cycle) are alternatively in  $M$  and not in  $M$ . All multigraphs are assumed to be connected.

We will use two families of multigraphs without proper hamiltonian paths. First, let  $H_{k,k+3}^2$  denote a 2-edge-colored multigraph on  $2k + 3$  vertices,  $k \geq 1$ , defined as follows. Consider a complete red graph on  $k$  vertices and join it with red edges to an independent set on  $k + 3$  vertices. Finally, superpose a complete blue graph on  $2k + 3$  vertices. For the second family, let  $H_{k,k+2}^c$  denote a  $c$ -edge-colored multigraph on  $2k + 2$  vertices,  $k \geq 1$  and  $c \geq 3$ . Consider a rainbow complete graph on  $k$  vertices and join it with edges of all possible colors to an independent set on  $k + 2$  vertices.

This paper is organized as follows: In Section 2 we present some preliminary results that will be useful for the rest of the paper. In Section 3 we study proper hamiltonian paths in 2-edge-colored multigraphs. In Section 4 we study proper hamiltonian paths in  $c$ -edge-colored multigraphs, for  $c \geq 3$ . We remark that this division is because of proper hamiltonian paths in 2-edge-colored multigraphs are just alternating paths in 2 colors, therefore the results are different of those for  $c \geq 3$  colors. Finally, in Section 5 we present the conclusions.

## 2 Preliminary results

**Lemma 2.1** *Let  $G$  be a connected non-colored simple graph on  $n$  vertices,  $n \geq 9$ . If  $m \geq \frac{(n-2)(n-3)}{2} + 3$ , then  $G$  has a matching  $M$  of size  $|M| = \lfloor \frac{n}{2} \rfloor$ .*

**Proof.** By a theorem in [7], a 2-connected graph on  $n \geq 10$  vertices and  $m \geq \frac{(n-2)(n-3)}{2} + 5$  edges, has a hamiltonian cycle. So, if we add a new vertex  $v$  and we join it to all the vertices of  $G$ , we have that  $G + \{v\}$  has  $m \geq \frac{(n-2)(n-3)}{2} + 3 + n = \frac{(n-1)(n-2)}{2} + 5$  edges. Therefore,  $G + \{v\}$  has a hamiltonian cycle, i.e.,  $G$  has a hamiltonian path and this implies that there exists a matching  $M$  in  $G$  of size  $|M| = \lfloor \frac{n}{2} \rfloor$ .  $\square$

**Lemma 2.2 ([6])** *Let  $G$  be a simple non-colored graph on  $n \geq 14$  vertices. If  $m \geq \frac{(n-3)(n-4)}{2} + 4$  and for every vertex  $x$ ,  $d(x) \geq 1$ , then  $G$  has a matching  $M$  of size  $|M| \geq \lceil \frac{n-2}{2} \rceil$ .*

**Lemma 2.3** *Let  $G^c$  be a 2-edge-colored multigraph on  $n \geq 14$  vertices. Suppose that for every vertex  $x$  in  $G^c$ ,  $rd(x) = 2$ . If  $m \geq (n-3)(n-4) + 3n - 2$ , then  $G^c$  has two matchings  $M^r$  and  $M^b$  on colors, say red and blue respectively, such that  $|M^r| = \lfloor \frac{n}{2} \rfloor$  and  $|M^b| \geq \lceil \frac{n-2}{2} \rceil$ .*

**Proof.** Let us denote  $E^r(G^c)$  and  $E^b(G^c)$  the set of edges colored in red and blue, of sizes  $|E^r(G^c)| = m^r$  and  $|E^b(G^c)| = m^b$ , respectively. Observe that, as for every vertex  $x$  in  $G^c$ ,  $rd(x) = 2$ , we have that  $d^i(x) \geq 1$  for  $i \in \{r, b\}$ . Observe also that  $m^i \geq \frac{(n-3)(n-4)}{2} + 4$  for  $i \in \{r, b\}$ , since this threshold is tight when the multigraph is complete on one of the colors.

Let us see the case when  $n$  is odd. By Lemma 2.2, there exist two matchings  $M^r$  and  $M^b$ , each one of size  $\frac{n-1}{2}$ , so the result follows straightforward.

Let us see now the case when  $n$  is even. Then, again by Lemma 2.2, there exist two matchings,  $M^r$  and  $M^b$ , each one of size at least  $\frac{n-2}{2}$ . We shall prove the result by contradiction. Let us consider the monochromatic subgraphs in color  $r$  and  $b$  respectively, and suppose that  $|M^r| = |M^b| = \frac{n-2}{2}$ . Let  $U = \{u_1, u_2\}$  denote the independent set of unmatched vertices in  $M^r$ . The vertices  $u_1$  and  $u_2$  are connected to the edges in  $M^r$ . We claim that there exist two distinct vertices  $v_1, v_2$  in  $V(G) - \{u_1, u_2\}$  such that  $u_1v_1, u_2v_2 \in E^r(G^c)$ . Otherwise, if  $N^r(\{u_1, u_2\}) = v_1$ , this vertex is the extremity of some edge  $v_1w_1$  in  $M^r$  and then we distinguish two cases. First, if  $N^r(w_1) = v_1$ , we have three distinct vertices of degree one which leads us to a contradiction with the total number of edges. Second, there exists  $w_2 \in N^r(w_1) - \{v_1\}$  and  $v_2$ , such that  $v_2w_2 \in M^r$ . Then we can replace  $\{v_1w_1, v_2w_2\}$  by  $\{u_1v_1, w_1w_2\}$ . After this permutation, we have a new matching on same size but  $u_1$  is replaced by  $v_2$ . So, as we claimed, there always exist two distinct vertices  $v_1, v_2$  in  $V(G) - \{u_1, u_2\}$  such that  $u_1v_1, u_2v_2 \in E^r(G^c)$  ( $u_2$  and  $v_2$  are those vertices in the case we have just seen).

Now, the edge  $v_1v_2$  is not in  $M^r$ , since otherwise, we can replace it by  $\{u_1v_1, u_2v_2\}$  and get a larger matching in color  $r$ , contradicting that  $M^r$  is maximum. Consequently, there are two vertices  $w_1$  and  $w_2$  in  $V(G) - \{u_1, u_2, v_1, v_2\}$  such that  $v_1w_1, v_2w_2 \in M^r$ . Observe that there can be at most two edges connecting the endpoints of any edge in  $M^r$  to the set  $\{u_1, u_2\}$ , i.e., there are at least two missing edges for each edge in  $M^r$ , in total  $2\frac{n-2}{2}$ . Similarly, for  $w_1$  and  $w_2$ , we have same constraints and therefore, this also means two missing edges for each edge in  $M^r - \{v_1w_1, v_2w_2\}$ , in total  $2\frac{n-6}{2}$ . Otherwise, suppose that there is an edge  $v_3w_3$  with three edges between its endpoints to the vertices  $w_1$  and  $w_2$ , so we can replace  $v_1w_1$  and  $v_2w_2$  by say  $w_1v_3, w_2v_3, v_1u_1$  and  $v_2u_2$  to obtain a perfect matching. Finally, as  $u_1$  and  $u_2$  are inde-

pendent, the edge  $u_1u_2$  is missing. If we sum up these numbers, there are at least  $(n - 2) + (n - 6) + 1 = 2n - 7$  missing edges in color red.

Same reasoning can be done with the matching  $M^b$  to obtain  $2n - 7$  blue missing edges. So, the total number of missing edges in colors red and blue is  $4n - 14$ . Since the complement of  $G^c$  has edge set of size less than or equal to  $3n - 10$ , for  $n \geq 6$  we have contradiction. and therefore, the result holds.  $\square$

**Lemma 2.4** *Let  $G^c$ ,  $c \geq 2$ , be a connected  $c$ -edge-colored multigraph. Suppose that  $G^c$  contains a proper path  $P = x_1y_1x_2y_2 \dots x_py_p$ ,  $p \geq 3$ , such that each edge  $x_iy_i$  is red. If  $G^c$  does not contain a proper cycle  $C$  with vertex set  $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ , then there are at least  $(c - 1)(2p - 2)$  missing edges in  $G^c$ .*

**Proof.** Let  $P = x_1y_1x_2y_2 \dots x_py_p$  be a proper path,  $p \geq 2$ , such that each edge  $x_iy_i$  is red. Let blue be any other color different from red.

The blue edge  $x_1y_p$  can not be in  $G^c$ , otherwise  $C = x_1y_1 \dots x_py_px_1$  is a proper cycle.

Suppose that the blue edges  $x_1x_i$  are present in  $G^c$ , for  $i = 2, \dots, p$ . Then, the blue edges  $y_{i-1}y_p$  cannot be in  $G^c$ , otherwise we have the proper cycle  $C = x_1x_i \dots y_py_{i-1} \dots x_1$  that contradicts our hypothesis. Therefore for each edge  $y_{i-1}x_i$  in the path, either the blue edge  $x_1x_i$  or the blue edge  $y_{i-1}y_p$  is missing. So there are  $\frac{2p-2}{2}$  blue missing edges.

Now, suppose that the blue edges  $x_1y_i$  are present in  $G^c$ , for  $i = 3, \dots, p - 2$ . Then, the blue edges  $x_{i+1}y_p$  cannot be in  $G^c$  at same time as  $x_iy_{i+1}$ ,  $y_{i-1}x_{i+2}$  or  $y_{i-1}y_{i+1}$ ,  $x_ix_{i+2}$ , otherwise we have the following proper cycles:  $x_1y_ix_iy_{i+1}x_{i+1}y_p \dots x_{i+2}y_{i-1} \dots x_1$  or  $x_1y_ix_ix_{i+2} \dots y_px_{i+1}y_{i+1}y_{i-1} \dots x_1$ . The minimum in this case corresponds to one missing edge  $x_{i+1}y_p$  for each edge  $y_{i-1}x_i$  in the path, for  $i = 2, \dots, p - 1$ . Therefore, there are  $\frac{2p-6}{2}$  blue missing edges.

For the moment we have  $2p - 3$  blue missing edges. To obtain the last missing edge suppose that the blue edge  $x_2y_p$  is present in  $G^c$ . Then, it cannot be at the same time with  $x_1y_2$ ,  $y_1x_3$  or  $x_1x_3$ ,  $y_1y_2$ , otherwise we obtain the proper cycles  $C = x_1y_2 \dots x_2y_p \dots x_3y_1x_1$  or  $C = x_1x_3 \dots y_px_2 \dots y_2y_1x_1$ . The minimum in this case corresponds to one missing edge  $x_2y_p$ . We remark that the blue edges  $x_2y_p$ ,  $y_1y_2$  and  $y_1x_3$  were not counted before. The edges  $x_1x_3$  and  $x_1y_2$  were supposed to exist, otherwise, to obtain the last missing edge we consider the symmetric case, i.e., using the blue edge  $y_{p-1}x_1$  (if exists).

In total there are  $\frac{2p-2}{2} + \frac{2p-6}{2} + 2 = (2p - 2)$  blue missing edges in  $G^c$ . As we have  $c - 1$  colors different from red, that gives  $(c - 1)(2p - 2)$  missing edges, as desired.

Note that this number of missing edges is the same as in the simplest case, this is, if all edges different from red  $x_1x_i$  and  $x_1y_i$ , for  $i = 2, \dots, p$  are not present in  $G^c$ .  $\square$

**Lemma 2.5** *Let  $G^c$  be a connected  $c$ -edge-colored multigraph,  $c \geq 2$ . Let  $M$  be a matching of  $G^c$  in one color, say red, of size  $|M| \geq \lceil \frac{n-2}{2} \rceil$ . Let  $P = x_1y_1x_2y_2 \dots x_py_p$ ,  $p \geq 3$ , be a longest proper path compatible with  $M$ . Then, the following holds:*

- **(1)**  $n$  is even,  $|M| = \frac{n}{2}$  and  $2p < n$ .
  - **(1a)** If there is no proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(n - 2 + pn - 2p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 4)(c - 1)$ , for  $p = \frac{n-2}{2}$ .
  - **(1b)** If there is a proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(2pn - 4p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 4)(c - 1)$ , for  $p = \frac{n-2}{2}$ .
- **(2)**  $n$  is odd,  $|M| = \frac{n-1}{2}$  and  $2p < n - 1$ .
  - **(2a)** If there is no proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(n - 3 - p + pn - 2p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 6)(c - 1)$ , for  $p = \frac{n-3}{2}$ .
  - **(2b)** If there is a proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(2pn - 2p - 4p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 6)(c - 1)$ , for  $p = \frac{n-3}{2}$ .
- **(3)**  $n$  is even,  $|M| = \frac{n-2}{2}$  and  $2p < n - 2$ .
  - **(3a)** If there is no proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(n - 4 - 2p + pn - 2p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 8)(c - 1)$ , for  $p = \frac{n-4}{2}$ .
  - **(3b)** If there is a proper cycle  $C$  such that  $V(C) = V(P)$ , then there are at least  $(2pn - 4p - 4p^2)(c - 1)$  missing edges in  $G^c$  different from red and the minimum value of this function is  $(2n - 8)(c - 1)$ , for  $p = \frac{n-4}{2}$ .

**Proof.** Before starting the proof, we remark that the edges  $x_1y_1$  and  $x_py_p$  are of color red. Otherwise, we can easily extend the path by adding an edge of the matching to  $P$ .

Suppose first that  $n$  is even and  $2p < n$ . Since  $M$  has size  $\frac{n}{2}$  there are  $\frac{n-2p}{2}$  red edges outside  $P$ . Let us denote these edges by  $e_i$  for  $i = 1, \dots, \frac{n-2p}{2}$ . Suppose there is no proper  $C$  cycle such that  $V(C) = V(P)$ . Let blue denote any other color different from red. By Lemma 2.4 there are  $(2p - 2)$  blue missing edges. As the path is maximum, we cannot extend  $P$  having an edge

$e_i$  neither at the beginning nor at the end of it, then there are no blue edges between the vertices  $x_1, y_p$  and the edges  $e_i$ . Therefore, there are  $4\frac{n-2p}{2}$  blue missing edges. Finally, As we cannot add any edge  $e_i$  in-between the path, then there at most 2 blue edges between the edges  $e_i$  and the edges  $y_i x_{i+1}$ ,  $i = 1, \dots, p-1$ . So, there are  $2\frac{n-2p}{2}\frac{2p-2}{2}$  blue missing edges different from red. Adding up and simplifying all these numbers and having  $c-1$  colors different from red, we arrive that there are  $(n-2+pn-2p^2)(c-1)$  missing edges in  $G^c$  different from red. If we search the minimum value of this function we arrive to  $(2n-4)(c-1)$  for  $p = \frac{n-2}{2}$  and case **(1a)** holds. Now if there is a proper cycle  $C$  such that  $V(C) = V(P)$  then there cannot exist any edge at all different from red between all vertices of  $C$  and the edges  $e_i$  and therefore there are  $2\frac{n-2p}{2}2p(c-1) = (2pn-4p^2)(c-1)$  missing edges different from red. Again, minimizing the function we obtain the same result as above and case **(1b)** holds.

Suppose now that  $n$  is odd,  $M = \frac{n-1}{2}$  and  $2p < n-1$ , or  $n$  is even,  $M = \frac{n-2}{2}$  and  $2p < n-2$ . In both cases, same arguments as before apply just replacing  $n$  with  $n-1$  or  $n-2$  respectively, in the number of missing edges. This is because we have  $n-1$  matched vertices and one non-matched vertex for the first case and  $n-2$  matched vertices and two non-matched vertices for the second one.  $\square$

### 3 2-edge-colored multigraphs

In this section we study the existence of proper hamiltonian paths in 2-edge-colored multigraphs. We present two main results. The first one involves the number of edges, and the second one involves the number of edges and the rainbow degree. Both results are tight.

**Theorem 3.1** *Let  $G^c$  be a 2-edge-colored multigraph on  $n \geq 8$  vertices. If  $m \geq (n-2)(n-3) + 2(n-2) + 2$ , then  $G^c$  has a proper hamiltonian path.*

**Proof.** The proof is by induction on  $n$ . For  $n = 8, 9$ , by a tedious analysis, the result can be shown. Suppose now that  $n \geq 10$ . Observe that  $|E(\overline{G^c})| \leq 2n-4$ . By a theorem in [1], if for every vertex  $v \in G^c$  we have that  $d^r(v) \geq \lceil \frac{n+1}{2} \rceil$  and  $d^b(v) \geq \lceil \frac{n+1}{2} \rceil$ , then  $G^c$  has a proper hamiltonian path. Suppose then that there exists a vertex  $v$  such that  $d^r(v) \leq \lceil \frac{n+1}{2} \rceil - 1$ , otherwise there is nothing to prove.

Suppose first that there exist two neighbors of  $v$ , say  $u$  and  $w$ , such that  $c(vu) = r$  and  $c(vw) = b$ . We construct then a new multigraph  $G'^c$  by replacing the vertices  $v, u$  and  $w$  with a new vertex  $z$  such that  $N^r(z) = N_{G^c - \{v, u, w\}}^r(w)$



and  $N^b(z) = N_{G^c - \{v, u, w\}}^r(u)$ . We remark that  $N_{G^c - \{v, u\}}^r(w)$  and  $N_{G^c - \{v, w\}}^r(u)$  cannot be both empty at the same time, otherwise  $|E(\overline{G^c})| \geq n - 3 + n - 3 + n - (\lceil \frac{n+1}{2} \rceil - 1) > 2n - 4$ , a contradiction in the total number of edges. So, by this we remove at most  $n - 1$  blue edges and  $\lceil \frac{n+1}{2} \rceil - 1$  red edges from  $v$ ,  $n - 3$  red edges from  $u$ ,  $n - 3$  blue edges from  $w$ , and one red and one blue between  $u$  and  $w$ . Therefore  $G'^c$  has at least  $(n - 2)(n - 3) + 2(n - 2) + 2 - (n - 1) - (\lceil \frac{n+1}{2} \rceil - 1) - (n - 3) - (n - 3) - 2$  edges. This number is greater or equal than  $(n - 4)(n - 5) + 2(n - 4) + 2$ , i.e., the number of edges required to have a proper hamiltonian path in a multigraph on  $n - 2$  vertices. So, by the inductive hypothesis  $G'^c$  has a proper hamiltonian path  $P$ . Finally, as we have chosen the appropriate edges to remove at  $u$  and  $w$  it is easy to extend  $P$  to a proper hamiltonian path for  $G^c$ .

Suppose now that there does not exist two neighbors of  $v$ , say  $u$  and  $w$ , such that  $c(vu) = r$  and  $c(vw) = b$ . So, we have two possible cases. First case is when  $v$  has one only neighbor  $w$  in both colors. It is easy to observe that  $G^c - \{v\}$  has  $(n - 2)(n - 3) + 2(n - 2) = (n - 1)(n - 2)$  edges, i.e., it is a rainbow complete multigraph. Therefore, we have any possible proper hamiltonian path, in particular the one that starts at  $w$  and then we easily extend the path to  $G^c$ . For the second case,  $v$  has just neighbors in one color, say  $b$ . Observe that for every vertex  $w \neq v$ ,  $w$  has a red neighbor different of  $v$ . Otherwise, if there is a vertex  $w$  without red neighbors, we have that  $|E(\overline{G^c})| \geq n - 1 + n - 2 > 2n - 4$ , a contradiction. Now, suppose first that  $v$  has at most  $n - 2$  blue neighbors. So, consider a neighbor  $w$  of  $v$  and remove all its blue incident edges. Remove then  $v$  from  $G^c$  and call this multigraph  $G'^c$ . In  $G'^c$ ,  $w$  is monochromatic in red, and  $G'^c$  has at least  $(n - 2)(n - 3) + 2(n - 2) + 2 - (n - 2) - (n - 2)$  edges. This number is exactly  $(n - 3)(n - 4) + 2(n - 3) + 2$ , i.e., the number of edges required to have a proper hamiltonian path in a multigraph on  $n - 1$  vertices. Then, by inductive hypothesis in  $G'^c$  we obtain a proper hamiltonian path that clearly starts at  $w$  since it was monochromatic. Therefore we have a proper hamiltonian path for  $G^c$ . Finally, if  $v$  has  $n - 1$  blue neighbors, suppose that one neighbor  $w$  of  $v$  has at most  $n - 3$  blue neighbors, then we proceed as before, we remove  $v$  from the multigraph and we remove all blue incident edges to  $w$ . This multigraph  $G'^c$  has at least  $(n - 2)(n - 3) + 2(n - 2) + 2 - (n - 1) - (n - 3) = (n - 3)(n - 4) + 2(n - 3) + 2$  edges. Again we obtain a proper hamiltonian path for  $G'^c$  and therefore a proper hamiltonian path for  $G^c$  (since  $w$  is monochromatic in red). Now, every vertex  $w$  has  $n - 2$  neighbors, i.e., the multigraph  $G^c - \{v\}$  is complete in blue. Call this multigraph  $G'^c$ . Now,  $G'^c$  has at least  $(n - 2)(n - 3) + 2(n - 2) + 2 - (n - 1) = n^2 - 4n + 5$  edges and

since this is bigger than  $(n-3)(n-4) + 2(n-3) + 2$  by inductive hypothesis we have a proper hamiltonian path in  $G'^c$ . Now, if  $n-1$  is odd, one of the extremities of the path is red and therefore we trivially add  $v$  to the path. If  $n-1$  is even, both extremities have the same color. If they are red we are done. Otherwise, remove all the blue edges from  $G'^c$ , this new (red) graph has  $n-1$  vertices and at least  $n^2 - 4n + 5 - \frac{(n-1)(n-2)}{2} = \frac{(n-2)(n-3)}{2} + 1$  edges therefore by a theorem in [7], it has a hamiltonian path  $P$ . Now, since  $G'^c$  is complete in blue we can use those blue edges along with  $P$  to form a proper hamiltonian  $P'$  for  $G'^c$  that starts and ends with color red. Finally, we can trivially add  $v$  to  $P'$ .

Since we covered all cases, the proof is finished.  $\square$

Theorem 3.1 is the best possible for  $n \geq 8$ . In fact, consider a rainbow complete 2-edge-colored multigraph on  $n-2$  vertices for  $n$  odd. Add two new vertices  $x_1$  and  $x_2$ . Then add a red edge  $x_1x_2$  and all red edges between  $\{x_1, x_2\}$  and the complete graph. Although the resulting graph has  $(n-1)(n-2) + 2(n-2) + 1$  edges, it has no proper hamiltonian path, since at least one of the vertices  $x_1$  or  $x_2$  cannot be attached to any such path. Indeed, for  $n$  odd, the first and last edge of any proper hamiltonian path must differ in colors. If  $n = 5, 7$ , Theorem 3.1 does not hold for the multigraphs  $H_{k, k+3}^2$ ,  $k = 1, 2$ .

**Theorem 3.2** *Let  $G^c$  be a 2-edge-colored multigraph on  $n \geq 14$  vertices. Suppose that for every vertex  $x$  in  $G^c$ ,  $rd(x) = 2$ . If  $m \geq (n-3)(n-4) + 3n - 2$ , then  $G^c$  has a proper hamiltonian path.*

**Proof.** Let us suppose that  $G^c$  has not a proper hamiltonian path. We will show that  $E(\overline{G^c})$  has more than  $3n - 10$  edges, i.e.,  $G^c$  has less than  $(n-3)(n-4) + 3n - 2$  edges, contradicting the hypothesis of the theorem.

We distinguish between two cases depending on the parity of  $n$ .

**Case A:**  $n$  is even. By Lemma 2.3,  $G^c$  has two matchings  $M^r$ ,  $M^b$ , such that  $|M^r| = \frac{n}{2}$  and  $|M^b| \geq \frac{n-2}{2}$ . Take the longest proper paths, say,  $P = x_1y_1x_2y_2 \dots x_p y_p$  and  $P' = x'_1y'_1x'_2y'_2 \dots x'_p y'_p$ , compatibles with  $M^r$  and  $M^b$ , respectively. Observe that as the length of  $P$  ( $P'$ ) is odd then both its first and last edges are on the same color. It follows that, since  $|M^r| = \frac{n}{2}$ ,  $c(x_1y_1) = c(x_p y_p) = r$ . Otherwise, we can easily extend  $P$  by adding an edge of  $M^r$ . It follows that the edges  $x_i y_i$  are red,  $i = 1, \dots, p$ . Similarly, we may suppose that  $c(x'_1 y'_1) = c(x'_p y'_p) = b$ . Indeed if  $c(x'_1 y'_1) = c(x'_p y'_p) = r$ , then either  $x'_1$  or  $y'_p$  is the endpoint of an edge of  $M^b - P'$  and therefore we obtain a path longer than  $P'$  compatible with  $M^b$ , a contradiction to the definition of  $P'$ .

Notice now that if  $2p = n$  or  $2p' = n$ , then we are finished. In addition, if

$2p < n - 2$  or  $2p' < n - 2$ , then by Lemma 2.5, there are at least  $2n - 4$  blue missing edges and  $2n - 8$  red ones. This gives a total of  $4n - 12 > 3n - 10$  missing edges, a contradiction. Consequently, in what follows we may suppose that  $2p = 2p' = n - 2$ .

Suppose first that there exists a proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ . Let  $e$  be the red edge of  $M^r$  in  $G^c - C$ . If there exists a blue edge, say  $e'$ , between an endpoint of  $e$  and  $C$ , we can easily obtain a proper hamiltonian path considering  $e, e'$  and a segment of  $C$  of length  $n - 3$  in the appropriate direction. Otherwise, as the multigraph is connected, all edges between the endpoints of  $e$  and  $C$  are red. Now, as  $rd(G^c) = 2$ , there must exist a blue edge, say  $e'$ , parallel to  $e$  and therefore we can obtain a proper hamiltonian path just as before but starting with  $e'$ .

Next suppose that there exists no proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ . By Lemma 2.5, there are at least  $2n - 4$  blue missing edges. Consider now the path  $P'$  and let  $v_1, w_1$  be the vertices in  $G^c - P'$ . It is clear that if there exists a blue edge joining  $v_1$  and  $w_1$ , then by symmetry on the colors there are at least  $2n - 4$  red missing edges. This gives a total of  $4n - 8$  missing edges, a contradiction. Otherwise, assume that there is no blue edge between  $v_1$  and  $w_1$ . In this case we will count the red missing edges assuming that we cannot extend  $P'$  to a proper hamiltonian path. If there exists no cycle  $C'$  in  $G^c$  such that  $V(C') = V(P')$  then by Lemma 2.4 there are  $2p' - 2 = n - 4$  red missing edges. By summing up, we obtain  $2n - 4 + n - 4 = 3n - 8 > 3n - 10$  missing edges, a contradiction. Finally, assume that there exists a proper cycle  $C'$  in  $G^c$  such that  $V(C') = V(P')$ . Set  $C = c_1c_2 \dots c_{2p'}c_1$  where  $c(c_i c_{i+1}) = r$  for  $i = 1, 3, \dots, 2p' - 1$ . If there are 3 or more red edges between  $\{v_1, w_1\}$  and  $\{c_i, c_{i+1}\}$ , for some  $i = 1, 3, \dots, 2p' - 1$ , then either the edges  $v_1c_i$  and  $w_1c_{i+1}$ , or  $v_1c_{i+1}$  and  $w_1c_i$  are red. Suppose  $v_1c_i$  and  $w_1c_{i+1}$  are red. In this case, the path  $v_1y'_ix'_i \dots y'_{i+1}x'_{i+1}w_1$  is a hamiltonian one. Otherwise, if there are no 3 or more red edges between  $\{v_1, w_1\}$  and  $\{c_i, c_{i+1}\}$ , for all  $i = 1, 3, \dots, 2p' - 1$ , then there are  $2\frac{2p'-2}{2} = n - 4$  red missing edges. If we sum up we obtain a total of  $2n - 4 + n - 4 = 3n - 8 > 3n - 10$  missing edges, a contradiction.

**Case B:**  $n$  is odd. By Lemma 2.3,  $G^c$  has two matchings  $M^r, M^b$ , such that  $|M^r| = |M^b| = \frac{n-1}{2}$ . As in Case A, we consider the longest proper paths  $P$  and  $P'$  compatibles with the matchings  $M^r$  and  $M^b$  respectively. Suppose first that  $2p < n - 1$  and  $2p' < n - 1$ . By Lemma 2.5, there are at least  $2n - 6$  blue missing edges and  $2n - 6$  red ones. We obtain a total of  $4n - 12 > 3n - 10$  missing edges, a contradiction.

Suppose next  $2p = 2p' = n - 1$  (the cases where  $2p < n - 1$  and  $2p' = n - 1$ , or  $2p = n - 1$  and  $2p' < n - 1$  are similar). In the rest of the proof, because

of symmetry, we will consider only the path  $P$  since same arguments may be applied as well for  $P'$ . In this case we will count the blue missing edges assuming that we cannot extend  $P$  to a proper hamiltonian path. Now, let  $v$  be the unique vertex in  $G^c - P$ . It is clear that if there is a proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ , we can trivially obtain a proper hamiltonian path since the multigraph is connected, i.e., there is at least one edge between  $v$  and  $C$ . So, as there is no proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ , we have by Lemma 2.4, that there are  $2p - 2 = n - 3$  blue missing edges. If there exists a blue edge between  $x_1$  and  $x_i$ , for some  $i = 2, \dots, p$ , then it cannot exist the blue edge  $vy_{i-1}$ , otherwise we obtain the proper hamiltonian path  $vy_{i-1} \dots x_1x_i \dots y_p$ . We can complete the argument in a similar way, if both edges  $y_py_i$  and  $vx_{i+1}$ ,  $i = 1, \dots, p - 1$  exist in  $G^c$  and are on color blue. Note that since there is no proper cycle  $C$  in  $G^c$  such that  $V(C) = V(P)$ , we cannot have at the same time the blue edges  $x_1x_i$  and  $y_py_{i-1}$ ,  $i = 2, \dots, p$ . Therefore, there are  $\frac{2p-2}{2} = \frac{n-3}{2}$  blue missing edges. If we make the sum and multiply it by two (since the same number of red missing edges is obtained with  $P'$ ), we conclude that there are  $2(n - 3 + \frac{n-3}{2}) = 3n - 9$  missing edges, a contradiction.

Since we covered all the cases, the theorem is proved.  $\square$

Theorem 3.2 is the best possible for  $n \geq 14$ . Indeed, for  $n$  odd,  $n \geq 14$ , consider a complete blue graph, say  $A$ , on  $n - 3$  vertices. Add 3 new vertices, say  $v_1, v_2, v_3$ , and join them to a same vertex  $v$  in  $A$  with blue edges. Finally, superpose the obtained graph with a complete red graph on  $n$  vertices. Although the resulting 2-edge-colored multigraph has  $(n - 3)(n - 4) + 3n - 3$  edges, it has no proper hamiltonian path since one of the vertices  $v_1, v_2, v_3$  cannot belong to such a path. For  $n = 7, 9, 11, 13$ , it is easy to see that the multigraphs  $H_{k,k+3}^2$ ,  $k = 2, 3, 4, 5$ , are exceptions for Theorem 3.2.

## 4 $c$ -edge-colored multigraphs, $c \geq 3$

In this section we study the existence of proper hamiltonian paths in  $c$ -edge-colored multigraphs, for  $c \geq 3$ . We present three main results. The first one involves the total number of edges. The second one, the total number of edges and the connectivity of the multigraph. The last one, the total number of edges and the rainbow degree. All results are best possible.

First, we present a key result that reduces the case  $c \geq 3$  just to the case  $c = 3$ .

**Lemma 4.1** *Let  $G^c$  be a  $c$ -edge-colored connected multigraph on  $n$  vertices,*

$c \geq 4$  and  $m \geq c f(n) + 1$  edges. There exists one color  $c_j$  such that if we color its edges with another used color and we delete parallel edges with the same color, then the resulting  $(c - 1)$ -edge-colored multigraph is connected and has  $m' \geq (c - 1) f(n) + 1$  edges, such that if  $G^{c-1}$  has a proper hamiltonian path then  $G^c$  has one too. Moreover, if  $rd(G^c) = k$ , then  $rd(G^{c-1}) = k - 1$  for  $1 \leq k \leq c$ .

**Proof.** Let  $c_i$  denote the color  $i$ , for  $i = 1, \dots, c$ , in  $G^c$ , and denote by  $|c_i|$  the number of edges with color  $i$ . Let  $c_j$  be the color with less number of edges. Color the edges on color  $c_j$  with another used color, say  $c_l$ , and delete (if necessary) parallel edges with that color. Call this multigraph  $G^{c-1}$ . By this, we delete at most  $|c_j|$  edges. It is clear that this multigraph is connected since we delete just parallel edges. Also, if  $G^{c-1}$  has a proper hamiltonian path, then, this path is also proper hamiltonian in  $G^c$  but maybe with some edges on color  $c_j$  (in the case that they have been recolored with  $c_l$ ). Observe also that if  $rd(G^c) = k$  then  $rd(G^{c-1}) = k - 1$  since only the color  $c_j$  disappeared. We will show now that  $m' \geq (c - 1) f(n) + 1$ . We have two cases. First, if  $|c_j| > f(n)$ , then clearly  $m' \geq (c - 1) f(n) + 1$  edges since for all  $i$ ,  $|c_i| > f(n)$ . For the second case, we have that  $|c_j| \leq f(n)$ . Now,  $m = \sum_{i=1}^c |c_i| \geq c f(n) + 1$  and therefore  $\sum_{i=1, i \neq j}^c |c_i| \geq c f(n) - |c_j| + 1 = (c - 1) f(n) + f(n) - |c_j| + 1$ . This last expression is greater or equal than  $(c - 1) f(n) + 1$  since  $f(n) - |c_j| \geq 0$ . Finally, we have that  $G^{c-1}$  has  $m' \geq (c - 1) f(n) + 1$  edges as desired.  $\square$

**Theorem 4.2** *Let  $G^c$  be a  $c$ -edge-colored multigraph on  $n$  vertices,  $n \geq 2$  and  $c \geq 3$ . If  $m \geq \frac{c(n-1)(n-2)}{2} + 1$ , then  $G^c$  has a proper hamiltonian path.*

**Proof.** By Lemma 4.1 we can assume that  $c = 3$  and  $m \geq \frac{3(n-1)(n-2)}{2} + 1$ . Furthermore, cases  $n \leq 7$  can be checked by exhaustive methods. Assume so,  $n \geq 8$ . Since there exists one color, say red, such that the number of red edges are at least  $\frac{(n-1)(n-2)}{2} + 1$  then by a theorem in [7], there is a hamiltonian red path and therefore a perfect or almost perfect matching  $M^r$ . Take the longest proper path  $P = x_1 y_1 x_2 y_2 \dots x_p y_p$  compatible with  $M^r$ . So we have that,  $c(x_i y_i) = r$  for  $i = 1, \dots, p$ . As  $m \geq \frac{3(n-1)(n-2)}{2} + 1$ ,  $|E(\overline{G^c})| \leq 3n - 4$ . Now, we will distinguish between two cases depending on the parity of  $n$ .

**Case A:**  $n$  even. Clearly  $|M^r| = \frac{n}{2}$ . By contradiction suppose that  $2p < n$ , otherwise we are finished. Assume first that there is no proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$ . By Lemma 2.5, there are at least  $2(n - 2 + pn - 2p^2)$  missing edges different from red and therefore the inequality  $2(n - 2 + pn - 2p^2) \leq 3n - 4$  must be satisfied. This inequality does not hold for  $n \geq 8$ . Therefore, we have a contradiction with the number of edges of  $G^c$ . Assume

next that there is a proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$ . Again by Lemma 2.5,  $2(2pn - 4p^2) \leq 3n - 4$  must be satisfied, and as before this is never possible for  $n \geq 8$ .

**Case B:**  $n$  odd. Clearly  $|M^r| = \frac{n-1}{2}$ .

Assume first that there is a proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$ . If  $2p = n - 1$  then we can obtain a proper hamiltonian path by just taking any edge between the unique vertex outside  $C$  and  $C$  and then following  $C$  in the appropriate direction. Otherwise by Lemma 2.5 there are  $2(2pn - 2p - 4p^2)$  missing edges different from red and therefore  $2(2pn - 2p - 4p^2) \leq 3n - 4$  must hold and this never happens for  $n \geq 9$ .

Assume next that there is no proper cycle  $C$  in  $G^c$  such that  $V(P) = V(C)$ . If  $2p < n - 1$  then, again by Lemma 2.5, the inequality  $2(n - 2 + pn - 2p^2) \leq 3n - 4$  must be satisfied. However this is not possible for  $n \geq 9$ . It remains to handle the case  $2p = n - 1$ . Let  $v$  be the unique vertex of  $G^c - P$ . We have the following cases depending on the degree and neighbors of  $v$ .

**Subcase B1:** *All edges incident to  $v$  have the same color, say red.*

Suppose first that  $d(v) \leq n - 2$ , consider the multigraph  $G^c - \{v\}$  and delete from a neighbor of  $v$ , say  $w$ , all the edges in two colors in order to have  $w$  monochromatic not in red. Call this multigraph  $G'^c$ . Observe that we can always do this since it is impossible to have 2 monochromatic vertices. So, by this, we delete at most  $n - 2 + n - 2 + n - 2 = 3n - 6$  edges. It is easy to see that the multigraph  $G'^c$  on  $n - 1$  vertices has at least  $\frac{3(n-1)(n-2)}{2} + 1 - (3n - 6) = \frac{3(n-2)(n-3)}{2} + 1$ , then, by the even case, we have a proper hamiltonian path  $P'$  in  $G'^c$ . Since  $w$  is monochromatic not in red,  $w$  is either in the beginning or in the end of  $P'$  and therefore it is trivial to add  $v$  to  $P'$  in order to obtain a proper hamiltonian path in  $G^c$ .

Suppose finally that  $d(v) = n - 1$ . Therefore the multigraph  $G^c - \{v\}$  on  $n - 1$  vertices has at least  $\frac{3(n-1)(n-2)}{2} + 1 - (n - 1) \geq \frac{3(n-2)(n-3)}{2} + 1$ , then, again, we have a proper hamiltonian path  $P'$  in  $G'^c$ . Now, if the path either starts or ends with a color different from red, we trivially add  $v$  to the path. If not, both of them finish with red. Now, if we can take any parallel edge of these without losing the property of being properly colored we have again that it is easy to add  $v$  to the path. Otherwise, we have, without losing generality, the degree in some color, say  $c_1$  of the first vertex of the path, say  $w$ , is at most  $n - 3$ . So, we are in the same case as before, since we take the multigraph  $G^c - \{v\}$  and we delete from  $w$  the edges in color  $c_1$  and in another color in order to have  $w$  monochromatic not in red. By this, we delete at most  $n - 1 + n - 2 + n - 3 = 3n - 6$  edges and finally, the result follows exactly as

in the first situation.

**Subcase B2:** *There exist two distinct edges incident to  $v$ , say  $vx$  and  $vy$ , such that  $c(vx) = c_1 \neq c_2 = c(vy)$ .*

Assume first that  $x = y$ . This case is similar to the previous one just taking the multigraph  $G^c - \{v\}$  and deleting from  $x$  the edges in the appropriate two colors in order to have it monochromatic. By this we delete at most 3 edges at  $v$  and  $2(n-2)$  at  $x$  to obtain a total of  $2n-1$  deleted edges and we have that the multigraph  $G^c - \{v\}$  has  $\frac{3(n-1)(n-2)}{2} + 1 - (2n-1)$  edges that is greater than or equal to  $\frac{3(n-2)(n-3)}{2} + 1$  for  $n \geq 5$ .

Assume next that  $x \neq y$ . We will prove the result by induction. If we cannot attach  $v$  to the path  $P$  we have that there are 4 missing edges in colors different from red between  $v$  and the vertices  $x_1, y_p$ . Also, we have at most 2 edges different from red between  $v$  and the edges  $y_i x_{i+1}$  of the path. Therefore there are  $2\frac{n-3}{2}$  missing edges. Adding up all this, we conclude that the degree of  $v$  in colors different from red is at most  $n+1$ . So, if we contract the three vertices  $v, x, y$  to a new one, say  $v'$ , such that  $N^{c_1}(v') = N_{G^c - \{v, x, y\}}^{c_1}(y)$ ,  $N^{c_2}(v') = N_{G^c - \{v, x, y\}}^{c_2}(x)$  and  $N^{c_3}(v') = N_{G^c - \{v, x, y\}}^{c_3}(x) \cap N_{G^c - \{v, x, y\}}^{c_3}(y)$ . By this we delete at most  $5n-6$  edges. Now, it is easy to see that the new 3-edge-colored multigraph on  $n-2$  vertices, say  $G'^c$ , has at least  $\frac{3(n-1)(n-2)}{2} + 1 - (5n-6)$  edges that is greater than or equal to  $\frac{3(n-3)(n-4)}{2} + 1$ , for  $n \geq 9$ , therefore, by inductive hypothesis, we have a proper hamiltonian path  $P'$  in  $G'^c$ . Now, because of the way we have chosen the edges to delete at  $x$  and  $y$ , it is easy to obtain from  $P'$  a proper hamiltonian path for  $G^c$ .

Since we covered all the cases, the theorem is proved.  $\square$

Theorem 4.2 is the best possible. Indeed, consider a rainbow complete multigraph on  $n-1$  vertices with  $c$  colors and add a new isolated vertex  $x$ . The resulting multigraph, although it has  $c\frac{(n-1)(n-2)}{2}$  edges, contains no proper hamiltonian path since it is not connected.

Notice that in the above theorem there is no condition guaranteeing the connectivity of the underlying graph. Next result adds this condition.

**Theorem 4.3** *Let  $G^c$  be a connected  $c$ -edge-colored multigraph on  $n$  vertices,  $n \geq 9$  and  $c \geq 3$ . If  $m \geq \frac{c(n-2)(n-3)}{2} + n$ , then  $G^c$  has a proper hamiltonian path.*

**Proof.** By Lemma 4.1 it is enough to prove the theorem for  $c = 3$ . The proof is by induction on  $n$ . For cases  $n = 9, 10$ , refer to [6]. Now, we have two cases, depending on whether  $G^c$  contains a monochromatic vertex or not.

**Case A:** *There exists a monochromatic vertex, say  $x$ , in  $G^c$ .* Let  $y$  be a

neighbor of  $x$ . If  $y$  is also monochromatic then the multigraph  $G^c - \{x, y\}$  has  $n-2$  vertices and at least  $\frac{3(n-2)(n-3)}{2} + n - (2n-3) = \frac{3(n-2)(n-3)}{2} - n + 3$  edges, i.e., it is almost rainbow complete and therefore it has a proper hamiltonian cycle. Then, it is easy to add  $x$  and  $y$  to the cycle to obtain a proper hamiltonian path in  $G^c$  otherwise we have a contradiction in the total number of edges. Suppose then that  $y$  is not monochromatic and that  $xy$  has color  $b$ . Let us contract the vertices  $x$  and  $y$  by a new vertex, say  $z$ , such that  $N^b(z) = N^r(z) = \emptyset$  and  $N^g(z) = N^g(y)$  (or  $N^b(z) = N^g(z) = \emptyset$  and  $N^r(z) = N^r(y)$ ). Observe that, if the resulting multigraph on  $n-1$  vertices is connected, (we show this later) and has enough edges for the induction hypothesis to hold, by induction it contains a proper hamiltonian path. Since  $z$  is monochromatic, it can only be the endpoint of the path. Therefore, using the blue edge  $xy$ , we can extend the path to a proper hamiltonian one in the initial multigraph.

We now count the maximum number of edges that may be deleted by the contraction. We delete all the edges incident to  $x$ , which is at most  $n-1$ , and the edges of color  $b$  and say  $g$  (or  $r$ ) incident to  $y$ , at most  $2(n-2)$ . This gives us a total of  $3n-5$  edges. We now show that we can choose  $x$  and  $y$  for the contraction process, such that we delete at most  $3\frac{(n-2)(n-3)}{2} + n - 3\frac{(n-3)(n-4)}{2} - n + 1 = 3n-8$  edges, necessary for the induction hypothesis to hold.

Let us consider now  $d^b(x)$ . Note that if  $d^b(x) \leq n-4$ , we delete at most  $n-4 + 2(n-2) = 3n-8$  edges from  $x$  and any selected neighbor  $y$  of  $x$  and we are done. Further, from a theorem in [1], if  $d^i(z) \geq \lceil \frac{n}{2} \rceil \forall z \in G^c - \{x\}, i \in \{r, g, b\}$ , then  $G^c - \{x\}$  has a proper hamiltonian cycle. This would imply a proper hamiltonian path in  $G$ . Thus, we may assume that there exists some vertex  $w \in G^c - \{x\}$  such that  $d^i(w) < \lceil \frac{n}{2} \rceil$  for some  $i \in \{r, g, b\}$ . We have the remaining cases:

**Subcase A1:**  $d^b(x) = n-1$ . Observe that  $w \in N^b(x)$ . In this case, the contraction process deletes  $n-1$  edges from  $x$ , and at most  $n-2 + \frac{n}{2} - 1$  from  $w$ , much less than  $3n-8$  for  $n > 10$ .

**Subcase A2:**  $d^b(x) = n-2$ . Let  $w$  be the only vertex not adjacent to  $x$ . Now, if there is a vertex  $y$  adjacent to  $x$  such that  $d_{G^c - \{x\}}^{r,b}(y) \leq 2n-6$  or  $d_{G^c - \{x\}}^{r,g}(y) \leq 2n-6$ . We choose  $x$  and  $y$  for the contraction process. Otherwise, for all  $y$  adjacent to  $x$  we have that  $d_{G^c - \{x\}}^{r,b}(y) \geq 2n-5$  and  $d_{G^c - \{x\}}^{r,g}(y) \geq 2n-5$ . Therefore, the multigraph  $G^c - \{x\}$  is almost rainbow complete and therefore, it has a proper hamiltonian cycle. Finally, we trivially add  $x$  to the cycle to form a proper hamiltonian path for  $G^c$ .

**Subcase A3:**  $d^b(x) = n-3$ . Similar to the last case. Just observe that if there is a vertex  $y$  adjacent to  $x$  such that  $d_{G^c - \{x\}}^{r,b}(y) \leq 2n-5$  or  $d_{G^c - \{x\}}^{r,g}(y) \leq 2n-5$ ,



we choose  $x$  and  $y$  for the contraction process. Otherwise, for all  $y$  adjacent to  $x$  we have that  $d_{G^c - \{x\}}^{r,b}(y) \geq 2n - 4$  and  $d_{G^c - \{x\}}^{r,g}(y) \geq 2n - 4$ , i.e., the multigraph  $G^c - \{x\}$  is rainbow complete and we proceed as before to obtain the proper hamiltonian path.

**Case B:** *There is no monochromatic vertex in  $G^c$ .*

First suppose that there exists a vertex  $x$  such that  $|N(x)| = 1$ . Pick  $x$  and some vertex  $y \in N(x)$ , for the contraction. We delete at most 3 edges at  $x$  and  $2n - 4$  at  $y$ , which guarantees the induction hypothesis and leaves  $y$  monochromatic.

In what follows, we suppose that  $|N(x)| \geq 2$  for all  $x \in G^c$ . We describe another contraction process, but now between 3 vertices. Consider a vertex  $x$  and  $y, z \in N(x)$  and suppose that  $c(xy) = b$  and  $c(xz) = r \neq c(xy)$ . For the contraction, replace  $x, y$  and  $z$  by a new vertex say  $s$ , such that  $N^r(s) = N_{G^c - \{x,y,z\}}^r(y)$ ,  $N^b(s) = N_{G^c - \{x,y,z\}}^b(z)$  and  $N^g(s) = N_{G^c - \{x,y,z\}}^g(y) \cap N_{G^c - \{x,y,z\}}^g(z)$ .

If the resulting multigraph has a proper hamiltonian path, it is easy to obtain a proper hamiltonian path for the initial multigraph since we chose the appropriate edges to delete at  $y$  and  $z$ . Let us count now the maximum number of edges that may be deleted in the contraction process.

We now need to select  $x, y$  and  $z$  so that we delete at most  $3 \frac{(n-2)(n-3)}{2} + n - (3 \frac{(n-4)(n-5)}{2} + n - 2) = 6n - 19$  edges, necessary for the induction hypothesis to hold.

Since we delete at most  $3n - 6$  edges incident to  $y$  and  $z$  in  $G^c - \{x\}$ , if the degree of  $x$  is at most  $6n - 19 - (3n - 6) = 3n - 13$ , the hypothesis holds. If not, we show that there exist  $x, y$  and  $z$  such that the total number of deleted edges in the contraction process is less than or equal to  $6n - 19$ .

In what follows we show how to find the desired triplet. We have two cases, depending on the parity of  $n$ .

Let  $E^i$  be the set of edges in color  $i$  and suppose that the color  $b$  maximizes  $|E^i|$ . The monochromatic subgraph in color  $b$  has at least  $m^b \geq \frac{(n-2)(n-3)}{2} + \lceil \frac{n}{3} \rceil$  edges. We distinguish two possibilities. If this subgraph is connected, then by Lemma 2.1 there is a perfect matching for  $n$  even and almost perfect matching for  $n$  odd. Otherwise, there is a matching of size  $\frac{n-2}{2}$  for  $n$  even and  $\frac{n-1}{2}$  if  $n$  is odd. Let  $M^b$  denote the maximum matching in  $E^b$ .

**Subcase B1:**  *$n$  is even.* Let  $P = x_1y_1x_2y_2 \dots x_py_p$  be the longest proper path compatible with  $M^b$ . It is easy to check that  $|P| \geq 4$ , otherwise there are not enough edges in  $G^c$ .

Suppose now that there is a proper cycle  $C$  such that  $V(C) = V(P)$ . We

have the following cases.

- (i)  $|M^b| = \frac{n}{2}$ . By Lemma 2.5, we can check that  $|P| \geq n - 2$ , otherwise there is a contradiction with the total number of edges of the multigraph. Now, if  $|P| = n - 2$ , we trivially add the edge of the matching outside the path since,  $P$  also defines a proper cycle, there are no monochromatic vertices and the multigraph is connected. If  $|P| = n$ ,  $P$  is a proper hamiltonian path and we are done.
- (ii)  $|M^b| = \frac{n-2}{2}$ . By Lemma 2.5, we can check that  $|P| \geq n - 4$ , otherwise, as before, there is a contradiction with the number of edges of the multigraph. Suppose first that  $|P| = n - 4$ . Let  $xy$  be the edge of the matching outside  $P$ . Clearly, there are no edges in color  $g$  or  $r$  between  $xy$  and the cycle otherwise we would have a longest proper path compatible with the matching. Then, we have that  $d^{r,g}(x) \leq 6$  and therefore  $d(x) \leq 6 + n - 1 < 3n - 13$ . So we take  $x$  with any two neighbors in different colors for the contraction process and we are done. Suppose now that  $|P| = n - 2$ . Let  $x, y$  be the unmatched vertices. For every edge  $y_i x_{i+1}$  on the cycle we can have at most 4 edges in colors  $r$  and  $g$  between their endpoints and the vertices  $x, y$ . Otherwise, we can add  $x$  and  $y$  to the cycle in order to obtain a proper hamiltonian path. Therefore, without losing of generality  $d^{r,g}(x) \leq 2\frac{n-2}{2} + 2$ . Same observation applies for the blue edges  $x_i y_i$ . This is, we can have at most 2 blue edges between their endpoints and the vertices  $x, y$ . Then  $d^b(x) \leq \frac{n-2}{2} - 1$ . Summing up we have  $d(x) \leq n + \frac{n-2}{2} - 1 < 3n - 13$ . Thus, we can take  $x$  for the contraction process.

Let us suppose now that there is no proper cycle  $C$  such that  $V(C) = V(P)$ . Consider the following cases, in a similar manner, as we did before.

- (i)  $|M^b| = \frac{n}{2}$ . By Lemma 2.5 we can check that  $|P| \geq n - 4$ . If  $P$  is hamiltonian we are done. Otherwise, there is at least one edge in  $M^b - (P \cap M^b)$ . Clearly, there are no edges in color  $r$  and  $g$  from the extremities of  $P$  to the edges in  $M^b - (P \cap M^b)$ . Now, for each edge outside the matching, there are at most 4 edges between their endpoints and the endpoints of the edges  $y_i x_{i+1}$ , otherwise we can obtain a longer path. Therefore, taking the vertex, say  $x$ , outside  $P$  with minimum degree, we have that  $d^{r,g}(x) \leq 2\frac{n-6}{2} + 6$ , for  $|P| = n - 4$ , and  $d^{r,g}(x) \leq 2\frac{n-4}{2} + 2$ , for  $|P| = n - 2$ . In both cases, if we consider  $d^b(x) = n - 1$ , we get  $d(x) \leq 3n - 13$  and we take it for the contraction.
- (ii)  $|M^b| = \frac{n-2}{2}$ . For  $|P| = n - 4$  the proof is similar as previous case.

Suppose then that  $|P| = n - 2$ . Let  $v, w$  be the unmatched vertices. We try to add these vertices to the path either at the extremities or between the vertices  $y_i$  and  $x_{i+1}$ . Suppose first that we cannot add any of them. Then  $d^{r,g}(v) \leq 2\frac{n-4}{2} + 2$ . Summing up this with at most  $n - 2$  blue edges we obtain  $d(v) \leq 2n - 4 < 3n - 13$  and we take  $v$  for the contraction. Suppose last that we can add  $v$  but we cannot add  $w$ . If  $v$  was added at one extremity of the path we obtain that  $d(w) \leq 2\frac{n-4}{2} + 1 + n - 2 = 2n - 5 < 3n - 13$ . If  $v$  was added inbetween the path, we have  $d(w) \leq 2\frac{n-6}{2} + 2 + n - 2 = 2n - 6 < 3n - 13$ . In both cases we can take  $w$  for the contraction.

**Subcase B2:**  $n$  is odd. Therefore  $|M^b| \geq \frac{n-1}{2}$ . Let  $P = x_1y_1x_2y_2 \dots x_py_p$  be the longest proper path compatible with  $M^b$ . It is easy to check that  $|P| \geq 4$ , otherwise there are not enough edges in  $G^c$ . Then we distinguish two cases:

There is a proper cycle  $C$  such that  $V(C) = V(P)$ . By Lemma 2.5 we can check that  $|P| \geq n - 5$ . If  $|P| = n - 1$  we are done since is trivial to add the unmatched vertex to the cycle to obtain a proper hamiltonian path. If  $n - 5 \leq |P| \leq n - 3$ , as in the even case, we can use for the contraction any vertex incident to any edge of the matching outside the path, since there are no red and green edges at all between the edges of the matching outside the path and the path.

There is no proper cycle  $C$  such that  $V(C) = V(P)$ . Again, by Lemma 2.5 we can check that  $|P| \geq n - 5$ . This case is also similar to the even case. For  $n - 5 \leq |P| \leq n - 3$ , we try to extend  $P$  with the edges of the matching outside the path. Since this is not possible we can chose one matched vertex outside the path to do the contraction. For  $|P| = n - 1$ , we try to extend  $P$  with the unique unmatched vertex. If this is not possible, we can use it for the contraction and we are done.

Since we cover all cases, we can always find three appropriate vertices to make the contraction process. We now check the connectivity of the resulting multigraph after the contraction of 2 or 3 vertices.

Assume that we contract two vertices  $x, y$  to a vertex  $s$  and the multigraph is disconnected. It can be easily shown that the graph has two components with one vertex and  $n - 2$  vertices respectively, otherwise, there is a contradiction on the total number of edges of the multigraph. Observe first that the isolated vertex cannot be  $s$  unless  $x$  and  $y$  are both monochromatic, but this case was solved independently. Then, as we have the choice at  $y$  of which colors to delete,  $s$  cannot be isolated in the contracted multigraph and therefore this case cannot occur. Suppose now that a vertex  $z \neq s$  is the isolated

vertex.

Suppose the first the case where  $x$  is monochromatic. Consider the original multigraph. There are at least  $2(n - 1)$  missing edges at  $x$  since it is monochromatic and  $3(n - 3)$  missing edges at  $z$  since  $z$  is isolated in the contracted multigraph. Also there are two more missing edges between  $y$  and  $z$  since we have the choice of which colors to delete at  $y$ . This gives us a total of  $5n - 9 = |E(\overline{G^c})|$  missing edges. Therefore, if there is one more missing edge, we have a contradiction with the total number of edges. Suppose now that there is another vertex  $y' \neq y$  adjacent to  $x$  and take  $x$  and  $y'$  for the contraction. If the contracted multigraph is connected we are done. Otherwise we have many more new missing edges and therefore a contradiction. Finally if there is no other vertex than  $y$  adjacent to  $x$  we obtain more missing edges at  $x$  and again a contradiction.

Suppose finally that  $x$  is not monochromatic. In this case  $x$  has  $y$  as its unique neighbor. So, there are  $3(n - 2)$  missing edges at  $x$  and  $3(n - 3)$  missing edges at  $z$  since  $z$  is isolated in the contracted multigraph. This gives us a total of  $6n - 15$  missing edges and this is greater than  $|E(\overline{G^c})| = 5n - 9$ . A contradiction.

Assume now that we contract three vertices  $x, y, z$  where  $c(xy) = b$  and  $c(xz) = r$  to a vertex  $s$  and the multigraph is disconnected. Again, this multigraph has exactly two components with one vertex, say  $u$ , and  $n - 3$  vertices respectively.

Suppose first that  $u \neq s$ . In the original multigraph  $u$  must have at least two different neighbors in two different colors among the vertices  $x, y$  and  $z$ , otherwise we would be in the case that either  $u$  is monochromatic or  $u$  has one unique neighbor and we have already solved these cases. Let  $y'$  and  $z'$  two neighbors of  $u$  among  $x, y$  and  $z$  such that  $c(uy') \neq c(uz')$ . Now we contract the vertices  $u, y', z'$  as before. Observe that at  $u$  we delete only 6 edges since  $u$  has only  $x, y$  and  $z$  as neighbors and the red edge  $uy$ , the blue edge  $uz$  and at least one green edge among  $uy$  and  $uz$  are missing. At  $y'$  and  $z'$  we delete  $3(n - 3) + 3$  edges as usual. With this contraction we delete in total  $6 + 3(n - 3) + 3 = 3n$  edges and therefore the contracted multigraph on  $n - 2$  vertices has at least  $\frac{3(n-2)(n-3)}{2} + n - 3n = \frac{3(n-3)(n-4)}{2} + n - 9$  edges which guarantees not only the inductive hypothesis but also the connectivity for  $n \geq 10$ .

Suppose finally that  $u = s$ . Then, there are no red edges between  $y$  and the multigraph  $G^c - \{x, y, z\}$  and no blue edges between  $z$  and  $G^c - \{x, y, z\}$ . Now, since we are not in the previous cases,  $y$  has at least two different neighbors, say  $y'$  and  $z'$ , such that  $c(yy') \neq c(yz')$ . Then we contract the vertices  $y, y', z'$ .

In the contraction process we delete at most  $2(n-3)$  edges between  $y$  and  $G^c - \{x, y, z\}$  (since there are no red edges), 6 between  $y$  and the vertices  $x, z$ , and  $3(n-3) + 3$  at  $y'$  and  $z'$ . We obtain in total at most  $5n - 6$  deleted edges. Now, this new contracted multigraph has  $n - 2$  vertices and at least  $\frac{3(n-2)(n-3)}{2} + n - (5n - 6) = \frac{3(n-3)(n-4)}{2} - n - 3$  edges. Clearly, if the multigraph is connected we are done. Otherwise, as before, it has two components with one vertex and  $n - 3$  vertices respectively. We can suppose that we are not in the previous case, i.e., we are in the case that the contracted vertex is the isolated one. Observe now that the component on  $n - 3$  vertices has at least  $\frac{3(n-3)(n-4)}{2} - n - 3$ , therefore it is almost rainbow complete and it has a proper hamiltonian cycle. Finally, in the original multigraph, it is easy to add the vertices  $y, y', z'$  to the proper cycle in order to obtain a proper hamiltonian path.

Now as the connectivity is proved the theorem holds.  $\square$

Theorem 4.3 is the best possible. Indeed, consider a rainbow complete multigraph on  $n - 2$  vertices with  $c$  colors and add two new vertices  $x$  and  $y$ . Add now the edge  $xy$  and also all edges between  $y$  and the complete multigraph, all on a same color. The resulting multigraph, although it has  $c \frac{(n-2)(n-3)}{2} + n - 1$  edges, it contains no proper hamiltonian path, as  $x$  cannot belong to such a path.

**Theorem 4.4** *Let  $G^c$  be a  $c$ -edge-colored multigraph on  $n$  vertices,  $n \geq 11$  and  $c \geq 3$ . Assume that for every vertex  $x$  of  $G^c$ ,  $rd(x) = c$ . If  $m \geq \frac{c(n-2)(n-3)}{2} + 2c + 1$ , then  $G^c$  has a proper hamiltonian path.*

**Proof.** By Lemma 4.1 it is enough to prove the theorem for  $c = 3$ . As  $m \geq \frac{3(n-2)(n-3)}{2} + 7$ , then  $|E(\overline{G^c})| \leq 6n - 16$ . The proof will be done by construction of a proper hamiltonian path or, if it is not possible, by a reduction to Theorem 4.3, i.e., to a connected 3-edge-colored multigraph on  $n' \geq 9$  vertices and  $m' \geq \frac{3(n'-2)(n'-3)}{2} + n'$  that has a proper hamiltonian path. We will do this reduction by contracting 2 or 3 vertices depending on if there exists a vertex  $x$  in  $G^c$ , such that  $|N^{r,g,b}(x)| = 1$  or not.

Suppose first that there exists a vertex  $x \in G^c$  such that  $|N^{r,g,b}(x)| = 1$ . Let  $y$  be the unique neighbor of  $x$ . We replace  $x$  and  $y$  by a new vertex  $s$ , such that  $N^r(s) = N^r(y)$  and  $N^b(s) = N^g(s) = \emptyset$ . Clearly, we delete 3 edges between  $x$  and  $y$  and at most  $2(n-2)$  edges in color  $b$  and  $g$  incident to  $y$ . This gives us a total of  $2n - 1$  deleted edges. For the reduction hypothesis we can delete at most  $\frac{3(n-2)(n-3)}{2} + 7 - \frac{3(n-3)(n-4)}{2} - n + 1 = 2n - 1$  edges. Observe that the resulting multigraph on  $n - 1$  vertices is connected since as  $rd(x) = 3$ , we can choose at  $y$  which two colors to delete, therefore since the

original multigraph is connected it is impossible that in all 3 possible choices the multigraph after contraction would be disconnected. So, as we deleted  $2n - 1$  edges and the multigraph is connected, by Theorem 4.3, it has a proper hamiltonian path. Now, since  $s$  is monochromatic, it can only be the endpoint of the path. We can replace back  $s$  with  $x$  and  $y$  using the edge  $xy$  in color  $b$  (or  $g$ ) and find the proper hamiltonian path for the initial multigraph.

Suppose next that there is no vertex  $x \in G^c$ , such that  $|N^{r,g,b}(x)| = 1$ . Let us suppose that there are 3 vertices  $x, y$  and  $z$ , such that  $xy$  is in color  $b$ , and  $xz$  in color  $r$ . Then, we contract  $x, y$  and  $z$  to a new vertex  $s$ , such that  $N^r(s) = N_{G^c - \{x,y,z\}}^r(y)$ ,  $N^b(s) = N_{G^c - \{x,y,z\}}^b(z)$  and  $N^g(s) = N_{G^c - \{x,y,z\}}^g(y) \cap N_{G^c - \{x,y,z\}}^g(z)$ . Clearly, we delete at most  $3(n - 1)$  edges incident to  $x$ ,  $n - 3$  edges in  $b$  incident to  $y$ ,  $n - 3$  edges in  $r$  incident to  $z$ ,  $n - 3$  in color  $g$  incident to  $y$  and  $z$ , and 3 edges between  $y$  and  $z$ . In total we delete at most  $6n - 9$  edges. Only  $\frac{3(n-2)(n-3)}{2} + 7 - \frac{3(n-4)(n-5)}{2} - n + 2 = 5n - 12$  edges can be deleted to make the reduction. Therefore, we need to find a vertex  $x$  such that its total degree is less than or equal to  $2n - 6$ .

Suppose without losing generality that  $|E^b| \geq |E^r| \geq |E^g|$ , then  $|E^b| \geq \frac{(n-2)(n-3)}{2} + 3$ . Since the subgraph in color  $b$  is connected because of the rainbow degree of the vertices, we have by Lemma 2.1 that there is a matching  $M^b$  such that  $|M^b| = \frac{n}{2}$  for  $n$  even and  $|M^b| = \frac{n-1}{2}$  for  $n$  odd. Let  $P = x_1y_1x_2y_2 \dots x_iy_i \dots x_py_p$  be the longest proper path compatible with  $M^b$ . The rest of the proof is now divided in two cases depending on the parity of  $n$ .

**Case A:**  $n$  is even.

Assume first that there is a proper cycle  $C$  such that  $V(C) = V(P)$ . By Lemma 2.5 we can check that  $|P| \geq n - 2$ , otherwise we have a contradiction with the number of edges. This case is trivial since, either  $P$  is a proper hamiltonian path or we can add the unique edge of the matching outside  $P$  directly to the cycle to obtain a proper hamiltonian path since the multigraph is connected.

Assume therefore that there is no proper cycle  $C$  such that  $V(C) = V(P)$ . By Lemma 2.5 we can check that  $|P| \geq n - 4$ . Let  $x$  be the vertex in  $M^b - (M^b \cap P)$  with minimum degree and let  $y$  be its adjacent vertex in the matching. Clearly, there cannot be edges in colors  $r$  and  $g$  between  $M^b - (M^b \cap P)$  and the extremities of  $P$ . Also, there can be at most 4 edges in colors  $r$  and  $g$  from each edge in  $M^b - (M^b \cap P)$  and the edges  $y_{i-1}x_i$  in  $P$ . Then, we can conclude that there are at most  $2\frac{2p-2}{2}$  edges in color  $r$  and  $g$ , between  $x$  and the vertices in  $P$ .

Suppose now that there is one parallel edge to the blue edge  $xy$  in color  $r$  or  $g$ , then using the edges  $xx_i, yy_i$  or  $yx_i, xy_i$  in color  $b$ , we can add the

edge  $xy$  in color  $r$  or  $g$  into  $P$ . Since this contradicts our hypothesis, we can conclude that there are 2 missing edges in color  $b$  from  $xy$  to each edge  $x_iy_i$  in  $P$ . Now, since there are  $\frac{2p}{2}$  edges  $x_iy_i$  we conclude that the vertex  $x$  has  $d^b(x) \leq \frac{2p}{2} + (n - 2p - 1)$ . In total we have that  $d(x) \leq 3(n - 2p - 1) + \frac{2p}{2} + (2p - 2) = 3n - 3p - 5$ , and for  $|P| = n - 2$ , this is less than or equal to  $2n - 6$  and we can make the contraction with it.

Now, if  $|P| = n - 2$  and there are no parallel edges in  $r$  and  $g$  at  $xy$ , we have that  $d(x) \leq 2n - 5$  and  $d(y) \leq 2n - 5$ . We can use one of these vertices unless both inequalities become equalities. In this case we try to replace the edges  $x_iy_i$  in  $P$  by  $x_ixy_i$  and  $x_jyy_j$ ,  $i \neq j$ . Suppose that we have the edges  $xx_i$  in colors  $r$  and  $g$ ,  $xy_i$  in color  $b$ ,  $yx_j$  in colors  $r$  and  $g$ , and  $yy_j$  in color  $b$ . Then we have a proper hamiltonian path  $P' = x_1y_1 \dots x_ixy_ix_{i+1} \dots x_jyy_j \dots x_py_p$ . Otherwise at least one of those edges is missing and therefore either  $d(x) \leq 2n - 6$  or  $d(y) \leq 2n - 6$  as desired.

If  $|P| = n - 4$  by Lemma 2.5 there are  $6n - 20$  missing edges in colors  $r$  and  $g$ . First, if there is one parallel edge to the blue edge  $xy$  in color  $r$  or  $g$  there are, as before,  $\frac{2p}{2} = \frac{n-4}{2}$  blue missing edges from  $xy$  to each edge  $x_iy_i$  in  $P$ , therefore since  $6n - 20 + \frac{n-4}{2} > 6n - 16$  we obtain a contradiction in the total number of edges.

Finally, if  $|P| = n - 4$  and there are no parallel edges in colors  $r$  and  $g$  to the blue edge  $xy$ , and the other edge of the matching outside the path. We obtain then, 4 more missing edges in colors  $r$  and  $g$  and therefore  $6n - 16$ . Observe now that if we can replace the edges  $xy$  and the other edge of the matching outside  $P$ , say  $uv$ , in  $M^b$  by  $xu$  and  $yv$ , then there are missing the 4 parallel edges in color  $r$  and  $g$ , obtaining  $6n - 12$ . A contradiction. Otherwise if we cannot, there are 2 more missing edges obtaining  $6n - 14$  edges. Again a contradiction.

**Case B:**  $n$  is odd.

Assume first that there is a proper cycle  $C$  such that  $V(C) = V(P)$ . By Lemma 2.5 we can check that  $|P| \geq n - 5$ . The case  $|P| = n - 1$  is trivial. Clearly, there are no edges in colors  $r$  and  $g$  between  $M^b - (M^b \cap P)$  and  $P$ . Therefore taking any vertex  $x$  in  $M^b - (M^b \cap P)$  we obtain that  $d(x) \leq 4 + n - 1 = n + 3$ , for  $|P| = n - 3$ , and  $d(x) \leq 8 + n - 1 = n + 7$ , for  $|P| = n - 5$ . In both cases, this is less than  $2n - 6$  so we can do the contraction with  $x$ .

Assume therefore that there is no proper cycle  $C$  such that  $V(C) = V(P)$ . By Lemma 2.5 we can check that  $|P| \geq n - 7$ . Suppose first  $|P| < n - 1$ . Let  $xy$  and  $uv$  be two edges in  $M^b - (M^b \cap P)$  where  $x$  has the minimum degree. There cannot be edges in colors  $r$  and  $g$  to the extremities of  $P$  and

there can be only 4 edges in colors  $r$  and  $g$  between  $M^b - (M^b \cap P)$  and  $y_{i-1}x_i$  in  $P$ . Suppose now that there exists an edge  $xy$  in color  $r$  or  $g$ , then we can use the edges in color  $b$  between  $xy$  and  $x_iy_i$  to extend  $P$ . Since it is not possible we conclude that there are only  $\frac{2p}{2} + (n - 2p - 1)$  edges in color  $b$  at  $x$ . For  $|P| = n - 7$  and  $|P| = n - 5$ , we have a contradiction in the total number of edges as in the even case  $|P| = n - 4$ . If  $|P| = n - 3$ , then  $d(x) \leq \frac{2p}{2} + (2p - 2) + 3(n - 2p - 1) = 3n - 3p - 5 \leq 2n - 6$  and we use  $x$  for the contraction.

Suppose there are no parallel edges in colors  $r$  and  $g$  at the edges in  $M^b - (M^b \cap P)$ . If  $|P| = n - 7$  then there are 6 missing edges. Now, if we can replace the three blue edges in  $M^b - (M^b \cap P)$  to another different three blue edges, we are, either in the previous case or there are 6 more parallel missing edges and by Lemma 2.5 there are  $8n - 48 - 6 - 6 = 8n - 36 > 6n - 16$  missing edges. A contradiction. Otherwise there are at least 5 missing edges in color  $b$  and again by Lemma 2.5 there are  $8n - 48 - 6 - 5 = 8n - 37$  missing edges. Again a contradiction.

If  $|P| = n - 5$  there are 4 parallel edges  $r$  and  $g$  missing at the edges in  $M^b - (M^b \cap P)$  and by Lemma 2.5 there are  $6n - 26$  more missing edges in color  $r$  and  $g$ . Therefore  $6n - 22$ . Now, if we cannot replace the blue edges  $xy$  and  $uv$  with the blue edges  $xu$ ,  $yv$  there are at least 2 missing edges in color  $b$ , otherwise we would miss more parallel edges. Also, there are 4 more missing edges in color  $b$  between the extremities of the edges in  $M^b - (M^b \cap P)$  and the unmatched vertex, since otherwise we can construct 4 different matchings and therefore 8 more missing edges. In conclusion we have at least 6 missing edges in color  $b$  and  $6n - 22$  in colors  $r$  and  $g$ , that gives us a total of  $6n - 16$  missing edges. To find the last one to get a contradiction suppose that there are the parallel edges  $x_1y_1$  and  $x_py_p$  in colors  $r$  and  $g$ , then we can make a proper cycle of length  $n - 1$ :  $x_1y_1 \dots x_py_pvyxux_1$ , if we also have the edges,  $y_pv$  and  $ux_1$  in color  $b$ ,  $xu$  and  $yv$  in color  $r$  or  $g$ . Now, as it is trivial to attach the unmatched vertex to the proper cycle there is at least one of those edges missing and therefore a contradiction.

If  $|P| = n - 3$ , we have as before that  $d_P^{r,g}(x) \leq n - 5$ . As always, there are no parallel edges to the edge  $xy$  in colors  $r$  and  $g$ . Now, if there are no edges between  $xy$  and the unmatched vertex  $z$  in color  $r$  and  $g$ , then  $d(x) \leq 2n - 6$  as desired for the contraction. Otherwise, if there are edges in colors  $r$  and  $g$  to the vertex  $z$ , then there cannot be the edges in color  $b$  parallel to them since we could replace  $xy$  by one of those and thus come back to the previous case. Therefore, we have that  $d(x) \leq 2n - 5$  and  $d(y) \leq 2n - 5$ . Then, we can choose for the contraction, either  $x$  or  $y$ , unless both inequalities become



equalities. Now, if we suppose that  $c(xz) = r$ , we contract the three vertices  $x, y$  and  $z$  such that we delete at most  $2n - 5$  edges at  $x$ ,  $n - 3$  blue edges at  $y$ ,  $n - 3$  red edges at  $z$ ,  $n - 3$  green edges with common endpoints at  $y$  and  $z$  and 2 edges, one red and one green between  $y$  and  $z$ . We remark that the blue edge  $yz$  is not present. By this, we delete at most  $5n - 12$  edges, i.e., the required number for the contraction.

Finally, suppose that  $|P| = n - 1$ . Let  $z$  be the unmatched vertex. Then, for same reasons as before,  $d^{r,g}(z) \leq n - 3$ . If we suppose that there are edges in colors  $r$  and  $g$  between  $z$  and  $y_i$ , then there cannot be the edges  $x_i z$  in color  $b$ , otherwise a hamiltonian path can be found. Suppose there are no such edges then, by Lemma 2.4, either  $x_1$  or  $y_p$  have degree in colors  $r$  and  $g$  at most  $n - 3$ . If there is at least one of the edges  $zy_1$  or  $zx_p$  in color  $b$ , then we can replace  $z$  by either  $x_1$  or  $y_p$ . In both cases, we arrive that  $d(x_1) \leq 2n - 6$  or  $d(y_p) \leq 2n - 6$ , otherwise  $d(z) \leq 2n - 6$  and the contraction can be done.

Since all cases were covered, there always exists a vertex  $x$  such that  $d(x) \leq 2n - 6$  and the contraction can be done. Therefore by Theorem 4.3 we obtain a proper hamiltonian path in this new multigraph. Then, it is simple to extend this path to a proper hamiltonian one in the initial multigraph because of the choice of the edges to delete at the contracted vertices.

We will check now the connectivity of the resulting multigraph after the contraction of 3 vertices. Suppose we contract three vertices  $x, y, z$ , such that  $c(xy) = b$  and  $c(xz) = r$ , to a vertex  $s$ , and the multigraph is disconnected. This multigraph has exactly two components with one vertex, say  $u$ , and  $n - 3$  vertices respectively.

First, if  $u \neq s$ , we proceed exactly as in the equivalent case in Theorem 4.3 and we find three other vertices  $u, y', z'$  to contract to a vertex  $s'$ , just deleting  $3n$  edges. This new contracted multigraph on  $n - 2$  vertices has at least  $\frac{3(n-2)(n-3)}{2} + 7 - 3n = \frac{3(n-3)(n-4)}{2} - 2$  edges. Then, if it is connected we are done, otherwise there is a component with one vertex, say  $u'$  and another one on  $n - 3$  vertices with at least  $\frac{3(n-3)(n-4)}{2} - 2$  edges, i.e., almost rainbow complete. Therefore, we obtain a proper hamiltonian cycle and then we can easily add either the isolated vertex  $u'$  (if  $u' \neq s'$ ) or the three  $u, y', z'$  (if  $u' = s'$ ) vertices to the cycle to obtain a proper hamiltonian path for the initial multigraph.

Finally, if  $u = s$  we will consider two cases. Suppose first that  $x$  has at most one neighbor with parallel edges. Therefore  $d(x) \leq n$ . Now, in this case, in the contraction process we delete  $4n - 6$  edges instead of  $5n - 12$ , so the contracted multigraph on  $n - 2$  vertices is disconnected and the component on  $n - 3$  vertices has at least  $\frac{3(n-2)(n-3)}{2} + 7 - (4n - 6) = \frac{3(n-3)(n-4)}{2} - n + 4$

edges. Then, this component is almost rainbow complete and, as before, we can obtain a proper hamiltonian cycle and easily add  $x, y$  and  $z$  to it in order to obtain a proper hamiltonian path to the initial multigraph. Suppose last that  $x$  has two different neighbors, say  $y'$  and  $z'$  with parallel edges. Consider the next two cases:

Assume first that the parallel edges are on the same two colors, this is,  $c(xy') = c(xz') = \{b, r\}$  (cases with other two colors are similar). Now, suppose that in both possible contractions the multigraph is disconnected and the contracted vertex is always the isolated one (since if not, that case was solved above). We can observe that the initial multigraph has  $n + 3$  missing edges at  $x$  (since  $d(x) \leq 2n - 6$ ),  $n - 3$  green edges and  $4(n - 3)$  blue and red edges at  $y'$  and  $z'$  (since in both contractions the multigraph is disconnected). By this we obtain a total of  $6n - 12 > 6n - 16$  missing edges. A contradiction.

Assume therefore that the parallel edges are not on the same two colors, this is,  $c(xy') = \{b, r\}$  and  $c(xz') = \{b, g\}$  (cases with other combinations are similar). Now since we are not in the case before we do not have neither the green edge  $xy'$  nor the red one  $xz'$ . Try any of the three possible contractions. Then, if the multigraph is disconnected and the contracted vertex is always the isolated one, we can observe that there can be just red edges between  $y'$  and  $G^c - \{x, y', z'\}$ , and green edges between  $z'$  and  $G^c - \{x, y', z'\}$ . Now since  $rd(G^c) = 3$  there must exist the green edge  $y'z'$  and the red edge  $z'y'$ , and again since we are not in the previous case, the blue edge  $y'z'$  is not present. We are in the precise situation that  $c(xy') = \{b, r\}$ ,  $c(xz') = \{b, g\}$  and  $c(y'z') = \{r, g\}$ . Now, we have nine different contractions to try, three for each triplet  $xy'z'$ ,  $y'xz'$  and  $z'xy'$ . So, if in all of them, we are in this same situation (the contracted multigraph is disconnected and the isolated vertex is the contracted one), we can conclude that in the original multigraph there can be just blue edges between  $x$  and  $G^c - \{x, y', z'\}$ , red edges between  $y'$  and  $G^c - \{x, y', z'\}$ , and green edges between  $z'$  and  $G^c - \{x, y', z'\}$ , therefore  $6(n - 3)$  missing ones. Finally, adding the 3 missing edges  $xy'$ ,  $xz'$  and  $y'z'$  in green, red and blue respectively, we obtain  $6n - 15$  missing edges. A contradiction.

Now as the connectivity is proved the theorem holds.  $\square$

Theorem 4.4 is the best possible for  $n \geq 11$ . In fact, consider a rainbow complete multigraph, say  $A$ , on  $n - 2$  vertices. Add 2 new vertices  $v_1, v_2$  and then join them to a vertex  $v$  of  $A$  with all possible colors. The resulting  $c$ -edge-colored multigraph has  $\frac{c(n-2)(n-3)}{2} + 2c$  edges and clearly has no proper hamiltonian path. If  $n = 6, 8$ , the multigraphs  $H_{k, k+2}^c$ ,  $k = 2, 3$ , are exceptions for Theorem 4.4.

## 5 Summary and remarks

In this work, we studied the existence of proper hamiltonian paths, in edge-colored multigraphs depending on the number of edges, the rainbow degree and the connectivity. Here, the notable fact is that the proofs were sometimes long and tedious despite the lower bounds for the edges in the considered multigraphs were really high. It should be also interesting to study similar conditions guaranteeing other patterns than proper paths, as for instance proper cycles, proper trees, etc.

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