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Mathieu Cathala

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# SHALLOW WATER WAVES OVER POLYGONAL BOTTOMS

MATHIEU CATHALA

The traditional shallow water model for waves propagating over varying bathymetry depends for its derivation on the asymptotic analysis of a Dirichlet-Neumann operator. This analysis however is restricted to smoothly varying topographies. We propose an adaptation to one dimensional polygonal bottoms using the conformal mapping idea of Hamilton and Nachbin. The asymptotic analysis of the Dirichlet-Neumann operator relies on an *ad hoc* transformation of the fluid domain into a flat bottom domain. We derive a new shallow water model which accounts for polygonal topographies.

## CONTENTS

1. Introduction	1
1.1. Water waves over polygonal topographies	1
1.2. Formulation of the water waves problem	2
1.3. Asymptotic expansion of the Dirichlet-Neumann operator	3
2. Reduction to a problem on the flat strip	4
2.1. Using conformal mappings to straighten the bottom	5
2.2. Transformed Laplace equation with flat bottom	8
2.3. Transformed problem on the flat strip	10
3. Shallow-water analysis of the Dirichlet-Neumann operator	10
3.1. The Dirichlet-Neumann operator on the flat strip	11
3.2. Asymptotic analysis of the Dirichlet-Neumann operator	12
4. A shallow-water model for polygonal bottoms	15
5. Conclusion	16
References	18

## 1. INTRODUCTION

**1.1. Water waves over polygonal topographies.** Studies of surface water wave dynamics in the presence of variable topographies are of great interest from coastal engineering point of view. Despite this importance, there is no general agreement about how to describe shallow water flows over rough topographies. Actually, it has been known for quite some time that the presence of strongly varying topographies introduces special problems for the formal derivation of shallow water models. In [9], Hamilton raises the limitations of the long wave models derived by Mei and Le Méhauté [15] and Peregrine [20] when the bottom is strongly sloping. For two dimensional flows, he used a conformal mapping technique (inspired from Kreisel [11]) to derive a long wave model on a fluid of strongly varying depth. The restriction of this method is that it requires knowledge of the conformal mapping between the fluid domain and a flat strip. In case the topography has polygonal shape,

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Nachbin [16] used Schwarz-Christoffel theory to compute this conformal mapping (numerically) and derived a weakly nonlinear, weakly dispersive, terrain-following Boussinesq system.

The difficulties pointed out by Hamilton to derive shallow water models in the presence of non smooth topographies also occur if one wants to use the more recent method based on the Zakharov/Craig-Sulem formulation of the water waves problem. The main task of this method is the asymptotic analysis of the Dirichlet-Neumann operator involved in this particular formulation of the water waves problem (see *e.g.* [1] or [10]). Now this analysis depends upon the transformation of the fluid layer into a flat strip. Unfortunately, as noted by Lannes (see [13, section 2.5.3]), the classical diffeomorphism between the fluid layer and a flat strip cannot be used when the bottom parametrization is not regular.

On the basis of these considerations, we intend in this paper to conduce the shallow water analysis of the Dirichlet-Neumann operator when the bottom has polygonal shape using the conformal mapping idea of Hamilton and Nachbin to straighten the fluid layer.

**1.2. Formulation of the water waves problem.** The water waves problem consists in describing the motion of the free surface, denoted by  $\zeta(t, x)$ , of an incompressible, homogeneous and inviscid fluid, under the influence of gravity. Through this paper, we assume that the topography of the bottom is polygonal, with a finite number of edges (that is the bottom is flat at infinity). The fluid

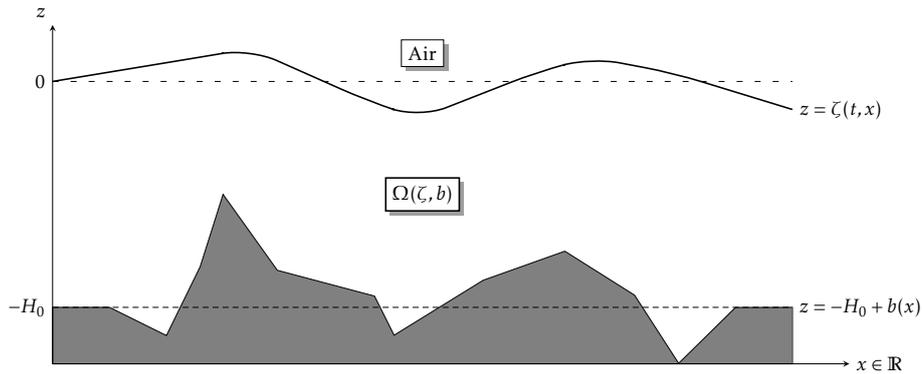


FIGURE 1. Sketch of the fluid domain.

domain is given by

$$\Omega(\zeta, b) = \{(x, z) \in \mathbb{R}^2 ; -H_0 + b(x) < z < \zeta(t, x)\},$$

where  $H_0$  is a reference depth and  $b(x)$  denotes the polygonal variations of the bottom (see Figure 1). With the usual assumption of irrotational flow, the fluid velocity is represented by the gradient of a potential  $\Phi$ .

The asymptotic analysis of the water waves problem requires the use of dimensionless quantities based on characteristics of the flow. More precisely, denoting by  $\lambda$  the typical wavelength of the waves, by  $a_{\text{surf}}$  their typical amplitude and by  $a_{\text{bott}}$  the typical amplitude of the bottom variations, we define dimensionless variables and unknowns as

$$x' = \frac{x}{\lambda}, \quad z' = \frac{z}{H_0}, \quad t' = \frac{\sqrt{gH_0}}{\lambda} t,$$

and

$$\zeta' = \frac{\zeta}{a_{\text{surf}}}, \quad b' = \frac{b}{a_{\text{bott}}}, \quad \Phi' = \frac{\Phi}{a_{\text{surf}} \lambda \sqrt{g/H_0}}.$$

To simplify the notations we omit the prime symbol in the rest of the paper. From the previous physical scales we also define three independent parameters:

$$\mu = \frac{H_0^2}{\lambda^2}, \quad \varepsilon = \frac{a_{\text{surf}}}{H_0}, \quad \beta = \frac{a_{\text{bott}}}{H_0}.$$

Our analysis focuses on the shallow water regime  $\mu \ll 1$ . The parameters  $\varepsilon$  and  $\beta$  respectively account for the relative amplitude of the waves and of the bathymetry.

As remarked by Zakharov [21], the evolution of the flow is characterized by the evolution of only two quantities located at the surface, namely the surface elevation  $\varepsilon\zeta$  and the trace of the velocity potential  $\psi = \Phi|_{z=\varepsilon\zeta}$ . The formulation of the water waves problem in the form of a system of two scalar evolution equations on  $(\zeta, \psi)$ , due to Craig and Sulem [6, 5], reads in dimensionless form

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\partial_x \psi|^2 - \varepsilon \mu \frac{(\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi + \varepsilon \partial_x \zeta \partial_x \psi)^2}{2(1 + \varepsilon^2 \mu |\partial_x \zeta|^2)} = 0. \end{cases} \quad (1.1)$$

The key point in this formulation is the introduction of the Dirichlet-Neumann operator

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b] : \psi \mapsto \sqrt{1 + \varepsilon^2 |\partial_x \zeta|^2} \partial_{\mathbf{n}} \Phi|_{z=\varepsilon\zeta}, \quad (1.2)$$

where the velocity potential  $\Phi$  is the solution to the non-dimensionalized elliptic problem

$$\begin{cases} \mu \partial_x^2 \Phi + \partial_z^2 \Phi = 0 & \text{in } \Omega(\varepsilon\zeta, \beta b), \\ \Phi = \psi & \text{on } \{z = \varepsilon\zeta\}, \\ \partial_{\mathbf{n}} \Phi = 0 & \text{on } \{z = -1 + \beta b\}, \end{cases} \quad (1.3)$$

in which  $\partial_{\mathbf{n}}$  stands for the upward conormal<sup>1</sup> derivative associated with the elliptic operator  $\mu \partial_x^2 + \partial_z^2$ . Using the Zakharov/Craig-Sulem formulation (1.1) as a starting point, approximating the water waves equation in shallow water regime then amounts to understand the asymptotic behavior of the Dirichlet-Neumann operator when the shallowness parameter  $\mu$  is small.

### 1.3. Asymptotic expansion of the Dirichlet-Neumann operator.

*Transforming the Laplace equation into an elliptic problem on a flat strip.* Since the Dirichlet-Neumann operator is explicitly defined in terms of the velocity potential, a natural way to derive asymptotic properties of this operator is by studying the asymptotic behavior of the potential. The issue is then to study a Laplace equation on the unknown fluid domain  $\Omega(\varepsilon\zeta, \beta b)$ . An efficient approach to get around this issue is to transform the fluid domain to the flat strip  $\mathcal{S} = (-1, 0) \times \mathbb{R}$  (see *e.g.* [2, 1, 10] or [18]). The main interest is that the resulting transformed potential on the flat strip then solves an elliptic boundary value problem with variable coefficients defined on the fixed domain  $\mathcal{S}$ . Since the Dirichlet-Neumann operator can be expressed in terms of the transformed potential, constructing a shallow water expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi$  reduces to finding an approximate solution to this new boundary value problem on  $\mathcal{S}$ .

<sup>1</sup>At the bottom boundary, the outward unit normal vector  $\mathbf{n}$  is well defined everywhere except at the vertices.

*Limitations of the classical approach.* In the previous approach, the choice of the diffeomorphism between the flat strip and the fluid domain is important because it governs the form of the resulting elliptic problem on  $\mathcal{S}$ . More precisely, given a diffeomorphism  $\Sigma$  mapping  $\mathcal{S}$  onto  $\Omega(\varepsilon\zeta, \beta b)$ , we know from Proposition 2.7 of [12] that the transformed velocity potential  $\phi = \Phi \circ \Sigma$  satisfies

$$\nabla_{x,z} \cdot P[\Sigma] \nabla_{x,z} \phi = 0 \quad \text{in } \mathcal{S},$$

where  $\nabla_{x,z} = [\partial_x, \partial_z]^T$  and where the matrix  $P[\Sigma]$  is defined in terms of the Jacobian matrix  $J_\Sigma$  of  $\Sigma$  as

$$P[\Sigma] = |\det J_\Sigma| J_\Sigma^{-1} \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} (J_\Sigma^{-1})^T.$$

To define this diffeomorphism, the simplest choice consists in transforming only the vertical coordinate:

$$\forall (x, z) \in \mathcal{S}, \quad \Sigma(x, z) = (x, \varepsilon\zeta(x) + z(1 + \varepsilon\zeta(x) - \beta b(x))).$$

Unfortunately, this choice requires some regularity on the bottom parametrization  $b$  since the coefficients of  $P$  involve, among others, the derivative of  $b$ .

*Schwarz-Christoffel mappings as an adaptation to polygonal topographies.* In this paper, we intend to adapt the previous approach to the particular case of a polygonal topography. Therefore, the first task is to construct a diffeomorphism between the flat strip and the fluid domain with polygonal bottom. This task is undertaken in section 2, in which we transform the Laplace problem (1.3) on the polygonal bottom domain  $\Omega(\varepsilon\zeta, \beta b)$  into a variable coefficients elliptic problem on the flat strip  $\mathcal{S}$ . Introducing complex canonical coordinates, we use a conformal mapping technique, namely Schwarz-Christoffel mapping theory (see [17] for example), to straighten the polygonal bottom. Section 3 is entirely devoted to the shallow water analysis of the Dirichlet-Neumann operator. Following the usual approach outlined above, we show that this operator can be expressed in terms of the solution of the boundary value problem on the flat strip. The asymptotic expansion of the Dirichlet-Neumann operator then hinges on the construction of an approximate solution to this boundary value problem. Using this asymptotic analysis, we finally derive in section 4 a shallow water model which accounts for polygonal topographies.

## 2. REDUCTION TO A PROBLEM ON THE FLAT STRIP

In view of the shallow water analysis of the Dirichlet-Neumann operator, it is important to address the recovering of the velocity potential  $\Phi$  from its trace  $\psi$  at the surface. For this reason, this section is devoted to the study of the Laplace equation on the physical domain  $\Omega = \Omega(\varepsilon\zeta, \beta b)$  with polygonal topography:

$$\begin{cases} \mu \partial_x^2 \Phi + \partial_z^2 \Phi = 0 & \text{in } \Omega, \\ \Phi = \psi & \text{on } \{z = \varepsilon\zeta\}, \\ \partial_{\mathbf{n}} \Phi = 0 & \text{on } \{z = -1 + \beta b\}. \end{cases} \quad (2.1)$$

In what follows, we assume that the water depth remains positive:

$$\exists h_{\min} > 0, \quad 1 + \varepsilon\zeta - \beta b \geq h_{\min}. \quad (2.2)$$

In this section, we explain how to transform the problem (2.1) into a variable coefficients elliptic problem on the flat strip. The main issue is to straighten the polygonal bottom using a smooth mapping. Section 2.1 is devoted to the construction of such straightening mappings. The transformation of the Laplace problem on  $\Omega$  into a boundary value problem on the flat strip then proceeds in two steps (see Figure 2):

- (1) Using the straightening of the bottom, we first transform in section 2.2 the Laplace problem (2.1) into a Laplace problem with flat bottom via a diffeomorphism, denoted by  $\Sigma_{\text{bott}}^{-1}$ .
- (2) Starting from this transformed Laplace problem with flat bottom, section 2.3 then addresses the flattening of the fluid boundary using the diffeomorphism  $\Sigma_{\text{surf}}^{-1}$ .

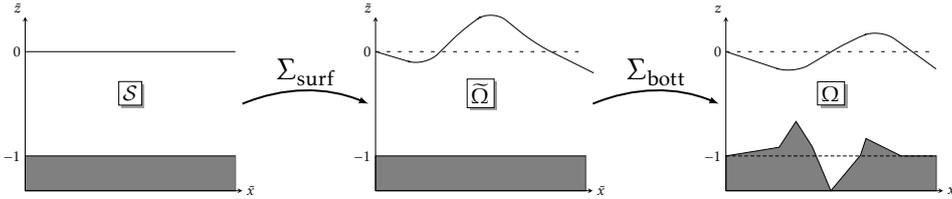


FIGURE 2. Two step straightening of the physical domain:  $\Sigma_{\text{bott}}^{-1}$  straightens the bottom, then  $\Sigma_{\text{surf}}^{-1}$  straightens the fluid boundary.

**2.1. Using conformal mappings to straighten the bottom.** We first aim at mapping the flat strip  $\mathcal{S}$  onto the domain at rest

$$\Omega_{\text{rest}} = \{(x, z) \in \mathbb{R}^2 ; -1 + \beta b(x) < z < 0\}.$$

To achieve this, we focus on  $\mu$ -conformal transformations that is transformations that are conformal for the metric  $\mu dx^2 + dz^2$ . As we will see, working with such mappings is convenient because they leave invariant the nondimensionalized Laplace equation  $\mu \partial_x^2 \Phi + \partial_z^2 \Phi = 0$ .

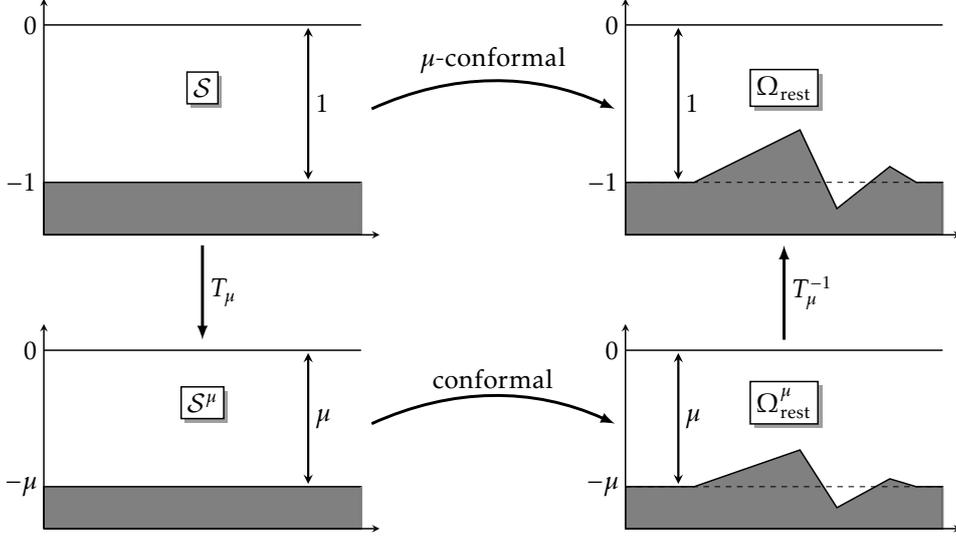
**2.1.1. Building  $\mu$ -conformal mappings.** Following Nachbin [16], a simple way to build  $\mu$ -conformal maps from conformal maps is to use a vertical scaling by a factor  $\sqrt{\mu}$ , namely  $T_\mu : (x, z) \mapsto (x, \sqrt{\mu}z)$ . As a matter of fact, a transformation  $\Sigma : \mathcal{S} \mapsto \Omega_{\text{rest}}$  is  $\mu$ -conformal if and only if the transformation  $\Sigma_\mu = T_\mu \circ \Sigma \circ T_\mu^{-1}$  is a conformal map. Building a  $\mu$ -conformal transformation further amounts to building a conformal map  $\Sigma_\mu$  between the scaled domains  $\mathcal{S}^\mu = T_\mu(\mathcal{S})$  and  $\Omega_{\text{rest}}^\mu = T_\mu(\Omega_{\text{rest}})$  (see Figure 3). The main interest of such a scaling is that conformal transformations between a strip and a polygonal domain can be found from using Schwarz-Christoffel mapping theory. The latter ensures that the problem of seeking a conformal map from a strip to the interior of a polygonal region can be reduced to solving a nonlinear system of equations, whose unknowns are the pre-images of the vertices of the polygonal boundary. This problem, known as the Schwarz-Christoffel parameter problem, is nontrivial and generally analytically intractable but it can be solved numerically with very efficient methods (see for instance [7]). As an illustration for the use of the Schwarz-Christoffel formula, we construct in the following example an analytic function which maps  $\mathcal{S}$  onto a step bottom domain.

*Example 2.1.* Let us consider the simple case of a step bottom, that is suppose that the bathymetry is parametrized by

$$b(x) = \begin{cases} 0 & \text{if } x < 0, \\ b_0 & \text{if } x > 0, \end{cases}$$

where  $b_0$  is some positive constant. Identifying  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$ , the rescaled strip  $\mathcal{S}^\mu$  reads

$$\mathcal{S}^\mu = \{\omega = x + iz ; (x, z) \in \mathbb{R}^2, -\sqrt{\mu} < z < 0\}$$

FIGURE 3. Getting  $\mu$ -conformal mappings from conformal ones.

and the rescaled domain at rest  $\Omega_{\text{rest}}^\mu$  is given as

$$\Omega_{\text{rest}}^\mu = \{ \omega = x + iz ; (x, z) \in \mathbb{R}^2, \sqrt{\mu}(-1 + \beta b(x)) < z < 0 \}.$$

We seek for an analytic function  $\Sigma_\mu$  from  $\mathcal{S}^\mu$  to  $\Omega_{\text{rest}}^\mu$  which maps the upper boundary

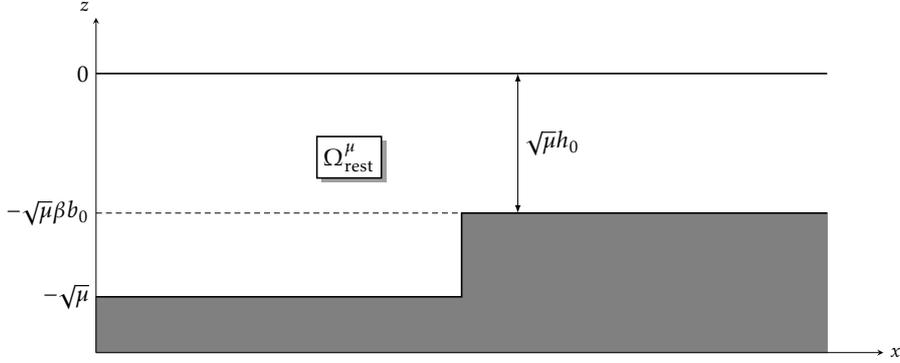


FIGURE 4. Rescaled domain at rest in the case of a step

$\{z = 0\}$  onto itself and the lower boundary  $\{z = -\sqrt{\mu}\}$  onto the rescaled bottom boundary of  $\Omega_{\text{rest}}^\mu$ . Using the Schwarz-Christoffel formula (see *e.g* [7, Theorem 2.1]), the desired mapping can be found by integrating the following expression

$$\frac{d\Sigma_\mu(\omega)}{d\omega} = \left( \frac{1 + h_0^2 \exp\left(\frac{\pi\omega}{\sqrt{\mu}}\right)}{1 + \exp\left(\frac{\pi\omega}{\sqrt{\mu}}\right)} \right)^{1/2},$$

where  $h_0 = (1 - \beta b_0)$  (see Figure 4). It follows that

$$\Sigma_\mu(\omega) = \sqrt{\mu} \frac{h_0}{\pi} \left( \log\left(\frac{1+s}{1-s}\right) - \frac{1}{h_0} \log\left(\frac{s+h_0}{s-h_0}\right) \right),$$

where  $s = h_0 \left( \frac{1 + \exp\left(\frac{\pi\omega}{\sqrt{\mu}}\right)}{1 + h_0^2 \exp\left(\frac{\pi\omega}{\sqrt{\mu}}\right)} \right)^{\frac{1}{2}}$ . See Figure 5 for a sketch of the behavior of the resulting conformal mapping  $\Sigma_\mu$ .

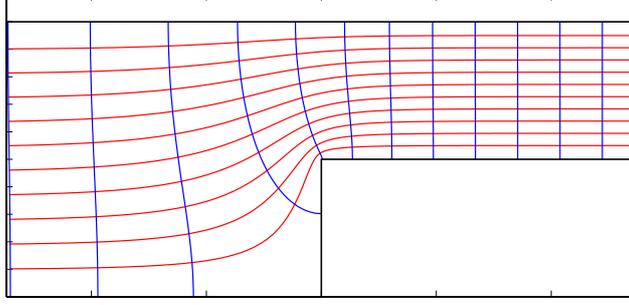


FIGURE 5. Level lines for the conformal map of Example 2.1

2.1.2. *Choice of a  $\mu$ -conformal transformation.* Let  $\Sigma_\mu$  be the Schwarz-Christoffel mapping from the rescaled strip  $\mathcal{S}^\mu$  to the rescaled physical domain at rest  $\Omega_{\text{rest}}^\mu$ . As explained above, a  $\mu$ -conformal transformation  $\Sigma_{\text{bott}}$  from  $\mathcal{S}$  to  $\Omega_{\text{rest}}$  is given by  $\Sigma_{\text{bott}} = T_\mu^{-1} \circ \Sigma_\mu \circ T_\mu$ , where we recall that  $T_\mu$  is the scaling defined as  $T_\mu(x, z) = (x, \sqrt{\mu}z)$ . Now, when solving the Schwarz-Christoffel parameter problem, the conformal map  $\Sigma_\mu$  can be chosen so as to map the upper boundary  $\{z = 0\}$  of  $\mathcal{S}^\mu$  onto the corresponding boundary of  $\Omega_{\text{rest}}^\mu$ . Thus, Schwarz reflection principle ensures that  $\Sigma_\mu$  can be analytically continued across  $\{z = 0\}$  to the reflected strip  $[0, \sqrt{\mu}] \times \mathbb{R}$ . It follows that the associated  $\mu$ -conformal transformation  $\Sigma_{\text{bott}}$  is actually a diffeomorphism between the strip  $(-1, 1) \times \mathbb{R}$  and the augmented domain  $\Omega_{\text{rest}} \cup \Omega_{\text{rest}}^*$  obtained as the union of  $\Omega_{\text{rest}}$  with the reflected domain at rest  $\Omega_{\text{rest}}^* = \{(x, z) \in \mathbb{R}^2 ; 0 < z < 1 - \beta b(x)\}$  (see Figure 6). We assume that the fluid

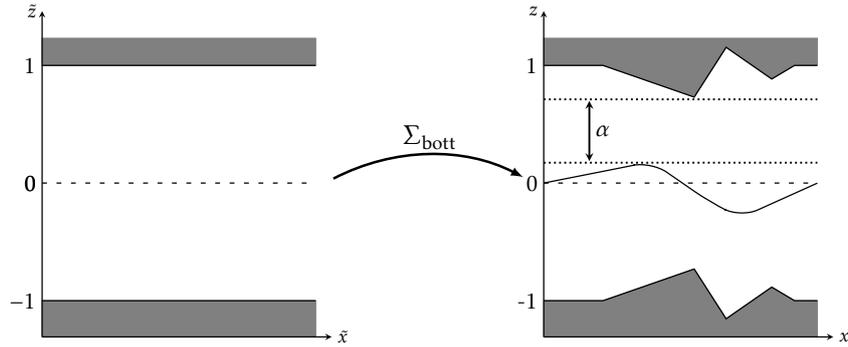


FIGURE 6. Reflected domains and assumption (2.3).

domain  $\Omega$  is contained within this augmented domain, more precisely

$$\exists \alpha > 0, \quad 1 - \beta b - \varepsilon \zeta \geq \alpha \quad \text{on } \mathbb{R}. \quad (2.3)$$

This allows us to set  $\tilde{\Omega} = \Sigma_{\text{bott}}^{-1}(\Omega)$  and to define functions  $\sigma_\mu$  and  $\rho_\mu$  through

$$\forall x \in \mathbb{R}, \quad (\sigma_\mu(x), \rho_\mu(x)) = \Sigma_{\text{bott}}^{-1}(x, \varepsilon \zeta(x)).$$

In other words, we get a transformed domain  $\tilde{\Omega}$  with flat bottom and whose free surface  $\tilde{\Gamma}$  is parametrized by  $(\sigma_\mu(x), \rho_\mu(x))^2$ . Moreover the diffeomorphism  $\Sigma_{\text{bott}}$  between  $\tilde{\Omega}$  and  $\Omega$  maps the bottom (resp. fluid) boundary of  $\tilde{\Omega}$  onto the bottom (resp. fluid) boundary of  $\Omega$  (see Figure 7).

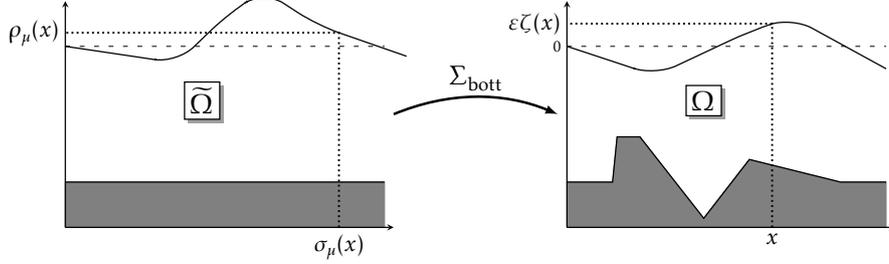


FIGURE 7. Straightening the polygonal topography with a  $\mu$ -conformal diffeomorphism.

*Remark 2.2.* Even if the physical elevation is parametrized by a graph (namely  $z = \varepsilon\zeta(x)$ ), the transformed free surface  $\tilde{\Gamma}$  of  $\tilde{\Omega}$  is not necessarily a graph too. However, for the sake of simplicity, we assume in what follows that  $\tilde{\Gamma}$  may be parametrized as a graph. More precisely, we make the following assumption

$$\exists \delta > 0, \quad \sigma'_\mu > \delta. \quad (2.4)$$

This assumption implies that  $\sigma_\mu$  is a diffeomorphism so that, setting  $\tilde{\zeta}_\mu = \rho_\mu \circ \sigma_\mu^{-1}$ , the transformed surface may be parametrized as

$$\tilde{\Gamma} = \left\{ (\tilde{x}, \tilde{\zeta}_\mu(\tilde{x})) ; \tilde{x} \in \mathbb{R} \right\}.$$

Note that, since  $\Sigma_{\text{bott}}^{-1}$  is smooth near  $\{z = 0\}$  and since this diffeomorphism maps this line onto itself, the previous assumption holds if  $\varepsilon$  is small enough.

**2.2. Transformed Laplace equation with flat bottom.** By carefully choosing the diffeomorphism  $\Sigma_{\text{bott}}$ , we have ensured that the transformed potential  $\tilde{\phi} = \Phi \circ \Sigma_{\text{bott}}$  still solves the (nondimensionalized) Laplace equation on the flat bottom domain  $\tilde{\Omega}$ . This actually follows from the fact that  $\Sigma_{\text{bott}}$  is a  $\mu$ -conformal transformation. More precisely, defining the inner product  $\langle \cdot, \cdot \rangle_\mu$  as

$$\forall u, v \in \mathbb{R}^2, \quad \langle u, v \rangle_\mu = I_\mu u \cdot v,$$

where

$$I_\mu = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix},$$

the Jacobian matrix  $J_{\Sigma_{\text{bott}}}$  of  $\Sigma_{\text{bott}}$  enjoys the following property.

**Lemma 2.3.** *For all vectors  $u, v \in \mathbb{R}^2$ ,*

$$\langle J_{\Sigma_{\text{bott}}}^T u, J_{\Sigma_{\text{bott}}}^T v \rangle_\mu = |\det J_{\Sigma_{\text{bott}}}| \langle u, v \rangle_\mu \quad (2.5)$$

<sup>2</sup>The subscript  $\mu$  on “ $\sigma$ ” and “ $\rho$ ” reminds one that the transformed free surface depends on  $\mu$ . Indeed, by definition, the transformation  $\Sigma_{\text{bott}}$  is dependent on  $\mu$ . Note that, as is clear from their definition,  $\sigma_\mu$  and  $\rho_\mu$  also depend on  $\varepsilon$  but this dependence is omitted.

*Proof.* The Jacobian matrix of the conformal transformation  $\Sigma_\mu$  can be written as

$$J_{\Sigma_\mu} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Differentiating the identity  $\Sigma_{\text{bott}} = T_\mu^{-1} \circ \Sigma_\mu \circ T_\mu$  then gives

$$J_{\Sigma_{\text{bott}}} = \begin{bmatrix} \tilde{a} & -\sqrt{\mu\tilde{b}} \\ \frac{\tilde{b}}{\sqrt{\mu}} & \tilde{a} \end{bmatrix}, \quad (2.6)$$

where  $\tilde{a} = a \circ T_\mu$  and  $\tilde{b} = b \circ T_\mu$ . Equality (2.5) then follows from direct computations using the latter expression for  $J_{\Sigma_{\text{bott}}}$ .  $\square$

Before stating the main proposition of this section let us recall that, with the notation above, the Laplace equation  $\mu\partial_x^2\Phi + \partial_z^2\Phi = 0$  holds in  $\mathcal{D}'(\Omega)$  if and only if

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \int_{\Omega} \langle \nabla_{x,z}\Phi, \nabla_{x,z}\varphi \rangle_\mu = 0.$$

**Proposition 2.4.** *Assume that  $\partial_x\psi \in H^{1/2}(\mathbb{R})$  and that  $\zeta \in W^{1,\infty}(\mathbb{R})$ . Then the transformed potential  $\tilde{\phi} = \Phi \circ \Sigma_{\text{bott}}$  is the solution of*

$$\begin{cases} \mu\partial_{\tilde{x}}^2\tilde{\phi} + \partial_{\tilde{z}}^2\tilde{\phi} = 0 & \text{in } \tilde{\Omega}, \\ \tilde{\phi} = \tilde{\psi}_\mu & \text{on } \{\tilde{z} = \tilde{\zeta}_\mu\}, \\ \partial_{\tilde{z}}\tilde{\phi} = 0 & \text{on } \{\tilde{z} = -1\}, \end{cases} \quad (2.7)$$

where the velocity  $\tilde{\psi}_\mu$  at the transformed free surface is defined as  $\tilde{\psi}_\mu = \psi \circ \sigma_\mu^{-1}$ .

*Proof.* Given the definitions of  $\tilde{\zeta}_\mu$  and  $\tilde{\psi}_\mu$ , it is straightforward that  $\tilde{\phi}$  satisfies the Dirichlet boundary condition on the fluid boundary  $\{\tilde{z} = \tilde{\zeta}_\mu(\tilde{x})\}$ . To prove that  $\tilde{\phi}$  solves (2.7), it therefore remains to show that we have

$$\int_{\tilde{\Omega}} \langle \nabla_{\tilde{x},\tilde{z}}\tilde{\phi}, \nabla_{\tilde{x},\tilde{z}}\tilde{v} \rangle_\mu = 0, \quad (2.8)$$

for any test function  $\tilde{v}$  in the functional space

$$\tilde{V} = \left\{ \tilde{v} \in H^1(\tilde{\Omega}) ; \tilde{v} = 0 \text{ on } \tilde{\Gamma} \right\}.$$

Let us prove first that (2.8) holds for  $\tilde{v}$  of the form  $\tilde{v} = v \circ \Sigma_{\text{bott}}$ , where  $v$  is any function in  $\mathcal{D}(\Omega \cup \{z = -1 + \beta b\})$ . Given such a function, we thus have to check that

$$\int_{\tilde{\Omega}} \langle \nabla_{\tilde{x},\tilde{z}}(\Phi \circ \Sigma_{\text{bott}}), \nabla_{\tilde{x},\tilde{z}}(v \circ \Sigma_{\text{bott}}) \rangle_\mu = 0. \quad (2.9)$$

Using the chain rule and then applying Lemma 2.3 yields

$$\begin{aligned} & \int_{\tilde{\Omega}} \langle \nabla_{\tilde{x},\tilde{z}}(\Phi \circ \Sigma_{\text{bott}}), \nabla_{\tilde{x},\tilde{z}}(v \circ \Sigma_{\text{bott}}) \rangle_\mu \\ &= \int_{\Sigma_{\text{bott}}(\Omega)} \langle \nabla_{x,z}\Phi \circ \Sigma_{\text{bott}}, \nabla_{x,z}v \circ \Sigma_{\text{bott}} \rangle_\mu |\det J_{\Sigma_{\text{bott}}}|. \end{aligned}$$

Using the mapping  $\Sigma_{\text{bott}}$  to perform a change of variable in the last integral, we get

$$\int_{\tilde{\Omega}} \langle \nabla_{\tilde{x},\tilde{z}}(\Phi \circ \Sigma_{\text{bott}}), \nabla_{\tilde{x},\tilde{z}}(v \circ \Sigma_{\text{bott}}) \rangle_\mu = \int_{\Omega} \langle \nabla_{x,z}\Phi, \nabla_{x,z}v \rangle_\mu$$

and thus, since  $\Phi$  solves (2.1),

$$\int_{\tilde{\Omega}} \langle \nabla_{\tilde{x},\tilde{z}}(\Phi \circ \Sigma_{\text{bott}}), \nabla_{\tilde{x},\tilde{z}}(v \circ \Sigma_{\text{bott}}) \rangle_\mu = 0.$$

It is important to note that, due to the behavior of  $\Sigma_{\text{bott}}$  near the prevertices (in  $\tilde{\Omega}$ ) of the polygonal bottom, every function of the form  $\tilde{v} = v \circ \Sigma_{\text{bott}}$  is not necessarily in  $H^1(\tilde{\Omega})$ . Nonetheless, by construction of  $\Sigma_{\text{bott}}$ , we know that  $v \circ \Sigma_{\text{bott}}$  is smooth on condition that  $v$  vanishes near the vertices of the polygonal bottom. Using such test function in (2.9), we deduce that (2.8) holds for all  $\tilde{v} \in \mathcal{D}(-1 \leq \tilde{z} < 1 + \tilde{\zeta}_\mu)$  supported away from the prevertices. Since this last set is dense in  $\tilde{V}$  (see [8, Lemma 2.1.2]) we finally conclude that (2.8) holds for any test function  $\tilde{v} \in \tilde{V}$ .  $\square$

**2.3. Transformed problem on the flat strip.** We have seen in Proposition 2.4 that the Laplace problem with polygonal bottom (2.1) can be reduced to the same equation posed in a transformed fluid domain  $\tilde{\Omega}$  with flat bottom. Therefore, to reduce the problem to a boundary value problem on the flat strip, it simply remains to straighten the transformed fluid boundary  $\{\tilde{z} = \tilde{\zeta}_\mu(\tilde{x})\}$  of  $\tilde{\Omega}$ . This can be done using the classical straightening diffeomorphism defined in section 1.3. Following this classical approach we define the diffeomorphism  $\Sigma_{\text{surf}}$  mapping  $\mathcal{S}$  onto the flat bottom domain  $\tilde{\Omega}$  as

$$\begin{aligned} \Sigma_{\text{surf}} : \mathcal{S} &\longrightarrow \tilde{\Omega} \\ (\tilde{x}, \tilde{z}) &\longmapsto (\tilde{x}, \tilde{\zeta}_\mu(\tilde{x}) + \tilde{z}(1 + \tilde{\zeta}_\mu(\tilde{x}))). \end{aligned}$$

Then we know (see *e.g.* [13, Proposition 2.26]) that the Laplace problem on  $\tilde{\Omega}$ , and thus the Laplace problem on  $\Omega$ , are equivalent to the following elliptic equation on  $\mathcal{S}$

$$\begin{cases} \nabla_{\tilde{x}, \tilde{z}} \cdot \tilde{P} \nabla_{\tilde{x}, \tilde{z}} \phi = 0 & \text{in } \mathcal{S}, \\ \phi = \tilde{\psi}_\mu & \text{on } \{\tilde{z} = 0\}, \\ \partial_{\mathbf{n}} \phi = 0 & \text{on } \{\tilde{z} = -1\}, \end{cases} \quad (2.10)$$

with  $\phi = \tilde{\phi} \circ \Sigma_{\text{surf}}$  and where the matrix  $\tilde{P}$  is given by

$$\tilde{P} = \begin{bmatrix} \mu(1 + \tilde{\zeta}_\mu) & -\mu(\tilde{z} + 1) \partial_{\tilde{x}} \tilde{\zeta}_\mu \\ -\mu(\tilde{z} + 1) \partial_{\tilde{x}} \tilde{\zeta}_\mu & \frac{1 + \mu(\tilde{z} + 1)^2 (\partial_{\tilde{x}} \tilde{\zeta}_\mu)^2}{1 + \tilde{\zeta}_\mu} \end{bmatrix}, \quad (2.11)$$

and  $\partial_{\mathbf{n}} \phi|_{\tilde{z}=-1} = \mathbf{e}_{\tilde{z}} \cdot \tilde{P} \nabla_{\tilde{x}, \tilde{z}} \phi|_{\tilde{z}=-1}$  denotes the upward conormal derivative.

### 3. SHALLOW-WATER ANALYSIS OF THE DIRICHLET-NEUMANN OPERATOR

In this section we concentrate on the asymptotic analysis of the Dirichlet-Neumann operator in shallow water regime ( $\mu \ll 1$ ). In the light of the previous section, the Laplace problem (2.1) with polygonal bottom boundary can be reduced to the elliptic problem (2.10) on the flat strip. This can then be used to build an asymptotic expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  with respect to  $\mu$  following the usual method for smooth topographies (see for instance [1] or [2, 14, 4] where the method has been used to derive long-wave models, see also [19] in which this change of variable approach is applied to address the analyticity of the Dirichlet-Neumann operator). As outlined in the introduction, this method consists of the following steps:

- (1) Express the Dirichlet-Neumann operator in terms of the solution  $\phi$  to the problem (2.10) on the flat strip;
- (2) Approximate the transformed potential  $\phi$  on  $\mathcal{S}$  using a BKW procedure;
- (3) Plug this approximate solution back into the expression of (1) to compute an approximation of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$ .

The first step is undertaken in section 3.1, while the second and third steps are detailed in section 3.2.

**3.1. The Dirichlet-Neumann operator on the flat strip.** As seen in the introduction, the Dirichlet-Neumann operator is given in terms of the velocity potential as

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \partial_z \Phi|_{z=\varepsilon\zeta} - \mu\varepsilon\partial_x\zeta\partial_x\Phi|_{z=\varepsilon\zeta} \quad (3.1)$$

$$= \langle \nabla_{x,z}\Phi|_{z=\varepsilon\zeta}, \mathbf{n} \rangle_\mu, \quad (3.2)$$

where  $\mathbf{n}$  is the (non-unit) normal vector to the free surface defined as  $\mathbf{n} = [-\varepsilon\partial_x\zeta, 1]^T$ . Using the  $\mu$ -conformal transformation  $\Sigma_{\text{bott}}$ , this operator can be similarly expressed in terms of the transformed velocity potential  $\tilde{\phi} = \Phi \circ \Sigma_{\text{bott}}$  on the flat bottom domain  $\tilde{\Omega}$ .

**Proposition 3.1.** *Assume that  $\partial_x\psi \in H^{1/2}(\mathbb{R})$  and that  $\zeta \in H^{t_0+1}(\mathbb{R})$  for some  $t_0 > 1/2$ . Then, the Dirichlet-Neumann operator can be written as*

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \left\langle \nabla_{\tilde{x},\tilde{z}}\tilde{\phi} \circ (\sigma_\mu, \rho_\mu), \tilde{\mathbf{n}} \right\rangle_\mu, \quad (3.3)$$

where  $\tilde{\mathbf{n}}$  is the (non-unit) normal vector to the transformed free surface  $\tilde{\Gamma}$  defined as  $\tilde{\mathbf{n}} = [-\rho'_\mu, \sigma'_\mu]^T$ .

*Proof.* The proof relies on the fact that the transformation  $\Sigma_{\text{bott}}$  is  $\mu$ -conformal. Indeed, from the chain rule, we have

$$(J_{\Sigma_{\text{bott}}}^T \circ (\sigma_\mu, \rho_\mu)) \nabla_{x,z}\Phi|_{z=\varepsilon\zeta} = \nabla_{\tilde{x},\tilde{z}}\tilde{\phi} \circ (\sigma_\mu, \rho_\mu).$$

Using Lemma 2.3 and the above equality in the definition (3.1) of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]$ , we obtain

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \left\langle \nabla_{\tilde{x},\tilde{z}}\tilde{\phi} \circ (\sigma_\mu, \rho_\mu), \left( \frac{J_{\Sigma_{\text{bott}}}^T}{|\det J_{\Sigma_{\text{bott}}}|} \circ (\sigma_\mu, \rho_\mu) \right) \mathbf{n} \right\rangle_\mu,$$

and so it remains to show that

$$\left( \frac{J_{\Sigma_{\text{bott}}}^T}{|\det J_{\Sigma_{\text{bott}}}|} \circ (\sigma_\mu, \rho_\mu) \right) \mathbf{n} = \tilde{\mathbf{n}}. \quad (3.4)$$

To do this, we first note that a tangent vector to the transformed fluid boundary  $\tilde{\Gamma}$  is given by  $\tilde{\mathbf{t}} = J_{\Sigma_{\text{bott}}}^{-1} \circ (\sigma_\mu, \rho_\mu) \mathbf{t}$ , where  $\mathbf{t} = [1, \varepsilon\partial_x\zeta]^T$  is a tangent vector to the (physical) free surface  $\{z = \varepsilon\zeta(x)\}$  (see Figure 8). Now, using the expression (2.6)

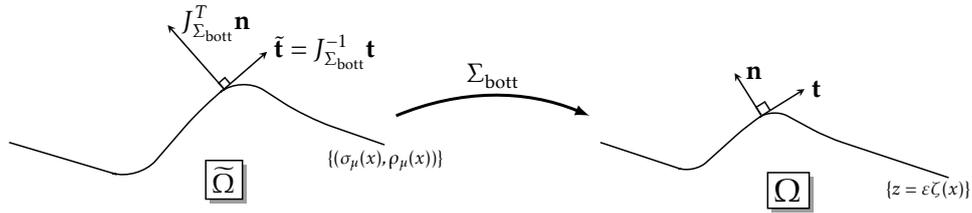


FIGURE 8. Tangent and normal vectors to the free surface and the transformed free surface.

for  $J_{\Sigma_{\text{bott}}}^T$ , we get

$$\left( \frac{J_{\Sigma_{\text{bott}}}^T}{|\det J_{\Sigma_{\text{bott}}}|} \circ (\sigma_\mu, \rho_\mu) \right) I_\mu^{-1} \mathbf{t} = I_\mu^{-1} \tilde{\mathbf{t}}.$$

Equality (3.4) is then obtained from this last relation by noting that  $\tilde{\mathbf{t}} = [\sigma'_\mu, \rho'_\mu]^T$  and using again the expression (2.6) for  $J_{\Sigma_{\text{bott}}}^T$ .  $\square$

*Remark 3.2.* A Dirichlet-Neumann operator  $\tilde{\mathcal{G}}_\mu[\tilde{\zeta}]\tilde{\psi}_\mu$  associated with the transformed Laplace problem with flat bottom (2.7) can be defined as

$$\tilde{\mathcal{G}}_\mu[\tilde{\zeta}]\tilde{\psi}_\mu = \partial_{\tilde{z}}\tilde{\phi}|_{\tilde{z}=\tilde{\zeta}_\mu} - \mu\partial_{\tilde{x}}\tilde{\zeta}_\mu\partial_{\tilde{x}}\tilde{\phi}|_{\tilde{z}=\tilde{\zeta}_\mu}. \quad (3.5)$$

It is worth noticing that both quantities  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  and  $\tilde{\mathcal{G}}_\mu[\tilde{\zeta}]\tilde{\psi}_\mu$  do not coincide since the  $\mu$ -conformal diffeomorphism  $\Sigma_{\text{bott}}$  acts on the horizontal coordinate when straightening the bottom. More precisely, we can rewrite the above result as

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \sigma'_\mu \left( \tilde{\mathcal{G}}_\mu[\tilde{\zeta}]\tilde{\psi}_\mu \right) \circ \sigma_\mu, \quad (3.6)$$

where we recall that  $\sigma_\mu$  is related to  $\Sigma_{\text{bott}}$  through

$$\forall x \in \mathbb{R}, \quad (\sigma_\mu(x), \rho_\mu(x)) = \Sigma_{\text{bott}}^{-1}(x, \varepsilon\zeta(x))$$

so that  $\sigma_\mu$  can be considered as the horizontal deformation of the free surface  $\{z = \varepsilon\zeta\}$  due to the straightening of the bottom.

Since we have transformed the Laplace problem with flat bottom (2.7) into the elliptic problem (2.10) on the flat strip using the trivial diffeomorphism  $\Sigma_{\text{surf}}$ , we know that the Dirichlet-Neumann operator  $\tilde{\mathcal{G}}_\mu[\tilde{\zeta}]\tilde{\psi}_\mu$  coincides with the Dirichlet-Neumann operator that comes from this straightened elliptic problem on the flat strip (see *e.g.* [13, Remark 3.7]). Combining this expression with the previous relationship (3.6) between  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  and  $\tilde{\mathcal{G}}_\mu[\tilde{\zeta}]\tilde{\psi}_\mu$  yields the desired expression of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  in terms of the solution  $\phi$  to the elliptic problem (2.10) on  $\mathcal{S}$ .

**Proposition 3.3.** *Assume that  $\partial_x\psi \in H^{1/2}(\mathbb{R})$  and that  $\zeta \in H^{t_0+1}(\mathbb{R})$  for some  $t_0 > 1/2$ . Then, the Dirichlet-Neumann operator can be written in terms of the transformed potential on the flat strip as*

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \sigma'_\mu \mathbf{e}_z \cdot \tilde{P}\nabla_{\tilde{x}, \tilde{z}}\phi \circ (\sigma_\mu, 0). \quad (3.7)$$

**3.2. Asymptotic analysis of the Dirichlet-Neumann operator.** Let us summarize the situation so far. Owing to an *ad hoc* straightening of the bottom, we have first reduced the Laplace problem with polygonal bottom (2.1) to the same Laplace equation on the flat bottom domain  $\tilde{\Omega}$  with transformed Dirichlet data  $\tilde{\psi}_\mu$  at the surface. Then we have seen that this transformed Laplace problem can in turn be reduced to a variable coefficients elliptic problem on the flat strip, namely equation (2.10), and that the Dirichlet-Neumann operator can be expressed in terms of the solution  $\phi$  of this elliptic problem. Our strategy here is to use this expression to derive an asymptotic expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  in shallow water regime. To do so, we first construct an approximate solution  $\phi_{\text{app}}$  to (2.10) and then replace the transformed potential  $\phi$  in (3.7) with this approximate potential.

**3.2.1. Asymptotic expansion of the transformed potential.** We look for an asymptotic expansion of the transformed potential  $\phi$  of the form

$$\phi_{\text{app}} = \phi_0 + \mu\phi_1. \quad (3.8)$$

This expansion is constructed as an approximate solution of the transformed elliptic problem on the flat strip, that is such that

$$\begin{cases} \nabla_{\tilde{x}, \tilde{z}} \cdot \tilde{P}\nabla_{\tilde{x}, \tilde{z}}\phi_{\text{app}} = O(\mu^2) & \text{in } \mathcal{S}, \\ \phi = \tilde{\psi}_\mu & \text{on } \{\tilde{z} = 0\}, \\ \partial_{\mathbf{n}}\tilde{\phi} = 0 & \text{on } \{\tilde{z} = -1\}. \end{cases} \quad (3.9)$$

To find this approximate solution, we plug the above expression for  $\phi_{\text{app}}$  into the elliptic operator from (3.9), expand the resulting expression in powers of  $\mu$  and

then choose  $\phi_0$  and  $\phi_1$  so as to cancel the leading order terms. Mimicking the computations of [13, Lemma 3.42], the resulting approximate solution is given by

$$\phi_0(\tilde{x}, \tilde{z}) = \tilde{\psi}_\mu(\tilde{x}), \quad (3.10)$$

$$\phi_1(\tilde{x}, \tilde{z}) = -\left(\frac{\tilde{z}^2}{2} + \tilde{z}\right)(1 + \tilde{\zeta}_\mu(\tilde{x}))^2 \partial_{\tilde{x}}^2 \tilde{\psi}_\mu(\tilde{x}). \quad (3.11)$$

*Remark 3.4.* It is noteworthy to note that, in the expansion (3.8), both functions  $\phi_0$  and  $\phi_1$  also depend on  $\mu$ .

*3.2.2. Asymptotic expansion of the Dirichlet-Neumann operator.* Given the previous approximate (transformed) potential  $\phi_{\text{app}}$ , we may now compute a formal expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$ . More precisely, using  $\phi_{\text{app}}$  in the expression of the Dirichlet-Neumann operator on the flat strip given in Proposition 3.3, we get

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \mathcal{G}_{\text{app}}\psi + O(\mu^2), \quad (3.12)$$

where  $\mathcal{G}_{\text{app}}$  is defined, according to (3.7), as

$$\mathcal{G}_{\text{app}}\psi = \sigma'_\mu \mathbf{e}_z \cdot \tilde{P}\nabla_{\tilde{x}, \tilde{z}}\phi_{\text{app}} \circ (\sigma_\mu, 0). \quad (3.13)$$

Computing the right hand side with the help of the explicit expressions (3.10)-(3.11) for the functions  $\phi_0$  and  $\phi_1$  and then using that, from their definitions,  $\tilde{\psi}_\mu \circ \sigma_\mu = \psi$  and  $\tilde{\zeta}_\mu \circ \sigma_\mu = \rho_\mu$ , we find

$$\mathcal{G}_{\text{app}}\psi = -\mu\partial_x \left( \frac{1 + \rho_\mu}{\sigma'_\mu} \partial_x \psi \right) + O(\mu^2). \quad (3.14)$$

Due to the dependence on  $\mu$  of the diffeomorphism  $\Sigma_{\text{surf}}$ , we need to make additional assumptions to establish an error estimate for the above approximate Dirichlet-Neumann operator. More precisely, we assume that the transformed free surface satisfies the following conditions uniformly with respect to  $\mu$ :

- (A1) There exists  $\tilde{h}_{\min} > 0$ , independent on  $\mu$ , such that  $1 + \rho_\mu \geq \tilde{h}_{\min}$ .
- (A2) There exists  $\tilde{r} > 0$ , independent on  $\mu$ , such that  $|\rho_\mu|_{W^{1,\infty}} \leq \tilde{r}$ .
- (A3) There exists  $\delta > 0$ , independent on  $\mu$ , such that  $\sigma'_\mu > \delta$ .

As will be made clear in the proof of the following proposition, these extra assumptions give control on both the coercivity of the elliptic operator in (3.9) and on the  $O(\mu^2)$  right hand side of this approximate equation.

**Proposition 3.5.** *Let  $s \in \mathbb{N}^*$ ,  $\zeta \in H^{s+5/2}(\mathbb{R})$  and  $\partial_x \psi \in H^{s+7/2}(\mathbb{R})$  be such that (A1)-(A3) are satisfied and that  $\sigma'_\mu \in W^{s,\infty}(\mathbb{R})$ . Then the following estimate on the remainder holds*

$$\left| \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \mu\partial_x \left( \frac{1 + \rho_\mu}{\sigma'_\mu} \partial_x \psi \right) \right|_{H^s} \leq \mu^2 C_0, \quad (3.15)$$

where  $C_0$  is a constant of the form

$$C_0 = C\left(\frac{1}{\delta}, \frac{1}{\tilde{h}_{\min}}, \tilde{r}, \left| \partial_{\tilde{x}} \tilde{\psi}_\mu \right|_{H^{s+7/2}}, \left| \tilde{\zeta}_\mu \right|_{H^{s+5/2}}, \left| \sigma'_\mu \right|_{W^{s,\infty}}\right),$$

and  $C$  is a nondecreasing function of its arguments.

*Proof.* From the definition of  $\mathcal{G}_{\text{app}}\psi$  and Proposition 3.3, we have

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi - \mathcal{G}_{\text{app}}\psi = \sigma'_\mu \mathbf{e}_z \cdot \tilde{P}\nabla_{\tilde{x}, \tilde{z}}(\phi - \phi_{\text{app}}) \circ (\sigma_\mu, 0).$$

Now, from the construction of the approximate potential  $\phi_{\text{app}}$ , one can check that  $u = \phi - \phi_{\text{app}}$  solves

$$\begin{cases} \nabla_{\tilde{x}, \tilde{z}} \cdot \tilde{P}\nabla_{\tilde{x}, \tilde{z}} u = \mu^2 R_\mu & \text{in } \mathcal{S}, \\ u = 0 & \text{on } \{\tilde{z} = 0\}, \\ \partial_{\mathbf{n}} u = 0 & \text{on } \{\tilde{z} = -1\}, \end{cases} \quad (3.16)$$

with  $R_\mu$  satisfying

$$|R_\mu|_{L^2(-1,0; H^{s+1/2}(\mathbb{R}))} \leq C \left( \left| \partial_{\tilde{x}} \tilde{\psi}_\mu \right|_{H^{s+7/2}}, \left| \tilde{\zeta}_\mu \right|_{H^{s+5/2}} \right). \quad (3.17)$$

Then, since

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi - \mathcal{G}_{\text{app}}\psi = \sigma'_\mu \mathbf{e}_{\tilde{z}} \cdot \tilde{P} \nabla_{\tilde{x}, \tilde{z}} u \circ (\sigma_\mu, 0), \quad (3.18)$$

one may feel inclined to deduce a control of the latter from (3.16) by resorting to elliptic estimates together with a trace inequality. Actually, since the coercivity constant of  $\tilde{P}$  depends on  $\mu$  (it is of order  $O(\mu)$ ), a straightforward application of such estimates does not directly yield (3.15). To face this difficulty, the idea is to consider the contribution of the shallowness parameter in the elliptic problem (3.16). More precisely, following for instance [1], the elliptic operator from (3.16) can be written as

$$\nabla_{\tilde{x}, \tilde{z}} \cdot \tilde{P} \nabla_{\tilde{x}, \tilde{z}} = \nabla_{\tilde{x}, \tilde{z}}^\mu \cdot \tilde{P}_\mu \nabla_{\tilde{x}, \tilde{z}}^\mu,$$

where the twisted gradient operator  $\nabla_{\tilde{x}, \tilde{z}}^\mu$  is defined as  $\nabla_{\tilde{x}, \tilde{z}}^\mu = [\sqrt{\mu} \partial_{\tilde{x}}, \partial_{\tilde{z}}]^T$  and with

$$\tilde{P}_\mu = \begin{bmatrix} 1 + \tilde{\zeta}_\mu & -\sqrt{\mu}(\tilde{z} + 1) \partial_{\tilde{x}} \tilde{\zeta}_\mu \\ -\sqrt{\mu}(\tilde{z} + 1) \partial_{\tilde{x}} \tilde{\zeta}_\mu & \frac{1 + \mu(\tilde{z} + 1)^2 (\partial_{\tilde{x}} \tilde{\zeta}_\mu)^2}{1 + \tilde{\zeta}_\mu} \end{bmatrix}.$$

The advantage is that the matrix  $\tilde{P}_\mu$  at the core of this twisted formulation satisfies the following coercivity estimate

$$\forall \theta \in \mathbb{R}^2, \quad k_\mu \tilde{P}_\mu \theta \cdot \theta \geq |\theta|^2,$$

with a constant  $k_\mu$  of the form

$$k_\mu = 2 \left( 1 + \left| \tilde{\zeta}_\mu \right|_{L^\infty} \right) + \frac{4}{\tilde{h}_{\min}} \left( 1 + \mu \left| \partial_{\tilde{x}} \tilde{\zeta}_\mu \right|_{L^\infty}^2 \right).$$

Now, since  $\mathbf{e}_{\tilde{z}} \cdot \tilde{P} \nabla_{\tilde{x}, \tilde{z}} u = \mathbf{e}_{\tilde{z}} \cdot \tilde{P}_\mu \nabla_{\tilde{x}, \tilde{z}}^\mu u$ , we can apply the aforementioned elliptic estimate and trace inequality to obtain from (3.17)

$$\left| \mathbf{e}_{\tilde{z}} \cdot \tilde{P} \nabla_{\tilde{x}, \tilde{z}} u \Big|_{\tilde{z}=0} \right|_{H^s} \leq \mu^2 C \left( k_\mu, \left| \partial_{\tilde{x}} \tilde{\psi}_\mu \right|_{H^{s+7/2}}, \left| \tilde{\zeta}_\mu \right|_{H^{s+5/2}} \right).$$

From (3.18), we deduce that

$$\left| \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi - \mathcal{G}_{\text{app}}\psi \right|_{H^s} \leq \mu^2 C \left( \frac{1}{\delta}, \left| \sigma'_\mu \right|_{W^{s, \infty}}, k_\mu, \left| \partial_{\tilde{x}} \tilde{\psi}_\mu \right|_{H^{s+7/2}}, \left| \tilde{\zeta}_\mu \right|_{H^{s+5/2}} \right). \quad (3.19)$$

To conclude, we first note that (3.14) can be written as

$$\mathcal{G}_{\text{app}}\psi = -\mu \partial_x \left( \frac{1 + \rho_\mu}{\sigma'_\mu} \partial_x \psi \right) - \mu^2 \sigma'_\mu r_\mu \circ \sigma_\mu,$$

where  $r_\mu = (\partial_{\tilde{x}} \tilde{\zeta}_\mu)^2 (1 + \tilde{\zeta}_\mu) \partial_{\tilde{x}}^2 \tilde{\psi}_\mu$ . From here, the desired estimate follows from (3.19) by applying the triangle inequality.  $\square$

*Comment.* The main drawback of the estimate furnished by the previous proposition is that the quantities that appear in the constant  $C_0$  in (3.15) depend on the shallowness parameter. Actually, the  $\mu$ -dependence is mainly due to the contribution of the parametrization  $(\sigma_\mu, \rho_\mu)$  of the transformed free surface in  $\tilde{\Omega}$ . Therefore, to improve this estimate, one should more carefully focus on how the parametrization  $(\sigma_\mu, \rho_\mu)$  depends on  $\mu$ . To do so, recall first that as we mentioned in section 2.1.1, the construction of the conformal mapping  $\Sigma_\mu$  associated with the straightening diffeomorphism  $\Sigma_{\text{bott}}$  hinges on the resolution of a Schwarz-Christoffel parameter problem. Identifying  $\mathbb{R}^2$  with the complex plane, denoting by  $a_1, a_2, \dots, a_n \in \mathbb{C}$  the vertices of the (rescaled) polygonal bottom and  $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$  its interior angles (see Figure 9), this parameter problem actually consists in finding prevertices

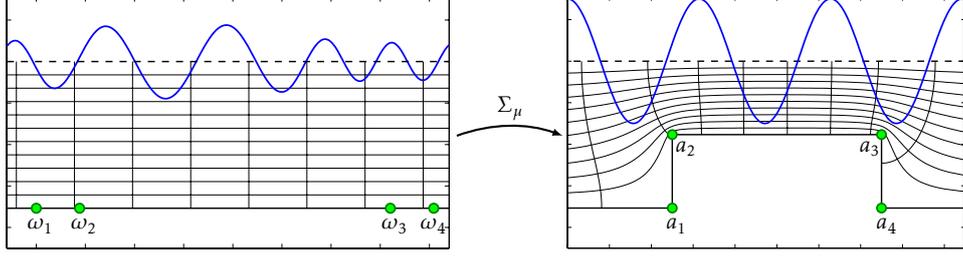


FIGURE 9. Solution of the Schwarz-Christoffel parameter problem for a bottom with a rectangular hump.

$\omega_1, \omega_2, \dots, \omega_n \in \{z = -\sqrt{\mu}\}$  and a constant  $\Lambda \in \mathbb{C}$  such that the desired conformal mapping  $\Sigma_\mu : \mathbb{C} \rightarrow \mathbb{C}$  satisfies

$$\frac{d\Sigma_\mu}{d\omega}(\omega) = \Lambda \prod_{k=1}^n \left( \exp\left(\frac{\pi\omega_k}{\sqrt{\mu}}\right) - \exp\left(\frac{\pi\omega}{\sqrt{\mu}}\right) \right)^{\alpha_k - 1}.$$

Hence, the parametrization  $(\sigma_\mu, \rho_\mu)$  of the transformed free surface  $\tilde{\Gamma}$  satisfies the ordinary differential equation

$$\sigma'_\mu + i\sqrt{\mu}\rho'_\mu = \Lambda^{-1} (1 + i\varepsilon\sqrt{\mu}\zeta') \prod_{k=1}^n \left( \exp\left(\frac{\pi\omega_k}{\sqrt{\mu}}\right) - \exp\left(\frac{\pi\sigma_\mu}{\sqrt{\mu}}\right) \exp(i\pi\rho_\mu) \right)^{1-\alpha_k}.$$

In the case of a step this differential equation reads (see Example 2.1)

$$\sigma'_\mu + i\sqrt{\mu}\rho'_\mu = (1 + i\varepsilon\sqrt{\mu}\zeta') \left( \frac{1 + \exp\left(\frac{\pi\sigma_\mu}{\sqrt{\mu}}\right) \exp(i\pi\rho_\mu)}{1 + h_0^2 \exp\left(\frac{\pi\sigma_\mu}{\sqrt{\mu}}\right) \exp(i\pi\rho_\mu)} \right)^{1/2}.$$

A possible (but far from obvious), way to improve estimate (3.15) might be to perform an asymptotic analysis of the solution of the previous differential equation as  $\mu \rightarrow 0$ .

#### 4. A SHALLOW-WATER MODEL FOR POLYGONAL BOTTOMS

In this last section, we focus on the study of shallow water waves over the polygonal bottom. Owing to the asymptotic analysis of the Dirichlet-Neumann operator conducted in the previous section, we derive a shallow water model that approximates, at order  $O(\mu)$ , the solutions of the water waves equations

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\partial_x \psi|^2 - \varepsilon \mu \frac{(\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi + \varepsilon \partial_x \zeta \partial_x \psi)^2}{2(1 + \varepsilon^2 \mu |\partial_x \zeta|^2)} = 0. \end{cases} \quad (4.1)$$

In the present case, since we consider a rough topography, we choose to work with variables located at the surface, that is away from the singularities of the bottom. Therefore the shallow water model is formulated in terms of the surface elevation  $\zeta$  and the horizontal velocity at the surface  $v_s = (\partial_x \Phi)|_{z=\varepsilon\zeta}$ . From the definition of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi$ , the vertical component of the velocity, on the other hand, can be written

$$(\partial_z \Phi)|_{z=\varepsilon\zeta} = \mu \frac{\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b] \psi + \varepsilon \partial_x \psi \partial_x \zeta}{1 + \mu \varepsilon^2 (\partial_x \zeta)^2}.$$

Since  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  gives first contributions at  $O(\mu)$ , we deduce that  $(\partial_z\Phi)|_{z=\varepsilon\zeta}$  is of size  $O(\mu)$ . Since, by definition of  $\psi = \Phi|_{z=\varepsilon\zeta}$ , we have  $\partial_x\psi = v_s + \varepsilon\partial_x\zeta(\partial_z\Phi)|_{z=\varepsilon\zeta}$ , we finally get  $\partial_x\psi = v_s + O(\mu)$ . Plugging the latter in (3.14) yields

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\partial_x\left(\frac{1+\rho_\mu}{\sigma'_\mu}v_s\right) + O(\mu), \quad (4.2)$$

where we recall that  $(\sigma_\mu, \rho_\mu)$  parametrizes the transformed free surface in  $\tilde{\Omega}$  and is defined from the straightening diffeomorphism  $\Sigma_{\text{bott}}$  by

$$\forall x \in \mathbb{R}, \quad (\sigma_\mu(x), \rho_\mu(x)) = \Sigma_{\text{bott}}^{-1}(x, \varepsilon\zeta(x)). \quad (4.3)$$

Defining the transformed (variable) free surface coefficient  $M_\mu = M_\mu[\varepsilon\zeta, \beta b]$  as

$$M_\mu = \frac{1+\rho_\mu}{\sigma'_\mu}, \quad (4.4)$$

then substituting expansion (4.2) into the first equation of (4.1), we get the following approximate evolution equation for the elevation

$$\partial_t\zeta + \partial_x(M_\mu v_s) = O(\mu).$$

Then, differentiating the second equation of (4.1) with respect to  $x$  and using both  $\partial_x\psi = v_s + O(\mu)$  and  $\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = O(1)$ , we are left with the following shallow water model with precision  $O(\mu)$

$$\begin{cases} \partial_t\zeta + \partial_x(M_\mu v_s) = 0, \\ \partial_t v_s + \partial_x\zeta + \varepsilon v_s \partial_x v_s = 0. \end{cases} \quad (4.5)$$

*Remark 4.1* (Flat bottom). In case the bottom is flat, the straightening diffeomorphism  $\Sigma_{\text{bott}}$  reduces to identity so that  $\sigma_\mu(x) = x$  and  $\rho_\mu(x) = \varepsilon\zeta(x)$ . Consequently  $M_\mu$  coincides with the water depth variable  $h = 1 + \varepsilon\zeta$  and we recover the classical Saint-Venant system

$$\begin{cases} \partial_t\zeta + \partial_x(hv_s) = 0, \\ \partial_t v_s + \partial_x\zeta + \varepsilon v_s \partial_x v_s = 0. \end{cases} \quad (4.6)$$

## 5. CONCLUSION

The main limitation of the shallow water model (4.5) is that the transformed free surface coefficient  $M_\mu$  depends on the variables  $(\sigma_\mu, \rho_\mu)$ . This is problematic since, due to the analytic intractability of the underlying Schwarz-Christoffel parameter problem, we do not have analytical expression for the mapping  $\Sigma_{\text{bott}}^{-1}$ . Therefore  $M_\mu = \frac{1+\rho_\mu}{\sigma'_\mu}$  cannot be explicitly written in terms of the variables  $\zeta$  and  $v_s$ . As said above, a possible improvement could be provided by an asymptotic analysis of both functions  $\sigma_\mu$  and  $\rho_\mu$ . Unfortunately, we have been unable thus far to find explicit asymptotic expansions for these coefficients, even in the simple case of a step where the expression of the straightening diffeomorphism  $\Sigma_{\text{bott}}$  is known. It is notable that, for weakly nonlinear waves  $\varepsilon \sim \mu \ll 1$ , Nachbin [16] proposes an interesting approximation of a coefficient similar to  $M_\mu$  by a time independent coefficient<sup>3</sup>.

Given these limitations, we develop in [3] a different approach to study shallow water flows over rough bathymetries. The starting point of this approach is the shape analyticity of the Dirichlet-Neumann operator. More precisely, we know how to find explicit expressions for the shape derivatives of the Dirichlet-Neumann operator around  $\zeta = 0$  and  $b = 0$ . We will see that these expressions only involve

<sup>3</sup>The idea at the basis of this approximation is to evaluate  $\Sigma_{\text{bott}}^{-1}$  in (4.3) not on the free surface points  $(x, \varepsilon\zeta(x))$  but on the undisturbed surface points  $(x, 0)$ . Setting  $(\sigma_0(x), 0) = \Sigma_{\text{bott}}^{-1}(x, 0)$ , this amounts to replace  $M_\mu$  in (4.5) by the time independent coefficient  $M_0 = \frac{1}{\sigma'_0}$ .

infinitely smooth contributions of the bottom. Therefore, we propose a more formal approach which consists in first approximating the Dirichlet-Neumann operator by its Taylor expansion around  $\zeta = 0$  and  $b = 0$  and then studying the shallow water asymptotic of the latter expansion.

Let us conclude by remarking that, even if the computation of the transformed free surface coefficient  $M_\mu$  is not obvious (it requires to evaluate the inverse of the Schwarz-Christoffel mapping  $\Sigma_\mu$  at the free surface), it can be achieved numerically by using for instance the Schwarz-Christoffel Toolbox of Driscoll and Trefethen [7, Appendix], thus making possible the development of a numerical method for (4.5). More details about the numerical method for (4.5) are given in the appendix of [3], in which we use equations (4.5) as a reference model to assess the performance of the formal approach developed therein. As an illustration, Figure 10 shows the time history of the surface elevation computed by (4.6) in the particular case of a rectangular bottom.

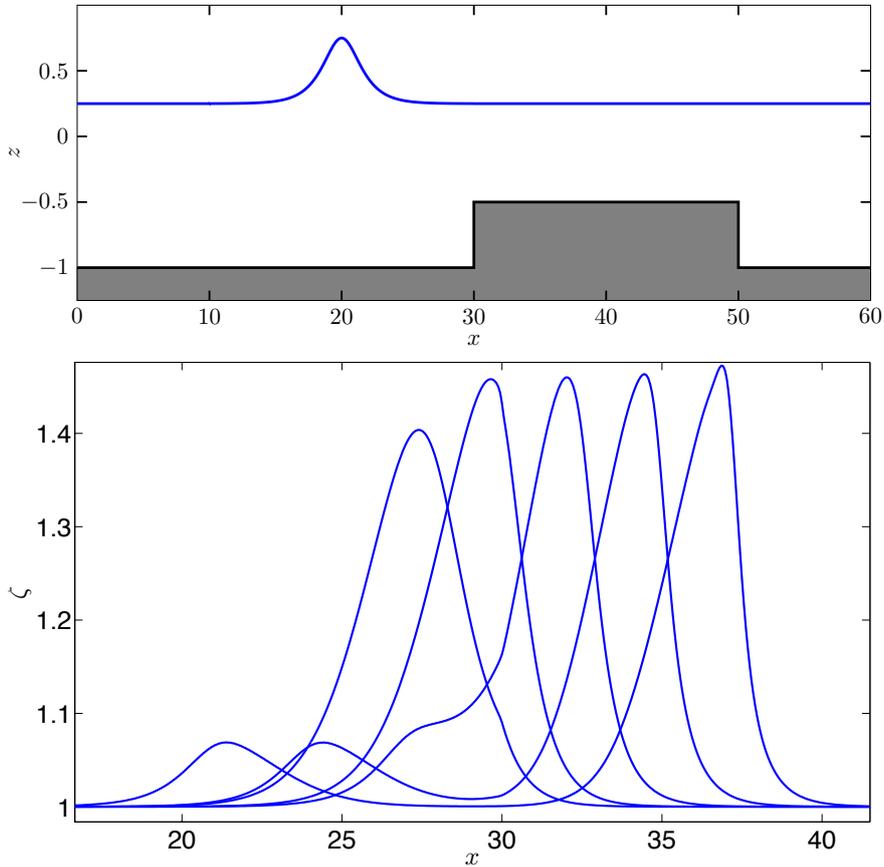


FIGURE 10. Wave passing over a rectangular hump. Top: topography and initial condition. Bottom: time series of surface elevation.

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UNIVERSITÉ MONTPELLIER 2, INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER, CC051, PLACE EUGÈNE BATAILLON, F-34095 MONTPELLIER.

*E-mail address:* mathieu.cathala@math.univ-montp2.fr