

Supplemental Material to Spin Drag of a Fermi Gas in a Harmonic Trap

I. DERIVATION OF THE TRANSPORT EQUATION FROM BOLTZMANN'S EQUATION

We consider an ensemble of spin 1/2 fermions of mass m . In the dilute limit, the statistical properties of the system are fully captured by the single-particle phase-space densities $f_s(\mathbf{r}, \mathbf{p}, t)$ of the spin species $s = \pm$. In the presence of a trapping potential V , the evolution of f_s is given by Boltzmann's equation

$$\partial_t f_s + \frac{\mathbf{p}}{m} \cdot \partial_{\mathbf{r}} f_s + \mathbf{F} \cdot \partial_{\mathbf{p}} f_s = I_{\text{coll}}[f_s, f_{-s}], \quad (1)$$

where $\mathbf{F} = -\partial_{\mathbf{r}} V$ is the trapping force and I_{coll} is the collision operator. For low-temperature fermions, collisions between same-spin particles are suppressed and at low phase-space densities, the collision operator is given by

$$I_{\text{coll}}[f_s, f_{-s}](\mathbf{r}, \mathbf{p}_1) = \int d^3 \mathbf{p}_2 d^2 \Omega' v_{\text{rel}} \frac{d\sigma}{d\Omega'} (f_{s,3} f_{-s,4} - f_{s,1} f_{-s,2}), \quad (2)$$

where \mathbf{p}_1 and \mathbf{p}_2 (\mathbf{p}_3 and \mathbf{p}_4) are ingoing (outgoing) momenta satisfying energy and momentum conservation, $v_{\text{rel}} = |\mathbf{p}_2 - \mathbf{p}_1|/m$ is the relative velocity, $d\sigma/d\Omega'$ is the differential cross-section towards the solid angle Ω' and $f_{s,i}$ stands for $f_s(\mathbf{r}, \mathbf{p}_i)$.

When the populations of the two spin states are equal, the equilibrium solution of Eq. (1) is given by the Maxwell-Boltzmann distribution $f_+ = f_- = f_0 = \mathcal{A} \exp[-\beta(p^2/2m + V)]$, where $\beta = 1/k_B T$ and \mathcal{A} is an integration constant such that $\int d^3 \mathbf{r} d^3 \mathbf{p} f_0$ is the population of one spin state. We consider a spin perturbation of the form $f_s(\mathbf{r}, \mathbf{p}, t) = f_0(\mathbf{r}, \mathbf{p}) (1 + s\alpha(\mathbf{r}, \mathbf{p}, t))$. Assuming the perturbation is small enough, we can expand Boltzmann's equation in α and to first order we obtain

$$\partial_t \alpha + \frac{\mathbf{p}}{m} \cdot \partial_{\mathbf{r}} \alpha + \mathbf{F} \cdot \partial_{\mathbf{p}} \alpha = -C[\alpha], \quad (3)$$

where, for s-wave collisions, the linearized collisional operator C is given by

$$C[\alpha](\mathbf{r}, \mathbf{p}_1) = \int d^3 \mathbf{p}_2 f_0(\mathbf{r}, \mathbf{p}_2) v_{\text{rel}} \sigma(v_{\text{rel}}) (\alpha_1 - \alpha_2), \quad (4)$$

and as above $\alpha_i = \alpha(\mathbf{r}, \mathbf{p}_i)$. In experiments, the trap can be described by a cylindrically-symmetric harmonic potential with frequency ω_z along the symmetry axis z and ω_{\perp} in the transverse (x, y) plane. In the rest of this Supplemental Material, we work in a unit system where $m = k_B T = \omega_{\perp} = 1$ and we then write $V(x, y, z) = (\omega_z^2 z^2 + \rho^2)/2$, with $\rho = (x, y)$.

We look for exponentially decaying solutions corresponding to small deviations from equilibrium, and there-

fore take $\alpha(\mathbf{r}, \mathbf{p}, t) = e^{-\gamma t} \tilde{\alpha}(\mathbf{r}, \mathbf{p})$. Eq. (3) then becomes

$$[-\gamma + p_z \partial_z - \omega_z^2 z \partial_{p_z}] \tilde{\alpha} = -[\mathbf{\Pi} \cdot \partial_{\rho} - \rho \cdot \partial_{\mathbf{\Pi}} + C] \tilde{\alpha}, \quad (5)$$

where $\mathbf{\Pi} = (p_x, p_y)$ is the projection of the momentum in the (x, y) plane. We note that in the rhs of Eq. (5) the only z -dependence is in the linearized collisional operator C , from $f_0 \propto \exp(-\omega_z^2 z^2/2)$. Let $C = \bar{n}_0(z) \tilde{C}$, where $\bar{n}_0(z) = \int d^2 \rho d^3 \mathbf{p} f_0(\mathbf{r}, \mathbf{p}) = \bar{n}_0(0) e^{-\omega_z^2 z^2/2}$ is the equilibrium 1D-density and \tilde{C} no longer acts on the coordinate z . Taking $z' = \omega_z z$, we obtain

$$[-\gamma + \omega_z (p_z \partial_{z'} - z' \partial_{p_z})] \tilde{\alpha} = -\mathcal{L}_{\bar{n}_0(z')} [\tilde{\alpha}], \quad (6)$$

with

$$\mathcal{L}_{\bar{n}_0(z')} = \bar{n}_0(z') \tilde{C} + \mathbf{\Pi} \cdot \partial_{\rho} - \rho \cdot \partial_{\mathbf{\Pi}}. \quad (7)$$

Note that $\mathcal{L}_{\bar{n}_0(z')}$ depends on the axial coordinate z' only through the axial density. In particular, z' is only a parameter of the operator since we neither integrate nor differentiate with respect to this coordinate.

Two properties of $\mathcal{L}_{\bar{n}}$ will be used below: (i) its kernel is spanned by the functions of z' only [3] and (ii) due to atom number conservation, we have for any $\tilde{\alpha}$, $\int d^2 \rho d^3 \mathbf{p} f_0 \mathcal{L}_{\bar{n}}[\tilde{\alpha}] = 0$. We look for solutions of Eq. (6) in the limit of a very elongated trap $\omega_z \rightarrow 0$. If we focus on the *slow* axial spin dynamics of the cloud studied experimentally in [1], we have also $\gamma \rightarrow 0$ and we can therefore expand γ and $\tilde{\alpha}$ as $\gamma = \sum_{n \geq 1} \gamma_n \omega_z^n$ and $\tilde{\alpha}(\mathbf{r}, \mathbf{p}) = \sum_{n \geq 0} \omega_z^n a_n(\mathbf{r}, \mathbf{p})$. Note that we are ultimately interested in the coefficient γ_2 , since we take $\Gamma_{\text{SD}} = \omega_z^2/\gamma$ as in [1]. Inserting these expansions in Eq. (6) we get to zero-th order $\mathcal{L}_{\bar{n}}[a_0] = 0$ [4]. According to property (i), a_0 is thus a function of z' only. It is determined explicitly by the study of the next order terms of the expansion. For $n = 1$ we obtain

$$-\gamma_1 a_0 + p_z \partial_{z'} a_0 = -\mathcal{L}_{\bar{n}_0(z')} [a_1]. \quad (8)$$

Using property (ii), we see readily that $\gamma_1 = 0$, which then leads to the following relation:

$$p_z \partial_{z'} a_0 = -\mathcal{L}_{\bar{n}_0(z')} [a_1]. \quad (9)$$

Consider a uniform density \bar{n} and assume for a moment that we know the solution $\chi_{\bar{n}}(\rho, \mathbf{p})$ of the integro-differential equation $p_z = \mathcal{L}_{\bar{n}}[\chi_{\bar{n}}]$ (the properties of $\chi_{\bar{n}}$ will be discussed below). Since $\bar{n}_0(z')$ is only a parameter, and by linearity of \mathcal{L} , the solution of Eq. (9) is thus $a_1 = -\chi_{\bar{n}_0(z')} \partial_{z'} a_0$.

Having expressed a_1 as a function of a_0 , we close the set of equations by considering the $n = 2$ term of the expansion. It reads:

$$(-\gamma_2 a_0 + (p_z \partial_{z'} - z' \partial_{p_z}) a_1) = -\mathcal{L}_{\bar{n}_0(z')} [a_2]. \quad (10)$$

Using the expression of a_1 as well as the property (ii), we obtain after integration by parts

$$\gamma_2 \bar{n}_0(z') a_0(z') + \partial_{z'} (G(z') \partial_{z'} a_0(z')) = 0, \quad (11)$$

where

$$G(z') \equiv \int d^2 \boldsymbol{\rho} d^3 \mathbf{p} f_0 p_z \chi_{\bar{n}_0(z')}(\boldsymbol{\rho}, \mathbf{p}). \quad (12)$$

We now show that G defined above can be identified with the spin conductance of an ideal gas of 1D density \bar{n} in a cylindrical harmonic trap. Indeed, by definition, the conductance is obtained by solving Boltzmann's equation in a cylindrical trap in the presence of a spin pulling force $\mathbf{F}_s = s F_0 \mathbf{e}_z$. Expanding Eq. (1) to first order in perturbation, and taking as above $f_s = f_0(1 + s\alpha(\boldsymbol{\rho}, \mathbf{p}))$, we see that α is solution of

$$F_0 p_z = \mathcal{L}_{\bar{n}}[\alpha]. \quad (13)$$

We recognize here the same equation as for the definition of χ and we then have $\alpha = F_0 \chi_{\bar{n}}$. Since the particle flux is given by $\Phi = \int d^3 \mathbf{p} d^2 \boldsymbol{\rho} f_0 \alpha p_z$, we see finally that, as claimed above, $G = \Phi/F_0 = \int d^2 \boldsymbol{\rho} d^3 \mathbf{p} f_0 p_z \chi_{\bar{n}}(\boldsymbol{\rho}, \mathbf{p})$.

II. SPIN DRAG COEFFICIENT FOR THE MAXWELLIAN GAS

Consider the special case of the radially trapped Maxwellian gas for which $\sigma(p) = \Lambda/p$ where Λ is some constant. This model is useful to interpolate between the weakly interacting ($\sigma = \text{const.}$) and the strongly interacting ($\sigma \propto 1/p^2$) limits. Taking the ansatz $\alpha(\mathbf{r}, \mathbf{p}) = F_0 p_z H(\boldsymbol{\rho}, \boldsymbol{\Pi})/2$, Boltzmann's equation for spin excitations turns into

$$\frac{1}{2} (\boldsymbol{\Pi} \cdot \partial_{\boldsymbol{\rho}} - \boldsymbol{\rho} \cdot \partial_{\boldsymbol{\Pi}}) H(\boldsymbol{\rho}, \boldsymbol{\Pi}) - 1 = -\frac{\Gamma_0}{2} e^{-\rho^2/2} H(\boldsymbol{\rho}, \boldsymbol{\Pi}), \quad (14)$$

with $\Gamma_0 = \Lambda n_0$ the damping rate of the spin excitations of a homogeneous gas of density n_0 . Using the rotational invariance around the z axis, the phase space density can be expressed using the new variables

$$\begin{aligned} h &= (\boldsymbol{\Pi}^2 + \rho^2)/2 \\ u &= (\boldsymbol{\Pi}^2 - \rho^2)/2 \\ v &= \boldsymbol{\Pi} \cdot \boldsymbol{\rho}. \end{aligned}$$

Let $u + iv = R e^{i\varphi}$. Eq. (14) then becomes

$$\partial_{\varphi} H(h, R, \varphi) - 1 = -\frac{\Gamma_0 e^{-h/2}}{2} e^{R \cos \varphi/2} H(h, R, \varphi). \quad (15)$$

Moreover, in these new variables, we have

$$\int d^2 \boldsymbol{\Pi} d^2 \boldsymbol{\rho} \dots = 2\pi \int_{h=0}^{\infty} \int_{x=0}^1 \int_{\varphi=0}^{2\pi} \frac{xdx h dh d\varphi}{\sqrt{1-x^2}} \dots \quad (16)$$

where $R = xh$ and the dots stand for any cylindrically symmetric function of $\boldsymbol{\Pi}$ and $\boldsymbol{\rho}$. In particular, the spin-conductance is given by

$$G = \frac{n_0}{2} \int \frac{xdx h dh d\varphi}{\sqrt{1-x^2}} e^{-h} H(h, R = xh, \varphi), \quad (17)$$

We now turn to the solution of Eq. (15) where we focus on the φ -dependence, since it is the only variable appearing in the differential operator. Eq. (15) takes the general form

$$H'(\varphi) + \mu A(\varphi) H(\varphi) = 1, \quad (18)$$

with $\mu = \Gamma_0 e^{-h/2}/2$ and $A = \exp(R \cos(\varphi)/2)$ is a 2π -periodic function. Take

$$K_{\mu}(\varphi) = \exp \left[-\mu \int_0^{\varphi} d\varphi' A(\varphi') \right], \quad (19)$$

the general solution of (18) is

$$H(\varphi) = K_{\mu}(\varphi) \int_{\varphi_0}^{\varphi} \frac{d\varphi'}{K_{\mu}(\varphi')}, \quad (20)$$

where φ_0 is an integration constant that can be determined by imposing the periodicity of H . Taking $H(0) = H(2\pi)$, we have finally

$$H(\varphi) = K_{\mu}(\varphi) \left[\int_0^{\varphi} \frac{d\varphi'}{K_{\mu}(\varphi')} + \frac{K_{\mu}(2\pi)}{1 - K_{\mu}(2\pi)} \int_0^{2\pi} \frac{d\varphi'}{K_{\mu}(\varphi')} \right]. \quad (21)$$

Let's now discuss the behavior of the solutions in the collisionless and hydrodynamic limits.

A. Collisionless limit $\mu \rightarrow 0$

In this limit, $K_{\mu} \simeq 1 - \mu \int_0^{\varphi} A(\varphi') d\varphi'$. The asymptotic behavior of H is then dominated by the singularity due to the denominator $1 - K_{\mu}(2\pi)$ that vanishes for $\mu = 0$. To leading order, we see that H does not depend on φ and is given by

$$H = \frac{1}{\mu \bar{A}}, \quad (22)$$

where

$$\bar{A} = \frac{1}{2\pi} \int_0^{2\pi} A(\varphi') d\varphi' = I_0(R/2) \quad (23)$$

is the average value of A (I_0 is the zeroth-order modified Bessel function of the first kind).

Using the actual values of μ and A we have

$$G \sim \frac{n_0}{\Gamma_0} \int_{h=0}^{\infty} \int_{x=0}^1 \frac{hdh x dx}{\sqrt{1-x^2}} \frac{2\pi e^{-h/2}}{I_0(xh/2)} \quad (24)$$

$$\sim \frac{15.87}{\Lambda}, \quad (25)$$

Note in particular that G does not depend on the density of the cloud.

Within this limit the velocity field in the collisionless regime is given by

$$v(\rho) = \frac{F_0}{2\pi\Gamma_0} \int \frac{e^{(\rho^2 - \Pi^2)/4} \Pi d\Pi d\theta}{I_0 \left(\frac{1}{2} \sqrt{\Pi^2 \rho^2 \cos^2 \theta + (\Pi^2 - \rho^2)^2/4} \right)} \quad (26)$$

B. Hydrodynamic limit $\mu \rightarrow \infty$

In the limit $\Gamma_0 \rightarrow \infty$ we may neglect the transport term H' in (18), yielding $H(\varphi) = 1/\mu A(\varphi) \propto 1/\Gamma_0$. Taking $\Gamma(\boldsymbol{\rho}) = \Gamma_0 e^{-\rho^2/2}$, the local spin damping rate, we recover the expected result $v(\rho) \propto 1/\Gamma(\boldsymbol{\rho})$, that was obtained in the main text using local density arguments.

III. A VARIATIONAL RESULT FOR THE COLLISIONLESS SPIN CONDUCTANCE

The spin conductance in a transverse trap is obtained by solving the linearized Boltzmann equation

$$(\boldsymbol{\Pi} \cdot \partial_{\boldsymbol{\rho}} - \boldsymbol{\rho} \cdot \partial_{\boldsymbol{\Pi}}) \alpha - F p_z = -C[\alpha], \quad (27)$$

where C is the linearized collisional operator defined by

$$C[\alpha](\boldsymbol{\rho}, \mathbf{p}_1) = \int d^3 \mathbf{p}_2 f_0(\boldsymbol{\rho}, \mathbf{p}_2) |\mathbf{p}_2 - \mathbf{p}_1| \sigma [\alpha(\boldsymbol{\rho}, \mathbf{p}_1) - \alpha(\boldsymbol{\rho}, \mathbf{p}_2)] \quad (28)$$

We recall that C is symmetric and positive for the scalar product

$$\langle g_1 | g_2 \rangle = \int d^2 \boldsymbol{\rho} d^3 \mathbf{p} g_1(\boldsymbol{\rho}, \mathbf{p}) g_2(\boldsymbol{\rho}, \mathbf{p}) f_0(\boldsymbol{\rho}, \mathbf{p}), \quad (29)$$

where $f_0(\boldsymbol{\rho}, \mathbf{p}) = n_0 e^{-(p^2 + \rho^2)/2} / \sqrt{2\pi}^3$ is the static phase-space density, and n_0 is the density at the center of the trap.

The spin current is defined by $\Phi = \int d^3 \mathbf{p} d^2 \boldsymbol{\rho} f_0 \alpha p_z = \langle \alpha | p_z \rangle$ and the spin conductance is then $G = \Phi/F$. Letting $\alpha(\boldsymbol{\rho}, \mathbf{p}) = F a(\boldsymbol{\rho}, \mathbf{p})$ we have more simply $G = \langle p_z | a \rangle$.

We work in the collisionless limit $\sigma \rightarrow 0$ and we thus take $\sigma(p_{\text{rel}}) = \varepsilon \hat{\sigma}(p_{\text{rel}})$ and $C = \varepsilon C_2$ where ε is small. Following the results obtained for the Maxwellian gas, we expand a as

$$a(\boldsymbol{\rho}, \mathbf{p}) = \frac{a_0(\boldsymbol{\rho}, \mathbf{p})}{\varepsilon} + a_1(\boldsymbol{\rho}, \mathbf{p}) + \varepsilon a_2(\boldsymbol{\rho}, \mathbf{p}) + \dots \quad (30)$$

Inserting this expansion in Boltzmann's equation, we obtain to leading order

$$(\boldsymbol{\Pi} \cdot \partial_{\boldsymbol{\rho}} - \boldsymbol{\rho} \cdot \partial_{\boldsymbol{\Pi}}) a_0 = 0. \quad (31)$$

This equation is solved readily by introducing the variables (p_z, h, x, φ) defined in the study of the Maxwellian gas.

In these coordinates, Eq. (31) becomes simply $\partial_{\varphi} a_0 = 0$. The set \mathcal{F}_0 of solutions of Eq. (31) is thus composed of functions whose value does not depend on the angle φ . To get the actual expression of a_0 we need to go one step further in the expansion. At this order in ε , we have

$$\partial_{\varphi} a_1 - p_z = -C_2[a_0]. \quad (32)$$

To get rid of a_1 , we integrate over φ and use the fact that the a_n are periodic functions of φ . We then obtain the equation

$$\bar{C}_2[a_0] = p_z, \quad (33)$$

with $\bar{C}_2[a_0] = \int d\varphi C_2[a_0]/2\pi$ and where a_0 is now the only unknown.

We define on \mathcal{F}_0 the new scalar product

$$(a|b) = 4\pi^2 \int \frac{x dx h dh dp_z}{\sqrt{1-x^2}} f_0 a(x, h, p_z) b(x, h, p_z), \quad (34)$$

which is equivalent to the old scalar product $\langle a|b \rangle$. We then see readily that $(a|\bar{C}[b]) = \langle a|C[b] \rangle$. Using the properties of C , we deduce that \bar{C}_2 is a symmetric, positive operator on \mathcal{F}_0 . Eq. (33) then has the same structure as the ones used to calculate transport coefficients in homogeneous systems. We can then use the usual tricks to get a bound on the spin conductance [2]. We indeed write that for any real λ and any function $b \in \mathcal{F}_0$, we have $(a_0 + \lambda b|\bar{C}_2[a_0 + \lambda b]) \geq 0$, and using the fact that this second order polynomial in λ is always positive, we obtain from the negativity of the discriminant that for any b ,

$$G \geq \frac{(p_z|b)^2}{(b|\bar{C}[b])} = \frac{\langle p_z|b \rangle^2}{\langle b|C[b] \rangle}, \quad (35)$$

the bound being reached for $b = a_0$. We take as a variational ansatz $b = p_z$, since as discussed earlier, the collisionless regime is associated with a rather flat velocity profile. We then obtain

$$G \geq \frac{\left(\int d^2 \boldsymbol{\rho} e^{-\rho^2/2} \right)^2 n_0}{\int d^2 \boldsymbol{\rho} e^{-\rho^2}} \frac{n_0}{\Gamma_0} = 4\pi \frac{n_0}{\Gamma_0}, \quad (36)$$

with Γ_0 the spin drag on the axis of the trap. The prefactor is $4\pi \simeq 12.56$, not far from the result 15.87 found analytically for the Maxwellian gas, and the bound is indeed satisfied.

An improved variational bound can be obtained by using the exact result Eq. (26) found for the Maxwellian gas to estimate the spin drag for constant-momentum or unitary-limited cross-sections. For the Maxwellian gas, this gives by definition the exact result, while for the constant cross-section and the unitary gases, we obtain respectively $\Gamma \geq 14.5/\Gamma_0$ and $\Gamma \geq 17/\Gamma_0$.

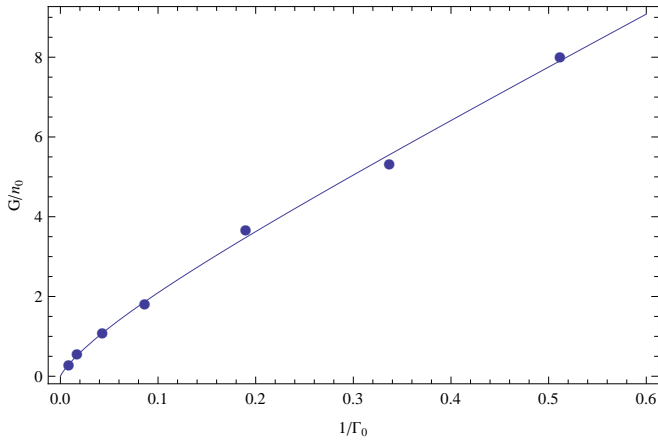


FIG. 1: Spin conductance G for a constant cross-section. Dots: Molecular Dynamics Simulation. Solid line: Padé's approximation.

IV. INTERPOLATION SCHEME FOR THE SPIN CONDUCTANCE

We know that for a power-law cross-section [5], the spin conductance G scales like $n_0(0)/\Gamma_0 f(1/\Gamma_0)$ where f

obeys the following asymptotic behaviors:

1. In the collisionless limit, f converges to a constant value ($\simeq 15.87$ for the Maxwellian gas).
2. In the hydrodynamic limit, f has a logarithmic singularity and scales like $2\pi \ln \Gamma_0$.

To interpolate between these two limits we make use of the Bessel function K_0 which vanishes at $+\infty$ and diverges as $-\ln x$ at $x = 0$. We thus approximate f by

$$f(x) = 2\pi K_0(x) + 15.87 \frac{x+a}{x+b}, \quad (37)$$

where a and b are determined by fitting the results of the molecular dynamics simulations (see Fig. 1). In the case of a constant cross-section, we obtain $a = 0.11$ and $b = 0.52$. We note that the largest relative error between the Padé interpolation and the result of the molecular simulation is observed for the largest values of Γ_0 and amounts to $\simeq 6\%$ for $\Gamma_0 \simeq 10$.

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- [1] A. Sommer, M. Ku, G. Roati, and M. W. Zwierlein, *Nature* **472**, 201 (2011).
- [2] H. Smith and H. H. Jensen, *Transport phenomena* (Oxford University Press, USA, 1989).
- [3] This result can be obtained by noting that if α belongs to the kernel of C we have first $\langle \alpha | C[\alpha] \rangle = \langle \alpha | \mathcal{L}[\alpha] \rangle = 0$ where the scalar product is defined as in Eq. (29). Moreover we have $\langle \alpha | C[\alpha] \rangle = \int d^2 \boldsymbol{\rho} d^3 \boldsymbol{p}_1 d^3 \boldsymbol{p}_2 f_{0,1} f_{0,2} v_{\text{rel}} \sigma (\alpha_1 - \alpha_2)^2 / 2$. So we see that α should not depend on the mo-

mentum and that $C[\alpha] = 0$. Using these properties in the equation $\mathcal{L}[\alpha] = 0$, we see that α is not a function of ρ either.

- [4] Note that, strictly speaking, ω_z still appears in C through the normalisation factor \mathcal{A} . We thus take the limit $\omega_z \rightarrow 0$ at constant peak-density to avoid this difficulty.
- [5] For a real cross-section, G should also depend on $k_{\text{th}} a$ - although this dependence is weak.