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Jonathan Chappelon*†[1], Luis Pedro Montejano‡[1] and Jorge Luis Ramírez Alfonsín§[1]

[1]Université Montpellier 2, Institut de Mathématiques et de Modélisation de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France

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Abstract. Let $K_{[k,t]}$ be the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped. In this note we give two explicit upper bounds for $R(K_{[k_1,t_1]},\dots,K_{[k_r,t_r]})$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i). Our first upper bound contains a classical one in the case when $k_1=\cdots=k_r$ and $t_i=1$ for all i. The second one is obtained by introducing a new edge coloring called χ_r-*colorings*. We finally discuss a conjecture claiming, in particular, that our second upper bound improves the classical one in infinitely many cases.

Keywords: Ramsey number, recursive formula.
MSC2010: 05C55, 05D10.

1. Introduction

Let K_n be a complete graph and let $r\geqslant 2$ be an integer. A r-edge coloring of a graph is a surjection from $E(G)$ to $\{0,\dots,r-1\}$ (and thus each color class is not empty). Let $k\geqslant t\geqslant 1$ be positive integers. We denote by $K_{[k,t]}$ the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped, see Figure 1.

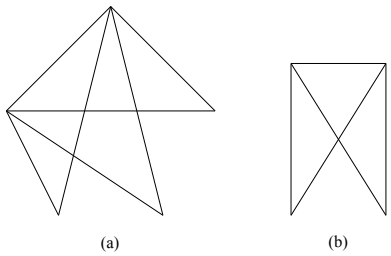

Figure 1. (a) $K_{[5,3]}$ and (b) $K_{[4,2]}$

Let $k_1,\dots,k_r$ and $t_1,\dots,t_r$ be positive integers with $k_i\geqslant t_i$ for all $i\in\{1,\dots,r\}$. Let $R([k_1,t_1],\dots,[k_r,t_r])$ be the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i.
In the case when $k_i=t_i$ for some i, we set

$$R([k_1,t_1],\dots,[k_{i-1},t_{i-1}],[t_i,t_i],[k_{i+1},t_{i+1}],\dots,[k_r,t_r])\leqslant t_i.$$

We note that equality is reached at $\min\limits_{1\leqslant i\leqslant r}\{t_i|t_i=k_i\}$. Since the set of all the edges of $K_{[t_i,t_i]}$ (which is empty) can always be colored with color i. We also notice that the case $R([k_1,1],\dots,[k_r,1])$ is exactly the classical Ramsey number $r(k_1,\dots,k_r)$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic

*Corresponding author
†E-mail address: jonathan.chappelon@um2.fr
‡E-mail address: lpmontejano@gmail.com
§E-mail address: jramirez@um2.fr

K_{k_i} for some i). We refer the reader to the excellent survey [8] on Ramsey numbers for small values.

In this note, we investigate general upper bounds for $R([k_1,t_1],\dots,[k_r,t_r])$. In the next section we present a recursive formula that yields to an explicit general upper bound (Theorem 2.2) generalizing the well-known explicit upper bound due to Graham and Rödl [5] (see equation 3). We also improve our explicit upper bound when $r=2$ for certain values of k_i, t_i (Theorems 2.4 and 2.5).

In Section 3, we shall present another general explicit upper bound for $R([k_1,t_1],\dots,[k_r,t_r])$ (Theorem 3.8) by introducing a new edge coloring called *χ_r-colorings*. We end by discussing a conjecture that is supported by graphical and numerical results.

2. Upper bounds

The following recursive inequality is classical in Ramsey theory

$$\begin{aligned} r(k_1,k_2,\dots,k_r) \leqslant\ & r(k_1-1,k_2,\dots,k_r)+r(k_1,k_2-1,\dots,k_r)+\cdots+ \\ & +r(k_1,k_2,\dots,k_r-1)-(r-2) \end{aligned} \tag{1}$$

In the same spirit, we have the following.

Lemma 2.1. *Let $r\geqslant 2$ and let $k_1,\dots,k_r$ and $t_1,\dots,t_r$ be positive integers with $k_i\geqslant t_i+1\geqslant 2$ for all i. Then,*

$$\begin{aligned} R([k_1,t_1],\dots,[k_r,t_r]) \leqslant\ & R([k_1-1,t_1],[k_2,t_2],\dots,[k_r,t_r]) \\ & +R([k_1,t_1],[k_2-1,t_2],\dots,[k_r,t_r]) \\ & \vdots \\ & +R([k_1,t_1],[k_2,t_2],\dots,[k_r-1,t_r])-(r-2). \end{aligned}$$

A similar recursive inequality has been treated in [9] in a more general setting (by considering a family of graphs intrinsically constructed via two operations *disjoin unions* and *joins*, see also [6] for the case $r=2$). Although the latter could be used to obtain Lemma 2.1, the arguments used here give a different and a more straight forward proof.

Proof of Lemma 2.1. Let us take any r-edge coloring of K_N with

$$N\geqslant R([k_1-1,t_1],[k_2,t_2],\dots,[k_r,t_r])+\cdots+R([k_1,t_1],[k_2,t_2],\dots,[k_r-1,t_r])-(r-2).$$

Let v a vertex of K_N and let $\Gamma_i(v)$ be the set of all vertices joined to v by an edge having color i for each $i=1,\dots,r$. We claim that there exists index $1\leqslant i\leqslant r$ such that

$$\Gamma_i(v)\geqslant R([k_1,t_1],\dots,[k_i-1,t_i],\dots,[k_r,t_r]).$$

Otherwise,

$$\begin{aligned} N-1=d(v)=\sum_{j=1}^{r}\Gamma_j(v) &\leqslant \sum_{j=1}^{r}(R([k_1,t_1],\dots,[k_i-1,t_i],\dots,[k_r,t_r])-1) \\ &= \sum_{j=1}^{r}(R([k_1,t_1],\dots,[k_i-1,t_i],\dots,[k_r,t_r])-r \\ &\leqslant N+(r-2)-r=N-2 \end{aligned}$$

which is a contradiction.

Now, suppose that $\Gamma_i(v)\geqslant R([k_1,t_1],\dots,[k_i-1,t_i],\dots,[k_r,t_r])$ for an index i. By definition of $R([k_1,t_1],\dots,[k_i-1,t_i],\dots,[k_r,t_r])$ we have that the complete graph induced by $\Gamma_i(v)$ contains either a subset of vertices inducing a copy $K_{[k_j,t_j]}$ having all edges with color j, for some $j\neq i$, and we are done or a subset of vertices inducing $K_{[k_i-1,t_i]}$ having all edges with color i. Adding vertex v to $K_{[k_i-1,t_i]}$ we obtain the desired copy of $K_{[k_i,t_i]}$ having all edges colored with color i. □

2.1. **Explicit general upper bound.** Lemma 2.1 yield us to the following general upper bound for $R([k_1,t_1],\dots,[k_r,t_r])$. The latter was not treated in [9] at all (in fact, suitable values/bounds needed to upper bound the recursion given in [9] for $R([k_1,t_1],\dots,[k_r,t_r])$ seem to be very difficult to estimate).

Theorem 2.2. *Let $r\geqslant 2$ be a positive integer and let $k_1,\dots,k_r$ and $t_1,\dots,t_r$ be positive integers such that $k_i\geqslant t_i$ for all $i\in\{1,\dots,r\}$. Then,*

$$R\left([k_1,t_1],\dots,[k_r,t_r]\right)\leqslant \max_{1\leqslant i\leqslant r}\{t_i\}\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)}{k_1-t_1,k_2-t_2,\dots\dots,k_r-t_r}$$

where $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\dots\dots,n_r}$ is the multinomial coefficient defined by $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\dots\dots,n_r}=\frac{(n_1+\cdots+n_r)!}{n_1!n_2!\cdots n_r!}$, for all nonnegative integers $n_1,\dots,n_r$.

Proof. We suppose that $t_1,\dots,t_r$ are fixed. We proceed by induction on $k_1+\cdots+k_r$, using Lemma 2.1. In the case where $k_j=t_j$, for some $j\in\{1,\dots,r\}$, we already know that

$$R\left([k_1,t_1],\dots,[k_{j-1},t_{j-1}],[t_j,t_j],[k_{j+1},t_{j+1}],\dots,[k_r,t_r]\right)=t_j,$$

and, since $k_i-t_i\geqslant 0$ for all i,

$$\binom{k_1+\cdots+k_{i-1}+k_{i+1}+\cdots+k_r-(t_1+\cdots+t_{i-1}+t_{i+1}+\cdots+t_r)}{k_1-t_1,\dots,k_{j-1}-t_{j-1},0,k_{j+1}-t_{j+1}\dots\dots,k_r-t_r}\geqslant 1.$$

Therefore

$$R\left([k_1,t_1],\dots,[k_r,t_r]\right)=t_j\leqslant \max_{1\leqslant i\leqslant r} t_i\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)}{k_1-t_1,k_2-t_2,\dots\dots,k_r-t_r}$$

in this case. Now, suppose that $k_i>t_i$ for all $i\in\{1,\dots,r\}$. By Lemma 2.1 and by induction hypothesis, we obtain that

$$\begin{aligned}
R([k_1,t_1],\dots,[k_r,t_r]) &\leqslant R([k_1-1,t_1],[k_2,t_2],\dots,[k_r,t_r])\\
&\quad +R([k_1,t_1],[k_2-1,t_2],\dots,[k_r,t_r])\\
&\qquad\vdots\\
&\quad +R([k_1,t_1],[k_2,t_2],\dots,[k_r-1,t_r])-(r-2)\\
&\leqslant \max_{1\leqslant i\leqslant r} t_i\left(\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)-1}{k_1-t_1-1,k_2-t_2,\dots\dots,k_r-t_r}\right.\\
&\quad +\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)-1}{k_1-t_1-1,k_2-t_2-1,\dots\dots,k_r-t_r}\\
&\qquad\vdots\\
&\quad \left.+\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)-1}{k_1-t_1-1,k_2-t_2,\dots\dots,k_r-t_r-1}\right)-(r-2)\\
&\leqslant \max_{1\leqslant i\leqslant r} t_i\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)}{k_1-t_1,k_2-t_2,\dots\dots,k_r-t_r},
\end{aligned}$$

since we have the following multinomial identity

$$\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\dots\dots,n_r}=\sum_{i=1}^{r}\binom{n_1+n_2+\cdots+n_r-1}{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r}$$

for all positive integers $n_1,n_2,\dots,n_r$. □

Theorem 2.2 is a natural generalization of the only known explicit upper bound for classical Ramsey numbers. Indeed, an immediate consequence of the above theorem (when $t = 1$) is the following classical upper bound due to Graham and Rödl [5, (2.48)] that was obtained by using (1).

$$R([k_1,1],\dots,[k_r,1]) \leqslant \binom{k_1+\cdots+k_r-r}{k_1-1,\dots,k_r-1}. \tag{2}$$

Let $R_r([k,t]) = R(\underbrace{[k,t],\dots,[k,t]}_{r})$.

Corollary 2.3. *Let $k \geqslant t \geqslant 2$ and $r \geqslant 2$ be integers. Then,*

$$R_r([k,t]) \leqslant t\binom{r(k-t)}{k-t,\dots,k-t}.$$

An immediate consequence of the above corollary (again when $t = 1$) is the following upper bound

$$R_r([k,1]) \leqslant \frac{(rk-r)!}{((k-1)!)^r}. \tag{3}$$

2.2. **Case $r = 2$.** When $r = 2$, it is the exact values of the recursive sequence generated from $u_{t,k} = u_{k,t} = t(= R_2([t,t]))$ for all $k \geqslant t$ and following the recursive identity $u_{k_1,k_2} = u_{k_1-1,k_2} + u_{k_1,k_2-1}$ for all $k_1, k_2 \geqslant t+1$.
We investigate with more detail the cases $R([s,2],[t,2])$ (resp. $R([s,2],[t,1])$), that is, the smallest integer n such that for any 2-edge coloring of K_n there always occurs a monochromatic $K_s - \{e\}$ or $K_t - \{e\}$ (resp. a monochromatic $K_s - \{e\}$ or K_t)). These cases have been extensively studied and values/bounds for specific s and t are known, see Table 1 obtained from [8].

	$K_3\setminus\{e\}$	$K_4\setminus\{e\}$	$K_5\setminus\{e\}$	$K_6\setminus\{e\}$	$K_7\setminus\{e\}$	$K_8\setminus\{e\}$	$K_9\setminus\{e\}$	$K_{10}\setminus\{e\}$	$K_{11}\setminus\{e\}$
$K_3\setminus\{e\}$	3	5	7	9	11	13	15	17	19
$K_4\setminus\{e\}$	5	10	13	17	28	[29,38]	34	41	
$K_5\setminus\{e\}$	7	13	22	[31,39]	[40,66]				
$K_6\setminus\{e\}$	9	17	[31,39]	[45,70]	[59,135]				
$K_7\setminus\{e\}$	13	28	[40,66]	[59,135]	251				
K_3	5	7	11	17	21	25	31	37	[42,45]
K_4	7	11	19	[30,33]	[37,52]	75	105	139	184
K_5	9	16	[30,34]	[43,67]	112	183	277	409	581
K_6	11	21	[37,53]	110	205	373	621	1007	1544
K_7	13	[28,30]	[51,83]	193	392	753	1336	2303	3751
K_8	15	42	123	300	657	1349	2558	4722	8200

TABLE 1. Known bounds and values of $R([s,2],[t,2])$ and $R([s,2],[t,1])$.

Lemma 2.1 allows to give (old) and new upper bounds for infinitely many cases.

Theorem 2.4.
(a) [8, 3.1 (a)] $R([3,2],[k,2]) = 2k-3$ *for all* $k \geqslant 2$,
(b) $R([4,2],[k,2]) \leqslant k^2-2k-39$ *for all* $k \geqslant 10$,
(c) $R([5,2],[8,2]) \leqslant 104$ *and* $R([5,2],[k,2]) \leqslant \frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{239}{6}k + 294$ *for all* $k \geqslant 9$,
(d) $R([6,2],[k,2]) \leqslant \frac{1}{12}k^4 - \frac{241}{12}k^2 + 274k - 1009$ *for all* $k \geqslant 8$,

(e) $R([7,2],[k,2]) \leqslant \frac{1}{60}k^5 + \frac{1}{24}k^4 - \frac{20}{3}k^3 + \frac{3047}{24}k^2 - \frac{17507}{20}k + 2064$ *for all* $k \geqslant 7$.

Proof.

(a) The result is obvious for $k = 2$. First, let us show that $R([3,2],[3,2]) = 3$. For, we notice that $K_{[3,2]}$ is the graph consisting of three vertices, one of degree 2 and two of degree 1, and so $R([3,2],[3,2]) > 2$. Now, for any 2-coloring of the edges of K_3 there is always a vertex with two incident edges with the same color, giving the desired $K_{[3,2]}$.

Suppose now that $k \geqslant 4$. We first prove that $R([3,2],[k,2]) \leqslant 2k-3$. For, we iterate inequality of Lemma 2.1 obtaining

$$\begin{aligned} R([3,2],[k,2]) &\leqslant R([2,2],[k,2]) + R([3,2],[k-1,2]) \\ &= 2 + R([3,2],[k-1,2]) \\ &\leqslant 2 + R([2,2],[k-1,2]) + R([3,2],[k-2,2]) \\ &= 2 + 2 + R([3,2],[k-2,2]) \\ &\leqslant \cdots \leqslant \underbrace{2 + \cdots + 2}_{k-3} + R([3,2],[3,2]) \\ &= 2(k-3) + 3 = 2k - 3. \end{aligned}$$

Now, we show that $R([3,2],[k,2]) > 2k-4$. For, take a perfect matching of $K_{2(k-2)}$. We color the edges belonging to the matching in red and all others in blue. We have neither a red $K_{[3,2]}$ red (since there are not vertex with two incident edges in red) nor a blue $K_{[k,2]}$ since any subset of k vertices forces to have at least two red edges.

(b) It is known [3] that $R([4,2],[10,2]) \leqslant 41$. By using the latter and the recurrence of Lemma 2.1, we obtain

$$\begin{aligned} R([4,2],[k,2]) &\leqslant \sum_{i=11}^{k} R([3,2],[i,2]) + R([4,2],[10,2]) \\ &\leqslant \sum_{i=11}^{k} (2i-3) + 41 = k^2 - 2k - 39, \end{aligned}$$

for all integers $k \geqslant 11$.

(c) It is known [6] that $R([5,2],[7,2]) \leqslant 66$. By using the latter and the recurrence of Lemma 2.1, we obtain

$$R([5,2],[8,2]) \leqslant R([4,2],[8,2]) + R([5,2],[7,2]) \leqslant 38 + 66 = 104,$$

$$R([5,2],[9,2]) \leqslant R([4,2],[9,2]) + R([5,2],[8,2]) \leqslant 34 + 104 = 138,$$

and, for all integers $k \geqslant 10$,

$$\begin{aligned} R([5,2],[k,2]) &\leqslant \sum_{i=10}^{k} R([4,2],[i,2]) + R([5,2],[9,2]) \\ &\leqslant \sum_{i=10}^{k} (i^2 - 2i - 39) + 138 \\ &= \frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{239}{6}k + 294. \end{aligned}$$

(d) It is known [6] that $R([6,2],[7,2]) \leqslant 135$. By using the latter and the recurrence of Lemma 2.1, we obtain

$$R([6,2],[8,2]) \leqslant R([5,2],[8,2]) + R([6,2],[7,2]) \leqslant 104 + 135 = 239,$$

and, for all integers $k \geqslant 9$,

$$\begin{aligned} R([6,2],[k,2]) &\leqslant \sum_{i=9}^{k} R([5,2],[i,2]) + R([6,2],[8,2]) \\ &\leqslant \sum_{i=9}^{k} \left(\frac{1}{3}i^3 - \frac{1}{2}i^2 - \frac{239}{6}i + 294 \right) + 239 \\ &= \frac{1}{12}k^4 - \frac{241}{12}k^2 + 274k - 1009. \end{aligned}$$

(e) It is known [10] that $R([7,2],[7,2]) \leqslant 251$. By using the latter and the recurrence of Lemma 2.1, we obtain for all integers $k \geqslant 8$,

$$\begin{aligned} R([7,2],[k,2]) &\leqslant \sum_{i=8}^{k} R([6,2],[i,2]) + R([7,2],[7,2]) \\ &\leqslant \sum_{i=8}^{k} \left(\frac{1}{12}i^4 - \frac{241}{12}i^2 + 274i - 1009 \right) + 251 \\ &= \frac{1}{60}k^5 + \frac{1}{24}k^4 - \frac{20}{3}k^3 + \frac{3047}{24}k^2 - \frac{17507}{20}k + 2064. \end{aligned}$$

□

Theorem 2.5.
(a) $R([3,2],[k,1]) = 2k-1$ *for all* $k \geqslant 2$,
(b) $R([4,2],[k,1]) \leqslant k^2 - 22$ *for all* $k \geqslant 8$,
(c) $R([5,2],[k,1]) \leqslant \frac{1}{3}k^3 + \frac{1}{2}k^2 - \frac{131}{6}k + 95$ *for all* $k \geqslant 8$,
(d) $R([6,2],[k,1]) \leqslant \frac{1}{12}k^4 + \frac{1}{3}k^3 - \frac{127}{12}k^2 + \frac{505}{6}k - 208$ *for all* $k \geqslant 8$,
(e) $R([7,2],[k,1]) \leqslant \frac{1}{60}k^5 + \frac{1}{8}k^4 - \frac{10}{3}k^3 + \frac{295}{8}k^2 - \frac{10061}{60}k + 287$ *for all* $k \geqslant 8$,
(f) $R([8,2],[k,1]) \leqslant \frac{1}{360}k^6 + \frac{1}{30}k^5 - \frac{55}{72}k^4 + \frac{32}{3}k^3 - \frac{11923}{180}k^2 + \frac{2093}{10}k - 239$ *for all* $k \geqslant 8$,
(g) $R([9,2],[k,1]) \leqslant \frac{1}{2520}k^7 + \frac{1}{144}k^6 - \frac{97}{720}k^5 + \frac{331}{144}k^4 - \frac{12241}{720}k^3 + \frac{2671}{36}k^2 - \frac{20351}{140}k + 24$ *for all* $k \geqslant 8$,
(h) $R([10,2],[k,1]) \leqslant \frac{1}{20160}k^8 + \frac{1}{840}k^7 - \frac{3}{160}k^6 + \frac{19}{48}k^5 - \frac{3031}{960}k^4 + \frac{4079}{240}k^3 - \frac{200713}{5040}k^2 - \frac{1019}{28}k + 408$ *for all* $k \geqslant 8$,
(i) $R([11,2],[k,1]) \leqslant \frac{1}{181440}k^9 + \frac{1}{5760}k^8 - \frac{31}{15120}k^7 + \frac{11}{192}k^6 - \frac{3827}{8640}k^5 + \frac{5443}{1920}k^4 - \frac{528539}{90720}k^3 - \frac{9761}{288}k^2 + \frac{965843}{2520}k - 1183$ *for all* $k \geqslant 8$.

Proof. For (a), the proof is analogous than the proof of Theorem 2.4 (a). For the other items, we use the upper bounds $R([4,2],[8,1]) \leqslant 42$, $R([5,2],[8,1]) \leqslant 123$, $R([6,2],[8,1]) \leqslant 300$, $R([7,2],[8,1]) \leqslant 657$, $R([8,2],[8,1]) \leqslant 1349$, $R([9,2],[8,1]) \leqslant 2558$, $R([10,2],[8,1]) \leqslant 4722$ and $R([11,2],[8,1]) \leqslant 8200$ and the recurrence of Lemma 2.1 as follows

$$R([i,2],[k,1]) \leqslant \sum_{j=9}^{k} R([i-1,2],[j,1]) + R([i,2],[8,1]),$$

for all integers $k \geqslant 9$ and for all $i \in \{4,5,\ldots,11\}$. For instance, for $i=4$, we obtain that

$$\begin{aligned} R([4,2],[k,1]) &\leqslant \sum_{i=9}^{k} R([3,2],[i,1]) + R([4,2],[8,1]) \\ &\leqslant \sum_{i=9}^{k} (2i-1) + 42 = k^2 - 22. \end{aligned}$$

for all integers $k \geqslant 9$. The proof for the other values of i is analogous. □

Unfortunately, when $r \geqslant 3$ (similar as in the classical case) bounds obtained from Theorem 2.2 (resp. obtained from (2), in the classical case) are worse than the bounds obtained from the recursion given in Lemma 2.1 (resp. from the recursion (1)).

3. χ_r-Colorings

An r-edge coloring of K_n is said to be a *χ_r-coloring*, if there exists a labeling of $V(K_n)$ with $\{1, \ldots, n\}$ and a function $\phi : \{1, \ldots, n\} \to \{0, \ldots, r-1\}$ such that for all $1 \leqslant i < j \leqslant n$ the edge $\{i, j\}$ has color t if and only if $\phi(i) = t$.

Remark 3.1. (a) Notice that the value $\phi(n)$ do not play any role in the coloring.
(b) A *monochromatic* edge coloring (all edges have the same color $0 \leqslant t \leqslant r-1$) of K_n is a χ_r-coloring. Indeed, it is enough to take any vertex labeling and to set $\phi(i) = t$ for all i.
(c) There exist r-edge colorings of K_n that are not χ_r-coloring. For instance, it can be checked that for any labeling of $V(K_3)$ there is not a suitable function ϕ giving three different colors to the edges of K_3.

Exemple 3.2. A 2-coloring of K_3 with two edges of the same color and the third one with different color is a χ_2-coloring. Indeed, If the edges $\{1, 2\}$ and $\{1, 3\}$ are colored with color 0 and the edge $\{2, 3\}$ with color 1 then we take $\phi(1) = 0, \phi(2) = 1$ and $\phi(3) = 1$.

Let $k \geqslant 1$ be an integer. Let $\chi_r(k)$ be the smallest integer n such that for any r-edge-coloring of K_N, $N \geqslant n$ there exist a clique of order k in which the induced r-edge coloring is a χ_r-coloring.

Remark 3.3. $\chi_r(k)$ always exists. Indeed, by Ramsey's Theorem, for any r-edge coloring of K_N, $N \geqslant R_r(K_k)$ there exist a clique order k that is monochromatic which, by Remark 3.1 (b), is a χ_r-coloring.

3.1. χ_r-colorings versus Erdős-Rado's colorings.

χ_r-colorings can be considered as a generalization of the classical Ramsey's Theorem. We notice that this generalization is different from the one introduced by Erdős and Rado [4] in which they consider colorings by using an *arbitrarily* number of colors (instead of fixing the number of colors r) of $\binom{[n]}{k}$ according to certain canonical patterns, see also [7]. Indeed, in the case when $k = 2$ the canonical patterns (the edge-colorings of the complete graph) considered by Erdős and Rado are those colorings that can be obtained as follows : there exists a (possibly empty) subset $I \subseteq \{1, 2\}$ such that the edges $e, f \in \binom{[n]}{2}$ have the same color if and only if $e_I = f_I$ where $\{x_1, x_2\}_I = \{x_i \in [n] \mid i \in I\}$. In this case we have the following 4 coloring patterns:

(a) If $I = \emptyset$ then two edges e, f have the same color if and only if $e_{\{\emptyset\}} = \emptyset = f_{\{\emptyset\}}$, that is, all the edges have the same color.
(b) If $I = \{1\}$ then two edges e, f have the same color if and only if $e_{\{1\}} = f_{\{1\}}$, that is, the smallest element of e is the same as the smallest element of f.
(c) If $I = \{2\}$ then two edges e, f have the same color if and only if $e_{\{2\}} = f_{\{2\}}$, that is, the largest element of e is the same as the largest element of f.
(d) If $I = \{1, 2\}$ then two edges e and f have the same color if and only if $e_{\{1,2\}} = e = f = f_{\{1,2\}}$, that is, all the edges have different colors.
Contrary to χ_r-colorings, the number of colors for Erdős-Rado's colorings is not fixed. So the existence of a Erdős-Rado's type coloring do not necessarily implies the existence of a χ_r-coloring. Nevertheless if the number of colors, say r, is fixed then the patterns (a), (b) and (c) can essentially be considered as χ_r-colorings (it is not the case for pattern (d)).

3.2. **Values and bounds for $\chi_r(k)$.** We clearly have that $\chi_r(2) = 2$. For $\chi_r(3)$, we first notice that $\chi_r(3) = R_r([3,2])$ and that $K_{[3,2]}$ is a *star* $K_{1,2}$(a graph on three vertices, one of degree 2 and two of degree one). Now, Burr and Roberts [2] proved that

$$R\left(K_{1,q_1},\ldots,K_{1,q_n}\right) = \sum_{j=1}^{n} q_j - n + \epsilon$$

where $\epsilon = 1$ if the number of even integers in the set $\{q_1,\ldots,q_n\}$ is even, $\epsilon = 2$ otherwise. Therefore, by applying the above formula when $q_i = 2$ for all i, we obtain

$$\chi_r(3) = \begin{cases} r+1 & \text{for } r \text{ even,} \\ r+2 & \text{for } r \text{ odd.} \end{cases} \tag{4}$$

Theorem 3.4. *Let $r \geqslant 2$ be a positive integer and let $k_1,\ldots,k_r$ and $t_1,\ldots,t_r$ be positive integers such that $k_i \geqslant t_i$ for all $i \in \{1,\ldots,r\}$. Then,*

$$R([k_1,t_1],\ldots,[k_r,t_r]) \leqslant \chi_r\left(\sum_{i=1}^{r}(k_i - t_i - 1) + 1 + \max_{1\leqslant i\leqslant r}\{t_i\}\right).$$

Proof. Consider a χ_r-coloring of $K_{\chi_r\left(\sum_{i=1}^{r}(k_i-t_i-1)+1+\max_{1\leqslant i\leqslant r}\{t_i\}\right)}$. Given the vertex labeling of the χ_r-coloring, we consider the complete graph K' induced by the vertices with labels $1,\ldots,\sum_{i=1}^{r}(k_i - t_i - 1) + 1$ (that is, we remove all the edges induced by the set of vertices T_1 with the $\max_{1\leqslant i\leqslant r}\{t_i\}$ largest labels). By the pigeonhole principle, there is a set T_2 of at least $k_i - t_i + 1 - 1$ vertices of K' with the same color for some i. Moreover, by definition of χ_r-coloring any edge $\{v_1, v_2\}$ with $v_1 \in T_1$ and $v_2 \in T_2$ has color i, giving the desired monochromatic $K_{[k_i,t_i]}$. □

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Let $r, k \geqslant 2$ be integers. Then,*

$$R_r([k,1]) \leqslant \chi_r(r(k-2)+2) \text{ and } R_r([k,2]) \leqslant \chi_r(r(k-3)+3).$$

Proposition 3.6. *Let $r, k \geqslant 2$ be integers. Then,*

$$\chi_r(k) \leqslant r\chi_r(k-1) - r + 2.$$

Proof. Consider a r-edge coloring of $K_{r\chi_r(k-1)-r+2}$ and let u be a vertex. Since $d(u) = r\chi_r(k-1) - r + 1$ then there are at least $\left\lceil \frac{r\chi_r(k-1)-r+1}{r} \right\rceil = \chi_r(k-1)$ set of edges with the same color all incident to u. Now, by definition of $\chi_r(k-1)$, there is a clique H of order $k-1$ which edge coloring is a χ_r- coloring. So, there is a labeling π of $V(H)$, $|V(H)| = k$ and a function ϕ giving such coloring. We claim that the r-edge coloring of the clique $H' = H \cup u$ is a χ_r-coloring. Indeed, by taking the label $\pi'(i) = \pi(i) + 1$ for all vertex $i \neq u$ and $\pi'(u) = 1$ and the function $\phi'(1) = 1$ and $\phi'(i) = \phi(i-1)$ for each $i = 2,\ldots,k$. □

Proposition 3.7. *Let $r, k \geqslant 2$ be integers. Then,*

$$\chi_r(k) \leqslant g(k,r) = \begin{cases} r^{k-2} + r^{k-3} + \cdots + r^2 + r + 2 = \dfrac{r^{k-1}-1}{r-1} + 1 & \text{for } r \text{ odd,} \\ r^{k-2} + r^{k-4} + r^{k-5} + \cdots + r^2 + r + 2 = \dfrac{r^{k-3}-1}{r-1} + r^{k-2} + 1 & \text{for } r \text{ even.} \end{cases}$$

Proof. By equality (4) and by successive applications of Proposition 3.6. □

Theorem 3.8. *Let $r \geqslant 2$ be a positive integer and let $k_1, \ldots, k_r$ and $t_1, \ldots, t_r$ be positive integers such that $k_i \geqslant t_i$ for all $i \in \{1, \ldots, r\}$. Then,*

$$R([k_1, t_1], \ldots, [k_r, t_r]) \leqslant g(k, r)$$

where

$$k := \sum_{i=1}^{r}(k_i - t_i - 1) + 1 + \max_{1 \leqslant i \leqslant r}\{t_i\}.$$

Proof. By Theorem 3.4 and Proposition 3.7. □

We believe that the above upper bound for $R_r([k, 1])$ is smaller than the one given by Corollary 2.3 (see equation (3)) for some values of k.

Conjecture 3.9. Let $r \geqslant 3$ be an integer. Then, for all $3 \leqslant k \leqslant r^{3/2} + r - 1$

$$g((r(k-2)+2, r) < \binom{r(k-1)}{k-1, k-1, \ldots\ldots, k-1} = \frac{(rk-r)!}{((k-1)!)^r}.$$

We have checked the validity of the above conjecture for all $3 \leqslant r \leqslant 150$ by computer calculations. Conjecture 3.9 is also supported graphically, by considering the continual behaviour of

$$f(k, r) = g((r(k-2)+2, r) - \frac{(rk-r)!}{((k-1)!)^r}.$$

To see that, we may use the fact that $\Gamma(z+1) = z!$ when z is a nonnegative integer, obtaining

$$f(k, r) = g((r(k-2)+2, r) - \frac{\Gamma(r(k-1)+1)}{\Gamma^r(k)}$$

where $\Gamma(z)$ is the well-known *gamma* function[5], see Figure 2.

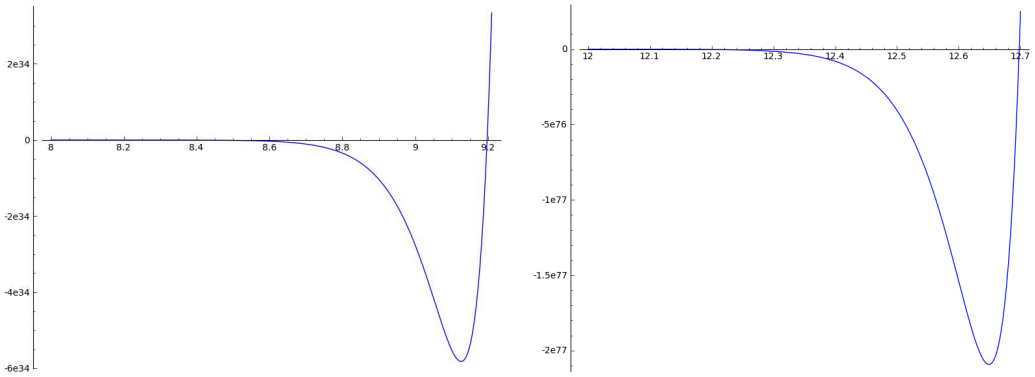

FIGURE 2. Behaviours of $f(4, k)$ with $8 \leqslant k < 10$ (left) and $f(5, k)$ with $12 \leqslant k < 13$ (right). We notice that due to the scaling used in the figures (in order to plot the minimum) the function f seems very close to zero but in fact it is very far apart, $f(4, 8) \leqslant -1,8 \times 10^{29}$ for the left one and $f(5, 12) \leqslant -5,7 \times 10^{72}$ for the right one.

We have also checked (by computer) that for each $3 \leqslant r \leqslant 150$ there is an interval I_r (increasing as r is growing) such that for each $k \geqslant 3, k \in I_r$ the function $g(r(k-3)+3, r)$ (resp. $g(r(k-4)+4, r)$) is a smaller upper bound for $R_r([k, 2])$ (resp. for $R_r([k, 3])$) than

[5]The gamma function is defined as $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ for any $z \in \mathbb{C}$ with $Re(z) > 0$. Moreover, $\Gamma(z+1) = z!$ when z is a nonnegative integer.

the corresponding ones obtained from Corollary 2.3. In view of the latter, we pose the following

Question 3.10. Let $t \geqslant 1$ and $r \geqslant 3$ be integers. Is there a function $c(r)$ such that for all $3 \leqslant k \leqslant c(r)$

$$g(r(k-t)+t, r) < t\binom{r(k-t)}{k-t, k-t, \ldots\ldots, k-t} \ ?$$

References

[1] L. Boza, Nuevas Cotas Superiores de Algunos Números de Ramsey del Tipo $r(K_m, K_n - e)$, in *Proceedings of the VII Jornada de Matemática Discreta y Algoritmica, JMDA 2010*, Castro Urdiales, Spain, July 2010.

[2] S.A. Burr, J.A. Roberts, On Ramsey numbers for stars, *Utilitas Math.*, **4** (1973), 217–220.

[3] V. Chvátal, F. Harary, Generalized Ramsey theory for graphs. II Small diagonal numbers, *Proc. Amer. Math. Soc.* **32** (1972), 389-394.

[4] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950), 249-255.

[5] R. Graham and V. Rödl, Numbers in Ramsey theory, *Surveys in Combinatorics* 1987, 123, London Mathematics Society Lecture Note Series (1987) 111–153.

[6] Y.R. Huang, K. Zhang, New upper bounds for Ramsey numbers, *European J. Combin.* **19**(3) (1998), 391-394.

[7] H. Lefmann and V. Rödl, On Erdős-Rado numbers, *Combinatorica* **15**(1) (1995), 85-104.

[8] S.P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* **1** (1994), Dynamic Survey 1, 30 pp (electronic) (revision #14 January 12, 2014).

[9] L. Shi, K. Zhang, A bound for multicolor Ramsey numbers, *Discrete Math.* **226**(1-3) (2001), 419-421.

[10] L. Shi, K. Zhang, An upper bound formula for Ramsey numbers, *manuscript* (2001).

Université Montpellier 2, Institut de Mathématiques et de Modélisation de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France.
E-mail address: jonathan.chappelon@um2.fr
E-mail address: lpmontejano@gmail.com
E-mail address: jramirez@um2.fr