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RAMSEY FOR COMPLETE GRAPHS WITH DROPPED CLIQUES

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ABSTRACT. Let $K_{[k,t]}$ be the complete graph on k vertices from which a set of edges, induced by a clique of order t , has been dropped. In this note we give two explicit upper bounds for $R(K_{[k_1,t_1]}, \dots, K_{[k_r,t_r]})$ (the smallest integer n such that for any r -edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i). Our first upper bound contains a classical one in the case when $k_1 = \dots = k_r$ and $t_i = 1$ for all i . The second one is obtained by introducing a new edge coloring called χ_r -colorings. We finally discuss a conjecture claiming, in particular, that our second upper bound improves the classical one in infinitely many cases.

1. INTRODUCTION

Let K_n be a complete graph and let $r \geq 2$ be an integer. A r -edge coloring of a graph is a surjection from $E(G)$ to $\{0, \dots, r-1\}$ (and thus each color class is not empty). Let $k \geq t \geq 1$ be positive integers. We denote by $K_{[k,t]}$ the complete graph on k vertices from which a set of edges, induced by a clique of order t , has been dropped, see Figure 1.

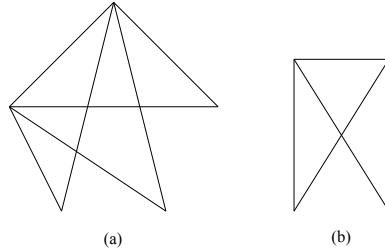


FIGURE 1. (a) $K_{[5,3]}$ and (b) $K_{[4,2]}$

Let k_1, \dots, k_r and t_1, \dots, t_r be positive integers with $k_i \geq t_i$ for all $i \in \{1, \dots, r\}$. Let $R([k_1, t_1], \dots, [k_r, t_r])$ be the smallest integer n such that for any r -edge coloring of K_n there always occurs a monochromatic $K_{[k_i, t_i]}$ for some i .

In the case when $k_i = t_i$ for some i , we set

$$R([k_1, t_1], \dots, [k_{i-1}, t_{i-1}], [t_i, t_i], [k_{i+1}, t_{i+1}], \dots, [k_r, t_r]) = t_i$$

since the set of all the edges of $K_{[t_i, t_i]}$ (which is empty) can always be colored with color i . We notice that the case $R([k_1, 1], \dots, [k_r, 1])$ is exactly the classical Ramsey number $r(k_1, \dots, k_r)$ (the smallest integer n such that for any r -edge coloring of K_n there always occurs a monochromatic K_{k_i} for some i). We refer the reader to the excellent survey [8] on Ramsey numbers for small values.

In this note, we investigate general upper bounds for $R([k_1, t_1], \dots, [k_r, t_r])$. In the next section we present a recursive formula that yields to an explicit general upper bound

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(Theorem 2.2) generalizing the well-known explicit upper bound due to Graham and Rödl [5] (see equation 2). We also improve our explicit upper bound when $r = 2$ for certain values of k_i, t_i (Proposition 2.4).

In Section 3, we shall present another general explicit upper bound for $R([k_1, t_1], \dots, [k_r, t_r])$ (Theorem 3.8) by introducing a new edge coloring called χ_r -colorings. We end by discussing a conjecture that is supported by graphical and numerical results.

2. THE UPPER BOUND

The following recursive inequality is classical in Ramsey theory

$$(1) \quad r(k_1, k_2, \dots, k_r) \leq r(k_1 - 1, k_2, \dots, k_r) + r(k_1, k_2 - 1, \dots, k_r) + \dots + r(k_1, k_2, \dots, k_r - 1) - (r - 2)$$

In the same spirit, we have the following.

Lemma 2.1. *Let $r \geq 2$ and let k_1, \dots, k_r and t_1, \dots, t_r be positive integers with $k_i \geq t_i + 1 \geq 2$ for all i . Then,*

$$\begin{aligned} R([k_1, t_1], \dots, [k_r, t_r]) &\leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) \\ &\quad + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) \\ &\quad \vdots \\ &\quad + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2). \end{aligned}$$

Proof. Let us take any r -edge coloring of K_N with

$$N \geq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) + \dots + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2).$$

Let v a vertex of K_N and let $\Gamma_i(v)$ be the set of all vertices joined to v by an edge having color i for each $i = 1, \dots, r$. We claim that there exists index $1 \leq i \leq r$ such that

$$|\Gamma_i(v)| \geq R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]).$$

Otherwise,

$$\begin{aligned} N - 1 = d(v) = \sum_{j=1}^r |\Gamma_j(v)| &\leq \sum_{j=1}^r (R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]) - 1) \\ &= \sum_{j=1}^r (R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]) - r) \\ &\leq N + (r - 2) - r = N - 2 \end{aligned}$$

which is a contradiction.

Now, suppose that $|\Gamma_i(v)| \geq R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r])$ for an index i . By definition of $R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r])$ we have that the complete graph induced by $\Gamma_i(v)$ contains either a subset of vertices inducing a copy $K_{[k_j, t_j]}$ having all edges with color j , for some $j \neq i$, and we are done or a subset of vertices inducing $K_{[k_i - 1, t_i]}$ having all edges with color i . Adding vertex v to $K_{[k_i - 1, t_i]}$ we obtain the desired copy of $K_{[k_i, t_i]}$ having all edges colored with color i . \square

A similar recursive inequality has been treated in [9] in a more general setting (by considering a family of graphs intrinsically constructed via two operations *disjoin unions* and *joins*, see also [6] for the case $r = 2$). Although the latter could be used to obtain Lemma 2.1, the arguments used here give a different and a more straight forward proof. Moreover, our approach yield us to the following general upper bound for $R([k_1, t_1], \dots, [k_r, t_r])$. The latter was not treated in [9] at all (suitable values/bounds needed to upper bound the recursion given in [9] for such general family seem to be very difficult to estimate).

Theorem 2.2. *Let $r \geq 2$ be a positive integer and let k_1, \dots, k_r and t_1, \dots, t_r be positive integers such that $k_i \geq t_i$ for all $i \in \{1, \dots, r\}$. Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \max_{1 \leq i \leq r} \{t_i\} \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

where $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r}$ is the multinomial coefficient defined by $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \frac{(n_1 + \dots + n_r)!}{n_1! n_2! \dots n_r!}$, for all nonnegative integers n_1, \dots, n_r .

Proof. We suppose that t_1, \dots, t_r are fixed. We proceed by induction on $k_1 + \dots + k_r$, using Lemma 2.1. In the case where $k_j = t_j$, for some $j \in \{1, \dots, r\}$, we already know that

$$R([k_1, t_1], \dots, [k_{j-1}, t_{j-1}], [t_j, t_j], [k_{j+1}, t_{j+1}], \dots, [k_r, t_r]) = t_j,$$

and, since $k_i - t_i \geq 0$ for all i ,

$$\binom{k_1 + \dots + k_{i-1} + k_{i+1} + \dots + k_r - (t_1 + \dots + t_{i-1} + t_{i+1} + \dots + t_r)}{k_1 - t_1, \dots, k_{j-1} - t_{j-1}, 0, k_{j+1} - t_{j+1}, \dots, k_r - t_r} \geq 1.$$

Therefore

$$R([k_1, t_1], \dots, [k_r, t_r]) = t_j \leq \max_{1 \leq i \leq r} t_i \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

in this case. Now, suppose that $k_i > t_i$ for all $i \in \{1, \dots, r\}$. By Lemma 2.1 and by induction hypothesis, we obtain that

$$\begin{aligned} R([k_1, t_1], \dots, [k_r, t_r]) &\leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) \\ &\quad + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) \\ &\quad \vdots \\ &\quad + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2) \\ &\leq \max_{1 \leq i \leq r} t_i \left(\binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \dots, k_r - t_r} \right) \\ &\quad + \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2 - 1, \dots, k_r - t_r} \\ &\quad \vdots \\ &\quad + \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \dots, k_r - t_r - 1} - (r - 2) \\ &\leq \max_{1 \leq i \leq r} t_i \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}, \end{aligned}$$

since we have the following multinomial identity

$$\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \sum_{i=1}^r \binom{n_1 + n_2 + \dots + n_r - 1}{n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_r}$$

for all positive integers n_1, n_2, \dots, n_r . □

Let $R_r([k, t]) = R(\underbrace{[k, t], \dots, [k, t]}_r)$.

Corollary 2.3. *Let $k \geq t \geq 2$ and $r \geq 2$ be integers. Then,*

$$R_r([k, t]) \leq t \binom{r(k-t)}{k-t, \dots, k-t}.$$

An immediate consequence of the above corollary (when $t = 1$) is the following classical upper bound due to Graham and Rödl [5] that was obtained by using (1).

$$(2) \quad R_r([k, 1]) \leq \frac{(rk - r)!}{((k-1)!)^r}.$$

2.1. Case $r = 2$. When $r = 2$, it is the exact values of the recursive sequence generated from $u_{t,k} = u_{k,t} = t (= R_2([t, t]))$ for all $k \geq t$ and following the recursive identity $u_{k_1, k_2} = u_{k_1-1, k_2} + u_{k_1, k_2-1}$ for all $k_1, k_2 \geq t + 1$.

We investigate with more detail the cases $R([s, 2], [t, 2])$ (resp. $R([s, 2], [t, 1])$), that is, the smallest integer n such that for any 2-edge coloring of K_n there always occurs a monochromatic $K_s - \{e\}$ or $K_t - \{e\}$ (resp. a monochromatic $K_s - \{e\}$ or K_t). These cases have been extensively studied and values/bounds for specific s and t are known, see Table 1 obtained from [8].

	$K_3 \setminus \{e\}$	$K_4 \setminus \{e\}$	$K_5 \setminus \{e\}$	$K_6 \setminus \{e\}$	$K_7 \setminus \{e\}$	$K_8 \setminus \{e\}$	$K_9 \setminus \{e\}$	$K_{10} \setminus \{e\}$	$K_{11} \setminus \{e\}$
$K_3 \setminus \{e\}$	3	5	7	9	11	13	15	17	19
$K_4 \setminus \{e\}$	5	10	13	17	28	[29,38]	34	41	
$K_5 \setminus \{e\}$	7	13	22	[31,39]	[40,66]				
$K_6 \setminus \{e\}$	9	17	[31,39]	[45,70]	[59,135]				
$K_7 \setminus \{e\}$	13	28	[40,66]	[59,135]	251				
K_3	5	7	11	17	21	25	31	[37,39]	[42,47]
K_4	7	11	19	[27,34]	[37,52]	77	105	143	187
K_5	9	16	[30,34]	[43,67]	112	186	277	418	586
K_6	11	21	[37,53]	114	205	385	621	1035	1551
K_7	13	[28,31]	[51,84]	197	394	768	1339	2355	3766
K_8	15	42	123	306	659	1382	2562	4844	8223

TABLE 1. Known bounds and values of $R([s, 2], [t, 2])$ and $R([s, 2], [t, 1])$.

Proposition 2.4. (a) $R([3, 2], [3, 2]) = R([3, 2], [2, 1]) = 3$,

(b) $R([3, 2], [k, 2]) = 2k - 3$ and $R([3, 2], [k, 1]) = 2k - 1$,

(c) $R([4, 2], [k, 2]) \leq k^2 - 2k - 39$ for each $k \geq 11$ and

(d) $R([4, 2], [k, 1]) \leq k^2 - 22$ for each $k \geq 9$.

Proof. (a) Let us show that $R([3, 2], [3, 2]) = 3$ (the proof for $R([3, 2], [2, 1]) = 3$, is analogous). For, we notice that $K_{[3,2]}$ is the graph consisting of three vertices, one of degree 2 and two of degree 1, and so $R([3, 2], [3, 2]) > 2$. Now, for any 2-coloring of the edges of K_3 there is always a vertex with two incident edges with the same color, giving the desired $K_{[3,2]}$.

(b) We first prove that $R([3, 2], [k, 2]) \leq 2k - 3$. For, we iterate inequality of Lemma 2.1 obtaining

$$\begin{aligned}
 R([3, 2], [k, 2]) &\leq R([2, 2], [k, 2]) + R([3, 2], [k - 1, 2]) \\
 &= 2 + R([3, 2], [k - 1, 2]) \\
 &\leq 2 + R([2, 2], [k - 1, 2]) + R([3, 2], [k - 2, 2]) \\
 &= 2 + 2 + R([3, 2], [k - 2, 2]) \\
 &\leq \cdots \leq \underbrace{2 + \cdots + 2}_{k-3} + R([3, 2], [3, 2]) \\
 &= 2(k - 3) + 3 = 2k - 3.
 \end{aligned}$$

Now, we show that $R([3, 2], [k, 2]) > 2k - 4$. For, take a perfect matching of $K_{2(k-2)}$. We color the edges belonging to the matching in red and all others in blue. We have neither a red $K_{[3,2]}$ (since there are not vertex with two incident edges in red) nor a blue $K_{[k,2]}$ since any subset of k vertices forces to have at least two red edges.

(It can be proved that $R([3, 2], [k, 1]) = 2k - 1$ by using completely similar arguments as above).

(c) It is known [3] that $R([4, 2], [10, 2]) = 41$. By using the latter and the recurrence of Lemma 2.1 we obtain

$$\begin{aligned}
 R([4, 2], [k, 2]) &\leq \sum_{i=11}^k R([3, 2], [i, 2]) + R([4, 2], [10, 2]) \\
 &\leq \sum_{i=11}^k (2i - 3) + 41 = (k - 10)(k + 11) - 3(k - 10) + 41 = k^2 - 2k - 39.
 \end{aligned}$$

(d) It is known [1] that $R([4, 2], [8, 1]) = 42$. By using the latter and the recurrence of Lemma 2.1 we obtain

$$\begin{aligned}
 R([4, 2], [k, 1]) &\leq \sum_{i=9}^k R([3, 2], [i, 1]) + R([4, 2], [8, 1]) \\
 &\leq \sum_{i=9}^k (2i - 1) + 42 = (k - 8)(k + 9) - (k - 8) + 41 = k^2 - 22.
 \end{aligned}$$

□

3. χ_r -COLORINGS

An r -edge coloring of K_n is said to be a χ_r -coloring, if there exists a labeling of $V(K_n)$ with $\{1, \dots, n\}$ and a function $\phi : \{1, \dots, n\} \rightarrow \{0, \dots, r - 1\}$ such that for all $1 \leq i < j \leq n$ the edge $\{i, j\}$ has color t if and only if $\phi(i) = t$.

Remark 3.1. (a) Notice that the value $\phi(n)$ do not play any role in the coloring.

(b) A *monochromatic* edge coloring (all edges have the same color $0 \leq t \leq r - 1$) of K_n is a χ_r -coloring. Indeed, it is enough to take any vertex labeling and to set $\phi(i) = t$ for all i .

(c) There exist r -edge colorings of K_n that are not χ_r -coloring. For instance, it can be checked that for any labeling of $V(K_3)$ there is not a suitable function ϕ giving three different colors to the edges of K_3 .

Example 3.2. A 2-coloring of K_3 with two edges of the same color and the third one with different color is a χ_2 -coloring. Indeed, If the edges $\{1, 2\}$ and $\{1, 3\}$ are colored with color 0 and the edge $\{2, 3\}$ with color 1 then we take $\phi(1) = 0, \phi(2) = 1$ and $\phi(3) = 1$.

Let $k \geq 1$ be an integer. Let $\chi_r(k)$ be the smallest integer n such that for any r -edge-coloring of $K_N, N \geq n$ there exist a clique of order k in which the induced r -edge coloring is a χ_r -coloring.

Remark 3.3. $\chi_r(k)$ always exists. Indeed, by Ramsey's Theorem, for any r -edge coloring of K_N , $N \geq R_r(K_k)$ there exist a clique order k that is monochromatic which, by Remark 3.1 (b), is a χ_r -coloring.

3.1. χ_r -colorings versus Erdős-Rado's colorings. χ_r -colorings can be considered as a generalization of the classical Ramsey's Theorem. We notice that this generalization is different from the one introduced by Erdős and Rado [4] in which they consider colorings by using an *arbitrarily* number of colors (instead of fixing the number of colors r) of $\binom{[n]}{k}$ according to certain canonical patterns, see also [7]. Indeed, in the case when $k = 2$ the canonical patterns (the edge-colorings of the complete graph) considered by Erdős and Rado are those colorings that can be obtained as follows : there exists a (possibly empty) subset $I \subseteq \{1, 2\}$ such that the edges $e, f \in \binom{[n]}{2}$ have the same color if and only if $e_I = f_I$ where $\{x_1, x_2\}_I = \{x_i \in [n] \mid i \in I\}$. In this case we have the following 4 coloring patterns:

- (a) If $I = \emptyset$ then two edges e, f have the same color if and only if $e_{\{\emptyset\}} = \emptyset = f_{\{\emptyset\}}$, that is, all the edges have the same color.
- (b) If $I = \{1\}$ then two edges e, f have the same color if and only if $e_{\{1\}} = f_{\{1\}}$, that is, the smallest element of e is the same as the smallest element of f .
- (c) If $I = \{2\}$ then two edges e, f have the same color if and only if $e_{\{2\}} = f_{\{2\}}$, that is, the largest element of e is the same as the largest element of f .
- (d) If $I = \{1, 2\}$ then two edges e and f have the same color if and only if $e_{\{1,2\}} = e = f = f_{\{1,2\}}$, that is, all the edges have different colors.

Contrary to χ_r -colorings, the number of colors for Erdős-Rado's colorings is not fixed. So the existence of a Erdős-Rado's type coloring do not necessarily implies the existence of a χ_r -coloring. Nevertheless if the number of colors, say r , is fixed then the patterns (a), (b) and (c) can essentially be considered as χ_r -colorings (it is not the case for pattern (d)).

3.2. Values and bounds for $\chi_r(k)$. We clearly have that $\chi_r(2) = 2$. For $\chi_r(3)$, we first notice that $\chi_r(3) = R_r([3, 2])$ and that $K_{[3,2]}$ is a *star* $K_{1,2}$ (a graph on three vertices, one of degree 2 and two of degree one). Now, Burr and Roberts [2] proved that

$$R(K_{1,q_1}, \dots, K_{1,q_n}) = \sum_{j=1}^n q_j - n + \epsilon$$

where $\epsilon = 1$ if the number of even integers in the set $\{q_1, \dots, q_n\}$ is even, $\epsilon = 2$ otherwise. Therefore, by applying the above formula when $q_i = 2$ for all i , we obtain

$$(3) \quad \chi_r(3) = \begin{cases} r + 1 & \text{for } r \text{ even,} \\ r + 2 & \text{for } r \text{ odd.} \end{cases}$$

Theorem 3.4. *Let $r \geq 2$ be a positive integer and let k_1, \dots, k_r and t_1, \dots, t_r be positive integers such that $k_i \geq t_i$ for all $i \in \{1, \dots, r\}$. Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \chi_r \left(\sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\} \right).$$

Proof. Consider a χ_r -coloring of $K_{\chi_r \left(\sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\} \right)}$. Given the vertex labeling of the χ_r -coloring, we consider the complete graph K' induced by the vertices with labels $1, \dots, \sum_{i=1}^r (k_i - t_i - 1) + 1$ (that is, we remove all the edges induced by the set of vertices

T_1 with the $\max_{1 \leq i \leq r} \{t_i\}$ largest labels). By the pigeonhole principle, there is a set T_2 of at least $k_i - t_i + 1 - 1$ vertices of K' with the same color for some i . Moreover, by definition of χ_r -coloring any edge $\{v_1, v_2\}$ with $v_1 \in T_1$ and $v_2 \in T_2$ has color i , giving the desired monochromatic $K_{[k_i, t_i]}$. \square

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *Let $r, k \geq 2$ be integers. Then,*

$$R_r([k, 1]) \leq \chi_r(r(k-2) + 2) \text{ and } R_r([k, 2]) \leq \chi_r(r(k-3) + 3).$$

Proposition 3.6. *Let $r, k \geq 2$ be integers. Then,*

$$\chi_r(k) \leq r\chi_r(k-1) - r + 2.$$

Proof. Consider a r -edge coloring of $K_{r\chi_r(k-1)-r+2}$ and let u be a vertex. Since $d(u) = r\chi_r(k-1) - r + 1$ then there are at least $\left\lceil \frac{r\chi_r(k-1)-r+1}{r} \right\rceil = \chi_r(k-1)$ set of edges with the same color all incident to u . Now, by definition of $\chi_r(k-1)$, there is a clique H of order $k-1$ which edge coloring is a χ_r -coloring. So, there is a labeling π of $V(H)$, $|V(H)| = k-1$ and a function ϕ giving such coloring. We claim that the r -edge coloring of the clique $H' = H \cup u$ is a χ_r -coloring. Indeed, by taking the label $\pi'(i) = \pi(i) + 1$ for all vertex $i \neq u$ and $\pi'(u) = 1$ and the function $\phi'(1) = 1$ and $\phi'(i) = \phi(i-1)$ for each $i = 2, \dots, k$. \square

Proposition 3.7. *Let $r, k \geq 2$ be integers. Then,*

$$\chi_r(k) \leq g(k, r) = \begin{cases} r^{k-2} + r^{k-3} + \dots + r^2 + r + 2 = \frac{r^{k-1} - 1}{r - 1} + 1 & \text{for } r \text{ odd,} \\ r^{k-2} + r^{k-4} + r^{k-5} + \dots + r^2 + r + 2 = \frac{r^{k-3} - 1}{r - 1} + r^{k-2} + 1 & \text{for } r \text{ even.} \end{cases}$$

Proof. By equality (3) and by successive applications of Proposition 3.6. \square

Theorem 3.8. *Let $r \geq 2$ be a positive integer and let k_1, \dots, k_r and t_1, \dots, t_r be positive integers such that $k_i \geq t_i$ for all $i \in \{1, \dots, r\}$. Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq g(k, r)$$

where

$$k := \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\}.$$

Proof. By Theorem 3.4 and Proposition 3.7. \square

We believe that the above upper bound for $R_r([k, 1])$ is smaller than the one given by Corollary 2.3 (see equation (2)) for some values of k .

Conjecture 3.9. Let $r \geq 3$ be an integer. Then, for all $3 \leq k \leq r^{3/2} + r - 1$

$$g((r(k-2) + 2), r) < \binom{r(k-1)}{k-1, k-1, \dots, k-1} = \frac{(rk-r)!}{((k-1)!)^r}.$$

We have checked the validity of the above conjecture for all $3 \leq r \leq 150$ by computer calculations. Conjecture 3.9 is also supported graphically, by considering the continual behaviour of

$$f(k, r) = g((r(k-2) + 2), r) - \frac{(rk-r)!}{((k-1)!)^r}.$$

To see that, we may use the fact that $\Gamma(z + 1) = z!$ when z is a nonnegative integer, obtaining

$$f(k, r) = g((r(k - 2) + 2), r) - \frac{\Gamma(r(k - 1) + 1)}{\Gamma^r(k)}$$

where $\Gamma(z)$ is the well-known *gamma* function¹, see Figure 2.

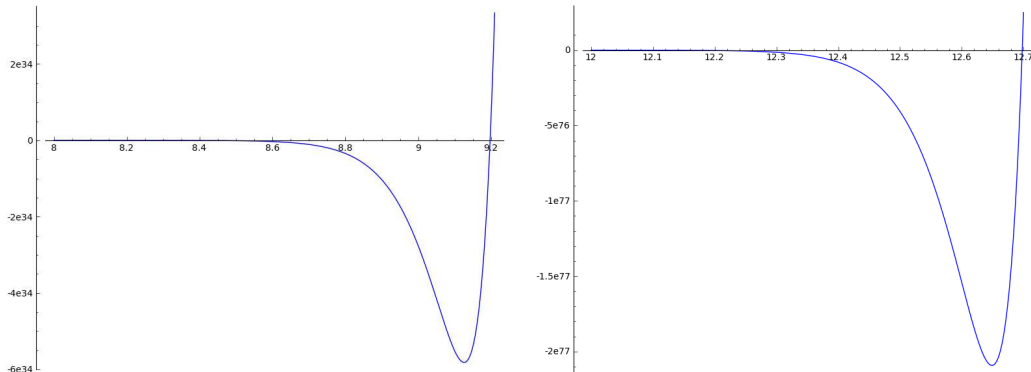


FIGURE 2. Behaviours of $f(4, k)$ with $8 \leq k < 10$ (left) and $f(5, k)$ with $12 \leq k < 13$ (right). We notice that due to the scaling used in the figures (in order to plot the minimum) the function f seems very close to zero but in fact it is very far apart, $f(4, 8) \leq -1,8 \times 10^{29}$ for the left one and $f(5, 12) \leq -5,7 \times 10^{72}$ for the right one.

We have also checked (by computer) that for each $3 \leq r \leq 150$ there is an interval I_r (increasing as r is growing) such that for each $k \geq 3, k \in I_r$ the function $g(r(k - 3) + 3, r)$ (resp. $g(r(k - 4) + 4, r)$) is a smaller upper bound for $R_r([k, 2])$ (resp. for $R_r([k, 3])$) than the corresponding ones obtained from Corollary 2.3. In view of the latter, we pose the following

Question 3.10. Let $t \geq 1$ and $r \geq 3$ be integers. Is there a function $c(r)$ such that for all $3 \leq k \leq c(r)$

$$g(r(k - t) + t, r) < t \binom{r(k - t)}{k - t, k - t, \dots, k - t} ?$$

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¹The gamma function is defined as $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ for any $z \in \mathbb{C}$ with $Re(z) > 0$. Moreover, $\Gamma(z + 1) = z!$ when z is a nonnegative integer.

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