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# RAMSEY FOR COMPLETE GRAPHS WITH DROPPED CLIQUES

JONATHAN CHAPPELON, LUIS PEDRO MONTEJANO,  
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ABSTRACT. Let  $K_{[k,t]}$  be the complete graph on  $k$  vertices from which a set of edges, induced by a clique of order  $t$ , has been dropped. In this note we give two explicit upper bounds for  $R(K_{[k_1,t_1]}, \dots, K_{[k_r,t_r]})$  (the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{[k_i,t_i]}$  for some  $i$ ). Our first upper bound contains a classical one in the case when  $k_1 = \dots = k_r$  and  $t_i = 1$  for all  $i$ . The second one is obtained by introducing a new edge coloring called  $\chi_r$ -colorings. We finally discuss a conjecture claiming, in particular, that our second upper bound improves the classical one in infinitely many cases.

## 1. INTRODUCTION

Let  $K_n$  be a complete graph and let  $r \geq 2$  be an integer. A  $r$ -edge coloring of a graph is a surjection from  $E(G)$  to  $\{0, \dots, r-1\}$  (and thus each color class is not empty). Let  $k \geq t \geq 1$  be positive integers. We denote by  $K_{[k,t]}$  the complete graph on  $k$  vertices from which a set of edges, induced by a clique of order  $t$ , has been dropped, see Figure 1.

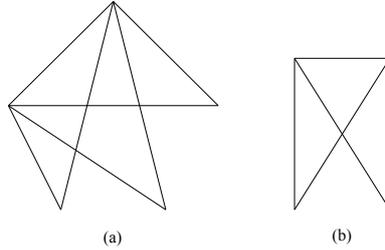


FIGURE 1. (a)  $K_{[5,3]}$  and (b)  $K_{[4,2]}$

Let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers with  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Let  $R([k_1, t_1], \dots, [k_r, t_r])$  be the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{[k_i, t_i]}$  for some  $i$ .

In the case when  $k_i = t_i$  for some  $i$ , we set

$$R([k_1, t_1], \dots, [k_{i-1}, t_{i-1}], [t_i, t_i], [k_{i+1}, t_{i+1}], \dots, [k_r, t_r]) = t_i$$

since the set of all the edges of  $K_{[t_i, t_i]}$  (which is empty) can always be colored with color  $i$ . We notice that the case  $R([k_1, 1], \dots, [k_r, 1])$  is exactly the classical Ramsey number  $r(k_1, \dots, k_r)$  (the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{k_i}$  for some  $i$ ). We refer the reader to the excellent survey [6] on Ramsey numbers for small values.

In this note, we investigate general upper bounds for  $R([k_1, t_1], \dots, [k_r, t_r])$ . In the next section we present a recursive formula that yields to an explicit general upper bound

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(Theorem 2.2) generalizing the well-known explicit upper bound due to Graham and Rödl [4] (see equation 2). We also improve our explicit upper bound when  $r = 2$  for certain values of  $k_i, t_i$  (Proposition 2.4).

In Section 3, we shall present another general explicit upper bound for  $R([k_1, t_1], \dots, [k_r, t_r])$  (Theorem 3.8) by introducing a new edge coloring called  $\chi_r$ -colorings. We end by discussing a conjecture that is supported by graphical and numerical results.

## 2. THE UPPER BOUND

The following recursive inequality is classical in Ramsey theory

$$(1) \quad r(k_1, k_2, \dots, k_r) \leq r(k_1 - 1, k_2, \dots, k_r) + r(k_1, k_2 - 1, \dots, k_r) + \dots + r(k_1, k_2, \dots, k_r - 1) - (r - 2)$$

In the same spirit, we have the following.

**Lemma 2.1.** *Let  $r \geq 2$  and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers with  $k_i \geq t_i + 1 \geq 2$  for all  $i$ . Then,*

$$\begin{aligned} R([k_1, t_1], \dots, [k_r, t_r]) &\leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) \\ &\quad + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) \\ &\quad \vdots \\ &\quad + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2). \end{aligned}$$

*Proof.* Let us take any  $r$ -edge coloring of  $K_N$  with

$$N \geq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) + \dots + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2).$$

Let  $v$  a vertex of  $K_N$  and let  $\Gamma_i(v)$  be the set of all vertices joined to  $v$  by an edge having color  $i$  for each  $i = 1, \dots, r$ . We claim that there exists index  $1 \leq i \leq r$  such that

$$|\Gamma_i(v)| \geq R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]).$$

Otherwise,

$$\begin{aligned} N - 1 = d(v) = \sum_{j=1}^r |\Gamma_j(v)| &\leq \sum_{j=1}^r (R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]) - 1) \\ &= \sum_{j=1}^r (R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]) - r) \\ &\leq N + (r - 2) - r = N - 2 \end{aligned}$$

which is a contradiction.

Now, suppose that  $|\Gamma_i(v)| \geq R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r])$  for an index  $i$ . By definition of  $R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r])$  we have that the complete graph induced by  $\Gamma_i(v)$  contains either a subset of vertices inducing a copy  $K_{[k_j, t_j]}$  having all edges with color  $j$ , for some  $j \neq i$ , and we are done or a subset of vertices inducing  $K_{[k_i - 1, t_i]}$  having all edges with color  $i$ . Adding vertex  $v$  to  $K_{[k_i - 1, t_i]}$  we obtain the desired copy of  $K_{[k_i, t_i]}$  having all edges colored with color  $i$ .  $\square$

A similar recursive inequality has been treated in [7] in a more general setting (by considering a family of graphs intrinsically constructed via two operations *disjoin unions* and *joins*, see also [5] for the case  $r = 2$ ). Although the latter could be used to obtain Lemma 2.1, the arguments used here give a different and a more straight forward proof. Moreover, our approach yield us to the following general upper bound for  $R([k_1, t_1], \dots, [k_r, t_r])$ . The latter was not treated in [7] at all (suitable values/bounds needed to upper bound the recursion given in [7] for such general family seem to be very difficult to estimate).

**Theorem 2.2.** *Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \max_{1 \leq i \leq r} \{t_i\} \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

where  $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r}$  is the multinomial coefficient defined by  $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \frac{(n_1 + \dots + n_r)!}{n_1! n_2! \dots n_r!}$ , for all nonnegative integers  $n_1, \dots, n_r$ .

*Proof.* We suppose that  $t_1, \dots, t_r$  are fixed. We proceed by induction on  $k_1 + \dots + k_r$ , using Lemma 2.1. In the case where  $k_j = t_j$ , for some  $j \in \{1, \dots, r\}$ , we already know that

$$R([k_1, t_1], \dots, [k_{j-1}, t_{j-1}], [t_j, t_j], [k_{j+1}, t_{j+1}], \dots, [k_r, t_r]) = t_j,$$

and, since  $k_i - t_i \geq 0$  for all  $i$ ,

$$\binom{k_1 + \dots + k_{i-1} + k_{i+1} + \dots + k_r - (t_1 + \dots + t_{i-1} + t_{i+1} + \dots + t_r)}{k_1 - t_1, \dots, k_{j-1} - t_{j-1}, 0, k_{j+1} - t_{j+1}, \dots, k_r - t_r} \geq 1.$$

Therefore

$$R([k_1, t_1], \dots, [k_r, t_r]) = t_j \leq \max_{1 \leq i \leq r} t_i \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

in this case. Now, suppose that  $k_i > t_i$  for all  $i \in \{1, \dots, r\}$ . By Lemma 2.1 and by induction hypothesis, we obtain that

$$\begin{aligned} R([k_1, t_1], \dots, [k_r, t_r]) &\leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) \\ &\quad + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) \\ &\quad \vdots \\ &\quad + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2) \\ &\leq \max_{1 \leq i \leq r} t_i \left( \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \dots, k_r - t_r} \right) \\ &\quad + \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2 - 1, \dots, k_r - t_r} \\ &\quad \vdots \\ &\quad + \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \dots, k_r - t_r - 1} - (r - 2) \\ &\leq \max_{1 \leq i \leq r} t_i \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}, \end{aligned}$$

since we have the following multinomial identity

$$\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \sum_{i=1}^r \binom{n_1 + n_2 + \dots + n_r - 1}{n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_r}$$

for all positive integers  $n_1, n_2, \dots, n_r$ . □

Let  $R_r([k, t]) = R(\underbrace{[k, t], \dots, [k, t]}_r)$ .

**Corollary 2.3.** *Let  $k \geq t \geq 2$  and  $r \geq 2$  be integers. Then,*

$$R_r([k, t]) \leq t \binom{r(k-t)}{k-t, \dots, k-t}.$$

An immediate consequence of the above corollary (when  $t = 1$ ) is the following classical upper bound due to Graham and Rödl [4] that was obtained by using (1).

$$(2) \quad R_r([k, 1]) \leq \frac{(rk-r)!}{((k-1)!)^r}.$$

**2.1. Case  $r = 2$ .** When  $r = 2$ , it is the exact values of the recursive sequence generated from  $u_{t,k} = u_{k,t} = t (= R_2([t, t]))$  for all  $k \geq t$  and following the recursive identity  $u_{k_1, k_2} = u_{k_1-1, k_2} + u_{k_1, k_2-1}$  for all  $k_1, k_2 \geq t + 1$ .

We investigate with more detail the cases  $R([s, 2], [t, 2])$  (resp.  $R([s, 2], [t, 1])$ ), that is, the smallest integer  $n$  such that for any 2-edge coloring of  $K_n$  there always occurs a monochromatic  $K_s - \{e\}$  or  $K_t - \{e\}$  (resp. a monochromatic  $K_s - \{e\}$  or  $K_t$ ). These cases have been extensively studied and values/bounds for specific  $s$  and  $t$  are known, see Table 1 obtained from [6].

	$K_3 \setminus \{e\}$	$K_4 \setminus \{e\}$	$K_5 \setminus \{e\}$	$K_6 \setminus \{e\}$	$K_7 \setminus \{e\}$	$K_8 \setminus \{e\}$	$K_9 \setminus \{e\}$	$K_{10} \setminus \{e\}$	$K_{11} \setminus \{e\}$
$K_3 \setminus \{e\}$	3	5	7	9	11	13	15	17	19
$K_4 \setminus \{e\}$	5	10	13	17	28	[29,38]	34	41	
$K_5 \setminus \{e\}$	7	13	22	[31,39]	[40,66]				
$K_6 \setminus \{e\}$	9	17	[31,39]	[45,70]	[59,135]				
$K_7 \setminus \{e\}$	13	28	[40,66]	[59,135]	251				
$K_3$	5	7	11	17	21	25	31	[37,39]	[42,47]
$K_4$	7	11	19	[27,34]	[37,52]	77	105	143	187
$K_5$	9	16	[30,34]	[43,67]	112	186	277	418	586
$K_6$	11	21	[37,53]	114	205	385	621	1035	1551
$K_7$	13	[28,31]	[51,84]	197	394	768	1339	2355	3766
$K_8$	15	42	123	306	659	1382	2562	4844	8223

TABLE 1. Known bounds and values of  $R([s, 2], [t, 2])$  and  $R([s, 2], [t, 1])$ .

**Proposition 2.4.** (a)  $R([3, 2], [3, 2]) = R([3, 2], [2, 1]) = 3$ ,

(b)  $R([3, 2], [k, 2]) = 2k - 3$  and  $R([3, 2], [k, 1]) = 2k - 1$ ,

(c)  $R([4, 2], [k, 2]) \leq k^2 - 2k - 39$  for each  $k \geq 11$  and

(d)  $R([4, 2], [k, 1]) \leq k^2 - 22$  for each  $k \geq 9$ .

*Proof.* (a) Let us show that  $R([3, 2], [3, 2]) = 3$  (the proof for  $R([3, 2], [2, 1]) = 3$ , is analogous). For, we notice that  $K_{[3,2]}$  is the graph consisting of three vertices, one of degree 2 and two of degree 1, and so  $R([3, 2], [3, 2]) > 2$ . Now, for any 2-coloring of the edges of  $K_3$  there is always a vertex with two incident edges with the same color, giving the desired  $K_{[3,2]}$ .

(b) We first prove that  $R([3, 2], [k, 2]) \leq 2k - 3$ . For, we iterate inequality of Lemma 2.1 obtaining

$$\begin{aligned}
 R([3, 2], [k, 2]) &\leq R([2, 2], [k, 2]) + R([3, 2], [k - 1, 2]) \\
 &= 2 + R([3, 2], [k - 1, 2]) \\
 &\leq 2 + R([2, 2], [k - 1, 2]) + R([3, 2], [k - 2, 2]) \\
 &= 2 + 2 + R([3, 2], [k - 2, 2]) \\
 &\leq \cdots \leq \underbrace{2 + \cdots + 2}_{k-3} + R([3, 2], [3, 2]) \\
 &= 2(k - 3) + 3 = 2k - 3.
 \end{aligned}$$

Now, we show that  $R([3, 2], [k, 2]) > 2k - 4$ . For, take a perfect matching of  $K_{2(k-2)}$ . We color the edges belonging to the matching in red and all others in blue. We have neither a red  $K_{[3,2]}$  (since there are not vertex with two incident edges in red) nor a blue  $K_{[k,2]}$  since any subset of  $k$  vertices forces to have at least two red edges.

(It can be proved that  $R([3, 2], [k, 1]) = 2k - 1$  by using completely similar arguments as above).

(c) It is known [3] that  $R([4, 2], [10, 2]) = 41$ . By using the latter and the recurrence of Lemma 2.1 we obtain

$$\begin{aligned}
 R([4, 2], [k, 2]) &\leq \sum_{i=11}^k R([3, 2], [i, 2]) + R([4, 2], [10, 2]) \\
 &\leq \sum_{i=11}^k (2i - 3) + 41 = (k - 10)(k + 11) - 3(k - 10) + 41 = k^2 - 2k - 39.
 \end{aligned}$$

(d) It is known [1] that  $R([4, 2], [8, 1]) = 42$ . By using the latter and the recurrence of Lemma 2.1 we obtain

$$\begin{aligned}
 R([4, 2], [k, 1]) &\leq \sum_{i=9}^k R([3, 2], [i, 1]) + R([4, 2], [8, 1]) \\
 &\leq \sum_{i=9}^k (2i - 1) + 42 = (k - 8)(k + 9) - (k - 8) + 41 = k^2 - 22.
 \end{aligned}$$

□

### 3. $\chi_r$ -COLORINGS

An  $r$ -edge coloring of  $K_n$  is said to be a  $\chi_r$ -coloring, if there exists a labeling of  $V(K_n)$  with  $\{1, \dots, n\}$  and a function  $\phi : \{1, \dots, n\} \rightarrow \{0, \dots, r - 1\}$  such that for all  $1 \leq i < j \leq n$  the edge  $\{i, j\}$  has color  $t$  if and only if  $\phi(i) = t$ .

**Remark 3.1.** (a) Notice that the value  $\phi(n)$  do not play any role in the coloring.

(b) A *monochromatic* edge coloring (all edges have the same color  $0 \leq t \leq r - 1$ ) of  $K_n$  is a  $\chi_r$ -coloring. Indeed, it is enough to take any vertex labeling and to set  $\phi(i) = t$  for all  $i$ .

(c) There exist  $r$ -edge colorings of  $K_n$  that are not  $\chi_r$ -coloring. For instance, it can be checked that for any labeling of  $V(K_3)$  there is not a suitable function  $\phi$  giving three different colors to the edges of  $K_3$ .

**Example 3.2.** A 2-coloring of  $K_3$  with two edges of the same color and the third one with different color is a  $\chi_2$ -coloring. Indeed, If the edges  $\{1, 2\}$  and  $\{1, 3\}$  are colored with color 0 and the edge  $\{2, 3\}$  with color 1 then we take  $\phi(1) = 0, \phi(2) = 1$  and  $\phi(3) = 1$ .

Let  $k \geq 1$  be an integer. Let  $\chi_r(k)$  be the smallest integer  $n$  such that for any  $r$ -edge-coloring of  $K_N, N \geq n$  there exist a clique of order  $k$  in which the induced  $r$ -edge coloring is a  $\chi_r$ -coloring.

**Remark 3.3.**  $\chi_r(k)$  always exists. Indeed, by Ramsey's theorem, for any  $r$ -edge coloring of  $K_N$ ,  $N \geq R_r(K_k)$  there exist a clique order  $k$  that is monochromatic which, by Remark 3.1 (b), is a  $\chi_r$ -coloring.

We clearly have that  $\chi_r(2) = 2$ . For  $\chi_r(3)$ , we first notice that  $\chi_r(3) = R_r([3, 2])$  and that  $K_{[3,2]}$  is a *star*  $K_{1,2}$  (a graph on three vertices, one of degree 2 and two of degree one). Now, Burr and Roberts [2] proved that

$$R(K_{1,q_1}, \dots, K_{1,q_n}) = \sum_{j=1}^n q_j - n + \epsilon$$

where  $\epsilon = 1$  if the number of even integers in the set  $\{q_1, \dots, q_n\}$  is even,  $\epsilon = 2$  otherwise. Therefore, by applying the above formula when  $q_i = 2$  for all  $i$ , we obtain

$$(3) \quad \chi_r(3) = \begin{cases} r+1 & \text{for } r \text{ even,} \\ r+2 & \text{for } r \text{ odd.} \end{cases}$$

**Theorem 3.4.** *Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \chi_r \left( \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\} \right).$$

*Proof.* Consider a  $\chi_r$ -coloring of  $K_{\chi_r \left( \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\} \right)}$ . Given the vertex labeling of the  $\chi_r$ -coloring, we consider the complete graph  $K'$  induced by the vertices with labels  $1, \dots, \sum_{i=1}^r (k_i - t_i - 1) + 1$  (that is, we remove all the edges induced by the set of vertices  $T_1$  with the  $\max_{1 \leq i \leq r} \{t_i\}$  largest labels). By the pigeonhole principle, there is a set  $T_2$  of at least  $k_i - t_i + 1 - 1$  vertices of  $K'$  with the same color for some  $i$ . Moreover, by definition of  $\chi_r$ -coloring any edge  $\{v_1, v_2\}$  with  $v_1 \in T_1$  and  $v_2 \in T_2$  has color  $i$ , giving the desired monochromatic  $K_{[k_i, t_i]}$ .  $\square$

The following result is an immediate consequence of Theorem 3.4.

**Corollary 3.5.** *Let  $r, k \geq 2$  be integers. Then,*

$$R_r([k, 1]) \leq \chi_r(r(k-2) + 2) \text{ and } R_r([k, 2]) \leq \chi_r(r(k-3) + 3).$$

**Proposition 3.6.** *Let  $r, k \geq 2$  be integers. Then,*

$$\chi_r(k) \leq r\chi_r(k-1) - r + 2.$$

*Proof.* Consider a  $r$ -edge coloring of  $K_{r\chi_r(k-1) - r + 2}$  and let  $u$  be a vertex. Since  $d(u) = r\chi_r(k-1) - r + 1$  then there are at least  $\left\lceil \frac{r\chi_r(k-1) - r + 1}{r} \right\rceil = \chi_r(k-1)$  set of edges with the same color all incident to  $u$ . Now, by definition of  $\chi_r(k-1)$ , there is a clique  $H$  of order  $k-1$  which edge coloring is a  $\chi_r$ -coloring. So, there is a labeling  $\pi$  of  $V(H)$ ,  $|V(H)| = k-1$  and a function  $\phi$  giving such coloring. We claim that the  $r$ -edge coloring of the clique  $H' = H \cup u$  is a  $\chi_r$ -coloring. Indeed, by taking the label  $\pi'(i) = \pi(i) + 1$  for all vertex  $i \neq u$  and  $\pi'(u) = 1$  and the function  $\phi'(1) = 1$  and  $\phi'(i) = \phi(i-1)$  for each  $i = 2, \dots, k$ .  $\square$

**Proposition 3.7.** *Let  $r, k \geq 2$  be integers. Then,*

$$\chi_r(k) \leq g(k, r) = \begin{cases} r^{k-2} + r^{k-3} + \dots + r^2 + r + 2 = \frac{r^{k-1} - 1}{r - 1} + 1 & \text{for } r \text{ odd,} \\ r^{k-2} + r^{k-4} + r^{k-5} + \dots + r^2 + r + 2 = \frac{r^{k-3} - 1}{r - 1} + r^{k-2} + 1 & \text{for } r \text{ even.} \end{cases}$$

*Proof.* By equality (3) and by successive applications of Proposition 3.6.  $\square$

**Theorem 3.8.** *Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq g(k, r)$$

where

$$k := \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\}.$$

*Proof.* By Theorem 3.4 and Proposition 3.7.  $\square$

We believe that the above upper bound for  $R_r([k, 1])$  is smaller than the one given by Corollary 2.3 (see equation (2)) for some values of  $k$ .

**Conjecture 3.9.** Let  $r \geq 3$  be an integer. Then, for all  $3 \leq k \leq r^{3/2} + r - 1$

$$g((r(k-2) + 2), r) < \binom{r(k-1)}{k-1, k-1, \dots, k-1} = \frac{(rk-r)!}{((k-1)!)^r}.$$

We have checked the validity of the above conjecture for all  $3 \leq r \leq 150$  by computer calculations. Conjecture 3.9 is also supported graphically, by considering the continual behaviour of

$$f(k, r) = g((r(k-2) + 2), r) - \frac{(rk-r)!}{((k-1)!)^r}.$$

To see that, we may use the fact that  $\Gamma(z+1) = z!$  when  $z$  is a nonnegative integer, obtaining

$$f(k, r) = g((r(k-2) + 2), r) - \frac{\Gamma(r(k-1) + 1)}{\Gamma^r(k)}$$

where  $\Gamma(z)$  is the well-known *gamma* function<sup>1</sup>, see Figure 2.

We have also checked (by computer) that for each  $3 \leq r \leq 150$  there is an interval  $I_r$  (increasing as  $r$  is growing) such that for each  $k \geq 3, k \in I_r$  the function  $g(r(k-3) + 3, r)$  (resp.  $g(r(k-4) + 4, r)$ ) is a smaller upper bound for  $R_r([k, 2])$  (resp. for  $R_r([k, 3])$ ) than the corresponding ones obtained from Corollary 2.3. In view of the latter, we pose the following

**Question 3.10.** Let  $t \geq 1$  and  $r \geq 3$  be integers. Is there a function  $c(r)$  such that for all  $3 \leq k \leq c(r)$

$$g(r(k-t) + t, r) < t \binom{r(k-t)}{k-t, k-t, \dots, k-t} ?$$

<sup>1</sup>The gamma function is defined as  $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$  for any  $z \in \mathbb{C}$  with  $\text{Re}(z) > 0$ . Moreover,  $\Gamma(z+1) = z!$  when  $z$  is a nonnegative integer.

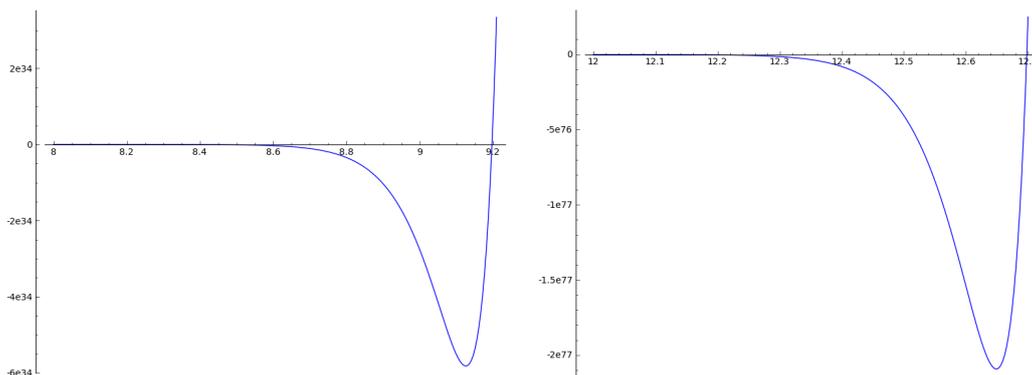


FIGURE 2. Behaviours of  $f(4, k)$  avec  $8 \leq k < 10$  (left) and  $f(5, k)$  avec  $12 \leq k < 13$  (right). We notice that due to the scaling used in the figures (in order to plot the minimum) the function  $f$  seems very close to zero but in fact it is very far apart,  $f(4, 8) \leq -1,8 \times 10^{29}$  for the left one and  $f(5, 12) \leq -5,7 \times 10^{72}$  for the right one.

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