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# Ramsey for complete graphs with dropped cliques

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**ABSTRACT.** Let  $K_{[k,t]}$  be the complete graph on  $k$  vertices from which a set of edges, induced by a clique of order  $t$ , has been dropped. In this note we give two explicit upper bounds for  $R(K_{[k_1,t_1]}, \dots, K_{[k_r,t_r]})$  (the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{[k_i,t_i]}$  for some  $i$ ). Our first upper bound contains a classical one in the case when  $k_1 = \dots = k_r$  and  $t_i = 1$  for all  $i$ . The second one is obtained by introducing a new edge coloring called  $\chi_r$ -colorings. We finally discuss a conjecture claiming, in particular, that our second upper bound improves the classical one in infinitely many cases.

**Keywords:** Ramsey number, recursive formula.

**MSC2010:** 05C55, 05D10.

## 1. INTRODUCTION

Let  $K_n$  be a complete graph and let  $r \geq 2$  be an integer. A  $r$ -edge coloring of a graph is a surjection from  $E(G)$  to  $\{0, \dots, r-1\}$  (and thus each color class is not empty). Let  $k \geq t \geq 1$  be positive integers. We denote by  $K_{[k,t]}$  the complete graph on  $k$  vertices from which a set of edges, induced by a clique of order  $t$ , has been dropped, see Figure 1.

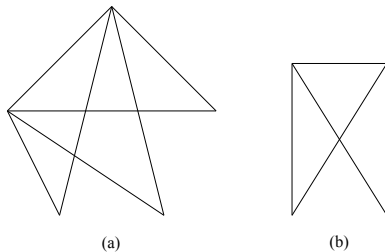


FIGURE 1. (a)  $K_{[5,3]}$  and (b)  $K_{[4,2]}$

Let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers with  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Let  $R([k_1, t_1], \dots, [k_r, t_r])$  be the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{[k_i, t_i]}$  for some  $i$ .

In the case when  $k_i = t_i$  for some  $i$ , we set

$$R([k_1, t_1], \dots, [k_{i-1}, t_{i-1}], [t_i, t_i], [k_{i+1}, t_{i+1}], \dots, [k_r, t_r]) \leq t_i.$$

We note that equality is reached at  $\min_{1 \leq i \leq r} \{t_i | t_i = k_i\}$ . Since the set of all the edges of  $K_{[t_i, t_i]}$  (which is empty) can always be colored with color  $i$ . We also notice that the case  $R([k_1, 1], \dots, [k_r, 1])$  is exactly the classical Ramsey number  $r(k_1, \dots, k_r)$  (the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic

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$K_{k_i}$  for some  $i$ ). We refer the reader to the excellent survey [8] on Ramsey numbers for small values.

In this note, we investigate general upper bounds for  $R([k_1, t_1], \dots, [k_r, t_r])$ . In the next section we present a recursive formula that yields to an explicit general upper bound (Theorem 2.2) generalizing the well-known explicit upper bound due to Graham and Rödl [5] (see equation 3). We also improve our explicit upper bound when  $r = 2$  for certain values of  $k_i, t_i$  (Theorems 2.4 and 2.5).

In Section 3, we shall present another general explicit upper bound for  $R([k_1, t_1], \dots, [k_r, t_r])$  (Theorem 3.8) by introducing a new edge coloring called  $\chi_r$ -colorings. We end by discussing a conjecture that is supported by graphical and numerical results.

## 2. UPPER BOUNDS

The following recursive inequality is classical in Ramsey theory

$$(1) \quad r(k_1, k_2, \dots, k_r) \leq r(k_1 - 1, k_2, \dots, k_r) + r(k_1, k_2 - 1, \dots, k_r) + \dots + r(k_1, k_2, \dots, k_r - 1) - (r - 2)$$

In the same spirit, we have the following.

**Lemma 2.1.** *Let  $r \geq 2$  and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers with  $k_i \geq t_i + 1 \geq 2$  for all  $i$ . Then,*

$$\begin{aligned} R([k_1, t_1], \dots, [k_r, t_r]) &\leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) \\ &\quad + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) \\ &\quad \vdots \\ &\quad + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2). \end{aligned}$$

A similar recursive inequality has been treated in [9] in a more general setting (by considering a family of graphs intrinsically constructed via two operations *disjoin unions* and *joins*, see also [6] for the case  $r = 2$ ). Although the latter could be used to obtain Lemma 2.1, the arguments used here give a different and a more straight forward proof.

*Proof of Lemma 2.1.* Let us take any  $r$ -edge coloring of  $K_N$  with

$$N \geq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) + \dots + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2).$$

Let  $v$  a vertex of  $K_N$  and let  $\Gamma_i(v)$  be the set of all vertices joined to  $v$  by an edge having color  $i$  for each  $i = 1, \dots, r$ . We claim that there exists index  $1 \leq i \leq r$  such that

$$|\Gamma_i(v)| \geq R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r]).$$

Otherwise,

$$\begin{aligned} N - 1 = d(v) = \sum_{j=1}^r |\Gamma_j(v)| &\leq \sum_{j=1}^r (R([k_1, t_1], \dots, [k_j - 1, t_j], \dots, [k_r, t_r]) - 1) \\ &= \sum_{j=1}^r (R([k_1, t_1], \dots, [k_j - 1, t_j], \dots, [k_r, t_r]) - r) \\ &\leq N + (r - 2) - r = N - 2 \end{aligned}$$

which is a contradiction.

Now, suppose that  $|\Gamma_i(v)| \geq R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r])$  for an index  $i$ . By definition of  $R([k_1, t_1], \dots, [k_i - 1, t_i], \dots, [k_r, t_r])$  we have that the complete graph induced by  $\Gamma_i(v)$  contains either a subset of vertices inducing a copy  $K_{[k_j, t_j]}$  having all edges with color  $j$ , for some  $j \neq i$ , and we are done or a subset of vertices inducing  $K_{[k_i - 1, t_i]}$  having all edges with color  $i$ . Adding vertex  $v$  to  $K_{[k_i - 1, t_i]}$  we obtain the desired copy of  $K_{[k_i, t_i]}$  having all edges colored with color  $i$ .  $\square$

**2.1. Explicit general upper bound.** Lemma 2.1 yield us to the following general upper bound for  $R([k_1, t_1], \dots, [k_r, t_r])$ . The latter was not treated in [9] at all (in fact, suitable values/bounds needed to upper bound the recursion given in [9] for  $R([k_1, t_1], \dots, [k_r, t_r])$  seem to be very difficult to estimate).

**Theorem 2.2.** *Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \max_{1 \leq i \leq r} \{t_i\} \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

where  $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r}$  is the multinomial coefficient defined by  $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \frac{(n_1 + \dots + n_r)!}{n_1! n_2! \dots n_r!}$ , for all nonnegative integers  $n_1, \dots, n_r$ .

*Proof.* We suppose that  $t_1, \dots, t_r$  are fixed. We proceed by induction on  $k_1 + \dots + k_r$ , using Lemma 2.1. In the case where  $k_j = t_j$ , for some  $j \in \{1, \dots, r\}$ , we already know that

$$R([k_1, t_1], \dots, [k_{j-1}, t_{j-1}], [t_j, t_j], [k_{j+1}, t_{j+1}], \dots, [k_r, t_r]) = t_j,$$

and, since  $k_i - t_i \geq 0$  for all  $i$ ,

$$\binom{k_1 + \dots + k_{i-1} + k_{i+1} + \dots + k_r - (t_1 + \dots + t_{i-1} + t_{i+1} + \dots + t_r)}{k_1 - t_1, \dots, k_{j-1} - t_{j-1}, 0, k_{j+1} - t_{j+1}, \dots, k_r - t_r} \geq 1.$$

Therefore

$$R([k_1, t_1], \dots, [k_r, t_r]) = t_j \leq \max_{1 \leq i \leq r} t_i \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

in this case. Now, suppose that  $k_i > t_i$  for all  $i \in \{1, \dots, r\}$ . By Lemma 2.1 and by induction hypothesis, we obtain that

$$\begin{aligned} R([k_1, t_1], \dots, [k_r, t_r]) &\leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) \\ &\quad + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) \\ &\quad \vdots \\ &\quad + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2) \\ &\leq \max_{1 \leq i \leq r} t_i \left( \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \dots, k_r - t_r} \right) \\ &\quad + \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2 - 1, \dots, k_r - t_r} \\ &\quad \vdots \\ &\quad + \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r) - 1}{k_1 - t_1 - 1, k_2 - t_2, \dots, k_r - t_r - 1} - (r - 2) \\ &\leq \max_{1 \leq i \leq r} t_i \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}, \end{aligned}$$

since we have the following multinomial identity

$$\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \sum_{i=1}^r \binom{n_1 + n_2 + \dots + n_r - 1}{n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_r}$$

for all positive integers  $n_1, n_2, \dots, n_r$ . □

Theorem 2.2 is a natural generalization of the only known explicit upper bound for classical Ramsey numbers. Indeed, an immediate consequence of the above theorem (when  $t = 1$ ) is the following classical upper bound due to Graham and Rödl [5, (2.48)] that was obtained by using (1).

$$(2) \quad R([k_1, 1], \dots, [k_r, 1]) \leq \binom{k_1 + \dots + k_r - r}{k_1 - 1, \dots, k_r - 1}.$$

Let  $R_r([k, t]) = R(\underbrace{[k, t], \dots, [k, t]}_r)$ .

**Corollary 2.3.** *Let  $k \geq t \geq 2$  and  $r \geq 2$  be integers. Then,*

$$R_r([k, t]) \leq t \binom{r(k-t)}{k-t, \dots, k-t}.$$

An immediate consequence of the above corollary (again when  $t = 1$ ) is the following upper bound

$$(3) \quad R_r([k, 1]) \leq \frac{(rk - r)!}{((k-1)!)^r}.$$

**2.2. Case  $r = 2$ .** When  $r = 2$ , it is the exact values of the recursive sequence generated from  $u_{t,k} = u_{k,t} = t (= R_2([t, t]))$  for all  $k \geq t$  and following the recursive identity  $u_{k_1, k_2} = u_{k_1-1, k_2} + u_{k_1, k_2-1}$  for all  $k_1, k_2 \geq t + 1$ .

We investigate with more detail the cases  $R([s, 2], [t, 2])$  (resp.  $R([s, 2], [t, 1])$ ), that is, the smallest integer  $n$  such that for any 2-edge coloring of  $K_n$  there always occurs a monochromatic  $K_s - \{e\}$  or  $K_t - \{e\}$  (resp. a monochromatic  $K_s - \{e\}$  or  $K_t$ ). These cases have been extensively studied and values/bounds for specific  $s$  and  $t$  are known, see Table 1 obtained from [8].

	$K_3 \setminus \{e\}$	$K_4 \setminus \{e\}$	$K_5 \setminus \{e\}$	$K_6 \setminus \{e\}$	$K_7 \setminus \{e\}$	$K_8 \setminus \{e\}$	$K_9 \setminus \{e\}$	$K_{10} \setminus \{e\}$	$K_{11} \setminus \{e\}$
$K_3 \setminus \{e\}$	3	5	7	9	11	13	15	17	19
$K_4 \setminus \{e\}$	5	10	13	17	28	[29,38]	34	41	
$K_5 \setminus \{e\}$	7	13	22	[31,39]	[40,66]				
$K_6 \setminus \{e\}$	9	17	[31,39]	[45,70]	[59,135]				
$K_7 \setminus \{e\}$	13	28	[40,66]	[59,135]	251				
$K_3$	5	7	11	17	21	25	31	37	[42,45]
$K_4$	7	11	19	[30,33]	[37,52]	75	105	139	184
$K_5$	9	16	[30,34]	[43,67]	112	183	277	409	581
$K_6$	11	21	[37,53]	110	205	373	621	1007	1544
$K_7$	13	[28,30]	[51,83]	193	392	753	1336	2303	3751
$K_8$	15	42	123	300	657	1349	2558	4722	8200

TABLE 1. Known bounds and values of  $R([s, 2], [t, 2])$  and  $R([s, 2], [t, 1])$ .

Lemma 2.1 allows to give (old) and new upper bounds for infinitely many cases.

**Theorem 2.4.**

- (a) [8, 3.1 (a)]  $R([3, 2], [k, 2]) = 2k - 3$  for all  $k \geq 2$ ,
- (b)  $R([4, 2], [k, 2]) \leq k^2 - 2k - 39$  for all  $k \geq 10$ ,
- (c)  $R([5, 2], [8, 2]) \leq 104$  and  $R([5, 2], [k, 2]) \leq \frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{239}{6}k + 294$  for all  $k \geq 9$ ,
- (d)  $R([6, 2], [k, 2]) \leq \frac{1}{12}k^4 - \frac{241}{12}k^2 + 274k - 1009$  for all  $k \geq 8$ ,

$$(e) R([7, 2], [k, 2]) \leq \frac{1}{60}k^5 + \frac{1}{24}k^4 - \frac{20}{3}k^3 + \frac{3047}{24}k^2 - \frac{17507}{20}k + 2064 \text{ for all } k \geq 7.$$

*Proof.*

(a) The result is obvious for  $k = 2$ . First, let us show that  $R([3, 2], [3, 2]) = 3$ . For, we notice that  $K_{[3,2]}$  is the graph consisting of three vertices, one of degree 2 and two of degree 1, and so  $R([3, 2], [3, 2]) > 2$ . Now, for any 2-coloring of the edges of  $K_3$  there is always a vertex with two incident edges with the same color, giving the desired  $K_{[3,2]}$ .

Suppose now that  $k \geq 4$ . We first prove that  $R([3, 2], [k, 2]) \leq 2k - 3$ . For, we iterate inequality of Lemma 2.1 obtaining

$$\begin{aligned} R([3, 2], [k, 2]) &\leq R([2, 2], [k, 2]) + R([3, 2], [k-1, 2]) \\ &= 2 + R([3, 2], [k-1, 2]) \\ &\leq 2 + R([2, 2], [k-1, 2]) + R([3, 2], [k-2, 2]) \\ &= 2 + 2 + R([3, 2], [k-2, 2]) \\ &\leq \cdots \leq \underbrace{2 + \cdots + 2}_{k-3} + R([3, 2], [3, 2]) \\ &= 2(k-3) + 3 = 2k - 3. \end{aligned}$$

Now, we show that  $R([3, 2], [k, 2]) > 2k - 4$ . For, take a perfect matching of  $K_{2(k-2)}$ . We color the edges belonging to the matching in red and all others in blue. We have neither a red  $K_{[3,2]}$  (since there are not vertex with two incident edges in red) nor a blue  $K_{[k,2]}$  since any subset of  $k$  vertices forces to have at least two red edges.

(b) It is known [3] that  $R([4, 2], [10, 2]) \leq 41$ . By using the latter and the recurrence of Lemma 2.1, we obtain

$$\begin{aligned} R([4, 2], [k, 2]) &\leq \sum_{i=11}^k R([3, 2], [i, 2]) + R([4, 2], [10, 2]) \\ &\leq \sum_{i=11}^k (2i - 3) + 41 = k^2 - 2k - 39, \end{aligned}$$

for all integers  $k \geq 11$ .

(c) It is known [6] that  $R([5, 2], [7, 2]) \leq 66$ . By using the latter and the recurrence of Lemma 2.1, we obtain

$$R([5, 2], [8, 2]) \leq R([4, 2], [8, 2]) + R([5, 2], [7, 2]) \leq 38 + 66 = 104,$$

$$R([5, 2], [9, 2]) \leq R([4, 2], [9, 2]) + R([5, 2], [8, 2]) \leq 34 + 104 = 138,$$

and, for all integers  $k \geq 10$ ,

$$\begin{aligned} R([5, 2], [k, 2]) &\leq \sum_{i=10}^k R([4, 2], [i, 2]) + R([5, 2], [9, 2]) \\ &\leq \sum_{i=10}^k (i^2 - 2i - 39) + 138 \\ &= \frac{1}{3}k^3 - \frac{1}{2}k^2 - \frac{239}{6}k + 294. \end{aligned}$$

(d) It is known [6] that  $R([6, 2], [7, 2]) \leq 135$ . By using the latter and the recurrence of Lemma 2.1, we obtain

$$R([6, 2], [8, 2]) \leq R([5, 2], [8, 2]) + R([6, 2], [7, 2]) \leq 104 + 135 = 239,$$

and, for all integers  $k \geq 9$ ,

$$\begin{aligned} R([6, 2], [k, 2]) &\leq \sum_{i=9}^k R([5, 2], [i, 2]) + R([6, 2], [8, 2]) \\ &\leq \sum_{i=9}^k \left( \frac{1}{3}i^3 - \frac{1}{2}i^2 - \frac{239}{6}i + 294 \right) + 239 \\ &= \frac{1}{12}k^4 - \frac{241}{12}k^2 + 274k - 1009. \end{aligned}$$

(e) It is known [10] that  $R([7, 2], [7, 2]) \leq 251$ . By using the latter and the recurrence of Lemma 2.1, we obtain for all integers  $k \geq 8$ ,

$$\begin{aligned} R([7, 2], [k, 2]) &\leq \sum_{i=8}^k R([6, 2], [i, 2]) + R([7, 2], [7, 2]) \\ &\leq \sum_{i=8}^k \left( \frac{1}{12}i^4 - \frac{241}{12}i^2 + 274i - 1009 \right) + 251 \\ &= \frac{1}{60}k^5 + \frac{1}{24}k^4 - \frac{20}{3}k^3 + \frac{3047}{24}k^2 - \frac{17507}{20}k + 2064. \end{aligned}$$

□

**Theorem 2.5.**

- (a)  $R([3, 2], [k, 1]) = 2k - 1$  for all  $k \geq 2$ ,
- (b)  $R([4, 2], [k, 1]) \leq k^2 - 22$  for all  $k \geq 8$ ,
- (c)  $R([5, 2], [k, 1]) \leq \frac{1}{3}k^3 + \frac{1}{2}k^2 - \frac{131}{6}k + 95$  for all  $k \geq 8$ ,
- (d)  $R([6, 2], [k, 1]) \leq \frac{1}{12}k^4 + \frac{1}{3}k^3 - \frac{127}{12}k^2 + \frac{505}{6}k - 208$  for all  $k \geq 8$ ,
- (e)  $R([7, 2], [k, 1]) \leq \frac{1}{60}k^5 + \frac{1}{8}k^4 - \frac{10}{3}k^3 + \frac{295}{8}k^2 - \frac{10061}{60}k + 287$  for all  $k \geq 8$ ,
- (f)  $R([8, 2], [k, 1]) \leq \frac{1}{360}k^6 + \frac{1}{30}k^5 - \frac{55}{72}k^4 + \frac{32}{3}k^3 - \frac{11923}{180}k^2 + \frac{2093}{10}k - 239$  for all  $k \geq 8$ ,
- (g)  $R([9, 2], [k, 1]) \leq \frac{1}{2520}k^7 + \frac{1}{144}k^6 - \frac{97}{720}k^5 + \frac{331}{144}k^4 - \frac{12241}{720}k^3 + \frac{2671}{36}k^2 - \frac{20351}{140}k + 24$  for all  $k \geq 8$ ,
- (h)  $R([10, 2], [k, 1]) \leq \frac{1}{20160}k^8 + \frac{1}{840}k^7 - \frac{3}{160}k^6 + \frac{19}{48}k^5 - \frac{3031}{960}k^4 + \frac{4079}{240}k^3 - \frac{200713}{5040}k^2 - \frac{1019}{28}k + 408$  for all  $k \geq 8$ ,
- (i)  $R([11, 2], [k, 1]) \leq \frac{1}{181440}k^9 + \frac{1}{5760}k^8 - \frac{31}{15120}k^7 + \frac{11}{192}k^6 - \frac{3827}{8640}k^5 + \frac{5443}{1920}k^4 - \frac{528539}{90720}k^3 - \frac{9761}{288}k^2 + \frac{965843}{2520}k - 1183$  for all  $k \geq 8$ .

*Proof.* For (a), the proof is analogous than the proof of Theorem 2.4 (a). For the other items, we use the upper bounds  $R([4, 2], [8, 1]) \leq 42$ ,  $R([5, 2], [8, 1]) \leq 123$ ,  $R([6, 2], [8, 1]) \leq 300$ ,  $R([7, 2], [8, 1]) \leq 657$ ,  $R([8, 2], [8, 1]) \leq 1349$ ,  $R([9, 2], [8, 1]) \leq 2558$ ,  $R([10, 2], [8, 1]) \leq 4722$  and  $R([11, 2], [8, 1]) \leq 8200$  and the recurrence of Lemma 2.1 as follows

$$R([i, 2], [k, 1]) \leq \sum_{j=9}^k R([i-1, 2], [j, 1]) + R([i, 2], [8, 1]),$$

for all integers  $k \geq 9$  and for all  $i \in \{4, 5, \dots, 11\}$ . For instance, for  $i = 4$ , we obtain that

$$\begin{aligned} R([4, 2], [k, 1]) &\leq \sum_{i=9}^k R([3, 2], [i, 1]) + R([4, 2], [8, 1]) \\ &\leq \sum_{i=9}^k (2i - 1) + 42 = k^2 - 22. \end{aligned}$$

for all integers  $k \geq 9$ . The proof for the other values of  $i$  is analogous. □

Unfortunately, when  $r \geq 3$  (similar as in the classical case) bounds obtained from Theorem 2.2 (resp. obtained from (2), in the classical case) are worse than the bounds obtained from the recursion given in Lemma 2.1 (resp. from the recursion (1)).

### 3. $\chi_r$ -COLORINGS

An  $r$ -edge coloring of  $K_n$  is said to be a  $\chi_r$ -coloring, if there exists a labeling of  $V(K_n)$  with  $\{1, \dots, n\}$  and a function  $\phi : \{1, \dots, n\} \rightarrow \{0, \dots, r-1\}$  such that for all  $1 \leq i < j \leq n$  the edge  $\{i, j\}$  has color  $t$  if and only if  $\phi(i) = t$ .

- Remark 3.1.** (a) Notice that the value  $\phi(n)$  do not play any role in the coloring.  
 (b) A *monochromatic* edge coloring (all edges have the same color  $0 \leq t \leq r-1$ ) of  $K_n$  is a  $\chi_r$ -coloring. Indeed, it is enough to take any vertex labeling and to set  $\phi(i) = t$  for all  $i$ .  
 (c) There exist  $r$ -edge colorings of  $K_n$  that are not  $\chi_r$ -coloring. For instance, it can be checked that for any labeling of  $V(K_3)$  there is not a suitable function  $\phi$  giving three different colors to the edges of  $K_3$ .

**Example 3.2.** A 2-coloring of  $K_3$  with two edges of the same color and the third one with different color is a  $\chi_2$ -coloring. Indeed, If the edges  $\{1, 2\}$  and  $\{1, 3\}$  are colored with color 0 and the edge  $\{2, 3\}$  with color 1 then we take  $\phi(1) = 0, \phi(2) = 1$  and  $\phi(3) = 1$ .

Let  $k \geq 1$  be an integer. Let  $\chi_r(k)$  be the smallest integer  $n$  such that for any  $r$ -edge-coloring of  $K_N, N \geq n$  there exist a clique of order  $k$  in which the induced  $r$ -edge coloring is a  $\chi_r$ -coloring.

**Remark 3.3.**  $\chi_r(k)$  always exists. Indeed, by Ramsey's Theorem, for any  $r$ -edge coloring of  $K_N, N \geq R_r(K_k)$  there exist a clique order  $k$  that is monochromatic which, by Remark 3.1 (b), is a  $\chi_r$ -coloring.

**3.1.  $\chi_r$ -colorings versus Erdős-Rado's colorings.**  $\chi_r$ -colorings can be considered as a generalization of the classical Ramsey's Theorem. We notice that this generalization is different from the one introduced by Erdős and Rado [4] in which they consider colorings by using an *arbitrarily* number of colors (instead of fixing the number of colors  $r$ ) of  $\binom{[n]}{k}$  according to certain canonical patterns, see also [7]. Indeed, in the case when  $k = 2$  the canonical patterns (the edge-colorings of the complete graph) considered by Erdős and Rado are those colorings that can be obtained as follows : there exists a (possibly empty) subset  $I \subseteq \{1, 2\}$  such that the edges  $e, f \in \binom{[n]}{2}$  have the same color if and only if  $e_I = f_I$  where  $\{x_1, x_2\}_I = \{x_i \in [n] \mid i \in I\}$ . In this case we have the following 4 coloring patterns:

- (a) If  $I = \emptyset$  then two edges  $e, f$  have the same color if and only if  $e_{\{\emptyset\}} = \emptyset = f_{\{\emptyset\}}$ , that is, all the edges have the same color.  
 (b) If  $I = \{1\}$  then two edges  $e, f$  have the same color if and only if  $e_{\{1\}} = f_{\{1\}}$ , that is, the smallest element of  $e$  is the same as the smallest element of  $f$ .  
 (c) If  $I = \{2\}$  then two edges  $e, f$  have the same color if and only if  $e_{\{2\}} = f_{\{2\}}$ , that is, the largest element of  $e$  is the same as the largest element of  $f$ .  
 (d) If  $I = \{1, 2\}$  then two edges  $e$  and  $f$  have the same color if and only if  $e_{\{1,2\}} = e = f = f_{\{1,2\}}$ , that is, all the edges have different colors.

Contrary to  $\chi_r$ -colorings, the number of colors for Erdős-Rado's colorings is not fixed. So the existence of a Erdős-Rado's type coloring do not necessarily implies the existence of a  $\chi_r$ -coloring. Nevertheless if the number of colors, say  $r$ , is fixed then the patterns (a), (b) and (c) can essentially be considered as  $\chi_r$ -colorings (it is not the case for pattern (d)).

**3.2. Values and bounds for  $\chi_r(k)$ .** We clearly have that  $\chi_r(2) = 2$ . For  $\chi_r(3)$ , we first notice that  $\chi_r(3) = R_r([3, 2])$  and that  $K_{[3,2]}$  is a *star*  $K_{1,2}$  (a graph on three vertices, one of degree 2 and two of degree one). Now, Burr and Roberts [2] proved that

$$R(K_{1,q_1}, \dots, K_{1,q_n}) = \sum_{j=1}^n q_j - n + \epsilon$$

where  $\epsilon = 1$  if the number of even integers in the set  $\{q_1, \dots, q_n\}$  is even,  $\epsilon = 2$  otherwise. Therefore, by applying the above formula when  $q_i = 2$  for all  $i$ , we obtain

$$(4) \quad \chi_r(3) = \begin{cases} r + 1 & \text{for } r \text{ even,} \\ r + 2 & \text{for } r \text{ odd.} \end{cases}$$

**Theorem 3.4.** *Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,*

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \chi_r \left( \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\} \right).$$

*Proof.* Consider a  $\chi_r$ -coloring of  $K_{\chi_r \left( \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\} \right)}$ . Given the vertex labeling of the  $\chi_r$ -coloring, we consider the complete graph  $K'$  induced by the vertices with labels  $1, \dots, \sum_{i=1}^r (k_i - t_i - 1) + 1$  (that is, we remove all the edges induced by the set of vertices  $T_1$  with the  $\max_{1 \leq i \leq r} \{t_i\}$  largest labels). By the pigeonhole principle, there is a set  $T_2$  of at least  $k_i - t_i + 1 - 1$  vertices of  $K'$  with the same color for some  $i$ . Moreover, by definition of  $\chi_r$ -coloring any edge  $\{v_1, v_2\}$  with  $v_1 \in T_1$  and  $v_2 \in T_2$  has color  $i$ , giving the desired monochromatic  $K_{[k_i, t_i]}$ .  $\square$

The following result is an immediate consequence of Theorem 3.4.

**Corollary 3.5.** *Let  $r, k \geq 2$  be integers. Then,*

$$R_r([k, 1]) \leq \chi_r(r(k-2) + 2) \text{ and } R_r([k, 2]) \leq \chi_r(r(k-3) + 3).$$

**Proposition 3.6.** *Let  $r, k \geq 2$  be integers. Then,*

$$\chi_r(k) \leq r\chi_r(k-1) - r + 2.$$

*Proof.* Consider a  $r$ -edge coloring of  $K_{r\chi_r(k-1) - r + 2}$  and let  $u$  be a vertex. Since  $d(u) = r\chi_r(k-1) - r + 1$  then there are at least  $\left\lceil \frac{r\chi_r(k-1) - r + 1}{r} \right\rceil = \chi_r(k-1)$  set of edges with the same color all incident to  $u$ . Now, by definition of  $\chi_r(k-1)$ , there is a clique  $H$  of order  $k-1$  which edge coloring is a  $\chi_r$ -coloring. So, there is a labeling  $\pi$  of  $V(H)$ ,  $|V(H)| = k-1$  and a function  $\phi$  giving such coloring. We claim that the  $r$ -edge coloring of the clique  $H' = H \cup u$  is a  $\chi_r$ -coloring. Indeed, by taking the label  $\pi'(i) = \pi(i) + 1$  for all vertex  $i \neq u$  and  $\pi'(u) = 1$  and the function  $\phi'(1) = 1$  and  $\phi'(i) = \phi(i-1)$  for each  $i = 2, \dots, k$ .  $\square$

**Proposition 3.7.** *Let  $r, k \geq 2$  be integers. Then,*

$$\chi_r(k) \leq g(k, r) = \begin{cases} r^{k-2} + r^{k-3} + \dots + r^2 + r + 2 = \frac{r^{k-1} - 1}{r - 1} + 1 & \text{for } r \text{ odd,} \\ r^{k-2} + r^{k-4} + r^{k-5} + \dots + r^2 + r + 2 = \frac{r^{k-3} - 1}{r - 1} + r^{k-2} + 1 & \text{for } r \text{ even.} \end{cases}$$

*Proof.* By equality (4) and by successive applications of Proposition 3.6.  $\square$

**Theorem 3.8.** Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq g(k, r)$$

where

$$k := \sum_{i=1}^r (k_i - t_i - 1) + 1 + \max_{1 \leq i \leq r} \{t_i\}.$$

*Proof.* By Theorem 3.4 and Proposition 3.7.  $\square$

We believe that the above upper bound for  $R_r([k, 1])$  is smaller than the one given by Corollary 2.3 (see equation (3)) for some values of  $k$ .

**Conjecture 3.9.** Let  $r \geq 3$  be an integer. Then, for all  $3 \leq k \leq r^{3/2} + r - 1$

$$g((r(k-2) + 2, r) < \binom{r(k-1)}{k-1, k-1, \dots, k-1} = \frac{(rk-r)!}{((k-1)!)^r}.$$

We have checked the validity of the above conjecture for all  $3 \leq r \leq 150$  by computer calculations. Conjecture 3.9 is also supported graphically, by considering the continual behaviour of

$$f(k, r) = g((r(k-2) + 2, r) - \frac{(rk-r)!}{((k-1)!)^r}.$$

To see that, we may use the fact that  $\Gamma(z+1) = z!$  when  $z$  is a nonnegative integer, obtaining

$$f(k, r) = g((r(k-2) + 2, r) - \frac{\Gamma(r(k-1) + 1)}{\Gamma^r(k)}$$

where  $\Gamma(z)$  is the well-known *gamma* function<sup>5</sup>, see Figure 2.

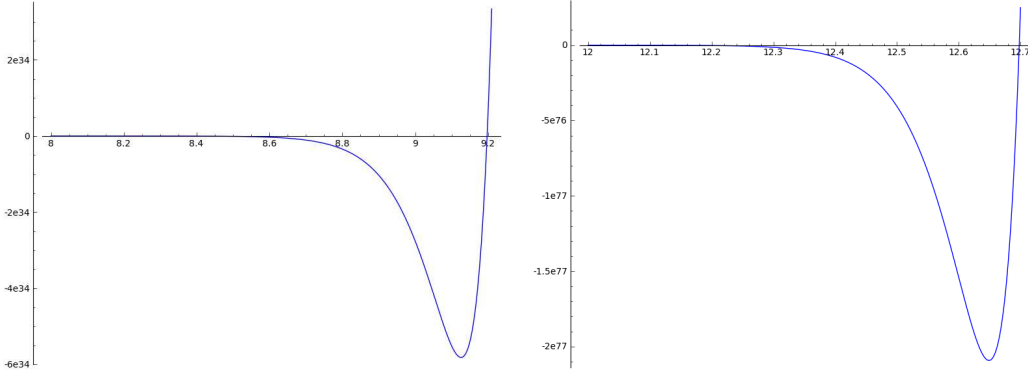


FIGURE 2. Behaviours of  $f(4, k)$  with  $8 \leq k < 10$  (left) and  $f(5, k)$  with  $12 \leq k < 13$  (right). We notice that due to the scaling used in the figures (in order to plot the minimum) the function  $f$  seems very close to zero but in fact it is very far apart,  $f(4, 8) \leq -1,8 \times 10^{29}$  for the left one and  $f(5, 12) \leq -5,7 \times 10^{72}$  for the right one.

We have also checked (by computer) that for each  $3 \leq r \leq 150$  there is an interval  $I_r$  (increasing as  $r$  is growing) such that for each  $k \geq 3, k \in I_r$  the function  $g(r(k-3) + 3, r)$  (resp.  $g(r(k-4) + 4, r)$ ) is a smaller upper bound for  $R_r([k, 2])$  (resp. for  $R_r([k, 3])$ ) than

<sup>5</sup>The gamma function is defined as  $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$  for any  $z \in \mathbb{C}$  with  $\text{Re}(z) > 0$ . Moreover,  $\Gamma(z+1) = z!$  when  $z$  is a nonnegative integer.

the corresponding ones obtained from Corollary 2.3. In view of the latter, we pose the following

**Question 3.10.** Let  $t \geq 1$  and  $r \geq 3$  be integers. Is there a function  $c(r)$  such that for all  $3 \leq k \leq c(r)$

$$g(r(k-t) + t, r) < t \binom{r(k-t)}{k-t, k-t, \dots, k-t} ?$$

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