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Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and Pseudodifferential calculus

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Abstract

Let G be a Lie groupoid. Recall that the associated adiabatic groupoid G_{ad} is the groupoid obtained by the deformation to the normal cone construction corresponding to the inclusion $G^{(0)} \to G$. In the present paper, we consider the crossed product G_{ga} of the natural action of $\mathbb{R}_+^*$ on the adiabatic groupoid G_{ad} associated with any Lie groupoid G. We construct an explicit Morita equivalence between the exact sequence of order 0 pseudodifferential operators on G and (a restriction of) the natural exact sequence associated with G_{ga}. As an important intermediate step, we express a pseudodifferential operator on G as an integral associated to a smoothing operator on the adiabatic groupoid G_{ad} of G.

Keywords: Noncommutative geometry; groupoids; pseudodifferential calculus.

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0 Introduction

Smooth groupoids are intimately linked to pseudodifferential calculi. Indeed, to every smooth groupoid is naturally associated a pseudodifferential calculus and therefore an analytic index. Furthermore many pseudodifferential calculi have been shown to be the ones associated to naturally defined groupoids. The groupoid approach may then give a natural geometric insight to these calculi and the corresponding index theorems. See [10] for an overview on the subject.

The use of groupoids in relation with index theory may be traced back to [7] where A. Connes introduced the longitudinal pseudodifferential calculus on a foliation (M, F). He thus constructs an extension $\Psi^*(M, F)$ of the foliation C^*-algebra $C^*(M, F)$ which gives rise to an exact sequence

$$0 \to C^*(M, F) \to \Psi^*(M, F) \to C(S^*F) \to 0$$

It appeared quite naturally that Connes' construction only used the (longitudinal) smooth structure of the holonomy groupoid and could therefore be extended to any smooth (longitudinally) groupoid (*cf.* [22, 23, 17]).

Some previously defined pseudodifferential calculi were recognized as being the ones associated with natural groupoids as for instance the groupoid defined by B. Monthubert in [20, 21] (see also [23]) was shown to be suitable for R.B. Melrose's b-calculus [19]. Moreover, it appeared in a work of J-M. Lescure ([16]) that this calculus is the natural calculus associated to conical pseudo-manifolds.

In [9], A. Connes showed that the analytic index on a compact manifold can in fact be described in a way not involving (pseudo)differential operators at all, just by using a construction of a deformation groupoid, called the "tangent groupoid". This idea was used in [14], and extended in [22] to the general case of a smooth groupoid, where the authors associated to every smooth groupoid G an *adiabatic groupoid*, which is obtained applying the "deformation to the normal cone" construction to the inclusion $G^{(0)} \to G$ of the unit space of G into G. The groupoid constructed in this way is the union $G_{ad} = G \times \mathbb{R}^* \cup \mathfrak{A}G \times \{0\}$ endowed with a natural smooth structure, where $\mathfrak{A}G$ is the total space of the algebroid of G i.e. of the normal bundle to the inclusion $G^{(0)} \to G$. They then showed that the connecting map of the corresponding exact sequence

$$0 \to C_0(\mathbb{R}_+^*) \otimes C^*(G) \longrightarrow C^*(G_{ad}^+) \xrightarrow{ev_0} C^*(\mathfrak{A}G) \simeq C_0(\mathfrak{A}^*G) \to 0 \tag{1}$$

is the analytic index, where $G_{ad}^+ = G \times \mathbb{R}_+^* \cup \mathfrak{A}G \times \{0\}$ is the restriction of G_{ad} over $G^{(0)} \times \mathbb{R}_+$.

In the present paper, extending ideas of Aastrup, Melo, Monthubert and Schrohe [1], we go one step further in this direction, showing that the (order 0) pseudodifferential operators on a smooth groupoid can also be described as convolution operators by smooth functions on a suitable groupoid. The groupoid that we use is the crossed product of the adiabatic groupoid by the natural action of the group $\mathbb{R}_+^*$. We call it the *gauge adiabatic groupoid.*

More precisely, let $J(G)$ be the preimage of the ideal $C_0(\mathfrak{A}^*G \setminus G^{(0)})$ of functions that vanish on the 0-section $G^{(0)}$ of the bundle $\mathfrak{A}^*G$. The exact sequence (1) gives then an exact sequence

$$0 \to C_0(\mathbb{R}_+^*) \otimes C^*(G) \to J(G) \to C_0(\mathfrak{A}^*G \setminus G^{(0)}) \to 0 \tag{2}$$

Since $J(G)$ is invariant and exact sequence (2) is equivariant with respect to the action of $\mathbb{R}_+^*$, we find an exact sequence

$$0 \to \Big(C_0(\mathbb{R}_+^*) \otimes C^*(G)\Big) \rtimes \mathbb{R}_+^* \longrightarrow J(G) \rtimes \mathbb{R}_+^* \xrightarrow{ev_0} C_0(\mathfrak{A}^*G \setminus G^{(0)}) \rtimes \mathbb{R}_+^* \to 0 \tag{3}$$

Note that $\mathbb{R}_+^*$ acts on $\mathfrak{A}^*G \setminus G^{(0)}$ by multiplication and its orbit space $(\mathfrak{A}^*G \setminus G^{(0)})/\mathbb{R}_+^*$ is the set $S^*\mathfrak{A}G$ of half lines in $\mathfrak{A}^*G$. Since this action is free and proper, the crossed product $C_0(\mathfrak{A}^*G \setminus G^{(0)}) \rtimes \mathbb{R}_+^*$ is canonically isomorphic to $C(S^*\mathfrak{A}G) \otimes \mathcal{K}$. In the same way, we have a natural isomorphism $C^*(G) \otimes \mathcal{K} \simeq \Big(C_0(\mathbb{R}_+^*) \otimes C^*(G)\Big) \rtimes \mathbb{R}_+^*$.

The main result of this paper is that the exact sequence

$$0 \to C^*(G) \otimes \mathcal{K} \to J(G) \rtimes \mathbb{R}_+^* \to C(S^*\mathfrak{A}G) \otimes \mathcal{K} \to 0 \tag{3}$$

is related to the exact sequence of order 0 pseudodifferential operators

$$0 \to C^*(G) \longrightarrow \Psi^*(G) \xrightarrow{\sigma_0} C(S^*\mathfrak{A}G) \to 0$$

via a Morita equivalence. Recall (*cf.* [7, 22, 23]) that $\Psi^*(G)$ is the closure of the algebra of order zero pseudodifferential operators on G in the multiplier algebra of $C^*(G)$ and σ_0 is the principal symbol map.

Our groupoid is the generalization of the one constructed in [1] for the case of the ordinary pseudodifferential calculus on a compact manifold M. It was shown there that the algebra associated to this groupoid is isomorphic to the algebra of Green operators (of order 0 and class 0) in the Boutet de Monvel calculus (*cf.* [12, 27]) which in turn is known to be Morita equivalent to $\Psi^*(M)$. The proof in [1] is somewhat indirect, using Voiculescu's theorem to prove that two exact sequences coincide.

Our proof here is much more direct and therefore extends immediately to the general groupoid case: we explicitly construct a bimodule $\mathcal{E}$ which is a Morita equivalence between the algebras $\Psi^*(G)$ and $J(G) \rtimes \mathbb{R}_+^*$. As an important intermediate step, we express a pseudodifferential operator on G as an integral associated to a smoothing operator on the adiabatic groupoid G_{ad} of G (Theorem 3.7). We should point out that J-M Lescure had previously observed that pseudodifferential operators on $\mathbb{R}^n$ arise as integrals of some functions on the tangent groupoid of $\mathbb{R}^n$ (private communication).

Next, we show that the C^*-algebra $J(G) \rtimes \mathbb{R}_+^*$ is stable and therefore isomorphic to $\Psi^*(G) \otimes \mathcal{K}$, and the Hilbert module $\mathcal{E}$ is isomorphic to Kasparov's absorbing module $\mathcal{H}_{\Psi^*(G)}$. To that end, we use the fact, established by S. Vassout in [28], that positive order pseudodifferential operators define *regular* unbounded multipliers (in the sense of [4, 29] [1]). Actually, using

[1] It follows that, if P is the closure of an elliptic operator of positive order on a Lie groupoid $\mathcal{G}$, then $(1 + P^*P)^{-1} \in C^*(\mathcal{G})$.

pseudodifferential calculus of complex order and complex powers also from [28], we construct an isomorphism of $J(G)$ onto a crossed product $\Psi^*(G) \rtimes \mathbb{R}$ which intertwines the action α of $\mathbb{R}_+^*$ on $J(G)$ with the dual action on the crossed product $\Psi^*(G) \rtimes \mathbb{R}$.

In a forthcoming paper ([11]), we will actually show that the algebra $J(G) \rtimes \mathbb{R}_+^*$ generalizes to the case of Lie groupoids the algebra of Green operators. In particular, we will show that in the case of the usual pseudodifferential calculus on a compact manifold, it is not only abstractly isomorphic to the algebra of Green operators as established in [1], but in fact the two algebras are *equal*.

The paper is organized as follows:

- In the first section, we recall the construction of the deformation to the normal cone and the space of Schwartz functions on it. We also characterize the space of those functions whose Fourier transform vanishes at infinite order at 0.
- In the second section we recall some definitions concerning Lie groupoids, and in particular the construction of the associated "adiabatic" groupoid and its crossed product by the canonical action of $\mathbb{R}_+^*$.
- Finally, in the last two sections, we construct the bimodule which is used for the Morita equivalence: in the third section we construct the smooth module $\mathcal{E}^\infty$, whose C^*-completion the Hilbert C^*-module $\mathcal{E}$ is shown to indeed define the desired Morita equivalence in section 4. At the end of section 4, we establish stability of the module $\mathcal{E}$ and actually identify the ideal $J(G)$ with a crossed product $\Psi^*(G) \rtimes \mathbb{R}$.

Remark on the general setting. *We will use the denomination "Lie groupoid" or "smooth groupoid" for the same object: a groupoid whose set of objects and morphisms are both smooth manifolds, source and range (target) maps are smooth submersions and the maps composition and inverse are smooth.*
For the simplicity of our exposition, we will assume that the groupoids involved are smooth and Hausdorff. We will further in some places reduce to the case where the unit space is compact. Of course one easily extends our constructions and results to more general settings: longitudinally smooth groupoids on manifolds with corners, continuous family groupoids (cf. [24]), non Hausdorff case (cf. [8]), and even groupoids associated with singular foliations (cf. [2])... To proceed, one only has to consider the appropriate spaces of functions on the groupoids - that appear in the above cited papers.
Furthermore, in what follows, $C^*(G)$ may be either the reduced or full C^*-algebra of the groupoid G. The choice is left to the reader!

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1 Schwartz spaces on deformations to normal cones

The adiabatic groupoid which is the main ingredient in our construction is a special case of a geometric object called *deformation to the normal cone.* This is a classical notion in algebraic geometry. It was first used in the context of Lie groupoids by A. Connes (see [9]) and since then by several authors.

1.1 Deformation to a normal cone

In this section, we recall this construction and define some function spaces on it: the associated space of functions of Schwartz decay *alla* P. Carrillo-Rouse ([5]) and some subspaces which will be needed in the sequel of the paper.

Let M_0 be a smooth compact submanifold of a smooth manifold M with normal bundle $\mathcal{N}$. As a set, the deformation to the normal cone is $D(M_0, M) = M \times \mathbb{R}^* \cup \mathcal{N} \times \{0\}$. In order to recall its smooth structure, we fix an exponential map $\theta : V' \to V$. Thus θ is a diffeomorphism from a neighborhood V' of the zero section M_0 in $\mathcal{N}$ to a neighborhood V of M_0 in M, for every $x \in M_0$, we have $\theta(x) = x$ and the differential $d\theta_x$ is the identity in the normal direction.

We may cover $D(M_0, M)$ with two open sets $M \times \mathbb{R}^*$ and $W = \mathcal{N} \times \{0\} \cup V \times \mathbb{R}^*$: we endow $D(M_0, M)$ with the smooth structure for which the map $\Theta : (x, X, t) \mapsto (\theta(x, tX), t)$ (for $t \neq 0$) and $\Theta : (x, X, 0) \mapsto (x, X, 0)$ is a diffeomorphism from $W' = \{(x, X, t) \in \mathcal{N} \times \mathbb{R};\ (x, tX) \in V'\}$ to W.

The group $\mathbb{R}_+^*$ acts smoothly on $D(M_0, M)$: for $t \in \mathbb{R}_+^*$ put $\alpha_t(z, \lambda) = (z, t\lambda)$ for $z \in M$ and $\lambda \in \mathbb{R}^*$ and $\alpha_t(x, U, 0) = (x, t^{-1}U, 0)$ for $x \in M_0$ and $U \in \mathcal{N}_x$.

1.2 Schwartz decay on $\mathbb{R}_+^* \times V$

If V is a smooth non necessarily compact manifold, define $\mathcal{S}(\mathbb{R}_+^*; C_c^\infty(V))$ to be the space of smooth functions $t \mapsto f_t$ from $\mathbb{R}_+^*$ to $C_c^\infty(V)$ such that all the f_t have support on a given compact subset of V and $t \mapsto f_t$ has rapid decay with respect to all natural norms of $C_c^\infty(V)$. More precisely (f_t) is such that the function $g : V \times \mathbb{R} \to \mathbb{C}$ defined by $g(z, t) = f_{\frac{t}{1-t}}(z)$ for $0 < t < 1$ and $g(z, t) = 0$ otherwise is smooth with compact support.

We will also consider $\mathcal{S}(\mathbb{R}^*; C_c^\infty(V)) = \mathcal{S}(\mathbb{R}_+^*; C_c^\infty(V)) \oplus \mathcal{S}(\mathbb{R}_-^*; C_c^\infty(V))$ which is the space of smooth functions $t \mapsto f_t$ from $\mathbb{R}^*$ to $C_c^\infty(V)$ such that all the f_t have support on a given compact subset of V and $t \mapsto f_t$ has rapid decay with respect of all norms at 0 and $\pm\infty$.

1.3 Schwartz functions on a vector bundle

There is a description of the space of Schwartz functions on the total space of a vector bundle in [5]. Here are two equivalent ways to defining this algebra in a way that it is obvious it is independent of choices of charts. Denote by S_E the set of half lines of the bundle $\mathbb{R} \times E$ over M and, choosing a metric q on E, embed E into S_E using the map which to (x, ξ) ($x \in M$, $\xi \in E_x$ assigns the half line through $(x, 1 - q(\xi), 2\xi)$. Then E sits in S_E as the complement of points at infinity i.e. the set M_∞ of half lines through $(x, -1, 0)$ where x runs through M.

Definition 1.1. Let E be a smooth real vector bundle over a smooth compact manifold M.

a) Consider E as an open subspace of the bundle of spheres S_E where M_∞ is the set of points at infinity. The algebra $\mathcal{S}(E)$ is the space of smooth functions on S_E vanishing at any point of M_∞ as well as all its derivatives.

b) For $t \in \mathbb{R}_+^*$, let β_t denote the map $(x, \xi) \mapsto (x, t\xi)$ from E to E. Then define $\mathcal{S}_\beta(E)$ to be the space of functions of the form $z \mapsto \displaystyle\int_0^{+\infty} f_t(\beta_t(z))\, dt$ where $f \in \mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E))$.

The fact that $\mathcal{S}_\beta(E) = \mathcal{S}(E)$ is quite obvious. Indeed if $f \in \mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E))$, then one defines $g \in C_c^\infty(S_E \times \mathbb{R})$ by $g(z,t) = f_{\frac{t}{1-t}}(\beta_{\frac{t}{1-t}}(z))$ if $0 < t < 1$ and $z \in E$ and $g(z,t) = 0$ elsewhere. Then one may integrate it and obtain that $z \mapsto \displaystyle\int_0^{+\infty} f_t(\beta_t(z))\,dt$ is smooth on S_E and vanishes as well as all its derivatives on M_∞.

Conversely, we have to show that the map $\varphi_\beta : \mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E)) \to \mathcal{S}(E)$ is onto, where $\varphi_\beta(f) : z \mapsto \displaystyle\int_0^{+\infty} f_t(\beta_t(z))\,dt$. Choose a metric on E; if $g \in \mathcal{S}(E)$, we may put $f(z,t) = \dfrac{h(\|z\|^2 + t^2)}{t} g(\beta_{t^{-1}}(z))$ where $h \in C_c^\infty(\mathbb{R})$ vanishes near 0 and $\displaystyle\int_0^{+\infty} h(s^2)\frac{ds}{s} = 1$. Then $\varphi_\beta(f)(z) = \displaystyle\int_0^{+\infty} h(t^2(1 + \|z\|^2))g(z)\,\frac{dt}{t} = g(z)$.

Remark 1.2. The Fourier transform $f \mapsto \hat{f}$ is an isomorphism of $\mathcal{S}_\beta(E)$ with $\mathcal{S}_{\beta^*}(E^*)$.

We will use the following rather easy result (where we consider $E \times \mathbb{R}$ as a bundle of dimension $\dim E + 1$ over M).

Proposition 1.3. *Let $f = (f_t)_{t\in\mathbb{R}} \in \mathcal{S}(E \times \mathbb{R})$. Thus, for $t \in \mathbb{R}$, $f_t \in \mathcal{S}(E)$ and $\hat{f}_t \in \mathcal{S}(E^*)$.*

a) *For all $g \in C_c^\infty(E)$ the function $F : M \times \mathbb{R} \to \mathbb{C}$ defined by $F(x,0) = \hat{f}_0(x,0)g(x,0)$ and $F(x,t) = t^{-p}\displaystyle\int_{E_x} g(x,U) f_t \circ \theta(x,U)\,dU$ for $t \neq 0$ is smooth on $M \times \mathbb{R}$.*

b) *The following are equivalent:*

   (i) *For all $g \in \mathcal{S}(E)$ the function $t \mapsto \displaystyle\int_E g(x,U) f(x, t^{-1}U, t)\,dx\,dU$ vanishes as well as all its derivatives at 0.*

   (ii) *For all $g \in \mathcal{S}(E)$ the (smooth) function $(x,t) \mapsto \displaystyle\int_{E_x} g(x,U) f(x,t^{-1}U,t)\,dU$ vanishes as well as all its derivatives on $M \times \{0\} \subset M \times \mathbb{R}$.*

   (iii) *The function $(x,\xi,t) \mapsto \hat{f}_t(x,\xi)$ vanishes as well as all its derivatives on $M \times \{0\}$ sitting in $E^* \times \mathbb{R}$ as zero section.*

Proof. Parseval's formula yields $t^{-p}\displaystyle\int_{E_x} g(x,U) f(x,t^{-1}U,t)\,dU = c\int_{E_x^*} \hat{g}(x,-\xi)\hat{f}_t(x,t\xi)\,d\xi$ (where c is a suitable constant and p is the dimension of E).

a) The function $(x,\xi,t) \mapsto \hat{g}(x,-\xi)\hat{f}_t(x,t\xi)$ lies in $\mathcal{S}(E^* \times \mathbb{R})$, thus (a) follows.

b) (ii)$\Rightarrow$(i) is obvious. Conversely if (i) is satisfied, writing $\displaystyle\int_{E_x} g(x,U) f(x,t^{-1}U,t)\,dU = t^k h_k(x) + o(t^k)$, and applying (i) to $g_1(x,U) = g(x,U)\overline{h_k(x)}$, we find $h_k = 0$; thus by induction, we get (ii).

   We may write the Taylor expansion $\hat{f}_t(x,t\xi) = \displaystyle\sum_{j=0}^{k} t^j a_j(x,\xi) + t^{k+1}R(x,\xi,t)$ where a_j are polynomials in ξ (of degree $\leq j$). It follows that $\displaystyle\int_{E_x} g(x,U) f(x,t^{-1}U,t)\,dU = o(t^{k+p})$ if and only if $a_j(x,\xi) = 0$, whence (ii) $\Longleftrightarrow$ (iii). $\square$

Remark 1.4. It is natural in this proposition to consider $f_t(x,U)$ as smooth families (indexed by M) of smooth measures on E_x (sections of the bundle of densities $\Omega^1(E)$) rather than functions and therefore introduce a factor t^{-p} in the integrals (i) and (ii). This factor of course has no consequence since rapid decay is not affected by this factor...

1.4 Trivial deformation to the normal cone

We consider here the deformation to the normal cone $D(M,E)$ (in the sense of paragraph 1.1) of the manifold M is sitting as zero section in the total space E of a vector bundle over M. Then $D(M,E) = E \times \mathbb{R}$. Here, $\mathbb{R}_+^*$ acts on $E \times \mathbb{R}$ by $\alpha_t(x,U,\lambda) = (x, t^{-1}U, t\lambda)$.

For $f \in \mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E \times \mathbb{R}))$, put $\varphi_\alpha(f)(z) = \displaystyle\int_0^{+\infty} f_t(\alpha_t(z))\,dt$. The image $\mathcal{S}_\alpha(E \times \mathbb{R})$ of φ_α is the set of $g \in \mathcal{S}(E \times \mathbb{R})$ such that the function $(x,U,t) \mapsto \|tU\|$ is bounded on the support of g.

Indeed, if $f \in \mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E \times \mathbb{R}))$, one checks immediately the support requirement and it is quite easy to check locally that $\varphi_\alpha(f) \in \mathcal{S}(E \times \mathbb{R})$. Conversely, let $g \in \mathcal{S}(E \times \mathbb{R})$ with the support requirements; take $\chi \in C_c^\infty(\mathbb{R})$ equal to 1 near 0 and $h \in C_c^\infty(\mathbb{R})$ a function which vanishes near 0 and satisfies $\displaystyle\int_0^{+\infty} h(s^2)\frac{ds}{s} = 1$ as previously. We may set $f_1(x,U,\lambda,t) = \dfrac{h(\lambda^2+t^2)}{t} g(x,tU,\dfrac{\lambda}{t})(1-\chi(\dfrac{\lambda^2}{t^2}))$ and $f_2(x,U,\lambda,t) = \dfrac{h(\|U\|^2+t^{-2})}{t} g(x,tU,\dfrac{\lambda}{t})\chi(\dfrac{\lambda^2}{t^2})$.

Note that for any $a > 0$, the function $t \mapsto \dfrac{1}{t} g(x,tU,\dfrac{\lambda}{t})$ obviously belongs to $\mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E_{\geq a} \times \mathbb{R}_{\geq a}))$ where $E_{\geq a} = \{U \in E\ ; \|U\| \geq a\}$ and $\mathbb{R}_{\geq a} = \mathbb{R} \backslash]-a,a[$.

For small enough $|\lambda|$ either $h(\lambda^2+t^2) = 0$ or $\chi(\dfrac{\lambda^2}{t^2}) = 1$ thus f_1 vanishes. For big enough $|\lambda|$ or t, $h(\lambda^2+t^2) = 0$. Moreover f_1 has rapid decay when $t \to 0$. Finally, f_1 belongs to $\mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E \times \mathbb{R}))$. Similarly f_2 vanishes for t near 0 or for small enough $\|U\|$ and $t \to \infty$. Anyway f_2 has rapid decay when $t \to \infty$ and thus f_2 belongs to $\mathcal{S}(\mathbb{R}_+^*; C_c^\infty(E \times \mathbb{R}))$. One can easily check that $\varphi_\alpha(f_1) + \varphi_\alpha(f_2) = g$.

1.5 Schwartz functions on a deformation to the normal cone

In the same way as for bundles, we define $\mathcal{S}_\alpha(D(M_0,M))$ to be the set of integrals $\varphi_\alpha(f) : z \mapsto \displaystyle\int_0^{+\infty} f_t(\alpha_t(z))\,dt$ where $t \mapsto f_t$ is in $\mathcal{S}(\mathbb{R}_+^*; C_c^\infty(D(M_0,M)))$.

Now, let $\theta : V' \to V$ be an "exponential map" which is a diffeomorphism of a (relatively compact) neighborhood V' of the 0 section M_0 in $\mathcal{N}$ onto a tubular neighborhood V of M_0 in M. We obtain a diffeomorphism $\Theta : W' \to W$ where $W' = \{(x,U,t) \in \mathcal{N} \times \mathbb{R};\ (x,tU) \in V'\}$ and $W = V \times \mathbb{R}^* \cup \mathcal{N} \times \{0\}$.

Since $D(M_0,M) = M \times \mathbb{R}^* \cup W$, it follows that $C_c^\infty(D(M_0,M)) = C_c^\infty(M \times \mathbb{R}^*) + C_c^\infty(W)$, and since both $M \times \mathbb{R}^*$ and W are invariant by α, it follows that $\mathcal{S}_\alpha(D(M_0,M))$ is the sum of $\mathcal{S}(\mathbb{R}^*; C_c^\infty(M))$ obtained as $\varphi_\alpha(\mathcal{S}(\mathbb{R}_+^*; C_c^\infty(M \times \mathbb{R}^*)))$ and $\{f \in C^\infty(W);\ f \circ \Theta \in \mathcal{S}_\alpha(\mathcal{N} \times \mathbb{R})\}$ (where $f \circ \Theta$ is extended by 0 outside W').

We denote by $\mathfrak{J}_0(M_0,M)$ the subspace $\mathcal{S}(\mathbb{R}^*; C_c^\infty(M))$ of $\mathcal{S}_\alpha(D(M_0,M))$.

For $f \in \mathcal{S}_\alpha(D(M_0,M))$ and $t \in \mathbb{R}$, we denote by $f_t : z \mapsto f(z,t)$ (with $z \in M$ if $t \neq 0$ and $z \in \mathcal{N}$ if $t = 0$).

Note that if $f \in \mathcal{S}_\alpha(D(M_0, M))$ and $\psi \in C^\infty(M)$ vanishes in the neighborhood of M_0, then the map $(z,t) \mapsto \psi(z)f(z,t)$ for $t \neq 0$ extends to a smooth map on $M \times \mathbb{R}$ vanishing at infinite order on $M \times \{0\}$.

As a direct consequence of Proposition 1.3 we have:

Proposition 1.5. *Let $\theta : V' \to V$ be an exponential diffeomorphism as above and $\chi \in C_c^\infty(V)$ equal to 1 near $M_0 \subset M$. Let $f \in \mathcal{S}_\alpha(D(M_0, M))$*

a) *For all $g \in C_c^\infty(\mathcal{N})$ the function $F : M_0 \times \mathbb{R} \to \mathbb{C}$ defined by $F(x,0) = \hat{f}_0(x,0)g(x,0)$ and $F(x,t) = t^{-p} \displaystyle\int_{\mathcal{N}_x} g(x,U) f_t \circ \theta(x,U)\, dU$ for $t \neq 0$ is smooth on $M_0 \times \mathbb{R}$.*

b) *The following are equivalent:*

   (i) *For all $g \in C_c^\infty(M)$ the function $t \mapsto \displaystyle\int_M g(x)f(x,t)\, dx$ vanishes as well as all its derivatives at 0.*

   (ii) *For all $g \in C_c^\infty(\mathcal{N})$ the function $(x,t) \mapsto \displaystyle\int_{\mathcal{N}_x} g(x,U) f_t \circ \theta(x,U)\, dU$ vanishes as well as all its derivatives on $M_0 \times \{0\} \subset M_0 \times \mathbb{R}$.*

   (iii) *The function $(x,\xi,t) \mapsto \widehat{(\chi f_t)} \circ \theta(x,\xi)$ vanishes as well as all its derivatives on $M_0 \times \{0\}$ sitting in $\mathcal{N}^* \times \mathbb{R}$ as zero section.*

In particular, it follows that conditions (ii) *and* (iii) *do not depend on the choice of θ.*
We denote by $\mathfrak{J}(M_0, M) \subset \mathcal{S}_\alpha(D(M_0, M))$ the set of functions satisfying the above equivalent conditions.

Proof. For every $f \in \mathcal{S}_\alpha(D(M_0, M))$, the function $(1-\chi)f_t$ has rapid decay when $t \to 0$. Replacing f by χf, we may assume that $f \in \mathcal{S}_\alpha(D(M_0, W))$. The result follows from Proposition 1.3 since $\mathcal{S}_\alpha(D(M_0, W)) \xrightarrow{\theta^*} \mathcal{S}_\alpha(D(M_0, W'))$ is an isomorphism. □

1.6 A family of semi-norms

We define a family $N_{k,\ell,j,m}$ of semi-norms on smooth functions on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}$, for $k \in \mathbb{N}^n$, $\ell \in \mathbb{N}^p$, $j \in \mathbb{N}$ and $m \in \mathbb{Z}$. Put

$$N_{k,\ell,j,m}(f) = \sup_{(x,\xi,t) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}} (\|\xi\|^2 + t^2)^{m/2} \left| \frac{\partial^{|k|+|\ell|+j} f}{\partial x^k \partial \xi^\ell \partial t^j}(x,\xi,t) \right|$$

We now use the notation introduced in previous subsection. Assume first that M_0 is compact. Fix a finite open cover $(\mathcal{O}_i)_{i \in I}$ of M_0 by subsets diffeomorphic to $\mathbb{R}^n$ over which the normal bundle is trivialized. Using a partition of the identity adapted to $(\mathcal{O}_i)_{i \in I}$, we obtain a family of semi-norms $N^i_{k,\ell,j,m}$ on $C^\infty(\mathcal{N}^* \times \mathbb{R})$ and thus, via the Fourier transform and the map $\Theta : W' \to W$ defined above, a family $\widetilde{N}^i_{k,\ell,j,m}$ of semi-norms on the space $C_c^\infty(W)$ defined by

$$\widetilde{N}^i_{k,\ell,j,m}(f) = N^i_{k,\ell,j,m}(\widehat{f \circ \Theta})$$

(where $f \circ \Theta$ is extended by 0 outside W').

Finally, if M_0 is not compact, we define similarly a family of semi-norms using a locally finite cover $(\mathcal{O}_i)_{i\in I}$.

The action of $\mathbb{R}_+^*$

The action of $\mathbb{R}_+^*$ on $D(M_0, M)$ leads to an action by automorphisms $u \mapsto \alpha_u$ on $\mathcal{S}_\alpha(D(M_0, M))$ and its subspaces $\mathfrak{J}(M_0, M)$ (see prop. 1.5) and $\mathfrak{J}_0(M_0, M)$ $(= \mathcal{S}(\mathbb{R}^*; C_c^\infty(M)))$. Note that $\widehat{\alpha_u(f)\circ\Theta}(x,\xi,t) = u^p\widehat{f\circ\Theta}(x, u\xi, ut)$ and therefore, the semi-norms $\widetilde{N}^i_{k,\ell,j,m}$ are multiplied by a suitable power of u by this action.

2 The groupoids

Convolution algebras and pseudodifferential calculus on smooth groupoids was first used by A. Connes in [7] in the case of the holonomy groupoid of a foliation (see also [8, 9]). Note that Connes uses the reduced C^*-algebras - full C^*-algebras were introduced in the general context of locally compact groupoids by J. Renault ([26]). Generalization to Lie groupoids, which was already implicit in Connes' work was given in [22, 23] (see also [10, 28] for an overview on the subject).

2.1 Main notation

We will use the denomination "Lie groupoid" or "smooth groupoid" for the same object: a groupoid whose set of objects and morphisms are both smooth manifolds, source and range (target) maps are smooth submersions and the maps composition and inverse are smooth.

Let us recall some standard constructions and notation on groupoids.

Densities. If M is a smooth manifold, E is a real vector bundle of dimension p on M and $q \in \mathbb{R}$, we denote by $\Omega^q(E) = |\Lambda^p E^*|^q$ the q-density bundle on E: for $x \in M$, $\Omega^q_x E$ is the set of maps $\psi : \Lambda^p E_x \setminus \{0\} \to \mathbb{R}$ such that $\psi(\lambda X) = |\lambda|^q\psi(X)$. A q-density is a section of $\Omega^q(E)$ and the product of a q-density with a q'-density leads to a $(q+q')$-density. There is a natural isomorphism $\Omega^q(E \oplus E') \to \Omega^q(E) \otimes \Omega^q(E')$.

The positivity of a density makes sense, and thus the bundle of densities is an oriented real line bundle which is therefore trivial(izable). It is however sometimes important to keep track of the natural normalizations they give rise to.

We put $\Omega^q(M) = \Omega^q(TM)$.

The main use of 1-densities is that their integral over M makes sense. The natural action of diffeomorphism takes into account the Radon-Nykodym derivative and therefore, there is a unique linear form $\int_M : C_c^\infty(M; \Omega^1(M)) \to \mathbb{R}$ which agrees in local coordinates with the Lebesgue integral. In this way one associates to a submersion $p : M \to M_1$ and a vector bundle E on M_1 a natural map $p_! : C_c^\infty(M; p^*E \otimes \Omega^1(\ker dp)) \to C_c^\infty(M_1; E)$ obtained by integrating one densities along the fibers of p.

Source, range, algebroid. When $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a Lie groupoid with source s and range r, we denote $\mathcal{G}_x := s^{-1}(x)$ and $\mathcal{G}^x := r^{-1}(x)$ for any $x \in \mathcal{G}^{(0)}$. The set of composable elements is $\mathcal{G}^{(2)} = \{(\gamma, \gamma');\ s(\gamma) = r(\gamma')\}$; the product $(\gamma, \gamma') \mapsto \gamma\gamma'$ is a smooth submersion $p : \mathcal{G}^{(2)} \to \mathcal{G}$.

The *s-vertical tangent bundle* is the tangent space to the s-fibers, that is $T_s\mathcal{G} := \ker ds = \bigcup_{x\in\mathcal{G}^{(0)}} T\mathcal{G}_x$. The r-vertical tangent bundle $T_r\mathcal{G} := \ker dr$ is defined similarly. Recall that the restriction of $T_s\mathcal{G}$ to the set $\mathcal{G}^{(0)}$ of units identifies with the total space $\mathfrak{A}\mathcal{G}$ of the Lie algebroid of $\mathcal{G}$, which can also be thought of as the normal bundle to the inclusion $\mathcal{G}^{(0)} \to \mathcal{G}$. As usual, we will denote by $T_s^*\mathcal{G}$ and $\mathfrak{A}^*\mathcal{G}$ the corresponding dual bundles.

The $*$-algebra of a groupoid. Let $\mathcal{G}$ be a Lie groupoid. As explained in [8], the natural $*$-algebra is obtained using half densities, namely $C_c^\infty(\mathcal{G};\Omega^{1/2}(\ker ds \oplus \ker dr))$ endowed with the following operations:

Involution. The map $\kappa : \gamma \mapsto \gamma^{-1}$ exchanges r and s and therefore it acts naturally on $\Omega^{1/2}(\ker ds \oplus \ker dr)$; also, the bundle $\Omega^{1/2}(\ker ds \oplus \ker dr)$ has a real structure, i.e. there is a natural complex conjugation $\omega \mapsto \overline{\omega}$ of this bundle. The adjoint of $f \in C_c^\infty(\mathcal{G};\Omega^{1/2}(\ker ds \oplus \ker dr))$ is defined by $f^*(\gamma) = \kappa_*\Big(\overline{f(\gamma^{-1})}\Big)$.

Product. If $f, g \in C_c^\infty(\mathcal{G};\Omega^{1/2}(\ker ds \oplus \ker dr))$, then the restriction of $f \otimes g$ to $\mathcal{G}^2$ is a section of the bundle $\Omega^{1/2}(p^*T_r\mathcal{G} \oplus \ker dp \oplus \ker dp \oplus p^*T_s\mathcal{G}) = p^*\Omega^{1/2}(\ker dr \oplus \ker ds) \otimes \Omega^1 \ker dp$, and by integration along the fibers of p we obtain $f * g \in C_c^\infty(\mathcal{G};\Omega^{1/2}(\ker ds \oplus \ker dr))$.

From now on, we just write $C_c^\infty(\mathcal{G};\Omega^{1/2})$ instead of $C_c^\infty(\mathcal{G};\Omega^{1/2}(\ker ds \oplus \ker dr))$. The C^*-algebra of the groupoid $\mathcal{G}$ is a completion of the $*$-algebra $C_c^\infty(\mathcal{G};\Omega^{1/2})$).

The reduced C^*-algebra (*cf.* [7, 26]) is obtained as the completion of $C_c^\infty(\mathcal{G};\Omega^{1/2})$ by the family of representations $(\lambda_x)_{x\in\mathcal{G}^{(0)}}$, where λ_x is the representation by left convolution on $L^2(\mathcal{G}_x)$ (which is the completion of $C_c^\infty(\mathcal{G}_x;\Omega^{1/2}\mathcal{G}_x)$).

The full C^*-algebra is obtained as the completion of $C_c^\infty(\mathcal{G};\Omega^{1/2})$ by the family of all continuous representations (*cf.* [26, 15]).

Pseudodifferential operators on Lie groupoids ([7, 22, 23]). Let $\mathcal{G}$ be a Lie groupoid and let $\theta : V' \to V$ be an "exponential map" which is a diffeomorphism of a (relatively compact) neighborhood V' of the 0 section $\mathcal{G}^{(0)}$ in $\mathfrak{A}\mathcal{G}$ (considered as the normal bundle to the inclusion $\mathcal{G}^{(0)} \subset \mathcal{G}$) onto a tubular neighborhood V of $\mathcal{G}^{(0)}$ in $\mathcal{G}$. We assume that $r(\theta(x,U)) = x$ for $x \in \mathcal{G}^{(0)}$ and $U \in \mathfrak{A}_x\mathcal{G}$.

Le $m \in \mathbb{Z}$. A classical pseudo-differential operator of order m is a multiplier $P = P_0 + K$ of the $*$-algebra $C_c^\infty(\mathcal{G};\Omega^{1/2})$ where $K \in C_c^\infty(\mathcal{G};\Omega^{1/2})$ and left multiplication by P_0 is given by an expression

$$P_0 * f(\gamma) = \int_{\xi\in\mathfrak{A}^*_{r(\gamma)}\mathcal{G}} \Big(\int_{\gamma_1\in\mathcal{G}^{r(\gamma)}} e^{i\langle\theta^{-1}(\gamma_1)|\xi\rangle} \varphi(r(\gamma),\xi)\chi(\gamma_1) f(\gamma_1^{-1}\gamma)\, d\gamma_1 \Big) d\xi$$

where $\chi \in C_c^\infty(V)$ is a bump function satisfying $\chi = 1$ on $\mathcal{G}^{(0)}$ and $\varphi \sim \sum_{k=0}^{+\infty} a_{m-k}$ is a polyhomogeneous symbol ($a_j(x,\xi)$ is homogeneous of order j in ξ).

We may then write $P_0(\gamma_1) = \int_{\xi\in\mathfrak{A}^*_{r(\gamma_1)}\mathcal{G}} e^{i\langle\theta^{-1}(\gamma_1)|\xi\rangle}\varphi(r(\gamma_1),\xi)\chi(\gamma_1)\,d\xi$ meaning *e.g.* that, as a multiplier, P_0 is the limit when $R\to\infty$ of $P_0^R \in C_c^\infty(\mathcal{G};\Omega^{1/2})$ where

$$P_0^R(\gamma) = \int_{\xi\in\mathfrak{A}^*_{r(\gamma)}\mathcal{G};\ \|\xi\|\leq R} e^{i\langle\theta^{-1}(\gamma)|\xi\rangle}\varphi(r(\gamma),\xi)\chi(\gamma)\,d\xi.$$

Recall that if the order m of P is strictly negative, then P extends to an element of $C^*(\mathcal{G})$ and if $m=0$, then P extends to a multiplier of $C^*(\mathcal{G})$. The map $P\mapsto a_0$ is well defined and extends to an onto morphism $\sigma_0 : \Psi^*(\mathcal{G})\to C(S^*\mathfrak{A}\mathcal{G})$ with kernel $C^*(\mathcal{G})$ where $\Psi^*(\mathcal{G})$ is the closure of the algebra of pseudodifferential operators of order ≤ 0 in the multiplier algebra of $C^*(\mathcal{G})$ and $C(S^*\mathfrak{A}\mathcal{G})$ is the commutative C^*-algebra of continuous functions on the space $S^*\mathfrak{A}\mathcal{G}$ of half lines in $\mathfrak{A}^*\mathcal{G}$. In other words, we have an exact sequence of C^*-algebras

$$0\to C^*(\mathcal{G})\longrightarrow \Psi^*(\mathcal{G})\xrightarrow{\sigma_0} C(S^*\mathfrak{A}\mathcal{G})\to 0.$$

2.2 The adiabatic groupoid

We briefly recall the construction of the adiabatic groupoid mainly due to Connes (see [9]) and extended in [22, 23]. Let us start with a smooth groupoid $G\rightrightarrows G^{(0)}$ with source map s and range map r and denote by $\mathfrak{A}G$ its Lie algebroid: it is the normal bundle of the inclusion $G^{(0)}\to G$ as unit space.

Its adiabatic groupoid is the deformation to the normal cone $G_{ad} = D(G^{(0)}, G)$ as in section 1.1.

As a set, $G_{ad} = G\times\mathbb{R}^*\cup\mathfrak{A}G\times\{0\}$, and its set of objects $G_{ad}^{(0)}$ is $G^{(0)}\times\mathbb{R}$. Its groupoid structure is defined by the fact that inclusions $q\mapsto(q,\lambda)$ are groupoid morphisms from G to G_{ad} for $\lambda\neq 0$, and from $\mathfrak{A}G$ to G_{ad} for $\lambda=0$.

We fix an everywhere positive smooth density ω on G. We then get a smooth density ω_{ad} on G_{ad}: for $t\neq 0$, $\omega_{ad,t} = |t|^{-p}\omega$. We will now on fix the density ω (and therefore ω_{ad}) and consider all elements of the groupoid algebra $C_c^\infty(G_{ad})$ as functions.

2.3 The action of $\mathbb{R}_+^*$ and the gauge adiabatic groupoid

The action of the group $\mathbb{R}_+^*$ on the deformation to a normal cone that we already used, is compatible with the groupoid structure of $G_{ad} = D(G^{(0)}, G)$:

$$\begin{array}{rcl} G_{ad}\times\mathbb{R}_+^* & \to & G_{ad} \\ (\gamma,t;\lambda) & \mapsto & (\gamma,\lambda t) \text{ when } t\neq 0 \\ (x,U,0,\lambda) & \mapsto & (x,\lambda^{-1}U,0) \end{array}$$

The *gauge adiabatic groupoid* is the (smooth) groupoid obtained as a crossed product of this action (see *e.g.* [18]). Namely $G_{ga} := G_{ad}\rtimes\mathbb{R}_+^*\rightrightarrows G^{(0)}\times\mathbb{R}_+$ with structural morphisms

$$\begin{array}{ll} \text{source and target :} & s_{ga}(\gamma,t;\lambda) = (s(\gamma),t) \text{ and } r_{ga}(\gamma,t;\lambda) = (r(\gamma),t\lambda) \text{ for } t\neq 0 \\ & s_{ga}(x,U,0;\lambda) = r_{ga}(x,U,0;\lambda) = (x,0) \\ \text{product :} & (\gamma,\lambda' t;\lambda)\cdot(\gamma',t;\lambda') = (\gamma\gamma',t;\lambda\lambda') \text{ for } t\neq 0 \\ & (x,U,0;\lambda)\cdot(x,U',0;\lambda') = (x,\lambda' U+U',0;\lambda\lambda')\ . \end{array}$$

Remarks 2.1. a) At the level of Lie algebroids, these constructions are very simple and natural. Let us denote $(\mathfrak{A}G, \sharp, [\ ,\ ]_{\mathfrak{A}G})$ the Lie algebroid of G, with its corresponding anchor and bracket.

The Lie algebroid of G_{ad} is $(\mathfrak{A}G \times \mathbb{R}, \sharp_{ad}, [\ ,\ ]_{ad})$ where $\sharp_{ad} : \mathfrak{A}G \times \mathbb{R} \to TG^{(0)} \times T\mathbb{R}$ is defined by $\sharp_{ad}(x, U, t) = (\sharp(x, tU), (t, 0))$ for (x, U, t) in $\mathfrak{A}G \times \mathbb{R}$, and $[\ ,\ ]_{ad}$ is the Lie bracket which satisfies $[X, Y]_{ad}(x, t) = t[X, Y]_{\mathfrak{A}G}(x)$ where $X,\ Y$ are smooth (local) sections of $\mathfrak{A}G$.

The Lie algebroid of G_{ga} is $(\mathfrak{A}G \times T\mathbb{R}, \sharp_{ga}, [\ ,\ ]_{ga})$ where $\sharp_{ga}(x, U, t, \lambda) = (\sharp(x, tU), (t, t\lambda))$ and $[\ ,\ ]_{ga}$ is the Lie bracket induced by $[(X, \tau), (Y, \sigma)]_{ga}(x, t) = (t[X, Y]_{\mathfrak{A}G}(x) + \tau(t)Y(x) - \sigma(t)X(x), t[\tau, \sigma](t))$ where $X,\ Y$ are smooth (local) sections of $\mathfrak{A}G$ and $\tau,\ \sigma$ are smooth local vector fields on $\mathbb{R}$.

b) This construction immediately extends to the case where G is only assumed to be longitudinally smooth (i.e. a continuous family groupoid in the sense of A. Paterson [24]).

c) Also, if $(M, \mathcal{F})$ is a singular foliation in the sense of [2] generated by vector fields $(X_i)_{1 \leq i \leq n}$, the *adiabatic foliation* was constructed in [3]. We may construct the *gauge adiabatic foliation* on $M \times \mathbb{R}$ to be the foliation generated by the vector fields $t(X_i \otimes 1)$ and $t(1 \otimes \partial/\partial t)$ (where t is the $\mathbb{R}$ coordinate in $M \times \mathbb{R}$).

3 Schwartz algebra and module

3.1 The Schwartz algebra of Carrillo-Rouse and the ideal $\mathcal{J}(G)$

Let G be a Lie groupoid and $G_{ad} = D(G^{(0)}, G)$ the corresponding adiabatic groupoid. We will use freely the notation introduced in the first section for general deformations to normal cones. In particular, we fix an exponential map yielding a tubular neighborhood construction $\theta : V' \tilde{\to} V$ and the corresponding diffeomorphism $\Theta : W' \to W$ where $W' = \{(x, \xi, t) \in \mathfrak{A}G \times \mathbb{R};\ (x, t\xi) \in V'\}$ and $W = \mathfrak{A} \times \{0\} \cup V \times \mathbb{R}^* \subset G_{ad}$.

We will use the Schwartz space described above for a general deformation to the normal cone; this leads to a slight modification of the Schwartz algebra of P. Carrillo-Rouse (*cf.* [5]).

We first fix notation for spaces of functions (half density sections) on G_{ad} which will be crucial in our study.

Definitions and Notation 3.1. a) *The ideal $\mathcal{J}_0(G)$ of functions with rapid decay at* 0.

We will denote by $\mathcal{J}_0(G)$ the space $\mathfrak{J}_0(G^{(0)}, G) = \mathcal{S}(\mathbb{R}^*; C_c^\infty(G))$ defined above (section 1.2 and 1.5). Recall that it consists of smooth half densities f with compact support on the groupoid $G \times \mathbb{R}$ such that, for every $k \in \mathbb{N}$, the function $(\gamma, t) \mapsto t^{-k} f(\gamma, t)$ extends smoothly on $G \times \mathbb{R}$.

b) *The (modified) Schwartz algebra $\mathcal{S}_c(G_{ad})$ of Carrillo-Rouse.*

We will denote by $\mathcal{S}_c(G_{ad})$ the algebra $\mathcal{S}_\alpha(D(G^{(0)}, G))$ defined above (section 1.5). Therefore, its elements are sums $f + g$ where $f \in \mathcal{J}_0(G)$ and, using the notation of section 1.5, $g \in C^\infty(W; \Omega^{1/2})$ is such that $\widetilde{N}^i_{k,\ell,j,m}(g) < +\infty$ for all i, k, ℓ, j, m with $m \geq 0$.

Note that there is a canonical groupoid morphism $G_{ad} \to G$ (the image of $\mathfrak{A}G \times \{0\}$ is $G^{(0)} \subset G$) and, by definition, the image under this morphism of the support of $f \in \mathcal{S}_c(G_{ad})$ is compact in G. On the other hand, unlike the original definition in [5], we had to drop here the conical support requirement.

c) *The ideal $\mathcal{J}(G)$.*

We will denote by $\mathcal{J}(G)$ the space $\mathfrak{J}(G^{(0)}, G)$ as defined in prop. 1.5. Its elements are sums $f + g$ where $f \in \mathcal{J}_0(G)$ and $g \in C^\infty(W; \Omega^{1/2})$ is such that $\widetilde{N}^i_{k,\ell,j,m}(g) < +\infty$ for all i, k, ℓ, j, m with any $m \in \mathbb{Z}$.

We now check that $\mathcal{S}_c(G_{ad})$ is indeed an algebra and $\mathcal{J}(G)$ an ideal. We begin with a remark:

Remark 3.2. An element $f \in \mathcal{S}_c(G_{ad})$ is a family $(f_t)_{t\in\mathbb{R}}$, where $f_t \in C_c^\infty(G)$ for $t \neq 0$ and $f_0 \in \mathcal{S}(\mathfrak{A}G)$. Note that, since $G \times \mathbb{R}^*$ is dense in G_{ad}, f is determined by $(f_t)_{t\neq 0}$. Roughly speaking, the definition implies that the support of f_t concentrates around $G^{(0)}$ when t goes to 0.

Proposition 3.3. *The space $\mathcal{S}_c(G_{ad})$ is a $*$-algebra: for $f, g \in \mathcal{S}_c(G_{ad})$, the families $(f_t^*)_{t\in\mathbb{R}}$ and $(f_t * g_t)_{t\in\mathbb{R}}$ belong to $\mathcal{S}_c(G_{ad})$. Moreover $\mathcal{J}_0(G)$ is a $*$-ideal of $\mathcal{S}_c(G_{ad})$.*

Proof. The function $(\gamma_1, \gamma_2) \mapsto f(\gamma_1)g(\gamma_2)$ defined on $G^{(2)}_{ad} = D(G^{(0)}, G^{(2)})$ is an element of $\mathcal{S}_\alpha(D(G^{(0)}, G^{(2)}))$. Since the composition $G^{(2)} \to G$ is a submersion, equivariant with respect to α (since α is an action of $\mathbb{R}^*_+$ by groupoid automorphisms), we find that integration along the fibers yields a continuous map $\mathcal{S}_\alpha(D(G^{(0)}, G^{(2)})) \to \mathcal{S}_\alpha(D(G^{(0)}, G))$.

By continuity of the product of $C_c^\infty(G)$, we find that if $f \in \mathcal{J}_0(G)$ or $g \in \mathcal{J}_0(G)$, then $f * g \in \mathcal{J}_0(G)$.

The assertions about the $*$-operations are obvious. □

Lemma 3.4. *Let $f \in \mathcal{S}_c(G_{ad})$.*

a) *For every $g \in C_c^\infty(G)$ the function $F : G \times \mathbb{R} \to \mathbb{C}$ defined by $F(\gamma, t) = f_t * g(\gamma)$ for $t \neq 0$ and $F(\gamma, 0) = \hat{f}_0(r(\gamma), 0)g(\gamma)$ is smooth.*

b) *We have $f \in \mathcal{J}(G)$ if and only if, for any $g \in C_c^\infty(G)$, the family $(f_t * g)_{t\in\mathbb{R}^*}$ is an element of $\mathcal{J}_0(G)$.*

Proof. Let $\theta : V' \to V$ be an "exponential map" which is a diffeomorphism of a (relatively compact) neighborhood V' of the 0 section $G^{(0)}$ in $\mathfrak{A}G$ onto a tubular neighborhood V of $G^{(0)}$ in G. We assume that $r(\theta(x, U)) = x$ for $x \in G^{(0)}$ and $U \in \mathfrak{A}_x G$.
Let $\chi \in C_c^\infty(G)$ with support in V, such that $\chi(\gamma) = 1$ for γ near $G^{(0)}$. Then $((1-\chi)f_t)_{t\in\mathbb{R}} \in \mathcal{J}_0(G)$, whence $((1-\chi)f_t * g)_{t\in\mathbb{R}} \in \mathcal{J}_0(G)$ for all $g \in C_c^\infty(G)$.
Furthermore,

$$(\chi f_t * g)(\gamma) = \int_{\mathfrak{A}_{r(\gamma)}G} f_t \circ \theta^{-1}(r(\gamma), U)h(\gamma, U)\, dU,$$

where $h(\gamma, U) = \chi(\theta(r(\gamma), U))g(\theta(r(\gamma), U)^{-1}\gamma)\delta(r(\gamma), U)$ - here δ is a suitable Radon-Nykodym derivative.
The Lemma follows now from Prop. 1.5. □

Proposition 3.5. *The space $\mathcal{J}(G)$ is a $*$-ideal of the algebra $\mathcal{S}_c(G_{ad})$.*

Proof. Follows immediately from Lemma 3.4.b). □

3.2 The smooth module $\mathcal{E}^\infty$

The following rather technical Lemma uses the action α of $\mathbb{R}_+^*$ and the semi-norms defined in section 1.6 and used in the Definitions 3.1.

Lemma 3.6. *Let $f, g \in \mathcal{J}(G)$. We assume that their support is small enough (in G) so that, for every $t, u \in \mathbb{R}^*$, the function $f_t * g_u$ has support in V. The function $u \mapsto (f_t * g_{tu})_{t\in\mathbb{R}}$ is a smooth map from $\mathbb{R}_+^*$ to $\mathcal{J}(G)$ with rapid decay as $u \to 0$ and as $u \to \infty$. More precisely, for every $i \in I$, $k \in \mathbb{N}^n$, $\ell \in \mathbb{N}^p$, $j \in \mathbb{N}$, $m \in \mathbb{Z}$ and $q \in \mathbb{Z}$, the function $u \mapsto u^q \widetilde{N}^i_{k,\ell,j,m}(f * \alpha_u(g))$ is bounded on $\mathbb{R}_+^*$.*

Proof. The adiabatic groupoid G_{ad} of G is a Lie groupoid; we may therefore construct its adiabatic groupoid $(G_{ad})_{ad}$. Since $g = (g_t)$ is an element of $\mathcal{J}(G)$, the function $(t, u) \mapsto \chi(t)\chi(u)g_{tu}$ is an element of $\mathcal{J}(G_{ad})$ where $\chi \in C_c^\infty(\mathbb{R})$ is equal to 1 near 0. By Lemma 3.4 applied to the groupoid G_{ad}, it follows that $(\chi(t)\chi(u)f_t * g_{tu})_{u\in\mathbb{R}_+^*}$ leads to an element of $\mathcal{J}_0(G_{ad})$ and thus has rapid decay when $u \to 0$, uniformly in t. In other words, the functions $u \mapsto u^{-q}\widetilde{N}^i_{k,\ell,j,m}(f * \alpha_u(g))$ are bounded for every $i \in I$, $k \in \mathbb{N}^n$, $\ell \in \mathbb{N}^p$, $j \in \mathbb{N}$, $m \in \mathbb{N}$ and $q \in \mathbb{N}$.

In the same way, $u \mapsto \alpha_u(f) * g$ has rapid decay when $u \to 0$, whence, using the compatibility of the semi-norms with the action of $\mathbb{R}_+^*$, $u \mapsto f * \alpha_u^{-1}(g)$ has rapid decay when $u \to 0$. From this, we deduce that the functions $u \mapsto u^q \widetilde{N}^i_{k,\ell,j,m}(f * \alpha_u(g))$ are bounded for every $i \in I$, $k \in \mathbb{N}^n$, $\ell \in \mathbb{N}^p$, $j \in \mathbb{N}$, $m \in \mathbb{N}$ and $q \in \mathbb{N}$, whence $q \in \mathbb{Z}$.

In other words, $u \mapsto f * \alpha_u(g)$ has rapid decay from $\mathbb{R}_+^*$ to $\mathcal{S}_c(G_{ad})$. To see that it also has rapid decay as a function from $\mathbb{R}_+^*$ to $\mathcal{J}(G)$, it is enough to check that, for $h \in C_c^\infty(G)$, the map $u \mapsto f * \alpha_u(g) * h$ has rapid decay from $\mathbb{R}_+^*$ to $\mathcal{J}_0(G)$. But by Lemma 3.4.b), $(t, u) \mapsto f_t * g_{tu} * h$ is an element of $\mathcal{S}(\mathbb{R}^2; C_c^\infty(G))$ and vanishes as well as all its derivatives when $t = 0$ or $u = 0$. □

Theorem 3.7. *For $f \in \mathcal{J}(G)$ and $m \in \mathbb{N}$, the operator $\displaystyle\int_0^{+\infty} t^m f_t \frac{dt}{t}$ is an order $-m$ pseudodifferential operator of the groupoid G i.e. an element of $\mathcal{P}_{-m}(G)$; its principal symbol σ is given by $\sigma(x, \xi) = \displaystyle\int_0^{+\infty} t^m \hat{f}(x, t\xi, 0)\frac{dt}{t}\cdot$ More precisely, there is a (classical) pseudodifferential operator P with principal symbol σ such that for every $g \in C_c^\infty(G)$, we have $P * g = \displaystyle\int_0^{+\infty} t^m f_t * g \frac{dt}{t}$ and $g * P = \displaystyle\int_0^{+\infty} t^m g * f_t \frac{dt}{t}\cdot$*

Proof. Of course, if $f \in \mathcal{J}_0(G)$ then $\displaystyle\int_0^{+\infty} t^m f_t \frac{dt}{t} \in C_c^\infty(G)$. In particular, this gives the meaning of $\displaystyle\int_0^{+\infty} t^m f_t * g \frac{dt}{t}$ and $\displaystyle\int_0^{+\infty} t^m g * f_t \frac{dt}{t}$ (thanks to Lemma 3.4).

We thus need only to treat the case of $f \in C_c^\infty(W; \Omega^{1/2})$ satisfying the above condition. We may trivialize the half densities using a positive half density ω on G; we then may write $f_t = t^{-p}h_t\omega$ where p is the dimension of the fibers of G and h is the restriction to $G \times \mathbb{R}_+^*$ of a smooth function with compact support in G_{ad}. Such a function can be written as $h(\gamma, t) = \chi(\gamma)\chi'(t)\varphi\Big(t^{-1}\theta^{-1}(\gamma), t\Big)$, where χ and χ' are bump-functions $\chi \in C_c^\infty(G)$ with support contained in V which is equal to 1 in a neighborhood of $G^{(0)}$, $\chi' \in C_c^\infty(\mathbb{R}_+)$ equal to 1 in a neighborhood of 0 and $\varphi \in C_c^\infty(\mathfrak{A}G \times \mathbb{R}_+)$ such that $\hat{\varphi}$ vanishes as well as all its derivatives at points of the form $(x, 0, 0)$ with $x \in G^{(0)}$.

Writing $\varphi(x,X,t) = (2\pi)^{-p}\int e^{i\langle X|\xi\rangle}\hat{\varphi}(x,\xi,t)\,d\xi$, we find

$$\begin{aligned} f_t(\gamma) &= (2\pi t)^{-p}\chi(\gamma)\chi'(t)\omega\int e^{i\langle t^{-1}\theta^{-1}(\gamma)|\xi\rangle}\hat{\varphi}(x,\xi,t)\,d\xi \\ &= (2\pi)^{-p}\chi(\gamma)\chi'(t)\omega\int e^{i\langle \theta^{-1}(\gamma)|\xi\rangle}\hat{\varphi}(x,t\xi,t)\,d\xi. \end{aligned}$$

Therefore, we have an equality (as multipliers of $C_c^\infty(G;\Omega^{1/2})$),

$$\int_0^{+\infty} t^m f_t(\gamma)\,\frac{dt}{t} = (2\pi)^{-p}\chi(\gamma)\omega\int e^{i\langle \theta^{-1}(\gamma)|\xi\rangle}a(x,\xi)\,d\xi,$$

where

$$a(x,\xi) = \int_0^{+\infty} t^m\chi'(t)\hat{\varphi}(x,t\xi,t)\,\frac{dt}{t}\cdot$$

Taking derivatives of a in x gives the same type of expression; taking derivatives in ξ increases m.

An expression $c(x,\xi) = \displaystyle\int_0^{+\infty}(t\|\xi\|)^m\chi'(t)b(x,t\xi,t)\,\frac{dt}{t}$ with b having rapid decay at infinity and points $(x,0,0)$ is bounded. In other words, a is a symbol of order $-m$ and type $(1,0)$

Now, given an expansion $\hat{\varphi}(x,\zeta,u)\sim\displaystyle\sum_{k=0}^{\infty}b_k(x,\zeta)u^k$, (for u small) we find an expansion (for ξ large), $a(x,\xi)\sim\displaystyle\sum_{k=0}^{\infty}a_{k+m}(x,\xi)$ where $a_{k+m}(x,\xi) = \displaystyle\int_0^{\infty}b_k(x,t\xi)t^{k+m}\,\frac{dt}{t}$ is homogeneous of degree $-k-m$ in ξ. □

Of course, the same computation works for $m\in\mathbb{Z}$ and even $m\in\mathbb{C}$ (*cf.* [28] for pseudodifferential operators on groupoids with complex order).

Note that Theorem 3.7 shows that every element $P\in\mathcal{P}_0(G)$ can be written as an integral $P_f = \displaystyle\int_0^{+\infty}f_t\,\frac{dt}{t}$ with $f\in\mathcal{J}(G)$ (using the standard Borel's Theorem-like techniques to construct f such that $P-P_f\in C_c^\infty(G)$; of course if $P\in C_c^\infty(G)$, then $P=P_f$ where $f_t=\chi(t)P$ with an obvious choice of $\chi\in\mathcal{S}(\mathbb{R}_+^*)$).

Lemma 3.8. *Let $f=(f_t)_{t\in\mathbb{R}}\in\mathcal{J}(G)$ and $P\in\mathcal{P}_0(G)$. There is a (unique) element $h=(h_t)_{t\in\mathbb{R}}$ of $\mathcal{J}(G)$ such that $h_t=f_t*P$ for $t\in\mathbb{R}^*$. Moreover $\widehat{h_0}=\widehat{f_0}\sigma_0(P)$.*

Proof. Uniqueness follows from density of $G\times\mathbb{R}^*$ in G_{ad}.

The family (f_t*P) is smooth in $G\times\mathbb{R}^*$ and the image in G of its support is compact.

Of course, if $f\in\mathcal{J}_0(G)$ then $(f_t*P)\in\mathcal{J}_0(G)$, we may thus assume that f has support in a neighborhood of $G^{(0)}$ in G as small as we wish.

In the same way, since P is quasi-local, we may assume, thanks to Lemma 3.4, that the support of P is also contained in a suitable neighborhood of $G^{(0)}$ in G.

Now, using Theorem 3.7, we may write $P=\displaystyle\int g_u\frac{du}{u}$ with $g\in\mathcal{J}(G)$.

By locality, we may assume that, for all $t, u \in \mathbb{R}^*$, the support of $f_t * g_u$ is contained in V.

We find $h_t = f_t * P = \int_0^{+\infty} f_t * g_u \frac{du}{u} = \int_0^{+\infty} f_t * g_{tu} \frac{du}{u}$. The conclusion follows immediately from Lemma 3.6.

We found $h = \int_0^{+\infty} f * \alpha_u(g) \frac{du}{u}$. Evaluating at $t = 0$, we find $h_0 = \int_0^{+\infty} f_0 * \alpha_u(g_0) \frac{du}{u}$, and thus $\widehat{h_0} = \widehat{f_0} \int_0^{+\infty} \alpha_u(\widehat{g_0}) \frac{du}{u} = \widehat{f_0} \sigma_0(P)$ (by Theorem 3.7). □

Lemma 3.8 asserts that $\mathcal{J}(G)$ is endowed with a (right) $\mathcal{P}_0(G)$-module structure; we will denote this smooth module by $\mathcal{E}^\infty$.

4 The Hilbert module $\mathcal{E}$

From now on, we will be interested in the restriction G_{ad}^+ of G_{ad} to $G^{(0)} \times \mathbb{R}_+$. Let $\mathrm{ev}_+ : C^*(G_{ad}) \to C^*(G_{ad}^+)$ be the morphism induced by the restriction map. Since $G^{(0)} \times \mathbb{R}_+$ is a closed saturated subspace of G_{ad} and G_{ad}^+ is invariant under the action of $\mathbb{R}_+^*$, all the previous results obviously remain true when one replaces $\mathcal{S}_c(G_{ad})$, $\mathcal{J}(G)$ and $\mathcal{J}_0(G)$ by their image under ev_+. For the simplicity of notations we keep the same notation for $\mathcal{J}(G)$ and $\mathcal{J}_0(G)$: they are the image under ev_+ of the previously defined $\mathcal{J}(G)$ and $\mathcal{J}_0(G)$. As noted in the introduction, $C^*(G)$ is the full groupoid C^*-algebra, but all the results here extend immediately to the reduced C^*-agebra case.

4.1 Completion of $\mathcal{E}^\infty$

Let $\Psi^*(G)$ denote the C^*-algebra of pseudodifferential operators, i.e. the norm closure of $\mathcal{P}_0(G)$ in the multiplier algebra of $C^*(G)$. Let also $\sigma_0 : \Psi^*(G) \to C(S^*\mathfrak{A}G)$ be the principal symbol map. We have the exact sequence of C^*-algebras

$$0 \to C^*(G) \longrightarrow \Psi^*(G) \xrightarrow{\sigma_0} C(S^*\mathfrak{A}G) \to 0 \tag{2}$$

For $P \in \Psi^*(G)$, the function $\sigma_0(P)$ is thought of as a homogeneous function defined outside the zero section in $\mathfrak{A}G$.

The elements of $C^*(G_{ad}^+)$ are families $(f_t)_{t \in \mathbb{R}_+}$ with $f_t \in C^*(G)$ for $t \neq 0$ and $f_0 \in C^*(\mathfrak{A}G) \simeq C_0(\mathfrak{A}^*G)$.

Put $J_0(G) = \{f \in C^*(G_{ad}^+);\ f_0 = 0\} \simeq C_0(\mathbb{R}_+^*; G)$. We have an exact sequence

$$0 \to J_0(G) \longrightarrow C^*(G_{ad}^+) \xrightarrow{\mathrm{ev}_0} C_0(\mathfrak{A}^*G) \to 0.$$

Finally, put $J(G) \subset C^*(G_{ad}^+) = \{f \in C^*(G_{ad}^+);\ \forall x \in G^{(0)},\ \widehat{f_0}(x, 0) = 0\}$.

Lemma 4.1. *An element $f \in C^*(G_{ad}^+)$ is determined by the family $(f_t)_{t \in \mathbb{R}_+^*}$: the ideal $J_0(G)$ is essential in $C^*(G_{ad}^+)$.*

Proof. Let $x \in G^{(0)}$. Put $G_{ad,x}^+ = \{(\gamma, t) \in G_{ad}^+;\ s(\gamma) = x\} = G_x \times \mathbb{R}_+^* \cup \mathfrak{A}_x G$. The map $(\gamma, t) \to t$ being a submersion, we obtain a continuous family $(H_{x,t})_{t \geq 0}$ of Hilbert spaces with $H_{x,0} = L^2(\mathfrak{A}_x G)$ and $H_{x,t} = L^2(G_x)$ for $t > 0$ as a completion of smooth half densities with compact support on $G_{ad,x}^+$.

Let $f \in C_c^\infty(G_{ad}^+;\Omega^{1/2})$ and $g \in C_c^\infty(G_{ad,x}^+;\Omega^{1/2})$; we have $f * g \in C_c^\infty(G_{ad,x}^+;\Omega^{1/2})$. It follows that $t \mapsto \|f_t * g_t\|$ is continuous; by density, this remains true for $f \in C^*(G_{ad}^+)$ and any continuous section $g = (g_t)$ of $(H_{x,t})$. But $\|f_0\| = \sup\{\|f_0 * g_0\|,\ x \in G^{(0)};\ g \in (H_{x,t});\ \sup_t \|g_t\| \leq 1\}$. It follows that $\|f_0\| \leq \sup_{t>0} \|f_t\|$. □

In the following, we consider $C^*(G_{ad}^+)$ in the multiplier algebra $\mathcal{M}(J_0(G))$ of $J_0(G)$. Since $J_0(G) = C_0(\mathbb{R}_+^*) \otimes C^*(G)$, the algebras $C^*(G)$, and $\Psi^*(G)$ sit also in $\mathcal{M}(J_0(G))$.

From lemmas 3.4 and 3.8, we immediately get.

Proposition 4.2. a) *For $f \in C^*(G)$ and $g \in J(G)$ we have $f * g \in J_0(G)$.*

b) *For $P \in \Psi_0^*(G)$ and $f \in J(G)$ we have $f * P \in J(G)$ and $(\widehat{f * P})_0 = \widehat{f}_0 \sigma_0(P)$.* □

Lemma 4.3. *For $f \in \mathcal{J}(G)$, the integral $\displaystyle\int_0^{+\infty} f_t \frac{dt}{t}$ of Theorem 3.7 converges strictly (i.e. in the topology of multipliers of $C^*(G)$).*

Proof. Let $g \in C_c^\infty(G)$. It follows from lemma 3.4 that $\displaystyle\int_0^{+\infty} f_t * g \frac{dt}{t}$ converges in norm. Taking adjoints, it follows that $\displaystyle\int_0^{+\infty} g * f_t \frac{dt}{t}$ converges also in norm.

From Theorem 3.7, it follows that $\displaystyle\int_0^{+\infty} f_t * g \frac{dt}{t} = P * g$ where P is a pseudodifferential operator of order 0 and therefore extends to a multiplier of $C^*(G)$.

Now, if f is a positive element in $\mathcal{J}(G)$, it follows that $\langle g | \Big(\displaystyle\int_s^{+\infty} f_t \frac{dt}{t}\Big) * g\rangle \leq \langle g | P * g\rangle$, therefore the family $\Big(\displaystyle\int_s^{+\infty} f_t \frac{dt}{t}\Big)_{s>0}$ is bounded. It follows that $\Big(\displaystyle\int_s^{+\infty} f_t \frac{dt}{t}\Big) * g$ converges to $P * g$ and $g * \Big(\displaystyle\int_s^{+\infty} f_t \frac{dt}{t}\Big)$ converges to $g * P$ for all $g \in C^*(G)$ (when $s \to 0$). Therefore $\displaystyle\int_0^{+\infty} f_t \frac{dt}{t}$ converges strictly.

By the polarization identity, we find that $\displaystyle\int_0^{+\infty} g_t^* * h_t \frac{dt}{t}$ converges in the multiplier algebra of $C^*(G)$ for $g, h \in \mathcal{J}(G)$.

Now, one can find $g, h \in \mathcal{J}(G)$ such that $f_0 = g_0^* * h_0$ (take for instance $\hat{g}_0(x,\xi) = |\hat{f}_0(x,\xi)|^2 + \exp(-\|\xi\|^2 - \|\xi\|^{-2})^{1/4}$ and $\hat{h}_0(x,\xi) = \hat{f}_0(x,\xi)\hat{g}_0(x,\xi)^{-1}$). It follows, that there exists $f^1 \in \mathcal{J}(G)$ with $f_t = t f_t^1 + g_t^* * h_t$. Now, since $f^1 \in C^*(G_{ad}^+)$, it follows that $\|f_1^t\|$ is bounded. Therefore, (using rapid decay at ∞) the integral $\displaystyle\int_0^{+\infty} f_t^1\, dt$ is norm convergent in $C^*(G)$. □

Lemma 4.4. *There exists $f \in \mathcal{J}(G)$ such that $\displaystyle\int_0^{+\infty} f_t^* * f_t \frac{dt}{t}$ is an invertible element of $\Psi^*(G)$ and $\Big(1 - \displaystyle\int_0^{+\infty} f_t^* * f_t \frac{dt}{t}\Big) \in C_c^\infty(G)$.*

Proof. Fix a smooth function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with support in $]1,2[$ such that $\displaystyle\int_0^{+\infty} \psi^2(t) \frac{dt}{t} = 1$. Let $g \in \mathcal{J}(G)$ be such that $\hat{g}_0(x,\xi) = \psi(\|\xi\|)$ and $g_t = 0$ for $t \geq 1$. The positive

pseudodifferential operator $P = \int_0^{+\infty} g_t^* * g_t \frac{dt}{t}$ has principal symbol equal to 1 by Theorem 3.7. By [28], there exists $Q \in \mathcal{P}_0(G)$ such that $1 - Q^*PQ \in C_c^\infty(G)$. Using the exact sequence (2), it follows that there exists $b \in C^*(G)$ such that $b^*b + Q^*PQ \geq 1$. By density of $C_c^\infty(G)$ in $C^*(G)$, there exists $h \in C_c^\infty(G)$ such that $h^* * h + Q^*PQ$ is invertible. Taking $f_t = g_t * Q$ for $t \leq 1$ and $f_t = \psi(t)h$ for $t \geq 1$, we find $\int_0^{+\infty} f_t^* * f_t \frac{dt}{t} = Q^*PQ + h^* * h$. □

We may now construct the main object of this section. Recall that given a C^*-algebra A, a Hilbert-A-module E is said to be *full* if the linear span of $\langle \xi | \eta \rangle$ when ξ, η run through E (which is a two sided ideal in A) is dense in A.

Theorem 4.5. *There is a Hilbert $\Psi^*(G)$-module $\mathcal{E}$ containing $\mathcal{J}(G)$ as a dense subset, with the following operations:*

- *For $f, g \in \mathcal{J}(G) \subset \mathcal{E}$, we have $\langle f | g \rangle = \int_0^{+\infty} f_t^* * g_t \frac{dt}{t}$.*
- *For $f \in \mathcal{J}(G) \subset \mathcal{E}$ and $P \in \mathcal{P}_0(G) \subset \Psi^*(G)$, we have $f * P \in \mathcal{J}(G) \subset \mathcal{E}$ and $(f * P)_t = f_t * P$ for $t \neq 0$.*

The module $\mathcal{E}$ is a full $\Psi^*(G)$ module.

Proof. Clearly, for fixed f, the map $g \mapsto \langle f | g \rangle = \int_0^{+\infty} f_t^* * g_t \frac{dt}{t}$ is linear and $\mathcal{P}_0(G)$-linear; also $\langle g | f \rangle = \langle f | g \rangle^*$. Furthermore $\langle f | f \rangle$ is the strict limit of elements of $C^*(G)_+$; therefore $\langle f | f \rangle \in \Psi^*(G)_+$.

For $f \in \mathcal{J}(G)$, we then may put $\|f\|_{\mathcal{E}} = \|\langle f | f \rangle\|_{C^*(G)}^{1/2}$. By the Cauchy-Schwarz inequality for C^*-modules, this defines a norm on $\mathcal{E}^\infty$. Now, using again the Cauchy-Schwarz inequality, for $f, g \in \mathcal{J}(G)$ we have $\|\langle f | g \rangle\|_{C^*(G)} \leq \|f\|_{\mathcal{E}} \|g\|_{\mathcal{E}}$ and, for $P \in \mathcal{P}_0(G)$, since $\langle f * P | f * P \rangle = P^* \langle f | f \rangle P \leq \|\langle f | f \rangle\|_{C^*(G)} P^* * P$, we find $\|f * P\|_{\mathcal{E}} \leq \|f\|_{\mathcal{E}} \|P\|_{\Psi^*(G)}$. It follows that the scalar product and the right action of $\mathcal{P}_0(G)$ extend and endow $\mathcal{E}$ with the desired Hilbert $\Psi^*(G)$-module structure.

It follows from Lemma 4.4 that $\mathcal{E}$ is full - and in fact, that there exists $\zeta \in \mathcal{E}$ ($\zeta = f \langle f | f \rangle^{-1/2}$) such that $\langle \zeta | \zeta \rangle = 1$. □

4.2 Computation of $\mathcal{K}(\mathcal{E})$

We now construct the desired natural isomorphism $J(G) \rtimes \mathbb{R}_+^* \to \mathcal{K}(\mathcal{E})$.

Note first, that if $f \in \mathcal{S}_c(G_{ad}^+)$ and $g \in \mathcal{J}(G)$, $f * g \in \mathcal{J}(G)$; furthermore, this left action is $\mathcal{P}_0(G)$-linear and $\langle f * g | f * g \rangle = \int g_t^* * f_t^* * f_t * g_t \frac{dt}{t} \leq \|f\|^2 \langle g | g \rangle$ (where $\|f\| = \sup \|f_t\|$ is the norm of f in $C^*(G_{ad}^+)$ - this holds as well for the reduced and the full C^*-norm on G and G_{ad}). Extending by continuity, we obtain a natural morphism $\pi_0 : C^*(G_{ad}^+) \to \mathcal{L}(\mathcal{E})$.

The action of $\mathbb{R}_+^*$ on G_{ad}^+ gives rise to a unitary action on $\mathcal{E}$ given by $(U_s(f))_t = f_{st}$ for $f \in \mathcal{J}(G)$ and $s, t \in \mathbb{R}_+^*$.

The couple (π_0, U) is an equivariant representation of $(C^*(G_{ad}^+), \mathbb{R}_+^*)$, and therefore gives rise to a representation of $C^*(G_{ad}^+) \rtimes \mathbb{R}_+^* \simeq C^*(G_{ga})$, i.e. a morphism

$$\pi : C^*(G_{ad}^+) \rtimes \mathbb{R}_+^* \to \mathcal{L}(\mathcal{E}).$$

Put $\mathcal{E}_0 = \mathcal{E}C^*(G)$. It is a closed submodule of $\mathcal{E}$.

Recall that if J is a closed two sided ideal in a C^*-algebra A and E is a Hilbert-A-module, then $EJ = \{x \in E;\ \langle x|x\rangle \in J\}$ is a closed submodule. The quotient E/EJ-is the Hilbert A/J-module $E \otimes_A A/J$. We clearly[2] have a short exact sequence

$$0 \to \mathcal{K}(EJ) \to \mathcal{K}(E) \to \mathcal{K}(E/EJ) \to 0.$$

Note that if J is an essential ideal of A, it follows that $EJ^\perp = \{0\}$. Indeed, for $x \in E$ non zero, there exists $b \in J$ such that $\langle x|x\rangle b \neq 0$, whence $\langle x|xb\rangle = \langle x|x\rangle b \neq 0$.

Theorem 4.6. *The morphism $\pi : J(G) \rtimes \mathbb{R}_+^* \to \mathcal{L}(\mathcal{E})$ is an isomorphism from $J(G) \rtimes \mathbb{R}_+^*$ onto $\mathcal{K}(\mathcal{E})$.*

Proof. Consider the exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & J_0(G) \rtimes \mathbb{R}_+^* & \to & J(G) \rtimes \mathbb{R}_+^* & \to & (J(G)/J_0(G)) \rtimes \mathbb{R}_+^* & \to & 0 \\ & & \pi \downarrow & & \pi \downarrow & & \overline{\pi} \downarrow & & \\ 0 & \to & \mathcal{K}(\mathcal{E}_0) & \to & \mathcal{K}(\mathcal{E}) & \to & \mathcal{K}(\mathcal{E}/\mathcal{E}_0) & \to & 0 \end{array}$$

We will show that:

a) π induces an isomorphism from $J_0(G) \rtimes \mathbb{R}_+^*$ onto $\mathcal{K}(\mathcal{E}_0)$;

b) $\pi(J(G) \rtimes \mathbb{R}_+^*) \supset \mathcal{K}(\mathcal{E})$;

c) The map $\overline{\pi}$ induced by π gives rise to an isomorphism from $(J(G)/J_0(G)) \rtimes \mathbb{R}_+^*$ onto $\mathcal{K}(\mathcal{E}/\mathcal{E}_0)$.

The Theorem then follows by diagram chasing.

We proceed with the proof of these facts:

a) According to Lemma 3.4, the module $\mathcal{E}_0$ is the closure of $\mathcal{J}_0(G)$. It is therefore canonically isomorphic to $C_0(\mathbb{R}_+^*) \otimes C^*(G)$ and, since $\mathbb{R}_+^*$ acts by translation in $C_0(\mathbb{R}_+^*)$ the statement follows.

b) Let $f, g \in \mathcal{J}(G)$; by Lemma 3.6, the map $s \mapsto f * \alpha_s(g^*)$ has rapid decay and thus defines an element $\displaystyle\int f * \alpha_s(g^*)\lambda_s ds/s$ in $J(G) \rtimes \mathbb{R}_+^*$. A direct computation then shows that $\vartheta_{f,g} = \pi\Big(\displaystyle\int f\alpha_s(g^*)\lambda_s ds/s\Big)$ where $\vartheta_{f,g}$ is the usual "rank one" operator $h \mapsto f\langle g|h\rangle$ on $\mathcal{E}$. The result follows from density of $\mathcal{J}(G)$ in $\mathcal{E}$.

c) The quotient $\Psi_*(G)/C^*(G)$ is isomorphic via the principal symbol map σ to $C(S^*\mathfrak{A}G)$. Theorem 3.7 and Lemma 3.8 give the computations of $\sigma(\langle f|g\rangle)$ and $(\widehat{f * P})_0$ for $f, g \in \mathcal{J}(G)$ and $P \in \mathcal{P}_0(G)$. It follows that $\mathcal{E}/\mathcal{E}_0 \simeq \mathcal{E} \otimes_\sigma C(S^*\mathfrak{A}G)$ is the Hilbert $C(S^*\mathfrak{A}G)$ module $C(S^*\mathfrak{A}G) \otimes L^2(\mathbb{R}_+^*)$ obtained as completion of $C_c^\infty(\mathfrak{A}^*G \backslash G^{(0)})$ with respect to the $C(S^*\mathfrak{A}G)$ valued scalar product given by $\langle f|g\rangle(x,\xi) = \displaystyle\int_0^{+\infty} \overline{f(x,t\xi)}g(x,t\xi)\frac{dt}{t}$ and right action $(fh)(x,\xi) = f(x,\xi)h(x, \dfrac{\xi}{\|\xi\|})$ (for $f, g \in C_c^\infty(\mathfrak{A}^*G \setminus G^{(0)})$ and $h \in C^\infty(S^*\mathfrak{A}G)$). Moreover, left action of $J(G)/J_0(G) \simeq C_0(\mathfrak{A}^*G \setminus G^{(0)})$ is given by pointwise multiplication and the action of $\mathbb{R}_+^*$ is by scaling. The result follows. $\square$

[2] Considering for instance the case $E = H_A$.

It is worth noting that thanks to Lemma 4.1 and the amenability of $\mathbb{R}_+^*$, the ideal $J_0(G) \rtimes \mathbb{R}_+^*$ of $C^*(G_{ga}) = C^*(G_{ad}^+) \rtimes \mathbb{R}_+^*$ is essential. In particular this remark gives an alternative proof of the injectivity of $\pi : J(G) \rtimes \mathbb{R}_+^* \to \mathcal{L}(E)$.

4.3 Pseudodifferential operators as convolution kernels

Using Theorem 4.6 together with Lemma 4.4, we can see $\Psi^*(G)$ as sitting as a corner in $C^*(G_{ga})$: Let $\zeta \in \mathcal{E}$ satisfying $\langle \zeta | \zeta \rangle = 1$ (as in Theorem 4.5). Then $\vartheta_{\zeta,\zeta} \in \mathcal{K}(E) \subset C^*(G_{ga})$ is a projection and $\Psi^*(G) \simeq \vartheta_{\zeta,\zeta} C^*(G_{ga}) \vartheta_{\zeta,\zeta}$.

Actually, taking $f \in \mathcal{E}^\infty$ such that $\langle f | f \rangle$ is invertible and $1 - \langle f | f \rangle \in C_c^\infty(G)$ (Lemma 4.4), we can really see the elements of $\mathcal{P}_0(G)$ as convolution operators on G_{ga}: the map $\vartheta_{T(f),f} \mapsto \langle f | T(f) \rangle$ is an isomorphism from a corner of the convolution algebra of smooth functions with Schwartz decay on G_{ga} onto a subalgebra of $\mathcal{P}_0(G)$, containing $\langle f | f \rangle \mathcal{P}_0(G) \langle f | f \rangle$. Note that for every $P \in \mathcal{P}_0(G)$, $P - \langle f | f \rangle * P * \langle f | f \rangle \in C_c^\infty(G)$. Finally $P = Q + R$ where Q is (the image of) a smooth function on G_{ga} and $R \in C_c^\infty(G)$.

4.4 Stability of $\mathcal{E}$

We now prove that the module $\mathcal{E}$ is stable, i.e. isomorphic to Kasparov's universal module $\mathcal{H}_{\Psi^*(G)}$.

We begin by recalling a few facts:

Facts 4.7. a) Recall first a few easy facts about Hilbert modules:

1. A Hilbert module E is countably generated if and only if $\mathcal{K}(E)$ is σ-unital (i.e. has a countable approximate unit).
2. For a countably generated Hilbert module E over a C^*-algebra B, the following are equivalent:
   (i) the Hilbert B-module E is stable i.e. isomorphic to $\ell^2(\mathbb{N}) \otimes E$;
   (ii) the C^*-algebra $\mathcal{K}(E)$ is stable i.e. isomorphic to $\mathcal{K} \otimes \mathcal{K}(E)$;
   (iii) there is a morphism $\psi : \mathcal{K} \to \mathcal{L}(E)$ such that $\psi(\mathcal{K})E = E$, or equivalently $\psi(\mathcal{K})\mathcal{K}(E) = \mathcal{K}(E)$.

   Note that by Cohen's theorem ([6, 13] - see *e.g.* [25] for the case of C^*-algebras) there is no linear span or closure.

   Indeed, implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are straightforward. If (iii) is satisfied, then $E = \bigoplus_k \psi(e_{kk})E$ and the various $\psi(e_{kk})E$ are isomorphic *via* $\psi(e_{jk})$ where e_{jk} are matrix units for $\mathcal{K}$, and (i) follows (using a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$).
3. For a countably generated Hilbert module E over a σ-unital C^*-algebra B, the following are equivalent:
   (i) E is stable and full;
   (ii) $E \simeq \mathcal{H}_B$.

   Indeed (ii) $\Rightarrow$ (i) is obvious.

   Conversely, let E^* be the Hilbert $\mathcal{K}(E)$-module $\mathcal{K}(E, B)$; if E is stable, then $\mathcal{K}(E)$ is isomorphic as a $\mathcal{K}(E)$-module to $\ell^2 \otimes \mathcal{K}(E) = \mathcal{H}_{\mathcal{K}(E)}$ and therefore $E' \oplus \mathcal{K}(E) \simeq$

$\mathcal{K}(E)$ for every countably generated Hilbert $\mathcal{K}(E)$-module E'; if E is full, then $B \simeq E^* \otimes_{\mathcal{K}(E)} E$; finally, if (ii) is satisfied, then

$$\mathcal{H}_B \simeq \mathcal{H}_B \oplus E \simeq ((\mathcal{H}_B \otimes_B E^*) \oplus \mathcal{K}(E)) \otimes_{\mathcal{K}(E)} E \simeq \mathcal{K}(E) \otimes_{\mathcal{K}(E)} E \simeq E.$$

b) Let $\mathcal{G}$ be a longitudinally smooth groupoid with compact space of objects $\mathcal{G}^{(0)}$.

Every element of the algebroid of $\mathcal{G}$, i.e. a section of $\mathfrak{A}\mathcal{G}$, defines a differential operator affiliated to $\mathcal{G}$, i.e. a multiplier of $\mathcal{G}$ (once chosen a trivialization of the longitunal half densities on $\mathcal{G}$).

Recall (*cf.* [28]) that the closure of an elliptic pseudodifferential operator D on $\mathcal{G}$ is a *regular* unbounded multiplier of $C^*(\mathcal{G})$ and $(1 + D^*\overline{D})^{-1}$ is a strictly positive element of $C^*(\mathcal{G})$; moreover, if D is formally self-adjoint, then $\overline{D}$ is self-adjoint.

In particular, if $(X_1, \ldots, X_m)$ are elements spanning the algebroid of $\mathcal{G}$ as a $C^\infty(\mathcal{G}^{(0)})$ module, then the closure of $\sum_i X_i^* X_i$ is such a self-adjoint elliptic operator whose spectrum is in $\mathbb{R}_+$ - i.e. it is positive.

Furthermore, let f be a smooth everywhere (strictly) positive function on $\mathcal{G}^{(0)}$. Write $f = k+g$ where $k \in \mathbb{R}_+^*$ and g is a non negative function. Then the operator $g+\sum_i X_i^* X_i$ is positive, whence $f + \sum_i X_i^* X_i$ is invertible and its inverse $\left(\sum_i X_i^* X_i + f\right)^{-1}$ is a strictly positive element in $C^*(\mathcal{G})$.

c) Recall (*cf.* [4, 29]) that a regular self-adjoint positive multiplier D of a C^*-algebra $\mathcal{A}$ with resolvent in $\mathcal{A}$ defines a morphism $\pi_D : f \mapsto f(D)$ from $C_0(\mathbb{R}_+^*)$ to $\mathcal{A}$. Note that, for $t \in \mathbb{R}_+^*$, we have $\pi_{tD} = \pi_D \circ \lambda_t$ where λ_t is the automorphism of $C_0(\mathbb{R}_+^*)$ induced by the regular representation. Since $t \mapsto \dfrac{t}{t^2+1}$ is a strictly positive element of $C_0(\mathbb{R}_+^*)$, it follows that $\pi_D(C_0(\mathbb{R}_+^*))\mathcal{A}$ is the closure of $D(D^2+1)^{-1}\mathcal{A}$.

Proposition 4.8. *Let G be a Lie groupoid with compact $G^{(0)}$ and G_{ad} its adiabatic groupoid; let $(Y_1, \ldots, Y_m)$ span $\mathfrak{A}G$ as a module over $C^\infty(G^{(0)})$. Let $D_1 = \left(\sum_i Y_i^* Y_i + 1\right)^{1/2}$. There is a unique morphism $\psi : C_0(\mathbb{R}_+^*) \to C^*(G_{ad}^+)$ such that*

(i) *$ev_1 \circ \psi = \pi_{D_1}$ where $ev_1 : C^*(G_{ad}) \to C^*(G)$ is evaluation at 1.*

(ii) *$\alpha_u \circ \psi = \psi \circ \lambda_u$ for all $u \in \mathbb{R}_+^*$.*

Moreover,

a) *$ev_0 \circ \psi(f) = f \circ q^{1/2} \in C_0(\mathfrak{A}^*G)$ where $q = \sum Y_i^2$ the Y_i's being considered as functions on $\mathfrak{A}^*G$ (linear on each fiber).*

b) *$\psi(C_0(\mathbb{R}_+^*)) \subset J(G)$ and $\psi(C_0(\mathbb{R}_+^*))J(G) = J(G)$.*

Proof. If ψ satisfies (i) and (ii), then $(\psi(f))_t = f(tD_1)$ for $t \neq 0$. This shows immediately uniqueness of ψ.

Choose $u_0 \geq 1$ and let $\mathcal{G}$ be the restriction of G_{ad} to $[0, u_0]$. Let $(X_1, \ldots, X_m)$ be the canonical extension of $(Y_1, \ldots, Y_m)$ to $\mathfrak{A}\mathcal{G}$. In particular for $u \neq 0$ we have $(X_i)_u = uY_i$. Put

$D = \Big(\sum_i X_i^* X_i + h^2\Big)^{1/2}$, where h is the function $(x,u) \mapsto u$ defined on $\mathcal{G}^{(0)} = G^{(0)} \times [0, u_0]$.

Note also $D_1 = \Big(\sum_i Y_i^* Y_i + 1\Big)^{1/2}$ its evaluation at 1. For $u \in]0, u_0]$, since $D_u = uD_1$, we have $ev_u \circ \pi_D = \pi_{D_1} \circ \lambda_u$.

Since the spectrum of D_1 is contained in $[1, +\infty[$, it follows that for $f \in C_0(\mathbb{R}_+^*)$ and $u \in \mathbb{R}_+^*$, $\|\pi_{D_1} \circ \lambda_u(f)\| \leq \sup\{|f(v)|;\ v \geq u\}$. In particular $\lim_{u \to +\infty} \pi_{D_1} \circ \lambda_u(f) = 0$. It follows that there is an element $\psi_f \in C^*(G_{ad}^+)$ such that, for $u \neq 0$ we have $(\psi_f)_u = \pi_{D_1} \circ \lambda_u$ and ψ_f restricted to $[0, u_0]$ is equal to $f(D)$.

We thus get a homomorphism $\psi : f \mapsto \psi_f$ from $C_0(\mathbb{R}_+^*)$ to $C^*(G_{ad})$.

It satisfies (i), and since $\alpha_u \circ \psi(f)$ and $\psi \circ \lambda_u(f)$ coincide on $\mathbb{R}_+^*$, they are equal. Property (ii) follows.

Now, $D_0^2 = q$ so we get property (a).

By uniqueness, the morphism ψ does not depend on the choice of u_0.

It follows from property (a) that, for all $f \in C_0(\mathbb{R}_+^*)$, since $f(0) = 0$, $ev_0 \circ \psi(f)$ vanishes on $M \subset \mathfrak{A}^* G$, whence $\psi(f) \in J(G)$.

Put $J' = \psi(C_0(\mathbb{R}_+^*))J(G)$.

As the restriction of h to the $[\varepsilon, u_0]$ is invertible for every u_0 and ε, the restriction of $D(1 + D^2)^{-1}$ to $[\varepsilon, u_0]$ is a strictly positive element; it follows that J' contains the functions $\mathbb{R}_+^* \to C^*(G)$ with compact support. It therefore contains the ideal $C_0(\mathbb{R}_+^*) \otimes C^*(G)$ of $C^*(G_{ad}^+)$.

As the quotient $C^*(G_{ad}^+)/C_0(\mathbb{R}_+^*) \otimes C^*(G) \simeq C_0(\mathfrak{A}G^*)$ is abelian, it follows that the right ideal J' is two sided. Finally $ev_0(J') = C_0(\mathfrak{A}G^* \setminus M)$ from property (a) since $\dfrac{q^{1/2}}{1+q}$ is a strictly positive element of $C_0(\mathfrak{A}G^* \setminus M)$. □

Corollary 4.9. *The C^*-algebra $J(G) \rtimes \mathbb{R}_+^* \simeq \mathcal{K}(\mathcal{E})$ is stable and therefore the Hilbert $\Psi^*(G)$-module is stable, i.e. isomorphic to $\ell^2 \otimes \Psi^*(G)$.*

Proof. Indeed, by condition (ii) in prop. 4.8, ψ induces a morphism $\hat{\psi}$ from $\mathcal{K} = C_0(\mathbb{R}_+^*) \rtimes_\lambda \mathbb{R}_+^*$ to $J(G) \rtimes_\alpha \mathbb{R}_+^*$ and by (b), $\hat{\psi}(\mathcal{K})\big(J(G) \rtimes_\alpha \mathbb{R}_+^*\big) = J(G) \rtimes_\alpha \mathbb{R}_+^*$. □

Remark 4.10. One can in fact show that $J(G)$ is isomorphic to a crossed product $\Psi^*(G) \rtimes_\beta \mathbb{R}$ in such a way that the $\mathbb{R}_+^*$ action on $J(G)$ is intertwined with the dual action $\hat{\beta}$. It follows that $J(G) \rtimes \mathbb{R}_+^* \simeq \Psi^*(G) \otimes \mathcal{K}$ - we use the duality of $\mathbb{R}$ with $\mathbb{R}_+^*$ given by $(t, u) \mapsto u^{it}$.

The action β is given by $\beta_t(P) = D_1^{it} P D_1^{-it}$. The operator D_1^{it} is pseudodifferential of complex order it (*cf.* [28]), therefore $\beta_t(P)$ is pseudodifferential of order 0; it follows also from [28] that it has the same principal symbol as P.

We may embed $\Psi^*(G)$ in the multiplier algebra of $J(G)$ setting $P.(f_u)_{u \in \mathbb{R}_+^*} = (P * f_u)_{u \in \mathbb{R}_+^*}$ and $(f_u)_{u \in \mathbb{R}_+^*}.P = (f_u * P)_{u \in \mathbb{R}_+^*}$ thanks to prop. 4.2.b); furthermore, we have a one parameter group $(D^{it})_{t \in \mathbb{R}}$ in the multipliers of $J(G)$. As D_u and D_1 are scalar multiples of each other, we find in this way a covariant representation of the pair $(\Psi^*(G), \mathbb{R})$. Associated to this covariant representation of $(\Psi^*(G), \mathbb{R})$ is a morphism from $\Psi^*(G) \rtimes \mathbb{R}$ into the multiplier algebra of $J(G)$, but since the image of $C^*(\mathbb{R}) \subset \Psi^*(G) \rtimes_\beta \mathbb{R}$ is contained in $J(G)$, we get a homomorphism $\varphi : \Psi^*(G) \rtimes_\beta \mathbb{R} \to J(G)$.

Note that the image of $\Psi^*(G)$ is translation invariant, i.e. invariant by the extension $\overline{\alpha}_u$ of α_u to the multiplier algebra, and that $\overline{\alpha}_u(D^{it}) = u^{it}D^{it}$. This shows that φ is an equivariant morphism from $(\Psi^*(G) \rtimes_\beta \mathbb{R}, \hat{\beta})$ to $(J(G), \alpha)$.

Now β_t restricts to an action of $\mathbb{R}$ on $C^*(G)$, and according to prop. 4.2.a) it follows that $\varphi(C^*(G) \rtimes_\beta \mathbb{R})$ is contained in the ideal $C_0(\mathbb{R}_+^*) \otimes C^*(G)$ of $J(G)$. As D_1^{it} is a multiplier of $C^*(G)$, this crossed product is trivial. More precisely: if $t \mapsto w_t$ is a continuous homomorphism from a group Γ to the multiplier algebra of a C^*-algebra A, the (full or reduced) crossed product $A \rtimes_{ad\,w} \Gamma$ is isomorphic to the (max or min) tensor product $A \otimes C^*(\Gamma)$ where A and Γ map to the multiplier algebra respectively by $a \mapsto a \otimes 1$ and $t \mapsto w_t \otimes \lambda_t$. It follows that $\varphi(C^*(G) \rtimes \mathbb{R}) = C_0(\mathbb{R}_+^*) \otimes C^*(G)$. Now, at the quotient level, the action β becomes trivial on symbols: we thus obtain equality $\varphi((\Psi^*(G) \rtimes_\beta \mathbb{R}) = J(G)$.

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