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Hamilton-Jacobi equations for optimal control on junctions and networks

Yves Achdou ^{*}, Salomé Oudet [†], Nicoletta Tchou [‡]

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Abstract

We consider continuous-state and continuous-time control problems where the admissible trajectories of the system are constrained to remain on a network. A notion of viscosity solution of Hamilton-Jacobi equations on the network has been proposed in earlier articles. Here, we propose a simple proof of a comparison principle based on arguments from the theory of optimal control.

Keywords Optimal control, networks, Hamilton-Jacobi equations, viscosity solutions

AMS 34H05, 49J15

1 Introduction

A network (or a graph) is a set of items, referred to as vertices (or nodes/crosspoints), with connections between them referred to as edges. In the recent years there has been an increasing interest in the investigation of dynamical system and differential equation on networks, in particular in connection with problem of data transmission and traffic management (see for example Garavello-Piccoli [9], Engel et al [6]). While control problems with state constrained in closures of open sets are well studied ([17, 18], [5], [12]) there is to our knowledge much fewer literature on problems on networks. The results of Frankowska and Plaskacz [8, 7] do apply to some closed sets with empty interior, but not to networks with crosspoints (except in very particular cases).

The literature on continuous-state and continuous-time control on networks is recent: the first two articles were published in 2012: control problems whose dynamics is constrained to a network and related Hamilton-Jacobi equations were studied in [1]: a Hamilton-Jacobi equation on the network was proposed, with a definition of viscosity solution, which reduces to the usual one if the network is a straight line (i.e. is composed of two parallel edges sharing an endpoint) and if the dynamics and cost are continuous; while in the interior of an edge, one can test the equation with a smooth test-function, the main difficulties arise at the vertices where the network does not have a regular differential structure. At a vertex, a notion of derivative similar to that of Dini's derivative (see for example [2]) was proposed: admissible test-functions are continuous functions whose restriction to each edge is \mathcal{C}^1 . With this definition, the intrinsic geodesic distance, fixed one argument, is an admissible test-function with respect to the other argument. The Hamiltonian at a vertex depends on all

^{*}Univ. Paris Diderot, Sorbonne Paris Cité, Laboratoire Jacques-Louis Lions, UMR 7598, UPMC, CNRS, F-75205 Paris, France. achdou@ljl.univ-paris-diderot.fr

[†]IRMAR, Université de Rennes 1, Rennes, France

[‡]IRMAR, Université de Rennes 1, Rennes, France, nicoletta.tchou@univ-rennes1.fr

directional derivatives in the directions of the edges containing the vertex, see § 3.3 below. Independently, Imbert, Monneau and Zidani [11] proposed an equivalent notion of viscosity solution for studying a Hamilton-Jacobi approach to junction problems and traffic flows. There is also the work by Schieborn and Camilli [16], in which the authors focus on eikonal equations on networks and on a less general notion of viscosity solution.

Both [1] and [11] contain the first comparison and uniqueness results: in [1], suitably modified geodesic distances are used in the doubling variables method for proving comparison theorems under rather strong continuity assumptions. In [11], Imbert, Monneau and Zidani used a completely different argument based on the explicit solution of a related optimal control problem, which could be obtained because it was assumed that the Hamiltonians associated with each edge did not depend on the state variable.

A very general comparison result has finally been obtained in the quite recent paper by Imbert-Monneau [10]. In the latter article, the Hamiltonians in the edges are completely independent from each other; the main assumption is that the Hamiltonian in each edge, say $H_i(x, p)$ for the edge indexed i , are bimonotone, i.e. non increasing (resp. non decreasing) for p smaller (resp. larger) than a given threshold $p_i^0(x)$. Of course, convex Hamiltonian coming from optimal control theory are bimonotone. Moreover, [10] handles more general transmission conditions at the vertices than the previous articles. The proof of the comparison result is rather involved and only uses arguments from the theory of partial differential equations: in the most simple case where all the Hamiltonians related to the edges are strictly convex and reach their minima at $p = 0$, the idea consists of doubling the variables and using a suitable test-function; then, in the general case, perturbation arguments are used for applying the results proved in the former case.

In coincidence with these research efforts about networks, Barles, Briani and Chasseigne, see [3, 4], have recently studied control problems with discontinuous dynamics and costs, obtaining comparison results for some Bellman equations arising in this context, with original and elegant arguments. Related problems were also recently addressed by Rao, Siconolfi and Zidani [15, 14].

The aim of the present paper is to focus on optimal control problems with independent dynamics and running costs in the edges, and to show that the arguments in [3] can be adapted to yield a simple proof of a comparison result.

Sections 2 to 5 are devoted to the case of a junction, i.e. a network with one vertex only. In Section 6, the results obtained for the junction are generalized for networks with more than one vertices. Section 2 contains a description of the geometry and of the optimal control problem. In Section 3, a Hamilton-Jacobi equation is proposed for the value function, together with a notion of viscosity solution. It is proved that the value function is a viscosity solution of the Hamilton-Jacobi equation. Also in Section 3, Lemma 3.1 on the structure of the Hamiltonian at the vertex will be important for obtaining the comparison principle. Some important properties of viscosity sub and supersolutions are given in Section 4, and the comparison principle is proved in Section 5.

2 The junction

2.1 The geometry

Let us focus on the model case of a junction in \mathbb{R}^d with N semi-infinite straight edges, $N > 1$. The edges are denoted by $(J_i)_{i=1, \dots, N}$. The edge J_i is the closed half-line $\mathbb{R}^+ e_i$. The vectors e_i are two by two distinct unit vectors in \mathbb{R}^d . The half-lines J_i are glued at the origin O to form the junction \mathcal{G} :

$$\mathcal{G} = \bigcup_{i=1}^N J_i.$$

The geodetic distance $d(x, y)$ between two points x, y of \mathcal{G} is

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same edge } J_i \\ |x| + |y| & \text{if } x, y \text{ belong to different branches } J_i \text{ and } J_j. \end{cases}$$

2.2 The optimal control problem

We consider infinite horizon optimal control problems which have different dynamics and running costs in the edges. We are going to describe the assumptions on the dynamics and costs in each edge J_i . The sets of controls are denoted by A_i and the system is driven by a dynamics f_i and the running cost is given by ℓ_i . Our main assumptions are as follows

[H0] A is a metric space (one can take $A = \mathbb{R}^m$). For $i = 1, \dots, N$, A_i is a non empty compact subset of A and $f_i : J_i \times A_i \rightarrow \mathbb{R}$ is a continuous bounded function. The sets A_i are disjoint. Moreover, there exists $L > 0$ such that for any i, x, y in J_i and $a \in A_i$,

$$|f_i(x, a) - f_i(y, a)| \leq L|x - y|.$$

We will use the notation $F_i(x)$ for the set $\{f_i(x, a)e_i, a \in A_i\}$.

[H1] For $i = 1, \dots, N$, the function $\ell_i : J_i \times A_i \rightarrow \mathbb{R}$ is a continuous and bounded function. There is a modulus of continuity ω_i such that for all x, y in J_i and for all $a \in A_i$, $|\ell_i(x, a) - \ell_i(y, a)| \leq \omega_i(|x - y|)$.

[H2] For $i = 1, \dots, N$, $x \in J_i$, the non empty and closed set

$$FL_i(x) \equiv \{(f_i(x, a)e_i, \ell_i(x, a)), a \in A_i\}$$

is convex.

[H3] There is a real number $\delta > 0$ such that for any $i = 1, \dots, N$,

$$[-\delta e_i, \delta e_i] \subset F_i(O).$$

In [H0] the assumption that the sets A_i are disjoint is not restrictive: it is made only for simplifying the proof of Theorem 2.2 below. The assumption [H2] is not essential: it is made in order to avoid the use of relaxed controls.

Here is a general version of Filippov implicit function lemma, see [13], which will be useful to prove Theorem 2.2 below.

Theorem 2.1. *Let I be an interval of \mathbb{R} and $\gamma : I \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ be a measurable function. Let K be a closed subset of $\mathbb{R}^d \times A$ and $\Psi : K \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ be continuous. Assume that $\gamma(I) \subset \Psi(K)$, then there is a measurable function $\Phi : I \rightarrow K$ with*

$$\Psi \circ \Phi(t) = \gamma(t) \quad \text{for a.a. } t \in I.$$

Proof. See [13]. \square

Let us denote by M the set:

$$M = \{(x, a); x \in \mathcal{G}, a \in A_i \text{ if } x \in J_i \setminus \{O\}, \text{ and } a \in \cup_{i=1}^N A_i \text{ if } x = O\}. \quad (2.1)$$

The set M is closed. We also define the function f on M by

$$\forall (x, a) \in M, \quad f(x, a) = \begin{cases} f_i(x, a)e_i & \text{if } x \in J_i \setminus \{O\}, \\ f_i(O, a)e_i & \text{if } x = O \text{ and } a \in A_i. \end{cases}$$

The function f is continuous on M because the sets A_i are disjoint. Let $\tilde{F}(x)$ be defined by

$$\tilde{F}(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to the edge } J_i \setminus \{O\} \\ \cup_{i=1}^N F_i(O) & \text{if } x = O. \end{cases}$$

For $x \in \mathcal{G}$, the set of admissible trajectories starting from x is

$$Y_x = \left\{ y_x \in Lip(\mathbb{R}^+; \mathcal{G}) : \begin{cases} \dot{y}_x(t) \in \tilde{F}(y_x(t)), & \text{for a.a. } t > 0, \\ y_x(0) = x, \end{cases} \right\}. \quad (2.2)$$

Theorem 2.2. Assume [H0],[H1],[H2] and [H3]. Then

1. For any $x \in \mathcal{G}$, Y_x is non empty.
2. For any $x \in \mathcal{G}$, for each trajectory y_x in Y_x , there exists a measurable function $\Phi : [0, +\infty) \rightarrow M$, $\Phi(t) = (\varphi_1(t), \varphi_2(t))$ with

$$(y_x(t), \dot{y}_x(t)) = (\varphi_1(t), f(\varphi_1(t), \varphi_2(t))), \quad \text{for a.e. } t,$$

which means in particular that y_x is a continuous representation of φ_1

3. Almost everywhere in $[0, +\infty)$,

$$\dot{y}_x(t) = \sum_{i=1}^N 1_{\{y_x(t) \in J_i \setminus \{O\}\}} f_i(y_x(t), \varphi_2(t)) e_i.$$

4. Almost everywhere on $\{t : y_x(t) = O\}$, $f(O, \varphi_2(t)) = 0$.

Proof. The proof of point 1 is easy, because $0 \in \tilde{F}(O)$.

The proof of point 2 is a consequence of Theorem 2.1, with $K = M$, $I = [0, +\infty)$, $\gamma(t) = (y_x(t), \dot{y}_x(t))$ and $\Psi(x, a) = (x, f(x, a))$.

From point 2, we deduce that

$$\dot{y}_x(t) = \sum_{i=1}^N 1_{\{y_x(t) \in J_i \setminus \{O\}\}} f_i(y_x(t), \varphi_2(t)) e_i + 1_{\{y_x(t) = O\}} f(O, \varphi_2(t)),$$

and from Stampacchia's theorem, $f(O, \varphi_2(t)) = 0$ almost everywhere in $\{t : y_x(t) = O\}$. This yields points 3 and 4. \square

It is worth noticing that in Theorem 2.2, a solution y_x can be associated with several control laws $\varphi_2(\cdot)$. We introduce the set of admissible controlled trajectories starting from the initial datum x :

$$\mathcal{T}_x = \left\{ \begin{array}{l} (y_x, \alpha) \in L_{\text{Loc}}^\infty(\mathbb{R}^+; M) : \quad y_x \in \text{Lip}(\mathbb{R}^+; \mathcal{G}), \\ \quad \quad \quad \quad \quad \quad \quad \quad y_x(t) = x + \int_0^t f(y_x(s), \alpha(s)) ds \quad \text{in } \mathbb{R}^+ \end{array} \right\}. \quad (2.3)$$

Remark 2.1. If two different edges are aligned with each other, say the edges J_1 and J_2 , many other assumptions can be made on the dynamics and costs:

- it is first possible to assume the continuity of the dynamics and costs at the origin, i.e. that $A_1 = A_2$, that f_1 and f_2 are respectively the restrictions to $J_1 \times A_1$ and $J_2 \times A_2$ of a continuous and bounded function $f_{1,2}$ defined $\mathbb{R}_{e_1} \times A_1$, which is Lipschitz continuous with respect to the first variable, and that ℓ_1 and ℓ_2 are respectively the restrictions to $J_1 \times A_1$ and $J_2 \times A_2$ of a continuous and bounded function $\ell_{1,2}$ defined $\mathbb{R}_{e_1} \times A_1$.
- following Barles et al, see [3, 4], one can allow for some mixing (relaxation) at the vertex, with several possible rules.

The cost functional The cost associated to the trajectory $(y_x, \alpha) \in \mathcal{T}_x$ is

$$J(x; (y_x, \alpha)) = \int_0^\infty \ell(y_x(t), \alpha(t)) e^{-\lambda t} dt,$$

where $\lambda > 0$ is a real number and the Lagrangian ℓ is defined on M by

$$\forall (x, a) \in M, \quad \ell(x, a) = \begin{cases} \ell_i(x, a) & \text{if } x \in J_i \setminus \{O\}, \\ \ell_i(O, a) & \text{if } x = O \text{ and } a \in A_i. \end{cases}$$

The value function The value function of the infinite horizon optimal control problem is

$$v(x) = \inf_{(y_x, \alpha) \in \mathcal{I}_x} J(x; (y_x, \alpha)). \quad (2.4)$$

Proposition 2.1. *Assume [H0],[H1],[H2] and [H3]. Then the value function v is bounded and continuous on \mathcal{G} .*

Proof. The proof essentially uses Assumption [H3]. Since it is classical, we skip it. \square

3 The Hamilton-Jacobi equation

3.1 Test-functions

For the definition of viscosity solutions on the irregular set \mathcal{G} , it is necessary to first define a class of the admissible test-functions

Definition 3.1. *A function $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ is an admissible test-function if*

- φ is continuous in \mathcal{G} and \mathcal{C}^1 in $\mathcal{G} \setminus \{O\}$
- for any $j, j = 1, \dots, N$, $\varphi|_{J_j} \in \mathcal{C}^1(J_j)$.

The set of admissible test-functions is noted $\mathcal{R}(\mathcal{G})$. If $\varphi \in \mathcal{R}(\mathcal{G})$ and $\zeta \in \mathbb{R}$, let $D\varphi(x, \zeta e_i)$ be defined by $D\varphi(x, \zeta e_i) = \zeta \frac{d\varphi}{dx_i}(x)$ if $x \in J_i \setminus \{O\}$ and $D\varphi(O, \zeta e_i) = \zeta \lim_{h \rightarrow 0^+} \frac{d\varphi}{dx_i}(he_i)$.

Property 3.1. *If $\varphi = g \circ \psi$ with $g \in \mathcal{C}^1$ and $\psi \in \mathcal{R}(\mathcal{G})$, then $\varphi \in \mathcal{R}(\mathcal{G})$ and*

$$D\varphi(O, \zeta) = g'(\psi(O))D\psi(O, \zeta).$$

3.2 Vector fields

For $i = 1, \dots, N$, we denote by $F_i^+(O)$ and $\text{FL}_i^+(O)$ the sets

$$F_i^+(O) = F_i(O) \cap \mathbb{R}^+ e_i, \quad \text{FL}_i^+(O) = \text{FL}_i(O) \cap (\mathbb{R}^+ e_i \times \mathbb{R}),$$

which are non empty thanks to assumption [H3]. Note that $0 \in \cap_{i=1}^N F_i(O)$. From assumption [H2], these sets are compact and convex. For $x \in \mathcal{G}$, the sets $F(x)$ and $\text{FL}(x)$ are defined by

$$F(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to the edge } J_i \setminus \{O\} \\ \cup_{i=1, \dots, N} F_i^+(O) & \text{if } x = O, \end{cases}$$

and

$$\text{FL}(x) = \begin{cases} \text{FL}_i(x) & \text{if } x \text{ belongs to the edge } J_i \setminus \{O\} \\ \cup_{i=1, \dots, N} \text{FL}_i^+(O) & \text{if } x = O. \end{cases}$$

3.3 Definition of viscosity solutions

We now introduce the definition of a viscosity solution of

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \text{FL}(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}. \quad (3.1)$$

Definition 3.2. • *An upper semi-continuous function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a subsolution of (3.1) in \mathcal{G} if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x , then*

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\} \leq 0; \quad (3.2)$$

- A lower semi-continuous function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a supersolution of (3.1) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x , then

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\} \geq 0; \quad (3.3)$$

- A continuous function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a viscosity solution of (3.1) in \mathcal{G} if it is both a viscosity subsolution and a viscosity supersolution of (3.1) in \mathcal{G} .

Remark 3.1. At $x \in J_i \setminus \{O\}$, the notion of sub, respectively super-solution in Definition 3.2 is equivalent to the standard definition of viscosity sub, respectively super-solution of

$$\lambda u(x) + \sup_{a \in A_i} \{-f_i(x, a) \cdot Du(x) - \ell_i(x, a)\} = 0.$$

3.4 Hamiltonians

We define the Hamiltonians $H_i : J_i \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$H_i(x, p) = \max_{a \in A_i} (-p f_i(x, a) - \ell_i(x, a)) \quad (3.4)$$

and the Hamiltonian $H_O : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$H_O(p_1, \dots, p_N) = \max_{i=1, \dots, N} \max_{a \in A_i \text{ s.t. } f_i(O, a) \geq 0} (-p_i f_i(O, a) - \ell_i(O, a)). \quad (3.5)$$

We also define what may be called the tangential Hamiltonian at O by

$$H_O^T = - \min_{i=1, \dots, N} \min_{a \in A_i \text{ s.t. } f_i(x, a) = 0} \ell_i(O, a). \quad (3.6)$$

The following definitions are equivalent to Definition 3.2:

Definition 3.3. • An upper semi-continuous function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a subsolution of (3.1) in \mathcal{G} if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x , then

$$\begin{aligned} \lambda u(x) + H_i(x, \frac{d\varphi}{dx_i}(x)) &\leq 0 && \text{if } x \in J_i \setminus \{O\}, \\ \lambda u(O) + H_O(\frac{d\varphi}{dx_1}(O), \dots, \frac{d\varphi}{dx_N}(O)) &\leq 0. \end{aligned} \quad (3.7)$$

- A lower semi-continuous function $u : \mathcal{G} \rightarrow \mathbb{R}$ is a supersolution of (3.1) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x , then

$$\begin{aligned} \lambda u(x) + H_i(x, \frac{d\varphi}{dx_i}(x)) &\geq 0 && \text{if } x \in J_i \setminus \{O\}, \\ \lambda u(O) + H_O(\frac{d\varphi}{dx_1}(O), \dots, \frac{d\varphi}{dx_N}(O)) &\geq 0. \end{aligned} \quad (3.8)$$

The Hamiltonian H_i are continuous with respect to $x \in J_i$, convex with respect to p . Moreover $p \mapsto H_i(O, p)$ is coercive, i.e. $\lim_{|p| \rightarrow +\infty} H_i(O, p) = +\infty$ from the controlability assumption [H3]. Following Imbert-Monneau [10], we introduce the nonempty compact interval \mathcal{P}_0^i

$$\mathcal{P}_0^i = \{p_0^i \in \mathbb{R} \text{ s.t. } H_i(O, p_0^i) = \min_{p \in \mathbb{R}} H_i(O, p)\}. \quad (3.9)$$

Lemma 3.1. Assume [H0],[H1],[H2] and [H3], then

1. $p_0^i \in \mathcal{P}_0^i$ if and only if there exists $a^* \in A_i$ such that $f_i(O, a^*) = 0$ and $H_i(O, p_0^i) = -p_0^i f_i(O, a^*) - \ell_i(O, a^*) = -\ell_i(O, a^*)$

2.

$$\min_{p \in \mathbb{R}} H_i(O, p) = - \min_{a \in A_i \text{ s.t. } f_i(O, a) = 0} \ell_i(O, a) \quad (3.10)$$

3. For all $p \in \mathbb{R}$, if $p \geq p_0^i$ for some $p_0^i \in \mathcal{P}_0^i$ then

$$\max_{a \in A_i \text{ s.t. } f_i(O,a) \geq 0} (-pf_i(O,a) - \ell_i(O,a)) = \min_{q \in \mathbb{R}} H_i(O,q) = - \min_{a \in A_i \text{ s.t. } f_i(O,a)=0} \ell_i(O,a).$$

Proof. The Hamiltonian H_i reaches its minimum at p_0^i if and only if $0 \in \partial H_i(O, p_0^i)$. The subdifferential of $H_i(O, \cdot)$ at p_0^i is characterized by

$$\partial H_i(O, p_0^i) = \overline{\text{co}}\{-f_i(O, a); a \in A_i \text{ s.t. } H_i(O, p_0^i) = -p_0^i f_i(O, a) - \ell_i(O, a)\},$$

see [19]. But from [H2],

$$\{(f_i(O, a), \ell_i(O, a)); a \in A_i \text{ s.t. } H_i(O, p_0^i) = -p_0^i f_i(O, a) - \ell_i(O, a)\}$$

is compact and convex. Hence,

$$\partial H_i(O, p_0^i) = \{-f_i(O, a); a \in A_i \text{ s.t. } H_i(O, p_0^i) = -p_0^i f_i(O, a) - \ell_i(O, a)\}.$$

Therefore, $0 \in \partial H_i(O, p_0^i)$ if and only if there exists $a^* \in A_i$ such that $f_i(O, a^*) = 0$ and $H_i(O, p_0^i) = -\ell_i(O, a^*)$. We have proved point 1.

Point 2 is a direct consequence of point 1.

If p is greater than or equal to some $p_0^i \in \mathcal{P}_0^i$, then

$$\max_{a \in A_i: f_i(O,a) \geq 0} (-pf_i(O, a) - \ell_i(O, a)) \leq \max_{a \in A_i: f_i(O,a) \geq 0} (-p_0^i f_i(O, a) - \ell_i(O, a)) = H_i(O, p_0^i)$$

where the last identity comes from point 1.

On the other hand,

$$\max_{a \in A_i: f_i(O,a) \geq 0} (-pf_i(O, a) - \ell_i(O, a)) \geq - \min_{a \in A_i: f_i(O,a)=0} \ell_i(O, a).$$

Point 3 is obtained by combining the two previous observations and point 2. \square

3.5 Existence

Theorem 3.1. *Assume [H0],[H1],[H2] and [H3]. The value function v defined in (2.4) is a bounded viscosity solution of (3.1) in \mathcal{G} .*

The proof of Theorem 3.1 is made in several steps, namely Proposition 3.1 and Lemmas 3.2 and 3.3 below: the first step consists of proving that the value function is a viscosity solution of a Hamilton-Jacobi equation with a more general definition of the Hamiltonian: for that, we introduce larger relaxed vector fields: for $x \in \mathcal{G}$,

$$\tilde{f}(x) = \left\{ \eta \in \mathbb{R}^d : \begin{array}{l} \exists (y_{x,n}, \alpha_n)_{n \in \mathbb{N}}, \\ (y_{x,n}, \alpha_n) \in \mathcal{T}_x, \text{ s.t.} \\ \exists (t_n)_{n \in \mathbb{N}} \end{array} \quad \begin{array}{l} t_n \rightarrow 0^+ \text{ and} \\ \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} f(y_{x,n}(t), \alpha_n(t)) dt = \eta \end{array} \right\}$$

and

$$\tilde{fl}(x) = \left\{ (\eta, \mu) \in \mathbb{R}^d \times \mathbb{R} : \begin{array}{l} \exists (y_{x,n}, \alpha_n)_{n \in \mathbb{N}}, \\ (y_{x,n}, \alpha_n) \in \mathcal{T}_x, \text{ s.t.} \\ \exists (t_n)_{n \in \mathbb{N}} \end{array} \quad \begin{array}{l} t_n \rightarrow 0^+, \\ \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} f(y_{x,n}(t), \alpha_n(t)) dt = \eta, \\ \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y_{x,n}(t), \alpha_n(t)) dt = \mu \end{array} \right\}.$$

Proposition 3.1. *Assume [H0],[H1],[H2] and [H3]. The value function v defined in (2.4) is a viscosity solution of*

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \tilde{fl}(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}, \quad (3.11)$$

where the definition of viscosity solution is exactly the same as Definition 3.2, replacing $FL(x)$ with $\tilde{fl}(x)$.

Proof. See [1]. \square

For all $\varphi \in \mathcal{R}(\mathcal{G})$, it is clear that if $x \in J_i \setminus \{O\}$, then $H_i(x, D\varphi) = \sup_{(\zeta, \xi) \in \tilde{f}\ell(x)} \{-D\varphi(x, \zeta) - \xi\}$. We are left with comparing $\sup_{(\zeta, \xi) \in \text{FL}(O)} \{-D\varphi(O, \zeta) - \xi\}$ and $\sup_{(\zeta, \xi) \in \tilde{f}\ell(O)} \{-D\varphi(O, \zeta) - \xi\}$. The two quantities are the same. This is a consequence of the following lemma

Lemma 3.2.

$$\tilde{f}\ell(O) = \bigcup_{i=1, \dots, N} \overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}.$$

Proof. The proof being a bit long, we postpone it to the appendix. \square

Lemma 3.3. *Assume [H0],[H1],[H2] and [H3]. For any function φ in $\mathcal{R}(\mathcal{G})$,*

$$\sup_{(\zeta, \xi) \in \tilde{f}\ell(O)} \{-D\varphi(O, \zeta) - \xi\} = \max_{(\zeta, \xi) \in \text{FL}(O)} \{-D\varphi(O, \zeta) - \xi\}. \quad (3.12)$$

Proof. It was proved in [1] that $\text{FL}(O) \subset \tilde{f}\ell(O)$. Hence

$$\max_{(\zeta, \xi) \in \text{FL}(O)} \{-D\varphi(O, \zeta) - \xi\} \leq \sup_{(\zeta, \xi) \in \tilde{f}\ell(O)} \{-D\varphi(O, \zeta) - \xi\}.$$

From the piecewise linearity of the function $(\zeta, \mu) \mapsto -D\varphi(O, \zeta) - \mu$, we infer that

$$\begin{aligned} & \sup_{(\zeta, \mu) \in \overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}} (-D\varphi(O, \zeta) - \mu) \\ &= \max \left(\max_{(\zeta, \mu) \in \text{FL}_i^+(O)} (-D\varphi(O, \zeta) - \mu), \max_{j \neq i} \max_{(0, \mu) \in \text{FL}_j(O)} -\mu \right) \\ &\leq \max_{j=1, \dots, N} \max_{(\zeta, \mu) \in \text{FL}_j^+(O)} -D\varphi(O, \zeta) - \mu = \max_{(\zeta, \xi) \in \text{FL}(O)} \{-D\varphi(O, \zeta) - \xi\}. \end{aligned}$$

We conclude by using Lemma 3.2. \square

4 Properties of viscosity sub and supersolutions

In this part, we study sub and supersolutions of (3.1), transposing ideas coming from Barles-Briani-Chasseigne [3, 4] to the present context.

Lemma 4.1. *Assume [H0],[H1],[H2] and [H3]. Any bounded and continuous viscosity sub-solution u of (3.1) is Lipschitz continuous in a neighborhood of O .*

Proof. From [2], Theorem III.2.33 and Remark III.2.34, u satisfies a global suboptimality principle in J_i for any $i \in \{1, \dots, N\}$. From the assumptions, there exists a positive number σ , positive real numbers r_i and controls $a_{i, \pm} \in A_i$ for $i = 1, \dots, N$, such that for all $x \in [O, re_i]$, $f_i(x, a_{i, -}) \leq -\sigma$ and $f_i(x, a_{i, +}) \geq \sigma$. We infer from this and the suboptimality principle, that for x, y in $[O, re_i]$

$$|u(x) - u(y)| \leq (\|\ell_i\|_\infty + \lambda \|u\|_\infty) \frac{|y - x|}{\sigma},$$

which yields the desired result. \square

The following lemma can be found in [3, 4] in a different context:

Lemma 4.2. *Let $v : \mathcal{G} \rightarrow \mathbb{R}$ be a viscosity supersolution of (3.1) in \mathcal{G} and w be a continuous viscosity subsolution of (3.1) in \mathcal{G} . Then if $x \in J_i \setminus \{0\}$, we have for all $t > 0$,*

$$v(x) \geq \inf_{\alpha_i(\cdot), \theta_i} \left(\int_0^{t \wedge \theta_i} \ell_i(y_{x^i}^i(s), \alpha_i(s)) e^{-\lambda s} ds + v(y_{x^i}^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \quad (4.1)$$

and

$$w(x) \leq \inf_{\alpha_i(\cdot), \theta_i} \left(\int_0^{t \wedge \theta_i} \ell_i(y_{x^i}^i(s), \alpha_i(s)) e^{-\lambda s} ds + w(y_{x^i}^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \quad (4.2)$$

where $\alpha_i \in L^\infty(0, \infty; A_i)$, y_x^i is the solution of $y_x^i(t) = x + \left(\int_0^t f_i(y_x^i(s), \alpha_i(s)) ds \right) e_i$ and θ_i is such that $y_x^i(\theta_i) = 0$ and θ_i lies in $[\tau_i, \bar{\tau}_i]$, where τ_i is the exit time of y_x^i from $J_i \setminus \{O\}$ and $\bar{\tau}_i$ is the exit time of y_x^i from J_i .

Proof. See [3]. \square

The following theorem is reminiscent of Theorem 3.3 in [3]:

Theorem 4.1. *Assume [H0],[H1],[H2] and [H3]. Let $v : \mathcal{G} \rightarrow \mathbb{R}$ be a viscosity supersolution of (3.1), bounded from below by $-c|x| - C$ for two positive numbers c and C . Either [A] or [B] below is true:*

[A] *There exists $\eta > 0$, $i \in \{1, \dots, N\}$ and a sequence $x_k \in J_i$, $\lim_{k \rightarrow +\infty} x_k = O$ such that $\lim_{k \rightarrow +\infty} v(x_k) = v(O)$ and for each k , there exists a control law α_i^k such that the corresponding trajectory $y_{x_k}(s) \in J_i$ for all $s \in [0, \eta]$ and*

$$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + v(y_{x_k}(\eta)) e^{-\lambda \eta} \quad (4.3)$$

[B]

$$\lambda v(O) + H_O^T \geq 0. \quad (4.4)$$

Proof. Let us assume that [B] does not hold.

For any i in $\{1, \dots, N\}$, take for example

$$q_i = \min_{p_0^i \in \mathcal{P}_0^i} p_0^i,$$

and $q = (q_1, \dots, q_N)$. From Lemma 3.1,

$$H_O(q) = H_O^T. \quad (4.5)$$

Consider the function

$$v(x) - q_i |x| + \frac{|x|^2}{\varepsilon^2} \quad \text{if } x \in J_i.$$

Standard arguments show that this function reaches its minimum near O and any sequence of such minimum points x_ε converges to O and that $v(x_\varepsilon)$ converges to $v(O)$.

It is not possible that x_ε be O , because since v is a viscosity supersolution of (3.1), we would have that

$$\lambda v(O) + H_O(q) \geq 0,$$

and therefore $\lambda v(O) + H_O^T \geq 0$, which is a contradiction since [B] does not hold.

Therefore, there exists $i \in \{1, \dots, N\}$ such that, up to the extraction of a subsequence, $x_\varepsilon \in J_i \setminus \{O\}$, for all ε . We can therefore apply Lemma 4.2: for any $t > 0$,

$$v(x_\varepsilon) \geq \inf_{\alpha_i(\cdot), \theta_i} \left(\int_0^{t \wedge \theta_i} \ell_i(y_{x_\varepsilon}^i(s), \alpha_i(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \quad (4.6)$$

where y_x^i is the solution of $y_x^i(t) = x + \left(\int_0^t f_i(y_x^i(s), \alpha_i(s)) ds \right) e_i$.

Take $t = 1$ for example. From [H0] and [H2], the minimum in (4.6) is reached for some $\alpha_{i,\varepsilon}$ and $\theta_{i,\varepsilon} > 0$, see [3] :

$$v(x_\varepsilon) \geq \int_0^{1 \wedge \theta_{i,\varepsilon}} \ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(1 \wedge \theta_{i,\varepsilon})) e^{-\lambda(1 \wedge \theta_{i,\varepsilon})}. \quad (4.7)$$

Assume by contradiction that [A] does not hold, then $\lim_{\varepsilon \rightarrow 0} \theta_{i,\varepsilon} = 0$.

Since x_ε is a minimum of $v(x) - q_i|x| + \frac{|x|^2}{\varepsilon^2}$, we deduce from (4.7) that

$$0 \geq \int_0^{\theta_{i,\varepsilon}} \ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(\theta_{i,\varepsilon})) (e^{-\lambda \theta_{i,\varepsilon}} - 1) - q_i|x_\varepsilon| + \frac{|x_\varepsilon|^2}{\varepsilon^2}, \quad (4.8)$$

and therefore

$$0 \geq \int_0^{\theta_{i,\varepsilon}} \ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(\theta_{i,\varepsilon})) (e^{-\lambda \theta_{i,\varepsilon}} - 1) - q_i|x_\varepsilon|. \quad (4.9)$$

We can write (4.9) as

$$0 \leq \int_0^{\theta_{i,\varepsilon}} (-\ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} - q_i f_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s))) ds - v(y_{x_\varepsilon}^i(\theta_{i,\varepsilon})) (e^{-\lambda \theta_{i,\varepsilon}} - 1). \quad (4.10)$$

Dividing by $\theta_{i,\varepsilon}$ and letting ε tend to 0, we obtain that $\lambda v(O) + H_i(O, q_i) \geq 0$. This implies that $\lambda v(O) + H_O^T \geq 0$, which is a contradiction since [B] does not hold. \square

5 Comparison principle and Uniqueness

Theorem 5.1. *Assume [H0],[H1],[H2] and [H3]. Let $u : \mathcal{G} \rightarrow \mathbb{R}$ be a bounded continuous viscosity subsolution of (3.1), and $v : \mathcal{G} \rightarrow \mathbb{R}$ be a bounded viscosity supersolution of (3.1). Then $u \leq v$ in \mathcal{G} .*

Proof. It is a simple matter to check that there exists a positive real number M such that the function $\psi(x) = -|x|^2 - M$ is a viscosity subsolution of (3.1). For $0 < \mu < 1$, μ close to 1, the function $u_\mu = \mu u + (1 - \mu)\psi$ is a viscosity subsolution of (3.1), which tends to $-\infty$ as $|x|$ tends to $+\infty$. Let M_μ be the maximal value of $u_\mu - v$ which is reached at some point \bar{x}_μ . We want to prove that $M_\mu \leq 0$.

1. If $\bar{x}_\mu \neq O$, then we introduce the function $u_\mu(x) - v(x) - d^2(x, \bar{x}_\mu)$, which has a strict maximum at \bar{x}_μ , and we double the variables, i.e. for $0 < \varepsilon \ll 1$, we consider

$$u_\mu(x) - v(y) - d^2(x, \bar{x}_\mu) - \frac{d^2(x, y)}{\varepsilon^2}.$$

Classical arguments then lead to the conclusion that $u_\mu(\bar{x}_\mu) - v(\bar{x}_\mu) \leq 0$, thus $M_\mu \leq 0$.

2. If $\bar{x}_\mu = O$. We use Theorem 4.1; we have two possible cases:

$$[B] \lambda v(O) \geq -H_O^T.$$

Since, from Lemma 4.1, u is Lipschitz continuous in a neighborhood of O , we know that there exists a test-function φ in $\mathcal{R}(\mathcal{G})$ which touches u from above at O . Since u is a subsolution of (3.1), we see that $\lambda u(O) + H_O(\frac{d\varphi}{dx_1}(O), \dots, \frac{d\varphi}{dx_N}(O)) \leq 0$, which implies that $\lambda u(O) + H_O^T \leq 0$. Therefore, we obtain that $u_\mu(O) \leq v(O)$, thus $M_\mu \leq 0$.

[A] With the notations of Theorem 4.1, we have that

$$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + v(y_{x_k}(\eta)) e^{-\lambda \eta}.$$

Moreover, from Lemma 4.2,

$$u_\mu(x_k) \leq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + u_\mu(y_{x_k}(\eta)) e^{-\lambda \eta}.$$

Therefore

$$u_\mu(x_k) - v(x_k) \leq (u_\mu(y_{x_k}(\eta)) - v(y_{x_k}(\eta))) e^{-\lambda \eta}.$$

Letting k tend to $+\infty$, we find that $M_\mu \leq M_\mu e^{-\lambda \eta}$, which implies that $M_\mu \leq 0$

We conclude by letting μ tend to 1. \square

Corollary 5.1. *Assume [H0],[H1],[H2] and [H3]. The value function u of the optimal control problem (2.4) is the unique bounded viscosity solution of (3.1).*

6 The case of a network

6.1 The geometrical setting and the optimal control problem

We consider a network in \mathbb{R}^d with a finite number of edges and vertices. A network in \mathbb{R}^d is a pair $(\mathcal{V}, \mathcal{E})$ where

- i) \mathcal{V} is a finite subset of \mathbb{R}^d whose elements are said vertices
- ii) \mathcal{E} is a finite set of edges, which are either closed straight line segments between two vertices, or a closed straight half-lines whose endpoint is a vertex. The intersection of two edges is either empty or a vertex of the network. The union of the edges in \mathcal{E} is a connected subset of \mathbb{R}^d . For a given edge $e \in \mathcal{E}$, the notation ∂e is used for the set of endpoints of e , and $e^* = e \setminus \partial e$ stands for the interior of e . Let also u_e be a unit vector aligned with e . There are two possible such vectors: if the boundary of e is made of one vertex x only, then u_e will be oriented from x to the interior of e ; if the boundary of e is made of two vertices, then the choice of the orientation is arbitrary.

We say that two vertices are adjacent if they are connected by an edge. For a given vertex x , we denote by \mathcal{E}_x the set of the edges for which x is an endpoint, and N_x the cardinality of \mathcal{E}_x . We denote by \mathcal{G} the union of all the edges in \mathcal{E} .

We consider infinite horizon optimal control problems which have different dynamics and running cost in the edges. We are going to describe the assumptions on the dynamics and costs in each edge e . The sets of controls are denoted by A_e and the system is driven by a dynamics f_e and the running cost is given by ℓ_e . Our main assumptions are as follows

[H0_n] A is a metric space (one can take $A = \mathbb{R}^m$). For $e \in \mathcal{E}$, A_e is a non empty compact subset of A and $f_e : e \times A_e \rightarrow \mathbb{R}$ is a continuous bounded function. The sets A_e are disjoint. Moreover, there exists $L > 0$ such that for any $e \in \mathcal{E}$, x, y in e and $a \in A_e$,

$$|f_e(x, a) - f_e(y, a)| \leq L|x - y|.$$

We will use the notation $F_e(x)$ for the set $\{f_e(x, a)u_e, a \in A_e\}$.

[H1_n] For $e \in \mathcal{E}$, the function $\ell_e : e \times A_e \rightarrow \mathbb{R}$ is a continuous and bounded function. There is a modulus of continuity ω_e such that for all x, y in e and for all $a \in A_e$, $|\ell_e(x, a) - \ell_e(y, a)| \leq \omega_e(|x - y|)$.

[H2_n] For $e \in \mathcal{E}$, $x \in e$, the non empty and closed set $FL_e(x) \equiv \{(f_e(x, a)u_e, \ell_e(x, a)), a \in A_e\}$ is convex.

[H3_n] There is a real number $\delta > 0$ such that for any $e \in \mathcal{E}$, for all endpoints x of e ,

$$[-\delta u_e, \delta u_e] \subset F_e(x).$$

Let us denote by M the set:

$$M = \{(x, a); x \in \mathcal{G}, a \in A_e \text{ if } x \in e^*, \text{ and } a \in \cup_{e \in \mathcal{E}_x} A_e \text{ if } x \in \mathcal{V}\}. \quad (6.1)$$

The set M is closed. We also define the function f on M by

$$\forall (x, a) \in M, \quad f(x, a) = \begin{cases} f_e(x, a)u_e & \text{if } x \in e^*, \\ f_e(x, a)u_e & \text{if } x \in \mathcal{V} \text{ and } a \in A_e \text{ for } e \in \mathcal{E}_x. \end{cases}$$

The set of admissible controlled trajectories starting from the initial datum $x \in \mathcal{G}$ can be defined by

$$\mathcal{T}_x = \left\{ (y_x, \alpha) \in L_{\text{loc}}^\infty(\mathbb{R}^+; M) : \begin{array}{l} y_x \in \text{Lip}(\mathbb{R}^+; \mathcal{G}), \\ y_x(t) = x + \int_0^t f(y_x(s), \alpha(s)) ds \quad \text{in } \mathbb{R}^+ \end{array} \right\}, \quad (6.2)$$

exactly as in § 2.1.

The cost associated to the trajectory $(y_x, \alpha) \in \mathcal{T}_x$ is

$$J(x; (y_x, \alpha)) = \int_0^\infty \ell(y_x(t), \alpha(t)) e^{-\lambda t} dt,$$

where $\lambda > 0$ is a real number and the Lagrangian ℓ is defined on M by

$$\forall (x, a) \in M, \quad \ell(x, a) = \begin{cases} \ell_e(x, a) & \text{if } x \in e^*, \\ \ell_e(x, a) & \text{if } x \in \mathcal{V} \text{ and } a \in A_e \text{ for } e \in \mathcal{E}_x. \end{cases}$$

The value function of the infinite horizon optimal control problem is

$$v(x) = \inf_{(y_x, \alpha) \in \mathcal{T}_x} J(x; (y_x, \alpha)). \quad (6.3)$$

6.2 The Hamilton-Jacobi equation

For each edge e , $x \in e^*$, let x_e be the coordinate of x in the system (O_e, u_e) where O_e is an arbitrary origin on e .

For the definition of viscosity solutions on the irregular set \mathcal{G} , it is necessary to first define a class of the admissible test-functions

Definition 6.1. A function $\varphi : \mathcal{G} \rightarrow \mathbb{R}$ is an admissible test-function if

- φ is continuous in \mathcal{G} and C^1 in $\mathcal{G} \setminus \mathcal{V}$
- for any e , $\varphi|_e \in C^1(e)$.

The set of admissible test-function is noted $\mathcal{R}(\mathcal{G})$. If $\varphi \in \mathcal{R}(\mathcal{G})$ and $\zeta \in \mathbb{R}$, let $D\varphi(x, \zeta u_e)$ be defined by $D\varphi(x, \zeta u_e) = \zeta \frac{d\varphi}{dx_e}(x)$ if $x \in e^*$, and $D\varphi(x, \zeta u_e) = \zeta \lim_{y \rightarrow x, y \in e^*} \frac{d\varphi}{dx_e}(y)$, if x is an endpoint of e .

We define the Hamiltonians $H_e : e \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$H_e(x, p) = \max_{a \in A_e} (-pf_e(x, a) - \ell_e(x, a)). \quad (6.4)$$

For a vertex $x \in \mathcal{V}$, for a given indexing of \mathcal{E}_x : $\mathcal{E}_x = \{e_1, \dots, e_{N_x}\}$, we use the notation $A_i = A_{e_i}$, $f_i = f_{e_i}$, $\ell_i = \ell_{e_i}$ for simplicity. Let also σ_i be 1 if u_{e_i} is oriented from x to the interior of e_i and -1 in the opposite case. The Hamiltonian $H_x : \mathbb{R}^{N_x} \rightarrow \mathbb{R}$ is defined by

$$H_x(p_1, \dots, p_{N_x}) = \max_{i=1, \dots, N_x} \max_{a \in A_i \text{ s.t. } \sigma_i f_i(x, a) \geq 0} (-p_i f_i(x, a) - \ell_i(x, a)). \quad (6.5)$$

We wish to define viscosity solutions of the following equations

$$\lambda v(x) + H_e(x, Dv(x)) = 0 \quad \text{if } x \in e^*, \quad (6.6)$$

$$\lambda v(x) + H_x(Dv(x)) = 0 \quad \text{if } x \in \mathcal{V}. \quad (6.7)$$

Definition 6.2. • *An upper semi-continuous function $w : \mathcal{G} \rightarrow \mathbb{R}$ is a subsolution of (6.6)-(6.7) in \mathcal{G} if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $w - \varphi$ has a local maximum point at x , then*

$$\begin{aligned} \lambda w(x) + H_e(x, \frac{d\varphi}{dx_e}(x)) &\leq 0 && \text{if } x \in e^*, \\ \lambda w(x) + H_x(\frac{d\varphi}{dx_1}(x), \dots, \frac{d\varphi}{dx_{N_x}}(x)) &\leq 0 && \text{if } x \in \mathcal{V}, \end{aligned} \quad (6.8)$$

where in the last case, $\frac{d\varphi}{dx_i}(x) = D\varphi(x, u_{e_i}(x))$, for $i = 1, \dots, N_x$.

• *A lower semi-continuous function $w : \mathcal{G} \rightarrow \mathbb{R}$ is a supersolution of (6.6)-(6.7) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $w - \varphi$ has a local minimum point at x , then*

$$\begin{aligned} \lambda w(x) + H_e(x, \frac{d\varphi}{dx_e}(x)) &\geq 0 && \text{if } x \in e^*, \\ \lambda w(x) + H_x(\frac{d\varphi}{dx_1}(x), \dots, \frac{d\varphi}{dx_{N_x}}(x)) &\geq 0 && \text{if } x \in \mathcal{V}. \end{aligned} \quad (6.9)$$

6.3 Comparison principle

Since all the arguments used in the junction case are local, we can replicate them in the case of a network and obtain:

Theorem 6.1. *Assume $[H0_n], [H1_n], [H2_n]$ and $[H3_n]$. Let $v : \mathcal{G} \rightarrow \mathbb{R}$ be a bounded continuous viscosity subsolution of (6.6)-(6.7), and $w : \mathcal{G} \rightarrow \mathbb{R}$ be a bounded viscosity supersolution of (6.6)-(6.7). Then $v \leq w$ in \mathcal{G} .*

6.4 Existence and uniqueness

By the same arguments as in the junction case, we can prove that v is a bounded viscosity solution of (6.6)-(6.7). From the Theorem 6.1, it is the unique bounded viscosity solution.

Proposition 6.1. *Assume $[H0_n], [H1_n], [H2_n]$ and $[H3_n]$. The value function v of the optimal control problem (6.3) is the unique bounded viscosity solution of (6.6)-(6.7).*

A Proof of Lemma 3.2

For any $i \in \{1, \dots, N\}$, the inclusion $\overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\} \subset \tilde{f}\ell(O)$ is proved by explicitly constructing trajectories, see [1]. We skip this part. This leads to

$$\bigcup_{i=1, \dots, N} \overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\} \subset \tilde{f}\ell(O).$$

We now prove the other inclusion. For any $(\zeta, \mu) \in \tilde{f}\ell(O)$, there exists a sequence of admissible trajectories $(y_n, \alpha_n) \in \mathcal{T}_O$ and a sequence of times $t_n \rightarrow 0+$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} f(y_n(t), \alpha_n(t)) dt = \zeta, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) dt = \mu.$$

• If $\zeta \neq 0$, then there must exist an index i in $\{1, \dots, N\}$ such that $\zeta = |\zeta|e_i$: in this case, $y_n(t_n) \in J_i \setminus \{O\}$. Hence,

$$y_n(t_n) = \int_0^{t_n} f(y_n(t), \alpha_n(t)) dt = \sum_{j=1}^N e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \setminus \{O\}} dt \quad (\text{A.1})$$

with

$$\begin{aligned} \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt &= 0 & \text{if } j \neq i, \\ \int_0^{t_n} f_i(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt &= |y_n(t_n)|. \end{aligned}$$

These identities are a consequence of Stampacchia's theorem: consider for example $j \neq i$ and the function $\kappa_j : y \mapsto |y| \mathbf{1}_{y \in J_j}$. It is easy to check that $t \mapsto \kappa_j(y_n(t))$ belongs to $W_0^{1,\infty}(0, t_n)$ and that its weak derivative coincides almost everywhere with $t \mapsto f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}}$. Hence,

$$\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt = 0.$$

For $j = 1, \dots, N$, let $T_{j,n}$ be defined by

$$T_{j,n} = \left| \left\{ t \in [0, t_n] : y_n(t) \in J_j \setminus \{O\} \right\} \right|.$$

If $j \neq i$ and $T_{j,n} > 0$ then

$$\begin{aligned} & \frac{1}{T_{j,n}} \left(\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt \right) \\ &= \frac{1}{T_{j,n}} \left(\int_0^{t_n} f_j(O, \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt, \int_0^{t_n} \ell_j(O, \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt \right) + o(1) \end{aligned}$$

where $o(1)$ is a vector tending to 0 as $n \rightarrow \infty$. Therefore, the distance of

$\frac{1}{T_{j,n}} \left(e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt \right)$ to the set $\text{FL}_j(O)$ tends to 0. Moreover, $\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt = 0$. Hence, the distance of $\frac{1}{T_{j,n}} \left(e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_j \setminus \{O\}} dt \right)$ to the set $\left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right)$ tends to zero as n tends to ∞ .

If the set $\{t : y_n(t) = O\}$ has a nonzero measure, then

$$\left(0, \frac{1}{|\{t : y_n(t) = O\}|} \int_0^{t_n} \ell(O, \alpha_n(t)) \mathbf{1}_{\{t : y_n(t) = O\}} dt \right) \in \overline{\text{co}} \left\{ \bigcup_{j=1}^N \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}.$$

Finally, we know that $T_{i,n} > 0$.

$$\begin{aligned} & \frac{1}{T_{i,n}} \left(\int_0^{t_n} f_i(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt, \int_0^{t_n} \ell_i(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt \right) \\ &= \frac{1}{T_{i,n}} \left(\int_0^{t_n} f_i(O, \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt, \int_0^{t_n} \ell_i(O, \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt \right) + o(1) \end{aligned}$$

so the distance of

$\frac{1}{T_{i,n}} \left(e_i \int_0^{t_n} f_i(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt, \int_0^{t_n} \ell_i(y_n(t), \alpha_n(t)) \mathbf{1}_{y_n(t) \in J_i \setminus \{O\}} dt \right)$ to the set $\text{FL}_i^+(O)$ tends to zero as n tends to ∞ .

Combining all the observations above, we see that the distance of

$\left(\frac{1}{t_n} \int_0^{t_n} f(y_n(t), \alpha_n(t)) dt, \frac{1}{t_n} \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) dt \right)$ to $\overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}$ tends to 0 as $n \rightarrow \infty$.

Therefore $(\zeta, \mu) \in \overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}$.

- If $\zeta = 0$, either there exists i such that $y_n(t_n) \in J_i \setminus \{O\}$ or $y_n(t_n) = O$:
 - If $y_n(t_n) \in J_i \setminus \{O\}$, then we can make exactly the same argument as above and conclude that $(\zeta, \mu) \in \overline{\text{co}} \left\{ \text{FL}_i^+(O) \cup \bigcup_{j \neq i} \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}$. Since $\zeta = 0$, we have in fact that $(\zeta, \mu) \in \overline{\text{co}} \bigcup_{j=1}^N \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right)$.
- if $y_n(t_n) = O$, we have that $\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \setminus \{O\}} dt = 0$ for all $j = 1, \dots, N$. We can repeat the argument above, and obtain that $(\zeta, \mu) \in \overline{\text{co}} \left\{ \bigcup_{j=1}^N \left(\text{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \right) \right\}$.

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