



**HAL**  
open science

## Local Algebraic K-Theory

Nicolae Teleman

► **To cite this version:**

| Nicolae Teleman. Local Algebraic K-Theory. 2013. hal-00846817v1

**HAL Id: hal-00846817**

**<https://hal.science/hal-00846817v1>**

Preprint submitted on 21 Jul 2013 (v1), last revised 28 Aug 2013 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LOCAL ALGEBRAIC $K$ -THEORY

NICOLAE TELEMAN

DIPARTIMENTO DI SCIENZE MATEMATICHE, UNIVERSITA' POLITECNICA DELLE MARCHE

E-MAIL: TELEMAN@DIPMAT.UNIVPM.IT

## 1. ABSTRACT

In this article we address the first part of the programme presented in [25], §2; we construct the *local  $K$ -theory level* of the index formula.

Our construction is sufficiently general to encompass the algebra of pseudo-differential operators of order zero on smooth manifolds, elliptic pseudo-differential operators of order zero, their *abstract symbol* (see Introduction §2. ) and their *local  $K$ -theory analytical and topological index classes*, see [25], §5, Definition 5 and 6. Our definitions are sufficiently general to apply to exact sequences of singular integral operators, which are of interest in the case of the index theorem on Lipschitz and quasi-conformal manifolds, see [21], [22], [9], [7].

In this article we introduce *localised algebras* (Definition 3)  $A$  and in §6 we define their *local algebraic  $K$ -theory*.

A localised algebra  $A$  is an algebra in which a decreasing filtration by vector sub-spaces  $A_\mu$  is introduced. The filtration  $A_\mu$  induces a filtration on the space of matrices  $\mathbb{M}(A_\mu)$ .

Although we define solely  $K_*^{loc}(A)$  for  $* = 0, 1$ , we expect our construction could be extended in higher degrees.

*We stress that our construction of  $K_0^{loc}(A)$  uses exclusively idempotent matrices and that the use of finite projective modules is totally avoided. (Idempotent matrices, rather than projective modules, contain less arbitrariness in the description of the  $K_0$  classes and allow a better filtration control).*

The group  $K_0^{loc}(A)$  is by definition the quotient space of the space of the Grothendieck completion of the space of idempotent matrices through three equivalence relations: -1) stabilisation  $\sim_s$ , -2) local conjugation  $\sim_l$ , and -3) projective limit with respect to the filtration.

By definition, the  $K_1^{loc}(A)$  is the projective limit of the local  $K_1(A_\mu)$  groups. The group  $K_1(A_\mu)$  is by definition the quotient of  $\mathbb{GL}(A_\mu)$  modulo the equivalence relation generated by: -1) stabilisation  $\sim_s$ , -2) local conjugation  $\sim_l$  and -3)  $\sim_{\mathbb{O}(A_\mu)}$ , where  $\mathbb{O}(A_\mu)$  is the submodule generated by elements of the form  $u \oplus u^{-1}$ , for any  $u \in \mathbb{GL}(A_\mu)$ . *The class of any invertible element  $u$  modulo conjugation (inner auto-morphisms) we call the Jordan canonical form of  $u$ . The local conjugation preserves the local Jordan canonical form of invertible elements. The equivalence relation  $\sim_{\mathbb{O}(A_\mu)}$  insures existence of opposite elements in  $K_1(A_\mu)$  and  $K_1^{loc}(A)$ .*

Our definition of  $K_1^{loc}(A)$  does not use the commutator sub-group  $[\mathbb{G}\mathbb{L}(A), \mathbb{G}\mathbb{L}(A)]$  nor elementary matrices in its construction.

We define short exact sequences of localised algebras. To get the corresponding (open) six terms exact sequence (Theorem 51) one has to take the tensor product of the expected six terms exact sequence by  $\mathbb{Z}[\frac{1}{2}]$ .

Our work shows that the basic structure of  $K_1$  resides in the sub-group generated by elements of the form  $u \oplus u^{-1}$ ,  $u \in \mathbb{G}\mathbb{L}(A)$ , rather than in the commutator sub-group.

## 2. INTRODUCTION

To motivate the next considerations, we make reference to algebras of pseudo-differential operators on smooth manifolds, having in mind the index formula. The index formula is a global statement whose ingredients may be computed by local data. Our leading idea is to localise  $K$ -theory and periodic cyclic homology along the lines of the Alexander-Spanier co-homology in such a way that the new tools operate naturally with the Alexander-Spanier co-homology, see [25].

This article deals with the first part of our programme. In this article we define localised algebras  $A$  and we define and study their local  $K$ -theory,  $K_i^{loc}(A)$ ,  $i = 0, 1$ . The main result of this article is the (open) six terms exact sequence Theorem 51 associated with short exact sequences of local algebras.

To facilitate the understanding of this paper, we say from the very beginning that we intend to formalise the algebraic phenomena behind the following short exact sequence of operator algebras

$$(1) \quad 0 \longrightarrow \Psi_{-1} \xrightarrow{\iota} \Psi_0 \xrightarrow{\pi} \Psi_0/\Psi_{-1} \longrightarrow 0,$$

where  $\Psi_0$  is the algebra of pseudo-differential operators whose pseudo-differential symbol is homogeneous of order  $\leq 0$  and  $\Psi_{-1}$  is the bi-lateral ideal of pseudo-differential operators whose symbol is homogeneous order  $\leq -1$ , on the smooth manifold  $M$ , see [1], §5.

Any such operator  $A$  has a Schwartz distributional kernel on  $M \times M$ . This is a singular integral operator. Cutting its distributional kernel by a smooth function  $\lambda$ , which has small support about the diagonal in  $M \times M$  and is identically 1 on a small neighbourhood of the diagonal, does not modify its symbol  $\sigma(A)$  and modifies the original operator by a smoothing operator, i.e. by an pseudo-differential operator of order  $-\infty$ ; smoothing operators are contained in the ideal  $\Psi_{-1}$ . In other words, any pseudo-differential operator of order zero has, modulo operators of order  $-\infty$ , a representative with the same symbol and whose support is arbitrarily small. The image through  $\pi$  of a pseudo-differential operators  $A$  into  $\Psi_0/\Psi_{-1}$  is called the abstract symbol of the operator  $A$  and denoted by  $\sigma(A)$ ; the passage from  $A$  to its abstract symbol  $\sigma(A)$  is a ring homomorphism

$$(2) \quad \pi : \Psi_0 \xrightarrow{\pi} \Psi_0/\Psi_{-1}.$$

We may introduce a Riemannian metric on  $M$  and measure the size of the supports of pseudo-differential (integral) operators on  $M$ . They satisfy the property

$$(3) \quad \text{Int}(r) \circ \text{Int}(s) \subset \text{Int}(r + s),$$

where  $\mathbb{I}nt(r)$  denotes integral operators with support in an  $r$ -neighbourhood of the diagonal in  $M \times M$ .

These entitle us to speak about the support of pseudo-differential operators. The support consideration leads to the possibility to introduce a filtration on the algebra of pseudo-differential operators. The property (3) justifies Definition 3 of localised algebras.

As said before, in this paper we define local algebraic  $K$ -theory. Here, for the index formula sake, we resume ourselves to define  $K_i^{loc}$ ,  $i = 0, 1$ , although, we do not exclude the extension of the constructions beyond these degrees.

Next, we discuss our construction of local  $K$  groups,  $K_i^{loc}(A)$ ,  $i = 0, 1$ , for localised algebras  $A$ .

With regard to the construction of  $K_0$ -theory groups, we mention that one or more of the following structures are used in the literature (in the pure algebraic context, Banach algebras or  $C^*$ -algebras), see [15], [12], [18] [17], [4], [26]: finite projective modules, idempotents and various equivalence relations: Murray-von Neumann equivalence, unitary equivalence, continuous homotopy equivalence, elementary matrix operations.

In the literature  $K_0(A)$  is built on finite projective modules  $P$  over the algebra  $A$  or idempotents  $p$  belonging to the matrix algebra  $\mathbb{M}_n(A)$ . The two descriptions are equivalent, but it is important to acknowledge the difference between them.

Our construction of  $K_0^{loc}$  uses exclusively idempotent matrices. There are a few reasons why we chose to avoid projective modules: -i) projective modules, in comparison with idempotent matrices, contain more arbitrariness in describing  $K_0$  classes and -ii) matrices are more suitable for controlling the algebra filtration data. Matrices are more prone to make calculations.

No reference to projective modules is used in our constructions.

Regarding our construction of  $K_1^{loc}$  we recall that the classical algebraic  $K$ -theory group  $K_1(A)$  of the unitary algebra  $A$ , see [27], [3], [15], [12], is by definition the Whitehead group

$$K_1(A) := \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)],$$

where  $[\mathrm{GL}(A), \mathrm{GL}(A)]$  is the commutator normal sub-group of the group of invertible matrices  $\mathrm{GL}(A)$ . Our definition of local  $K$ -theory groups needs to keep track of the number of multiplications performed inside the algebra  $A$ . In order for our constructions to hold it is necessary to use constructions which use a bounded number of multiplications. Unfortunately, in general, the number of multiplications needed to generate the whole commutator sub-group is not bounded. It is known that the commutator sub-group is also generated by the elementary matrices. This is the reason why our definition of  $K_1^{loc}(A)$  avoids entirely factorising  $\mathrm{GL}(A)$  through the commutator sub-group or the sub-group generated by elementary matrices.

$K_0^{loc}(A)$ , **resp.**  $K_1^{loc}(A)$ , is by definition the Grothendieck completion of the semi-group of idempotent matrices in  $\mathbb{M}_n(A_\mu)$  modulo three equivalence relations: -i) stabilisation, -ii) local conjugation  $\sim_l$  by invertible elements  $u \in \mathrm{GL}_n(A_\mu)$  and -iii) projective limits with respect to  $\mu \in \mathbb{N}$  (Alexander-Spanier type limit).

To understand the relation between our definition of the group  $K_1^{loc}$  and  $K_1$ , recall first the definition of the commutator sub-group  $[\mathbb{G}\mathbb{L}(A), \mathbb{G}\mathbb{L}(A)]$ ; it is generated by all multiplicative commutators

$$[A, B] := ABA^{-1}B^{-1}, \quad \text{for any } A, B \in \mathbb{G}\mathbb{L}_n(A).$$

Supposing that  $A$  and  $B$  are conjugated, i.e.  $A = UBU^{-1}$ , and we write  $A \sim B$ , we have

$$A = UBU^{-1} = UBU^{-1}B^{-1}B = [U, B]B.$$

This shows that if  $A$  and  $B$  are conjugated, they differ, multiplicatively, by a commutator. To complete this remark, we say that  $A$  and  $B$  are locally conjugated and write  $A \sim_l B$  provided  $A, B$  and  $U$  belong to some  $\mathbb{G}\mathbb{L}_n(A_\mu)$ ; here,  $A_\mu$  denote the terms of the filtration of  $A$ .

It is important to note that in the particular case  $A = \mathbb{C}$  the quotient of  $\mathbb{G}\mathbb{L}(\mathbb{M}(\mathbb{C}))$  through the commutator sub-group gives much less information than  $K_1^{loc}(\mathbb{M}(\mathbb{C}))$ , which is obtained as the quotient of the same space through the action of the inner auto-morphism group; in this last case one obtains a quotient of the set of conjugacy classes, i.e. the space of Jordan canonical forms of invertible matrices, modulo permutations of Jordan cells.

The main result of this article is the open the six terms exact sequence associated to short exact sequence, Theorem 51. We note, however, that given the higher amount of information kept by the local algebraic  $K_*$  groups we are defining here, exactness of the six terms sequence requires tensorising the expected six terms sequence by  $\mathbb{Z}[\frac{1}{2}]$ . This phenomenon is related to the difficulty of lifting elements of the form  $u \oplus u^{-1}$  over the quotient algebra  $A/A'$  to elements of the same form over the algebra  $A$ . The presence of the factor  $\mathbb{Z}[\frac{1}{2}]$  should have important consequences.

### 3. GENERALITIES AND NOTATION.

Let  $A$  be a complex algebra, with or without unit. If the unit will be needed, the unit will be adjoined.

**Definition 1.** Given the algebra  $A$  we denote by  $\mathbb{M}_n(A)$  the space of  $n \times n$  matrices with entries in  $A$ .  $\mathbb{M}_n(A)$  is a bi-lateral  $A$ -module.

Let  $\text{Idemp}_n \subset \mathbb{M}_n(A)$  be the subset of idempotents  $p$  ( $p^2 = p$ ) of size  $n$  with entries in  $A$ .

Suppose  $A$  has a unit. We denote by  $\mathbb{G}\mathbb{L}_n(A)$  the sub-space of matrices  $M$  of size  $n$ , with entries in  $A$  which are invertible, i.e. there exists the matrix  $M^{-1} \in \mathbb{M}_n(A)$  such that  $MM^{-1} = M^{-1}M = 1$ .  $\mathbb{G}\mathbb{L}_n(A)$  is a non-commutative group under multiplication.

**Definition 2.** The inclusions

$$-i) \text{Idemp}_n(A) \longrightarrow \text{Idemp}_{n+1}(A)$$

$$(4) \quad p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

-ii)  $\mathrm{GL}_n(A) \longrightarrow \mathrm{GL}_{n+1}(A)$

$$(5) \quad M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

are called stabilisations.

Stabilisations define two direct systems with respect to  $n \in \mathbb{N}$ .

#### 4. Localised ALGEBRAS

**Definition 3.** *Localised Algebras.*

Let  $A$  be an unital complex associative algebra with unit 1. The algebra  $A$  is called localised algebra provided it is endowed with an additional structure satisfying the axioms (1) - (4) below.

*Axiom 1.* The underlying vector space  $A$  has a decreasing filtration by vector sub-spaces  $\{A_\mu\}_{\mu \in \mathbb{N}} \subset A$ .

*Axiom 2.*  $\cup_{\mu \in \mathbb{N}} A_\mu = A$

*Axiom 3.*  $\mathbb{C} \cdot 1 \subset A_\mu$ , for any  $\mu \in \mathbb{N}$

*Axiom 4.* For any  $\mu, \mu' \in \mathbb{N}^+$ ,  $A_\mu \cdot A_{\mu'} \subset A_{\min(\mu, \mu')-1}$  ( $A_0 \cdot A_0 \subset A_0$ ).

**Remark 4.** An algebra  $A$  could have different localisations.

**Definition 5.** *Homomorphisms of localised algebras. Induced homomorphism.* -i) A homomorphism from the localised algebra  $A = \{A_n\}_{n \in \mathbb{N}}$  to the localised algebra  $B = \{B_\mu\}_{\mu \in \mathbb{N}}$  is an algebra homomorphism  $\phi : A \longrightarrow B$  such that  $\phi : A_n \longrightarrow B_\mu$ , for any  $\mu \in \mathbb{N}$ .

-ii) Let  $f : A \longrightarrow B$  be a localised algebra homomorphism.

Let  $f_* : \mathbb{M}_n(A_\mu) \longrightarrow \mathbb{M}_n(B_\mu)$  be the induced homomorphism which replaces any component  $a_{ij}$  of the matrix  $M$  with the component  $f(a_{ij})$  of the matrix  $f_*(M)$ .

**Remark 6.** The notion of localised algebra differs from the notion of  $m$ -algebras defined by Cuntz [8], in many respects. The sub-spaces  $A_n$  are not required to be algebras, or even more, topological algebras. In the Cuntz' definition of localised Banach algebras, the projective limit of these sub-algebras might be the zero Banach algebra. However, even in these cases, the corresponding local  $K$ -theory could not be trivial.

**Remark 7.** The immediate application of this theory regards pseudo-differential operators. The pseudo-differential operators of non-positive order on a compact smooth manifold form a localised (Banach) algebra. The filtration is defined in terms of the support of the operators; the bigger the filtration order is, the smaller the supports of the operators are towards the diagonal.

#### 5. MAYER-VIETORIS DIAGRAMS.

In this section we adapt Milnor's [15] construction of the first two algebraic  $K$ -theory groups to the case of localised rings.

Let  $\Lambda, \Lambda_1, \Lambda_2$  and  $\Lambda'$  be rings with unit 1 and let

$$(6) \quad \begin{array}{ccc} \Lambda & \xrightarrow{i_1} & \Lambda_1 \\ \downarrow i_2 & & \downarrow j_1 \\ \Lambda_2 & \xrightarrow{j_2} & \Lambda' \end{array}$$

be a commutative diagram of ring homomorphisms. All ring homomorphisms  $f$  are assumed to satisfy  $f(1) = 1$ . Any module in this paper is a left module.

We assume the diagram satisfies the three conditions below.

*Hypothesis 1.* All rings and ring homomorphisms are localised, see §3.

*Hypothesis 2.*  $\Lambda$  is a localised product of  $\Lambda_1$  and  $\Lambda_2$ , i.e. for any pair of elements  $\lambda_1 \in \Lambda_{1,\mu}$  and  $\lambda_2 \in \Lambda_{2,\mu}$  such that  $j_1(\lambda_{1,\mu}) = j_2(\lambda_{2,\mu}) = \lambda' \in \Lambda'_\mu$ , there exists only one element  $\lambda_\mu \in \Lambda_\mu$  such that  $i_1(\lambda_\mu) = \lambda_{1,\mu}$  and  $i_2(\lambda_\mu) = \lambda_{2,\mu}$ .

The ring structure in  $\Lambda$  is defined by

$$(7) \quad (\lambda_1, \lambda_2) + (\lambda'_1, \lambda'_2) := (\lambda_1 + \lambda'_1, \lambda_2 + \lambda'_2), \quad (\lambda_1, \lambda_2) \cdot (\lambda'_1, \lambda'_2) := (\lambda_1 \cdot \lambda'_1, \lambda_2 \cdot \lambda'_2),$$

i.e. the ring operations in  $\Lambda$  are performed component-wise. In § and § we will need to operate with matrices with entries in  $\Lambda$ .

*Hypothesis 3.* At least one of the homomorphisms  $j_1$  and  $j_2$  is surjective.

**Remark 8.** -i) Any matrix  $M \in \mathbb{M}_n(\Lambda)$  consists of a pair of matrices  $(M_1, M_2) \in \mathbb{M}_n(\Lambda_1) \times \mathbb{M}_n(\Lambda_2)$  subject to the condition  $j_{1,*}M_1 = j_{2,*}M_2$ . Any matrix  $M \in \mathbb{M}_n(\Lambda)$  is called double matrix.

-ii) if  $(M_1, M_2), (N_1, N_2)$  are double matrices, (**resp.** belong to  $\mathbb{M}(\Lambda_1) \times \mathbb{M}(\Lambda_2)$ ) and  $(\lambda_1, \lambda_2) \in \Lambda$ , (**resp.**  $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$ ) then relations (7) induce onto the space of double matrices, (**resp.** the space  $\mathbb{M}(\Lambda_1) \times \mathbb{M}(\Lambda_2)$ ) the following relations

$$\begin{aligned} (\lambda_1, \lambda_2) (M_1, M_2) &= (\lambda_1 M_1, \lambda_2 M_2) \\ (M_1, M_2) + (N_1, N_2) &= (M_1 + N_1, M_2 + N_2) \\ (M_1, M_2) \cdot (N_1, N_2) &= (M_1 \cdot N_1, M_2 \cdot N_2) \end{aligned}$$

**Definition 9.** A commutative diagram satisfying Hypotheses 1. 2. 3. will be called localised Mayer-Vietoris diagram.

## 6. $K_0^{loc}(A_\mu)$ AND $K_0^{loc}(A)$ .

**Definition 10.** We assume the algebra  $A$  is localised, see §4.

We consider the space of matrices with entries in  $A_\mu$  and we denote it by  $\mathbb{M}_n(A_\mu)$ .

Let  $\text{Idemp}_n(A_\mu)$  denote the space of idempotent matrices of size  $n$  with entries in  $A_\mu$ .

Let  $\text{GL}_n(A_\mu)$  denote the space of invertible matrices  $M$  of size  $n$  with the property that the entries of both  $M$  and  $M^{-1}$  belong to  $A_\mu$ .

Let  $\text{Idemp}(A_\mu) := \text{inj lim}_n \text{Idemp}_n(A_\mu)$ .

Let  $\text{GL}(A_\mu) := \text{inj lim}_n \text{GL}_n(A_\mu)$ .

**Definition 11.** -i) Two matrices  $s, t \in \mathbb{M}_n(A)$  will be called conjugated and we write  $s \sim t$  provided there exists  $u \in \mathbb{GL}_n(A)$  such that  $s = utu^{-1}$ .

-ii) Two matrices  $s, t \in \mathbb{M}_n(A_\mu)$  will be called locally conjugated and we write  $s \sim_l t$  provided there exists  $u \in \mathbb{GL}_n(A_\mu)$  such that  $s = utu^{-1}$ .

In particular,

-ii.1) two idempotents  $p, q \in \text{Idemp}_n(A_\mu)$  are locally isomorphic and we write  $p \sim_l q$  provided there exists  $u \in \mathbb{GL}_n(A_\mu)$  such that  $q = upu^{-1}$  and

-ii.2) two invertible matrices  $s, t \in \mathbb{GL}_n(A_\mu)$  are locally conjugated and we write  $s \sim_l t$  provided there exists  $u \in \mathbb{GL}_n(A_\mu)$  such that  $s = utu^{-1}$ .

**Proposition 12.**  $\text{Idemp}_n(A_\mu)$  and  $\mathbb{GL}_n(A_\mu)$  are semigroups with respect to the direct sum

$$(8) \quad A + B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

The spaces  $\text{Idemp}_n(A_\mu), \mathbb{GL}_n(A_\mu)$  are compatible with stabilisations.

The direct sum addition of idempotents, resp. invertibles, is compatible with the local conjugation equivalence relation. Indeed, if  $s_1, s_2$  are conjugated through an inner automorphism defined by the element  $u_1$  ( $s_1 \sim_l s_2$ ) and  $t_1, t_2$  are conjugated through an inner automorphism defined by the element  $u_2$  ( $t_1 \sim_l t_2$ ), then  $(s_1 + t_1) \sim_l (s_2 + t_2)$  are conjugated through the inner automorphism  $u_1 \oplus u_2$ .

With this observation, the associativity of the addition is now immediate.

These show that  $\text{Idemp}(A_\mu) / \sim_l$ , resp.  $\mathbb{GL}(A_\mu) / \sim_l$ , is an associative semi-group.

**Proposition 13.** The semi-groups  $\text{Idemp}(A_\mu) / \sim_l, \mathbb{GL}(A_\mu) / \sim_l$  are commutative.

*Proof.* The result follows from the following identity valid for any two matrices  $A, B \in \mathbb{M}_n(A_\mu)$

$$(9) \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1},$$

which tells that  $(A + B) \sim_l (B + A)$ . □

For more information about the relationship between the classical algebraic K-theory and the local K-theory, please see §9.

### 6.1. $K_0(A_\mu)$ and $K_0^{loc}(A)$ .

**Definition 14.** Suppose  $A$  is a localised unital associative algebra. We define

$$(10) \quad K_0(A_\mu) = \text{injlim}_{n \in \mathbb{N}} \mathbb{G}(\text{Idemp}_n(A_\mu) / \sim_l),$$

where  $\mathbb{G}$  means Grothendieck completion, and

$$(11) \quad K_0^{loc}(A) = \text{projlim}_{\mu \in \mathbb{N}} K_0(A_\mu).$$

Any local inner automorphism induces the identity on  $K_0^{loc}(A)$ .



6.2.  $K_1(A_\mu)$  and  $K_1^{loc}(A)$ .

In this sub-section  $[u]_{\sim_{sl}}$  denotes the class of the element  $u \in \mathbb{GL}(A_\mu)$  with respect to the equivalence relation generated by stabilisation and local conjugation  $\sim_l$ .

We call  $[u]_{\sim_{sl}}$  the abstract Jordan canonical form of the invertible element  $u$ . The  $K_1(A_\mu)$  we are going to define preserves the information provided by the abstract Jordan form. The classical definition of  $K_1$  extracts a minimal part of the abstract Jordan form. As the addition in the semi-group  $\mathbb{GL}(A)$  is given by direct sum and the Jordan canonical form  $J(u)$  (at least in the classical case of the algebra  $\mathbb{GL}_n(\mathbb{R})$ ) behaves additively ( $J(u \oplus v) = J(u) \oplus J(v)$  modulo permutations of the Jordan blocks), given an arbitrary element  $u \in \mathbb{GL}(A)$ , it is not reasonable to expect existence of an element  $\tilde{u}$  such that  $[u + \tilde{u}]_{\sim_{sl}} = [1_{2n}]_{\sim_{sl}}$ . Given that we want to define  $K_1(A_\mu)$  as a group, we introduce the group structure (opposite elements) forcibly. In the case of the classical  $K_1$ , the class of the element  $u^{-1}$  represents the opposite class,  $-[u] \in K_1(A)$ . We intend produce a theory in which this relation is preserved.

Note that if  $u_1 \sim_{sl} u_2$ , then  $u_1^{-1} \sim_{sl} u_2^{-1}$ . As said above, we would like the class of the element  $u^{-1}$  to represent the opposite class, i.e. to think of  $u + u^{-1} \sim_{sl} I_{2n}$  as representing the zero class. This need, along with other related facts, see Theorem 40.- i), proof of Theorem 27 and Theorem 35, lead us to concentrate our attention onto the space of matrices of the form  $u + u^{-1}$  and based on it to introduce a new equivalence relation.

**Definition 15.** We define, for any  $u \in \mathbb{GL}_n(A_\mu)$

$$(12) \quad \mathcal{O}(u) := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \mathbb{GL}_{2n}(A_\mu).$$

**Definition 16.** Let

$$(13) \quad \mathbb{O}_{2n}(A_\mu) := \{[\mathcal{O}(u)] \mid u \in \mathbb{GL}_n(A_\mu)\}$$

and

$$(14) \quad \mathbb{O}(A_\mu) := \text{inj lim}_{\mu \in \mathbb{N}} \mathbb{O}_{2n}(A_\mu) \{[\mathcal{O}(u)] \mid u \in \mathbb{GL}_n(A_\mu)\}$$

Next we analyse the properties of this space.

**Proposition 17.** -i) The space  $\mathbb{O}(A_\mu)$  is a commutative semi-group with zero element, given by the identity

-ii) the mapping

$$(15) \quad \mathcal{O} : \mathbb{GL}(A_\mu) \longrightarrow \mathbb{O}(A_\mu)$$

is additive and commutes with local conjugation

$$(16) \quad \mathcal{O}(u_1 + u_2) = \mathcal{O}(u_1) + \mathcal{O}(u_2)$$

$$(17) \quad \mathcal{O}(\lambda u \lambda^{-1}) = \lambda \mathcal{O}(u) \lambda^{-1}, \quad \lambda \in \mathbb{GL}(A_\mu),$$

i.e. if  $u_1 \sim_{sl} u_2$  then  $\mathcal{O}(u_1) \sim_{sl} \mathcal{O}(u_2)$

-iii)

$$(18) \quad \mathcal{O}(u^{-1}) \sim_{sl} \mathcal{O}(u).$$

-iv)

$$(19) \quad \mathcal{O}(u_1 u_2) \neq \mathcal{O}(u_1) \mathcal{O}(u_2),$$

**Proposition 18.**  $\mathbb{O}(A_\mu)/\sim_{sl}$  is a sub semi-group with zero element in the commutative semi-group  $\mathbb{GL}(A_\mu)/\sim_{sl}$ .

*Proof.* According to Proposition 13,  $\mathbb{GL}(A_\mu)/\sim_l$  is a commutative semi-group; the addition in this semi-group is given by the direct sum.

Proposition 18 shows that  $\mathbb{O}(A_\mu)$  is closed under addition and that it has zero element.  $\square$

**Definition 19.** Define

$$(20) \quad K_1(A_\mu) := (\mathbb{GL}(A_\mu)/\sim_{sl}) / (\mathbb{O}(A_\mu)/\sim_{sl}).$$

The next definition explains the meaning of the quotient in the formula (20) by introducing a new equivalence relation  $\sim_{\mathbb{O}}$ .

**Definition 20.** We say that two elements  $u_1, u_2 \in \mathbb{GL}(A_\mu)$  are equivalent modulo the sub semi-group  $\mathbb{O}(A_\mu)/\sim_{sl}$  (and write  $u_1 \sim_{\mathbb{O}_\mu} u_2$ ) provided there exist two elements  $\xi_1, \xi_2 \in \mathbb{O}(A_\mu)$  such that  $u_1 + \xi_1 = u_2 + \xi_2$ .

**Remark 21.** Let us denote by  $J(u)$  the equivalent class of the invertible matrix  $u \in \mathbb{GL}_n(A_\mu)$  with respect to the local conjugation equivalence relation  $\sim_l$ . The relation  $\sim_{\mathbb{O}_\mu}$  kills all pairs of abstract Jordan canonical forms  $(J(u), J(u^{-1}))$ , for any  $u \in \mathbb{GL}(A_\mu)$ .

Note that the two components of the pair  $(u, u^{-1})$  determine one each other.

**Proposition 22.** -i)  $\sim_{\mathbb{O}_\mu}$  is an equivalence relation.

Any local inner auto-morphism induces the identity on  $K_1^{loc}(A)$ .

-ii) If  $\xi \in \mathbb{O}_{2n}(A_\mu)$  then  $\xi \sim_{\mathbb{O}} 1_{2n}$ , i.e.  $\xi$  represents the zero element in  $K_1(A_\mu)$ .

*Proof.* -i)  $\sim_{\mathbb{O}_\mu}$  is reflexive. Indeed, if  $u \in \mathbb{GL}_n(A_\mu)$  and because  $1_n \in \mathbb{O}_n(A_\mu)$ , one has  $u + 1_n = u + 1_n \in \mathbb{GL}_{2n}(A_\mu)$ .

$\sim_{\mathbb{O}_\mu}$  is symmetric: if  $u_1 + \xi_1 = u_2 + \xi_2$ , then  $u_2 + \xi_2 = u_1 + \xi_1$ .

The transitivity is clear. If

$$u_1 + \xi_1 = u_2 + \xi_2 \quad \text{and} \quad u_2 + \tilde{\xi}_2 = u_3 + \xi_3$$

then

$$u_1 + (\xi_1 + \tilde{\xi}_2) = u_2 + \xi_2 + \tilde{\xi}_2 = u_3 + (\xi_2 + \xi_3).$$

-ii) For any element  $\xi \in \mathbb{O}(A_\mu)$ , the element  $1_n$  belongs already to  $\mathbb{O}(A_\mu)$  and then one gets

$$(21) \quad \xi + 1_n = 1_n + \xi,$$

which shows that  $\xi \sim_{\mathbb{O}_\mu} 1_n$ , i.e.  $\xi$  represents the zero element in  $K_1(A_\mu)$ .  $\square$

**Proposition 23.** -i) The equivalence relation  $\sim_{\mathbb{O}_\mu}$  is compatible with the semi-group structure: for any  $u_1, u_2 \in \mathbb{GL}(A)$

$$(22) \quad [u_1 + u_2] = [u_1] + [u_2] \in K_1(A_\mu)$$

-ii)  $K_1(A_\mu)$  is a (commutative) group; for any  $u \in \mathbb{GL}(A_\mu)$

$$(23) \quad -[u] = [u^{-1}] \in K_1(A_\mu).$$

-iii)  $K_1(A_\mu)$  and  $K_1^{loc}(A)$  consist of equivalence classes of invertible elements; the elements of this group are not virtual elements (as in the Grothendieck completion case).

-iv) Let  $u_1, u_2 \in \mathbb{GL}(A_\mu)$  such that  $[u_1] = [u_2] \in K_1(A_\mu)$ . Then there exist two elements  $\xi_1, \xi_2 \in \mathbb{O}(A_\mu)$  and  $u \in \mathbb{GL}(A_\mu)$  such that

$$(24) \quad u_1 + \xi_1 = u (u_2 + \xi_2) u^{-1} \in \mathbb{GL}(A_\mu)$$

*Proof.* -i) Suppose  $u_1 \sim_{\mathbb{O}_\mu} u'_1$  and  $u_2 \sim_{\mathbb{O}_\mu} u'_2$ . This means there exist four elements  $\xi_1, \xi'_1, \xi_2, \xi'_2 \in \mathbb{O}_\mu$  such that

$$(25) \quad u_1 + \xi_1 = u'_1 + \xi'_1, \quad u_2 + \xi_2 = u'_2 + \xi'_2,$$

which give by addition

$$(26) \quad (u_1 + u_2) + (\xi_1 + \xi_2) = (u'_1 + u'_2) + (\xi'_1 + \xi'_2),$$

i.e.

$$(27) \quad (u_1 + u_2) \sim_{\mathbb{O}_\mu} (u'_1 + u'_2)$$

-ii) Let  $u \in \mathbb{GL}(A_\mu)$ . Then  $(u + u^{-1}) + 1_{2n} = 1_{2n} + (u \oplus u^{-1})$ . In other words,

$$(28) \quad [u + u^{-1}] = [u] + [u^{-1}] \sim_{\mathbb{O}_\mu} 1_{2n} \sim_{\mathbb{O}_\mu} 0 \in K_1(A_\mu).$$

-iii) It is clear.

-iv) Once the factorisation through the semi-group  $\mathbb{O}(A_\mu)$  is used to define  $K_1(A_\mu)$ , it remains to involve the equivalence relation  $\sim_l$  (note  $\sim_l$  and  $\sim_s$  commute).  $\square$

**Remark 24.** The factorisation used in the formula (20) for the definition of  $K_1$  is weaker than the factorisation used in the classical definition of the  $K_1$ -theory group. In fact, the identity (23) is equivalent to the factorisation (20); on the other side, formula (23) holds in the classical  $K$ -theory, see e.g. [15].

We may be more explicit on this point. In the classical definition

$$(29) \quad K_1(A) := \mathbb{GL}(A)/[\mathbb{GL}(A), \mathbb{GL}(A)].$$

On the other side, the following identity holds

$$(30) \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix},$$

which shows that multiplicatively, modulo elements of the form  $\mathcal{O}(B)$  (which belong to the commutator sub-group  $\mathbb{O}(A)$ ), one has

$$(31) \quad [A + B] = [AB] \in K_1(A).$$

**Proposition 25.** -i) The relation (23) holds in  $K_1(A_\mu)$  and  $K_1^{loc}(A)$ .

-ii) The local algebraic  $K_i(A_\mu)$  and  $K_i(A^{loc})$  groups are Morita invariant.

*Proof.* The projective limit with respect to  $\mu$  assures the formula (23) remains valid in  $K_1^{loc}(A)$ .  $\square$

## 7. INDUCED HOMOMORPHISMS.

**Definition 26.** Let  $f : A \rightarrow B$  be a localised algebra homomorphism.

Then the homomorphism  $f_*$  (see Definition 5) induces homomorphisms

$$(32) \quad f_* : K_0(A_\mu) \rightarrow K_0(B_\mu), \quad f_* : K_0^{loc}(A) \rightarrow K_0^{loc}(B)$$

and

$$(33) \quad f_* : K_0(A_\mu) \rightarrow K_0(B_\mu), \quad f_* : K_1^{loc}(A) \rightarrow K_1^{loc}(B)$$

## 8. CONSTRUCTING IDEMPOTENTS AND INVERTIBLE MATRICES OVER $\Lambda_\mu$ .

We come back to the situation presented in §5 - Mayer-Vietoris diagrams.

### 8.1. Constructing idempotents over $\Lambda_\mu$ .

**Theorem 27. Definition.** Suppose  $p_1 \in \text{Idemp}_n(\Lambda_{1,\mu})$ , resp.  $p_2 \in \text{Idemp}_n(\Lambda_{2,\mu})$ , is an idempotent matrix with entries in  $\Lambda_{1,\mu}$ , resp.  $\Lambda_{2,\mu}$ , such that

$$(34) \quad j_{1*}(p_1) = u j_{2*}(p_2) u^{-1},$$

where  $u \in \text{GL}_n(\Lambda'_\mu)$  is an invertible matrix.

-i) Then there exists an idempotent double matrix  $p \in \text{Idemp}_{2n}(\Lambda_\mu)$  such that

$$(35) \quad i_{1*}(p) = p_1 \oplus 0_n \quad \text{and} \quad i_{2*}(p) = \tilde{p}_2$$

where the idempotent  $\tilde{p}_2 \in M_{2n}(\Lambda_{2,\mu})$  is conjugated to  $p_2 \oplus 0_n$  through an invertible matrix  $\tilde{U} \in \text{GL}_{2n}(\Lambda_{2,\mu})$ , that is  $\tilde{p}_2 = \tilde{U} p_2 \tilde{U}^{-1}$ .

The corresponding double matrix idempotent  $p$  is denoted  $p = (p_1, p_2, u)$ .

-ii)

$$(36) \quad j_{2,*}\tilde{p}_2 = (u j_{2,*}(p_2) u^{-1}) \oplus 0_n = j_{1,*}(p_1 \oplus 0_n)$$

-iii) Any idempotent matrix  $p \in \text{M}_{2n}(\Lambda_\mu)$  is conjugated through a localised inner automorphism defined by a matrix  $U \in \text{GL}_{2n}(\Lambda_\mu)$  to an idempotent matrix of the form  $p = (p_1, p_2, u)$  defined by -i).

Condition (34) says that  $[j_{1*}p_1] = [j_{2*}p_2] \in K_0(\Lambda'_\mu)$ . Part -i) says that the pair  $([p_1], [\tilde{p}_2]) \in K_0(\Lambda_1) \oplus K_0(\Lambda_2)$  belongs to the image of  $(i_{1*}, i_{2*})$ . This property is part of the proof of Theorem 51.-i) on exactness.

*Proof.* All considerations made to prove this theorem will be performed onto objects related to  $\Lambda_{2,\mu}$ .

**Lemma 28.** Let  $p_1 = (a_{ij}) \in \mathbb{I}demp_n(\Lambda_{1,\mu})$  and  $p_2 = (b_{ij}) \in \mathbb{I}demp_n(\Lambda_{2,\mu})$  be idempotents.

Suppose the idempotents  $j_{1*}(p_1)$ ,  $j_{2*}(p_2)$  are conjugate through an inner automorphism defined by  $u \in \mathbb{G}L_n(\Lambda'_\mu)$ , i.e.

$$(37) \quad j_{1*}(p_1) = u j_{2*}(p_2) u^{-1}.$$

Assume, additionally, that the invertible element  $u$  lifts to an invertible element  $\tilde{u} \in \mathbb{G}L_n(\Lambda_{2,\mu})$  (i.e.  $j_{2*}\tilde{u} = u$ ).

Then  $p = (p_1, p_2, u) \in \mathbb{I}demp_n(\Lambda_\mu)$  is an idempotent given by the double matrix

$$(38) \quad p = ((a_{ij}, c_{ij})),$$

where

$$(39) \quad (a_{ij}) = p_1 \in \mathbb{I}demp_n(\Lambda_{1,\mu}) \text{ and } p_2' = (c_{ij}) := \tilde{u} p_2 \tilde{u}^{-1} \in \mathbb{I}demp_n(\Lambda_{2,\mu}).$$

Remark that in this lemma the size of the double matrix  $p$  does not change.

*Proof of Lemma 28.* We use Remark 8.-ii). It is clear that the matrix  $p$  given by (38), (39) is an idempotent. In fact, to evaluate  $p^2$  amounts to compute separately the square of the first and second component matrices of the matrix  $p$ , i.e. the squares of  $(a_{ij})$  and  $(c_{ij})$ . These are

$$(40) \quad (a_{ij})^2 = (a_{ij}) \text{ and } (c_{ij})^2 = (\tilde{u} p_2 \tilde{u}^{-1})^2 = \tilde{u} p_2^2 \tilde{u}^{-1} = \tilde{u} p_2 \tilde{u}^{-1} = (c_{ij}).$$

It remains to verify that  $p \in \mathbb{M}_n(\Lambda_\mu)$ , i.e.  $j_{1*}(a_{ij}) = j_{2*}(c_{ij})$ . This follows from (40) combined with (37)

$$(41) \quad j_{1*}(p_1) = u j_{2*}(p_2) u^{-1} = j_{2*}(\tilde{u} p_2 \tilde{u}^{-1}) = j_{2*}(p_2').$$

This ends the proof of Lemma 28.

**Lemma 29.** Let  $p_1 = (a_{ij}) \in \mathbb{I}demp_n(\Lambda_{1,\mu})$  and  $p_2 = (b_{ij}) \in \mathbb{I}demp_n(\Lambda_{2,\mu})$  be idempotents.

Suppose the idempotents  $j_{1*}(p_1)$ ,  $j_{2*}(p_2)$  are conjugate through an inner automorphism defined by  $u \in \mathbb{G}L_n(\Lambda'_\mu)$ , i.e.

$$(42) \quad j_{1*}(p_1) = u j_{2*}(p_2) u^{-1}.$$

Then

-i)  $j_{1*}(p_1 \oplus 0_n)$  and  $j_{2*}(p_2 \oplus 0_n)$  are conjugated by  $U := u \oplus u^{-1} \in \mathbb{G}L_{2n}(\Lambda'_\mu)$ , i.e.

$$(43) \quad j_{1*}(p_1) \oplus 0_n = j_{1*}(p_1 \oplus 0_n) = U j_{2*}(p_2 \oplus 0_n) U^{-1} = (u j_{2*}(p_2) u^{-1}) \oplus 0_n.$$

-ii) Supposing that  $j_2$  is surjective, the invertible matrix  $U$  lifts to an invertible matrix  $\tilde{U} \in \mathbb{M}_{2n,\mu}(\Lambda_2)$ . Let

$$(44) \quad \tilde{p}_2 := \tilde{U} p_2 \tilde{U}^{-1}.$$

Then

$$(45) \quad j_{1,*}(p_1 \oplus 0_n) = (u j_{2,*}(p_2) u^{-1}) \oplus 0_n = j_{2,*}(\tilde{p}_2);$$

i.e. the matrices  $p_1 \oplus 0_n$ ,  $\tilde{p}_2$  form a double matrix idempotent in  $\mathbb{I}demp_{2n}(\Lambda_\mu)$ , denoted  $p := (p_1, p_2, u) \in \mathbb{I}demp_{2n}(\Lambda_\mu)$  and

$$(46) \quad (i_{1*}, i_{2*})p = (p_1, \tilde{p}_2)$$

-iii) The pair of idempotents  $p_1 \in \mathbb{I}demp_n(\Lambda_{1,\mu})$ ,  $p_2 \in \mathbb{I}demp_n(\Lambda_{2,\mu})$  is stably equivalent to the pair of idempotents  $p_1 \oplus 0_n$ ,  $p_2 \oplus 0_n$  and  $p_2 \oplus 0_n \sim_l \tilde{p}_2$ . In other words

$$(47) \quad ([p_1], [p_2]) = ([p_1], [\tilde{p}_2]) \in K_0(\Lambda_{1,\mu}) \oplus K_0(\Lambda_{2,\mu}).$$

Note that in this lemma, in comparison with the preceding Lemma 28, the size of the desired idempotent doubles; otherwise, the important modifications still occur onto matrices associated with  $\Lambda_{2,\mu}$ .

*Proof of Lemma 29.* Part -i) is clear.

The proof of -ii) uses  $\mathcal{O}_{2n}(u)$ , where  $u \in \mathbb{G}L_n(A_\mu)$ , Definition 15. The next formula provides a decomposition of  $\mathcal{O}_{2n}(u)$  in a product of elementary matrices and a scalar matrix

$$(48) \quad U := \mathcal{O}_{2n}(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_{2n}(\Lambda_{2,\mu}).$$

To complete the proof we need to show that the invertible matrix  $U$  has an invertible lifting  $\tilde{U} \in \Lambda_{2,\mu}$ . This follows from the properties of elementary matrices, discussed next.

**Definition 30.** *Elementary Matrices.*

A matrix  $E_{ij}(a) \in GL_n(A_\mu)$  having all entries equal to zero, except for the diagonal entries equal to 1 and just one  $(i, j)$ -entry  $a \in A_\mu$ ,  $0 \leq i \neq j \leq n$  is called elementary matrix with entry in  $A_\mu$ .

The space of elementary matrices with entries in  $A_\mu$  is by definition

$$(49) \quad E_n(A_\mu) := \{E_{ij}(a) \mid 1 \leq i \neq j \leq n, a \in A_\mu\}.$$

Let  $\mathbb{E}_n(A)$  the sub-group generated by all elementary matrices and

$$(50) \quad \mathbb{E}(A) := \operatorname{inj} \lim_{n \in \mathbb{N}} \mathbb{E}_n(A).$$

**Lemma 31.** *The elementary matrices satisfy*

$$(51) \quad E_{i,j}(a).E_{i,j}(b) = E_{i,j}(a+b)$$

$$(52) \quad E_{i,j}(a)^{-1} = E_{i,j}(-a),$$

therefore  $E_{i,j}(a) \in GL(A_\mu)$ , Any elementary matrix is a commutator

$$(53) \quad E_{ij}(A) = [E_{ik}(A), E_{kj}(1)], \text{ for any } i, j, k \text{ distinct indices.}$$

We come back to the proof of Lemma 29. -ii). As  $j_2$  is surjective, each of the entries of factors of the RHS of (48) has a lifting in  $\mathbb{M}_{2n}(\Lambda_{2,\mu})$  so as each of the elementary matrix factor lifts as an invertible elementary matrix. The last factor lifts as it is. Therefore,  $U$  has an invertible lifting  $\tilde{U} \in \mathbb{G}L_{2n}(\Lambda_{2,\mu})$ . Lemma 28 completes the proof of Lemma 29. -ii).

Lemma 29. -iii) follows from the definition of  $K_0^{loc}(\Lambda)$ . This completes the proof of Lemma 29.

Theorem 27 follows from Lemma 28 combined with Lemma 29.  $\square$

Theorem 27 refers to the construction and description of idempotents over  $\Lambda_\mu$ . We need to extend Theorem 27 to elements of  $K_0^{loc}(A)$ , i.e. to formal differences of local idempotents. A new difficulty occurs. This has to do with the ambiguity involved in the Grothendieck construction; without special precautions, we would have to lift idempotents over  $\Lambda'_\mu$  to idempotents over  $\Lambda_{2,\mu}$  which, in general, could not be done.

Now we proceed.

**Lemma 32.** (compare [15] Lemma 1.1)

Let  $p_1, p_2, q_1, q_2 \in \mathbb{I}demp_n(A_\mu)$  be idempotents and let  $[ \ ]$  denote their  $K_0(A_\mu)$  class. Suppose

$$(54) \quad [p_1] - [p_2] = [q_1] - [q_2] \in K_0(A_\mu).$$

Then  $p_1 + q_2$  and  $p_2 + q_1$  are local stably isomorphic,  $p_1 + q_2 \sim_{ls} p_2 + q_1$ .

*Proof.* As said above, the description of  $K_0(A_\mu)$  classes in terms of generators (idempotents) contains an ambiguity due to the Grothendieck completion construction and stabilisation. The Grothendieck completion implies that there exists an idempotent  $s \in \mathbb{I}demp_m(A_\mu)$  such that the idempotents

$$p_1 + q_2 + s, \quad p_2 + q_1 + s$$

are local stably isomorphic. We assume that the idempotent  $s$  is already sufficiently stabilised. This means there exists an invertible matrix  $u \in \mathbb{G}L_{2n+m}(A_\mu)$  such that

$$p_1 + q_2 + s, = u (p_2 + q_1 + s) u^{-1}.$$

We add to both sides of this equality the idempotent  $1_m - s$  and we extend  $u$  to be the identity on the last summand. We get

$$p_1 + q_2 + s + (1 - s), = u (p_2 + q_1 + s + (1 - s)) u^{-1}.$$

From this we get further

$$p_1 + q_2 + 1_{2m}, = u (p_2 + q_1 + 1_{2m}) u^{-1},$$

that is, the idempotents  $p_1 + q_2, p_2 + q_1$  are local stably isomorphic

$$p_1 + q_2 \sim_{sl} p_2 + q_1.$$

□

**Lemma 33.** Let  $p, q \in \mathbb{I}demp_n(A_\mu)$  be idempotents. Suppose

$$(55) \quad [p] - [1_n] = [q] - [1_n] \in K_0(A_\mu).$$

Then

- i)  $p$  and  $q$  are local stably isomorphic,  $p \sim_{ls} q$ ,
- ii) exists an  $N \in \mathbb{N}$  and an  $u \in \mathbb{G}L_{n+N}(A_\mu)$  such that

$$(56) \quad p + 1_N = u q u^{-1} + 1_N = u (q + 1_N) u^{-1}.$$

*Proof.* -i) Lemma 32 says that the idempotents  $p + 1_n$ ,  $q + 1_n$  are *local stably* isomorphic. This means that the idempotents  $p$  and  $q$  are *local stably* isomorphic. Part -ii) describes this.  $\square$

**Theorem 34.** *Let  $p_{ij}$  be idempotents*

$$[p_1] = [p_{11}] - [p_{12}] \in K_0(\Lambda_{1,\mu})$$

$$[p_2] = [p_{21}] - [p_{22}] \in K_0(\Lambda_{2,\mu})$$

*with the property that*

$$j_{1*}[p_1] = j_{2*}[p_2] \in K_0(\Lambda'_\mu).$$

*Then there exists  $[p] = [p_{01}] - [p_{02}] \in K_0(\Lambda_\mu)$  with the property that*

$$i_{1*}[p] = [p_1] \text{ and } i_{2*}[p] = [p_2].$$

*Proof.* We may describe the two  $K$ -theory classes differently

$$[p_1] = [p_{11}] - [p_{12}] = [p_{11} + (1 - p_{12})] - [p_{12} + (1 - p_{12})] = [p'_{12}] - [1_n] \in K_0(\Lambda_{1,\mu})$$

and

$$[p_2] = [p_{21}] - [p_{22}] = [p_{21} + (1 - p_{22})] - [p_{22} + (1 - p_{22})] = [p'_{22}] - [1_n] \in K_0(\Lambda_{2,\mu}).$$

Then

$$j_{1*}[p_1] = j_{1*}([p'_{12}] - [1_n]) = (j_{1*}[p'_{12}]) - [1_n]$$

and

$$j_{2*}[p_2] = j_{2*}([p'_{22}] - [1_n]) = (j_{2*}[p'_{22}]) - [1_n].$$

The hypothesis says that

$$(j_{1*}[p'_{12}]) - [1_n] = (j_{2*}[p'_{22}]) - [1_n].$$

Lemma 33 says that the idempotents  $j_{1*}[p'_{12}]$ ,  $j_{2*}[p'_{22}]$  are *local stably* isomorphic. Now we are in the position to use Theorem 27. Let  $u \in \mathbb{G}\mathbb{L}(\Lambda_2(A_\mu))$  be the conjugation

$$(57) \quad j_{1*}(p'_{12}) = u j_{2*}(p'_{22}) u^{-1}.$$

Theorem 27 provides the idempotent

$$p = (j_{1*}(p'_{12}), (j_{2*}(p'_{22}), u).$$

The desired idempotents are

$$p_{10} = p = (j_{1*}(p'_{12}), (j_{2*}(p'_{22}), u) \in \mathbb{I}demp(\Lambda_\mu)$$

$$p_{20} = 1_N \in \mathbb{I}demp(\Lambda_\mu).$$

$\square$



## 8.2. Constructing invertible matrices over $\Lambda_\mu$ .

**Theorem 35. Definition.** *Suppose  $s_1 \in \mathbb{GL}_n(\Lambda_{1,\mu})$ , resp.  $s_2 \in \mathbb{GL}_n(\Lambda_{2,\mu})$ , are invertible matrices with entries in  $\Lambda_{1,\mu}$ , resp.  $\Lambda_{2,\mu}$ , such that*

$$(58) \quad j_{1*}(u_1) = u j_{2*}(u_2) u^{-1},$$

where  $u \in \mathbb{GL}_n(\Lambda'_\mu)$  is an invertible matrix.

-i) Then there exists an invertible matrix  $s \in \mathbb{GL}_{2n}(\Lambda_\mu)$  such that

$$(59) \quad i_{1*}(s) = s_1 \oplus 1_n \quad \text{and} \quad i_{2*}(s) = \tilde{s}_2$$

where the invertible matrix  $\tilde{s}_2 \in \mathbb{GL}_{2n}(\Lambda_{2,\mu})$  is conjugated to  $s_2 \oplus 1_n$  through an inner auto-morphism defined by the invertible matrix  $\tilde{U} \in GL_{2n}(\Lambda_{2,\mu})$ , that is  $\tilde{p}_2 = \tilde{U} p_2 \tilde{U}^{-1}$ .

The corresponding invertible double matrix  $s$  is denoted  $s = (s_1, s_2, u)$ .

-ii)

$$(60) \quad j_{2,*}\tilde{s}_2 = (u j_{2,*}(s_2) u^{-1}) \oplus 1_n = j_{1,*}(s_1 \oplus 1_n)$$

-iii) Any invertible double matrix  $s \in \mathbb{GL}_{2n}(\Lambda_\mu)$  is conjugated through a localised inner automorphism defined by a matrix  $U \in \mathbb{GL}_{2n}(\Lambda_\mu)$  to an idempotent matrix of the form  $s = (s_1, s_2, u)$  defined by -i).

*Proof.* The proof of this theorem goes along the same manner as the proof of Theorem 27.

In retrospect, the proof of Theorem 27 is based on the following facts: -a) operations with double matrices respect Remark 8, -b) lifting of the invertible element  $U = \mathbb{O}_{2n}(u) := u \oplus u^{-1} \in \mathbb{GL}_{2n}(\Lambda_{2,\mu})$  associated to the invertible element  $u \in \mathbb{GL}_n(\Lambda_{2,\mu})$  by means of the factorisation of  $U$  by elementary matrices (identity (48)), -c) the fact that the inner auto-morphisms keep the mapping  $\mathbb{O}_n$  unchanged.

To prove Theorem 35 we use the same arguments -a), - b), -c) with the following changes: idempotents are replaced by invertible elements while for -c) we use the fact that the inner auto-morphisms transform the mapping  $1_n$  into itself.  $\square$

The next theorem is the analogue of Theorem 34 in the  $K_1(A_\mu)$  case.

**Theorem 36.** *Suppose  $j_1$  and  $j_2$  are epi-morphisms.*

Let  $u_1 \in \mathbb{GL}(\Lambda_{1,\mu})$  and  $u_2 \in \mathbb{GL}(\Lambda_{2,\mu})$  be such that

$$(61) \quad j_{1*}[u_1] = j_{2*}[u_2] \in K_1(\Lambda'_\mu) \otimes \mathbb{Z}[\frac{1}{2}].$$

Then there exists  $s \in \text{Idemp}(\Lambda_\mu)$  such that

$$(62) \quad i_{1*}[s] = [u_1] \in K_1(\Lambda_{1,\mu}) \otimes \mathbb{Z}[\frac{1}{2}] \quad \text{and} \quad i_{2*}[s] = [u_2] \in K_1(\Lambda_{2,\mu}) \otimes \mathbb{Z}[\frac{1}{2}].$$

**Remark 37.** *Tensor multiplication by  $\mathbb{Z}[\frac{1}{2}]$  reflects the difficulty to lift elements belonging to  $\mathcal{O}(\Lambda'_\mu)$  to elements belonging to  $\mathcal{O}(\Lambda_{1,\mu})$  and  $\mathcal{O}(\Lambda_{2,\mu})$ . The nature of elements belonging to  $\mathcal{O}(\Lambda'_\mu)$ , see formula (48), insures the existence of the lift as an invertible element; the factor  $1/2$  is needed to insure that the invertible lifts belong to  $\mathcal{O}(\Lambda_{1,\mu})$  and  $\mathcal{O}(\Lambda_{2,\mu})$ .*

The presence of the factor  $1/2$  has important consequences.

*Proof.* The definition of  $K_1(\Lambda'_\mu)$ , see Definition 19 - 20, tells that the description of its elements contains an ambiguity belonging to the sub-module  $\mathbb{O}(\Lambda'_\mu)$ . Equality (61) tells then that there exist two elements  $\xi_1, \xi_2 \in \mathbb{O}_{2n}(\Lambda'_\mu)$  such that the invertible matrices  $j_{1*}(u_1) + \xi_1, j_{2*}(u_2) + \xi_2$  are *local* conjugated by means of a matrix  $u \in \mathbb{GL}_{2n}(\Lambda')$

$$(63) \quad j_{1*}(u_1) + \xi_1 = u (j_{2*}(u_2) + \xi_2) u^{-1}.$$

At this point a new difficulty occurs. We would like to use the analogue of Lemma 29 and use the invertible matrix  $U = u \oplus u^{-1} \in \mathbb{O}_{2n}(\Lambda'_\mu)$  to lift and twist invertible elements belonging to  $\mathbb{GL}(\Lambda'_\mu)$ . With the exception of  $u$ , in order to apply Lemma 29 we have to start with invertible elements which are already defined over  $\Lambda_{1,\mu}$  and  $\Lambda_{2,\mu}$ . In this case we need to show two more things: -1) that the elements  $\xi_1, \xi_2$  have invertible lifts in  $\Lambda_{1,\mu}$  resp.  $\Lambda_{2,\mu}$  and that -2) such lifts belong to  $\mathbb{O}_{2n}(\Lambda_{i,\mu}), i = 1, 2$ .

Here we address these two problems. We illustrate the solution in the case of the element  $\xi_1$ .

We intend to lift the invertible matrix

$$(64) \quad \xi_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} \in \mathbb{O}_{2n}(\Lambda'_\mu)$$

to an invertible matrix  $\tilde{\xi}_1 \in \mathbb{O}_{2n}(\Lambda_{1,\mu})$ .

To produce the lift we use the decomposition (48)

$$(65) \quad \xi_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We suppose the homo-morphism  $j_1$  to be an epi-morphism. To produce the lift one replaces in this formula the elements  $\alpha_1$ , resp.  $\alpha_1^{-1}$ , by some elements  $\tilde{\alpha}_1$ , resp.  $\tilde{\beta}_1 \in \Lambda_{1,\mu}$  such that  $j_1(\tilde{\alpha}_1) = \alpha_1$ , resp.  $j_1(\tilde{\beta}_1) = \alpha_1^{-1}$ . The obtained element is

$$(66) \quad \tilde{\xi}_1 = \begin{pmatrix} 1 & \tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tilde{\beta}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We use property (52) of elementary matrices to find the inverse of  $\tilde{\xi}_1$

$$(67) \quad \tilde{\xi}_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\beta}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix}.$$

Although we do not use here the exact formula (66) of  $\tilde{\xi}_1$  and its inverse (67)  $\tilde{\xi}_1^{-1}$ , it is enough to know that such lifts exist and that  $j_1(\tilde{\xi}_1) = \xi_1$  and  $j_1(\tilde{\xi}_1^{-1}) = (j_1(\tilde{\xi}_1))^{-1} = \xi_1^{-1}$ , to produce the element

$$(68) \quad \tilde{U}_1 = \begin{pmatrix} \tilde{\xi}_1 & 0 \\ 0 & \tilde{\xi}_1^{-1} \end{pmatrix} \in \mathbb{O}_{4n}(\Lambda_{1,\mu}).$$

From this relation we obtain

$$(69) \quad j_1(\tilde{U}_1) = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1^{-1} \end{pmatrix} \sim_l 2\xi_1.$$

We have proved the following result.

**Lemma 38.** *Suppose the homo-morphism  $j_1$  is epimorphic.*

-1) *Then there exists a recipe which for any element  $\xi_1 \in \mathcal{O}_{2n}(\Lambda'_\mu)$  produces an invertible element  $\tilde{U}_1 \in \mathcal{O}_{4n}(\Lambda_{1,\mu})$  such that*

$$(70) \quad j_1(\tilde{U}_1) = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1^{-1} \end{pmatrix} \sim_l 2\xi_1.$$

-2) *Then  $j_{1*}(u_1 + \xi_1)$ , from formula (63), lifts in  $K_1(\Lambda_{1,\mu}) \otimes \mathbb{Z}[\frac{1}{2}]$  to  $u_1 + \frac{1}{2}\tilde{U}$  and*

$$(71) \quad [u_1 + \frac{1}{2}\tilde{U}] = [u_1] \in K_1(\Lambda_{1,\mu}) \otimes \mathbb{Z}[\frac{1}{2}]$$

The properties -1) and -2) above being satisfied, the proof of the Theorem 36 proceeds as in the case of Lemma 29.  $\square$

### 9. $K_1^{loc}(A)$ vrs. $K_1(A)$ .

*The following identities are well known and used as building blocks of  $K$ -theory, see [27], [15], [12], [4], [18], [17], [26].*

*To facilitate the reading of this article we summarise some basic facts from the classical algebraic  $K$ -theory and put them in prospective with some considerations discussed here.*

**Definition 39.** *Whitehead Group  $K_1(A)$ .*

*By definition*

$$(72) \quad K_1(A) := \text{inj lim}_{n \in \mathbb{N}} \text{GL}_n / [\text{GL}_n, \text{GL}_n]$$

*The group structure in  $K_1(A)$  is given by matrix multiplication*

$$(73) \quad [A] + [B] := [A.B].$$

**Theorem 40.** *-i) Any commutator is stably isomorphic to a product of elements of the form  $\mathcal{O}_{2n}(A)$ . More specifically, for any  $A, B \in \text{GL}(A_\mu)$*

$$(74) \quad \begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}.$$

*-ii) If  $A \in \text{GL}_n(A_\mu)$ , then (this repeats Definition 15 formula (12) )*

$$(75) \quad \mathcal{O}_{2n}(A) := \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

*-iii) Any elementary matrix is a commutator (this repeats formula (53) )*

$$(76) \quad E_{ij}(A) = [E_{ik}(A), E_{kj}(1)], \text{ for any } i, j, k \text{ distinct indices.}$$

*-iv) For any  $A, B \in \text{GL}_n(A_\mu)$ ,  $A + B$  is stably equivalent to  $AB$  and  $BA$  modulo (multiplicatively) elements of the form  $\mathcal{O}_{2n}(A)$*

$$(77) \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} BA & 0 \\ 0 & 1 \end{pmatrix}.$$

-v) For any  $A, B \in \mathbb{GL}_n(A_\mu)$  one has the identity  
(78)

$$\begin{pmatrix} ABA^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}^{-1}.$$

*Proof.* The proof is obtained by direct verification.  $\square$

**Theorem 41.** ([27], [3], [20], [15], [12], [18] )

- i)  $[\mathbb{GL}(A), \mathbb{GL}_n(A)] = \mathbb{E}_n(A)$
  - ii)  $[ABA^{-1}B^{-1}] = [E_{ij}(A)] = 0 \in K_1(A)$
  - iii)  $[A] + [B] = [AB] = [BA] = [B] + [A] \in K_1(A)$ .
- Therefore,  $\mathbb{GL}(A)$  is an Abelian group.
- iv)  $[\mathcal{O}_{2n}(A)] = [1_n] = 1 \in K_1(A)$ .
  - v)  $[A] = [A^{-1}]$  for any  $A \in \mathbb{GL}(A)$ .
  - vi)  $[ABA^{-1}] = [B] \in K_1(A)$ .

*Proof.* Relation (74) says that any commutator is a product of matrices of type  $\mathcal{O}_n(A)$ . Formula (75) says that any matrix of type  $\mathcal{O}_n(A)$  is a product of elementary matrices. Viceversa, (76) says that any elementary matrix is a commutator. This proves -i).

-ii) follows from the definition of  $K_1(A)$ .

-iii) Formula (77) implies the first part of -iii). Formula (75) combined with Theorem 41. -i) and the definition of  $K_1(A)$  imply that  $[\mathcal{O}(A)] = 0 \in K_1(A)$ , which proves -iv). Formula (77) shows that  $[A \oplus B] = [A].[B]$ , which completes the proof of -iii).

-v) This follows from formula (77) together with -iii) and -ii).  $\square$

**Remark 42.** -i) The construction of the classical  $K_1(A)$  does not require to use the Grothendieck completion. This is because the factorisation of  $\mathbb{GL}(A)$  through the commutator sub-group insures property -iv), which provides the group structure.

The group structure in  $K_1(A)$  is essentially due to the fact that the both the direct sum and product of invertible matrices define the addition/multiplication in  $K_1(A)$ . These properties imply that  $\mathcal{O}_n(A)$  represents the zero class in  $K_1(A)$ , which insures existence of opposite elements.

The next remark refers to the specific parts of the construction of local  $K_1$  which make our construction different from the classical one.

**Remark 43.** -i) In our construction the commutator sub-group and the group generated by elementary matrices are avoided because the number of multiplications needed to generate these sub-groups might be un-bounded. Any multiplication increases the size of the support (in the definition of localised algebras this means that the support of the elements could not be controlled).

In the whole article the multiplication of elements in the algebra is avoided for the same reason. The multiplication of elements is replaced by sums and direct sums. The increase of the size of matrices replaces in our construction the need to perform multiple multiplications. Where multiplications could not be avoided, we were careful to use not

more than three such multiplications. The corresponding increase in the size of the supports is absorbed by the projective limit used in the definition of  $K_1^{loc}(A)$ .

-ii) Our construction uses the elements  $\mathcal{O}(A)$  additively and not multiplicatively. These elements generate the commutator sub-group (see formula (74)) through an un-bounded number of multiplications; in our construction an un-bounded number of multiplications is not allowed.

-iii) The construction of  $K_1^{loc}(A)$  uses the factorisation of  $\mathbb{GL}(A)$  through the smaller sub-group of inner auto-morphisms. The class of an invertible element modulo inner auto-morphisms, which we called abstract Jordan form, contains much more information than the class of the invertible element modulo the commutator sub-group. Abstract Jordan canonical forms do not insure, in general, that the matrices  $\mathcal{O}_n(A)$  represent the null element. In the case of the algebra of complex matrices the Jordan canonical forms behave additively (modulo permutations of the Jordan cells) with respect to direct sums. For this reason it is not reasonable to expect that the Jordan canonical forms of  $u$  and  $u^{-1}$  would cancel out. This is why we should not expect that elements of the form  $\mathcal{O}(A)$  are equivalent, through an inner auto-morphism, to the null element. Given that the elements  $\mathcal{O}(A)$  play in the  $K$ -theory multiple roles (see above discussion), we find it natural, for the definition of the local  $K_1$ , to quotient  $\mathbb{GL}(A)$  through the additive sub-group generated by direct sums of  $\mathcal{O}(A)$ , rather than through the commutator sub-group. By this factorisation we decree that the Jordan canonical forms of the elements  $u$  and  $u^{-1}$  are opposite one to each other. The additive group generated by elements  $\mathcal{O}(A)$  is contained in the commutator sub-group; this property insures the fact that there exists a natural epi-morphism from  $K_1^{loc}$  to  $K_1$ .

-iv) Additionally, the projective limit (Alexander-Spanier type construction) makes the local algebraic  $K$ -theory reacher than the classical  $K$ -theory.

**Theorem 44.** -i) There is a canonical epi-morphism

$$(79) \quad \Pi : K_1^{loc}(A) \longrightarrow K_1(A)$$

$$(80) \quad Ker \Pi = [ \mathbb{GL}(A), \mathbb{GL}(A) ] / Inner(A).$$

## 10. CONNECTING HOMO-MORPHISM $\partial : K_1^{loc}(\Lambda') \longrightarrow K_0^{loc}(\Lambda)$ .

In this section we assume that the diagram (6) satisfies Hypotheses 1, 2, 3.

### 10.1. Connecting homo-morphism.

**Definition 45.** Connecting homomorphism-first form.

Let  $U \in \mathbb{GL}_n(\Lambda'_\mu)$ . Theorem 27 associates the idempotent  $p = p(1_n, 1_n, U)$ . By definition,  $\partial[U] = [p(1_n, 1_n, U)]$ .

$\partial[U]$  may be produced explicitly in a different manner. So, let  $[U] \in K_1^{loc}(\Lambda')$  be such that  $U \in \mathbb{GL}_n(\Lambda'_\mu)$  for some  $n$  and  $\mu$ .

Let  $A, B \in \mathbb{GL}_n(\Lambda_{1,\mu})$  be liftings of  $U$ , resp.  $U^{-1}$ , in  $\mathbb{M}_n(\Lambda_{1,\mu})$ . Such liftings exist because we assume  $j_1$  is surjective (note the change of the index). Here we do not assume that  $A$  and  $B$  are invertible.

With  $A$  and  $B$  one associates  $S_0 = 1 - BA$  and  $S_1 = 1 - AB$ ; these elements belong to  $\mathbb{M}_n(\Lambda_{1,\mu-1})$ . The matrices  $S_0, S_1$  satisfy

$$(81) \quad j_{2,*}(S_0) = j_{1,*}(S_1) = 0.$$

With these matrices one associates the invertible matrix (ref. [6])

$$(82) \quad L = \begin{pmatrix} S_0 & -(1 + S_0)B \\ A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\Lambda_{1,\mu-2});$$

the inverse of the matrix  $L$  is

$$(83) \quad L^{-1} = \begin{pmatrix} S_0 & (1 + S_0)B \\ -A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\Lambda_{1,\mu-2}).$$

Let  $e_1, e_2$  be the idempotents

$$(84) \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Idemp}_{2n}(\Lambda_{1,\mu}).$$

The invertible matrix  $L$  is used to produce the idempotent

$$(85) \quad P := Le_1L^{-1} = \begin{pmatrix} S_0^2 & S_0(1 + S_0)B \\ S_1A & 1 - S_1^2 \end{pmatrix} \in \text{Idemp}_{2n}(\Lambda_{1,\mu-2}).$$

A matrix whose entries belong to the double ring  $\Lambda$  will be called double-matrix.

The idempotent  $P$  is used to construct the double-matrix idempotent

$$(86) \quad \mathbb{P}_U := \begin{pmatrix} (S_0^2, 0) & (S_0(1 + S_0)B, 0) \\ (S_1A, 0) & (1 - S_1^2, 1) \end{pmatrix} \in \text{Idemp}_{2n}(\Lambda_{1,\mu-2} \oplus \Lambda_{2,\mu-2}).$$

The matrix  $\mathbb{P}_U$  is an idempotent in  $\mathbb{M}_{2n}(\Lambda_{\mu-2})$ . Indeed, using (84) one gets  $(j_{1,*} - j_{2,*})\mathbb{P}_U = 0$  and therefore  $\mathbb{P}_U \in \mathbb{M}_{2n}(\Lambda_{\mu-2})$ . The fact that the matrix  $\mathbb{P}_U$  is an idempotent in  $\mathbb{M}_{2n,\mu-2}(\Lambda)$  follows from the fact that  $P$  and  $e_2$  are idempotents along with Remark 8, §5.

**Definition 46.** Connecting homomorphism-second form. (compare [6] for non-localised  $K$ -theory).

For any  $[U] \in K_1^{\text{loc}}(\Lambda')$  one defines the connecting homomorphism  $\partial : K_1^{\text{loc}}(\Lambda') \rightarrow K_0^{\text{loc}}(\Lambda)$  by

$$(87) \quad \partial[U] := [\mathbb{P}_U] - [(e_2, e_2)] \in K_0^{\text{loc}}(\Lambda).$$

**10.2. Extension of the Connecting Homomorphism  $\partial$ .** In this paper we will need an extension of the formula for the definition of  $\partial$ . In the proof of the exactness, Theorem 51.-iii), we will need to know how the connecting homomorphism, formula (86), behaves with respect to stabilisations. For this purpose we consider a more general situation than that considered in the §10.1.

Let  $U \in \mathbb{GL}_{m+n}(\Lambda')$  and

$$(88) \quad e_{(0,n)} = \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix} \in \mathbb{M}_{m+n}(\Lambda')$$

be such that the diagram

$$(89) \quad \begin{array}{ccc} \Lambda'^{m+n} & \xrightarrow{U} & \Lambda'^{m+n} \\ \downarrow e_{(0,n)} & & \downarrow e_{(0,n)} \\ \Lambda'^{m+n} & \xrightarrow{U} & \Lambda'^{m+n} \end{array}$$

is commutative. This condition is used at the end of the construction of  $\partial[U]$ ; it is needed to show that  $\mathbb{P}_U \in \mathbb{Idemp}_{m+n}(\Lambda)$ , formula (97).

The case discussed in §10.1 corresponds in this subsection to  $m = 0$ .

We proceed as before. Let  $A, B \in \mathbb{GL}_{m+n}(\Lambda_{1,\mu})$  be liftings of  $U$ , resp.  $U^{-1}$ , in  $\mathbb{M}_{m+n,\mu}(\Lambda_1)$ . Such liftings exist because we assume  $j_1$  is surjective.

With  $A$  and  $B$  one associates  $S_0 = 1 - BA \in \mathbb{GL}_{m+n}(\Lambda_{1,\mu})$  and  $S_1 = 1 - AB \in \mathbb{GL}_{m+n}(\Lambda_{1,\mu})$ . The matrices  $S_0, S_1$  satisfy

$$(90) \quad j_{1,*}(S_0) = j_{1,*}(S_1) = 0.$$

With these matrices one associates the invertible matrix (ref. [6])

$$(91) \quad L = \begin{pmatrix} S_0 & -(1 + S_0)B \\ A & S_1 \end{pmatrix} \in \mathbb{GL}_{2(m+n)}(\Lambda_{1,\mu-2});$$

the inverse of the matrix  $L$  is

$$(92) \quad L^{-1} = \begin{pmatrix} S_0 & (1 + S_0)B \\ -A & S_1 \end{pmatrix} \in \mathbb{GL}_{2(m+n)}(\Lambda_{1,\mu-2}).$$

Let  $e_1$  be the idempotent

$$(93) \quad e_1 = \begin{pmatrix} e_{(0,n)} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{GL}_{2(m+n)}(\Lambda_{1,\mu}).$$

and

$$(94) \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & e_{(0,n)} \end{pmatrix} \in \mathbb{GL}_{2(m+n)}(\Lambda_{1,\mu}).$$

The invertible matrix  $L$  is used to produce the idempotent

$$(95) \quad P := Le_1L^{-1} = \begin{pmatrix} S_0 e_{(0,n)} S_0 & S_0 e_{(0,n)} (1 + S_0)B \\ A e_{(0,n)} S_0 & A e_{(0,n)} (1 + S_0)B \end{pmatrix} \in \mathbb{Idemp}_{2(m+n)}(\Lambda_{1,\mu-2}).$$

The idempotent  $P$  is used to construct the double-matrix idempotent

(96)

$$\mathbb{P}_U := \begin{pmatrix} (S_0 e_{(0,n)} S_0, 0) & (S_0 e_{(0,n)} (1 + S_0)B, 0) \\ (A e_{(0,n)} S_0, 0) & (A e_{(0,n)} (1 - S_0)B, e_{(0,n)}) \end{pmatrix} \in \text{Idemp}_{2(m+n)}(\Lambda_{1,\mu-2} \oplus \Lambda_{2,\mu-2}).$$

The matrix  $\mathbb{P}_U$  is an idempotent in  $\mathbb{M}_{2(m+n)}(\Lambda_{\mu-2})$ . We may verify directly that  $(j_{1,*} - j_{2,*})\mathbb{P}_U = 0$ . Indeed,  $j_{1*}$  is a ring homomorphism and  $j_{1*}(S_0) = j_{1*}(S_0) = 0$ ; finally, the hypothesis (89) gives

$$(97) \quad j_1(A e_{(0,n)} (1 - S_0)B) = U e_{(0,n)} U^{-1} = e_{(0,n)} = j_2(e_{(0,n)}).$$

Therefore  $\mathbb{P}_U \in \mathbb{M}_{2n}(\Lambda_{\mu-2})$ . The fact that the matrix  $\mathbb{P}_U$  is an idempotent in  $\mathbb{M}_{2(m+n)}(\Lambda_{\mu-2})$  follows from the fact that  $P$  and  $e_2$  are idempotents along with the discussion above.

**Definition 47.** Connecting homomorphism - third form.

We suppose the assumptions and constructions of §10.2 above are in place. For any  $[U] \in K_1^{\text{loc}}(\Lambda')$  one defines the connecting homomorphism  $\partial : K_1^{\text{loc}}(\Lambda') \rightarrow K_0^{\text{loc}}(\Lambda)$  by

$$(98) \quad \partial[U] := [\mathbb{P}_U] - [(e_2, e_2)] = \begin{pmatrix} (0, 0) & (0, 0) \\ (0, 0) & (A e_{(0,n)} B, e_{(0,n)}) \end{pmatrix} - [(e_2, e_2)] \in K_0^{\text{loc}}(\Lambda).$$

**Remark 48.** If the lifts  $A$ , resp.  $B$ , of the isomorphisms  $U$ , resp.  $U^{-1}$ , in  $\Lambda_1^{m+n}$  are inverse one to each other, then the corresponding matrices  $S_0 = S_1 = 0$  and

$$(99) \quad \partial[U] := [\mathbb{P}_U] - [(e_2, e_2)] \in K_0^{\text{loc}}(\Lambda).$$

**Proposition 49.** The connecting homomorphism  $\partial$  is well defined.

*Proof.* It is easy to see that  $\mathbb{P}_U$  behaves correctly with respect to stabilisation and sums.

Given that any  $\partial\xi$ , with  $\xi \in \mathbb{O}_\mu(\Lambda'_\mu)$ , represents the zero element in  $K_1(A_\mu)$ , see Proposition 22. -ii), we have to show that  $\partial\xi = 0$ . To do this we take into account the Definition 46 together with formula (48). In fact, this last formula shows that the invertible element  $\xi$  has an invertible lifting in  $\mathbb{GL}(\Lambda_{2,\mu})$ . This lifted element together with its inverse produce a pair of elements  $A, B$ , see Definition 45, for which the corresponding elements  $S_0 = S_1 = 0$ . Formulas (86), (87) show that  $\partial\xi = 0$ .

It remains to follow up how  $\mathbb{P}_U$  depends of the choice of the lifts  $A$  and  $B$  of  $U$  and  $U^{-1}$ . We observe that a different choice of  $A$  and  $B$  has the effect of modifying just the matrix  $L$ . We will show that if  $A'$  and  $B'$  are two such different lifts and  $L'$  is the corresponding matrix, then

$$(100) \quad L' = \tilde{L} L, \quad \text{with } \tilde{L} \in \mathbb{GL}_{2(m+n)}(\Lambda)$$

and hence the corresponding idempotents  $\mathbb{P}_U := L e_1 L^{-1}$ ,  $\mathbb{P}'_U := L' e_1 L'^{-1}$  are conjugate.

To better organise the computation, we change the liftings one at the time.

We begin with  $A$ . Let  $\tilde{A} = A + K$  with  $j_1(K) = 0$ . Let  $\tilde{S}_0 = 1 - B\tilde{A}$ ,  $\tilde{S}_1 = 1 - \tilde{A}B$ ,  $\tilde{L}$ ,  $\tilde{P}$  and  $\tilde{\mathbb{P}}_U$  be the corresponding elements. A direct computation gives

$$(101) \quad \tilde{L} L^{-1} = \begin{pmatrix} 1 - BK & -BK B \\ K & 1 + KB \end{pmatrix}$$



or

$$(102) \quad \tilde{L} = \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix} L.$$

We know that  $\tilde{L}$  and  $L$  are invertible matrices; therefore the RHS of (101) is an invertible matrix.

The corresponding idempotent  $\tilde{P}$  is

$$(103) \quad \tilde{P} = \tilde{L}.e_1.\tilde{L}^{-1} = \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix} L.e_1.L^{-1} \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix}^{-1} =$$

$$(104) \quad = \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix} P \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix}^{-1}$$

and furthermore

$$(105) \quad \tilde{\mathbb{P}}_U = \left( \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix}, 1 \right) \mathbb{P}_U \left( \begin{pmatrix} 1 - BK & -BKB \\ K & 1 + KB \end{pmatrix}, 1 \right)^{-1}.$$

Therefore,  $[\tilde{\mathbb{P}}_U] = [\mathbb{P}_U] \in K_0^{loc}(\Lambda)$ .

It remains to see what happens if  $A$  remains unchanged and the lifting  $B$  is changed. Let  $\tilde{B} = B + H$ , with  $j_1(H) = 0$ . Let  $\tilde{L}$ ,  $\tilde{P}$  and  $\tilde{\mathbb{P}}_U$  be the corresponding matrices. A direct computation gives

$$(106) \quad \tilde{L}.L^{-1} = \begin{pmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{pmatrix} \in \mathbb{M}_{2n}(\Lambda_{\mu-4})$$

where

$$\Delta_{11} = HA - HAH - BAH$$

$$\Delta_{12} = -2H + HAB + HAH + BAH - BAHAB - HAHAB$$

$$\Delta_{21} = AHA$$

$$\Delta_{22} = -AH + AHAB.$$

The RHS of (106) is a product of invertible matrices; therefore, it is an invertible matrix. Proceeding as above we get

$$(107) \quad \tilde{\tilde{\mathbb{P}}}_U = \left( \begin{pmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{pmatrix}, 1 \right) \mathbb{P}_U \left( \begin{pmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{pmatrix}, 1 \right)^{-1},$$

which completes the proof of the Proposition 49.  $\square$

**Remark 50.** *The connecting homomorphism  $\partial$  could be defined by the same formula (95), (96), (98) where  $L \in \mathbb{GL}_{2n}(\Lambda_{1,\mu})$  is any invertible lifting of the matrix*

$$(108) \quad \begin{pmatrix} 0 & -U^{-1} \\ U & 0 \end{pmatrix}.$$

If  $L, L'$  are two such liftings, the corresponding idempotents  $\mathbb{P}_U$  are conjugate

$$(109) \quad \mathbb{P}'_U = (L'L^{-1})\mathbb{P}_U(L'L^{-1})^{-1}.$$

## 11. SIX TERMS EXACT SEQUENCE.

**Theorem 51.** *The six terms sequence*

$$(110) \quad K_1^{loc}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \longrightarrow K_1^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus K_1^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{j_{1*}-j_{2*}} K_1^{loc}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\partial}$$

$$(111) \quad K_0^{loc}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{(i_{1*}, i_{2*})} K_0^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus K_0^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \longrightarrow K_0^{loc}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}].$$

is exact.

More precisely, the following sequences are exact

-i)

$$(112) \quad K_0^{loc}(\Lambda) \xrightarrow{(i_{1*}, i_{2*})} K_0^{loc}(\Lambda_1) \oplus K_0^{loc}(\Lambda_2) \xrightarrow{j_{1*}-j_{2*}} K_0^{loc}(\Lambda')$$

-ii)

$$(113) \quad K_1^{loc}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{(i_{1*}, i_{2*})} K_1^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus K_1^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{j_{1*}-j_{2*}} K_1^{loc}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}]$$

-iii)

$$(114) \quad K_1^{loc}(\Lambda_1) \oplus K_1^{loc}(\Lambda_2) \xrightarrow{j_{1*}-j_{2*}} K_1^{loc}(\Lambda') \xrightarrow{\partial} K_0^{loc}(\Lambda) \xrightarrow{(i_{1*}, i_{2*})} K_0^{loc}(\Lambda_1) \oplus K_0^{loc}(\Lambda_2).$$

*Proof.* -i) It is easy to verify that  $Im(i_{1*}, i_{2*}) \subset Ker(j_{1*} - j_{2*})$ .

We verify that  $Ker(j_{1*} - j_{2*}) \subset Im(i_{1*}, i_{2*})$ .

Let  $[p_1] - [p_2], [q_1] - [q_2] \in K_0^{loc}(\Lambda_{1,\mu}) \oplus K_0^{loc}(\Lambda_{2,\mu})$  be such that

$$0 = j_{1,*}([p_1] - [p_2]) - j_{2,*}([q_1] - [q_2]),$$

where  $p_1, p_2, q_1, q_2$  are idempotents. The pair of  $K$ -theory classes may be re-written

$$0 = j_{1,*}([p_1 + \bar{p}_2] - [p_2 + \bar{p}_2]) - j_{2,*}([q_1 + \bar{q}_2] - [q_2 + \bar{q}_2])$$

or

$$0 = j_{1,*}([p_1 + \bar{p}_2] - [1_r]) - j_{2,*}([q_1 + \bar{q}_2] - [1_s]).$$

By adding all sides a trivial idempotent of sufficiently large size, we may assume that  $r = s$  is large. This relation may be re-written

$$0 = j_{1,*}([p_1 + \bar{p}_2] - j_{2,*}([q_1 + \bar{q}_2])).$$

This means that there exists an idempotent  $\xi \in \mathbb{I}demp_N(\Lambda'_\mu)$  such that the idempotents

$$(j_{1,*}([p_1 + \bar{p}_2] + \xi) - (j_{2,*}([q_1 + \bar{q}_2] + \xi)))$$

are isomorphic. We add further the idempotent  $\bar{\xi} := 1_N - \xi$  to get isomorphic idempotents

$$(j_{1,*}([p_1 + \bar{p}_2] + \xi + \bar{\xi}), (j_{2,*}([q_1 + \bar{q}_2] + \xi + \bar{\xi})) \in \mathbb{I}demp_{\bar{N}}(\Lambda'_\mu)$$

( $\bar{N}$  being the size of these idempotents) or

$$(j_{1,*}([p_1 + \bar{p}_2] + 1_N), (j_{2,*}([q_1 + \bar{q}_2] + 1_N)).$$

This means there exists  $u \in \mathbb{GL}_{\bar{N}}(\Lambda')$  which conjugates these two idempotents.

Theorem 34 says that there exists the idempotent  $p \in \mathbb{I}demp_{2\bar{N}}$  such that

$$i_{1,*}p = (p_1 + \bar{p}_2 + 1_N) \oplus 1_{\bar{N}}$$

and

$$i_{2,*}p = U ((q_1 + \bar{q}_2 + 1_N) \oplus 1_{\bar{N}}) U^1$$

where

$$U \in \mathbb{GL}_{2\bar{N}}(\Lambda'_\mu).$$

This means the image of the  $K$ -theory class of  $p - 1_{\bar{N}}$  through the pair of homomorphisms  $(i_{1,*}, i_{2,*})$  is

$$\begin{aligned} & (i_{1,*}, i_{2,*})[p - 1_{\bar{N}}] = \\ & = ((p_1 + \bar{p}_2 + 1_N) \oplus 1_{\bar{N}} - 1_{\bar{N}}, U ((q_1 + \bar{q}_2 + 1_N) \oplus 1_{\bar{N}}) U^1 - 1_{\bar{N}}) = \\ & = ((p_1 + \bar{p}_2 + 1_N) \oplus 1_{\bar{N}} - 1_{\bar{N}}, U ((q_1 + \bar{q}_2 + 1_N) \oplus 1_{\bar{N}} - 1_{\bar{N}}) U^1) = \\ & = ([p_1 - p_2], [q_1 - q_2]), \end{aligned}$$

which completes the proof of the assertion.

**-ii)** The only problem might occur in verifying that  $Ker(j_{1*} - j_{2*}) \subset Im(i_{1*}, i_{2*})$ . The proof of it is provided by Theorem 36. The remaining verifications are obvious.

**-iii)**

**-iii.1)** Let  $u \in \mathbb{GL}(A_\mu)$ . Lemma 38 says that  $\mathcal{O}(u) \otimes \mathbb{Z}[\frac{1}{2}]$  lifts as an invertible element over  $\Lambda_{2,\mu}$ . Remark 48 completes the argument stating that  $\partial u = 0 \in K_0(\Lambda_\mu \otimes \mathbb{Z}[\frac{1}{2}])$ . This result takes care of the elements belonging to  $\mathbb{O}(\Lambda'_\mu)$ .

We verify that  $\partial \circ j_{1*} = 0$ . Indeed, let  $\tilde{U} \in \mathbb{GL}_n(\Lambda_{1,\mu})$  and  $U = j_{1*}\tilde{U}$ . Then  $\partial \circ j_{1*}[\tilde{U}] = \partial[U]$ . The elements  $U, U^{-1}$  have the liftings  $A = \tilde{U}$  and  $B = \tilde{U}^{-1}$ . Then the corresponding elements  $S_0$  and  $S_1$  are equal to zero. Therefore

$$(115) \quad \partial[U] = \left[ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (1,1) \end{pmatrix} \right] - \left[ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (1,1) \end{pmatrix} \right] = 0.$$

**-iii.2)**  $(i_{1*}, i_{2*}) \circ \partial = 0$ . Let  $[U] \in K_1^{loc}(\Lambda')$  and let  $\partial[U] = [\mathbb{P}_U] - [(e_2, e_2)]$ , where  $\mathbb{P}_U$  is given by formula (89). Then

$$(116) \quad i_{1,*}\partial[U] = [Le_1L^{-1}] - [e_2] = [e_1] - [e_2] = 0.$$

On the other side,

$$(117) \quad i_{2,*}\partial[U] = [e_2] - [e_2] = 0.$$

**-iii.3)** Suppose  $[u] \in K_1^{loc}(\Lambda')$  and  $\partial[u] = 0$ . We have to prove that there exist invertible matrices  $U_1 \in \mathbb{GL}(\Lambda_{1,\mu})$  and  $U_2 \in \mathbb{GL}(\Lambda_{1,\mu})$  such that

$$(118) \quad [u] = (j_{1,*} - j_{2,*})(U_1, U_2) = j_{1,*}(U_1) - j_{2,*}(U_2)$$

$\partial[u] = 0$  means (after stabilisation) that  $\mathbb{P}_u$  is l-conjugate to  $[(e_2, e_2)]$ , i.e. there exists

$$(119) \quad U = (U_1, U_2) \in \mathbb{GL}_n(\Lambda_{1,\mu}) \otimes \mathbb{GL}_n(\Lambda_{2,\mu}),$$

which, as components of an invertible matrix over  $\Lambda_\mu$ , satisfy the compatibility condition

$$(120) \quad j_1(U_1) = j_2(U_2)$$

such that

$$(121) \quad \mathbb{P}_U := \begin{pmatrix} (S_0^2, 0) & (S_0(1 + S_0)B, 0) \\ (S_1A, 0) & (1 - S_1^2, 1) \end{pmatrix} = U.(e_2, e_2).U^{-1}.$$

In other words,

$$(122) \quad \partial u = [U.(e_2, e_2).U^{-1}] - [(e_2, e_2)] = [U_1.e_2.U_1^{-1}, U_2.e_2.U_2^{-1}] - [(e_2, e_2)].$$

We assume  $\Lambda_{1,\mu} = \Lambda_{2,\mu}$ . The class  $[U.(e_2, e_2).U^{-1}] \in \mathbb{Idemp}(\Lambda_\mu)$  remains unchanged by further conjugating it by the double invertible matrix

$$(123) \quad \tilde{U} = (U_1^{-1}, U_1^{-1}) \in \mathbb{GL}(\Lambda_\mu).$$

Denote by  $Inn(A)$  the inner auto-morphism associated with the invertible element  $A$ . Noting that the inner auto-morphisms satisfy

$$(124) \quad Inn(A) \circ Inn(B) = Inn(A.B),$$

we have, after recalling Theorem 27. -i)

$$(125) \quad \begin{aligned} \partial u &= [U_1.e_2.U_1^{-1}, U_2.e_2.U_2^{-1}] - [(e_2, e_2)] = \\ &= [\tilde{U}(U_1.e_2.U_1^{-1}, U_2.e_2.U_2^{-1})\tilde{U}^{-1}] - [(e_2, e_2)] = \\ &= [U_1^{-1}(U_1.e_2.U_1^{-1}U_1, U_1^{-1}U_2.e_2.U_2^{-1}U_1)\tilde{U}^{-1}] - [(e_2, e_2)] = p(1_N, 1_N, U_1^{-1}U_2). \end{aligned}$$

Note that  $U_1^{-1}U_2 \in \mathbb{GL}_N(\Lambda_{1,\mu}) = \mathbb{GL}_N(\Lambda_{2,\mu})$ . This relation proves the assertion -iii.3).

**-iii.4)** Let  $[p] = ([p_1] - [p_2]) \in K_0(\Lambda_\mu)$  ( $p_1 \in \mathbb{Idemp}_m(\Lambda_\mu)$  and  $p_2 \in \mathbb{Idemp}_n(\Lambda_\mu)$ ) and suppose  $(i_{1*}, i_{2*})[p] = 0$ . We have to show that  $[p_1] - [p_2] = \partial[U]$ , with  $U \in \mathbb{GL}(\Lambda'_\mu)$ .

We may assume that the idempotent  $p_2$  is trivial; indeed, the matrix  $(1 - p_2, 1 - p_2)$ , seen as a matrix in  $\mathbb{M}_n(\Lambda_\mu)$ , is an idempotent over  $\Lambda_\mu$  (here we have assumed that  $\Lambda_{1,\mu} = \Lambda_{2,\mu}$ ) and therefore  $[p_1] - [p_2] = [p_1 \oplus (1 - p_2)] - [p_2 \oplus (1 - p_2)]$ . Here

$$(126) \quad p_1 \oplus (1 - p_2) := \bar{p}_1 \in \mathbb{Idemp}_{m+n}(\Lambda_\mu) \text{ and}$$

$$(127) \quad p_2 \oplus (1 - p_2) := \bar{p}_2 \in \mathbb{Idemp}_{2n}(\Lambda_\mu) \text{ is a trivial idempotent.}$$

Denote

$$(128) \quad e_{(k,l)} := \begin{pmatrix} 0 & 0 \\ 0 & 1_l \end{pmatrix} \in \mathbb{M}_{k+l}(\mathbb{C}).$$

Let

$$(129) \quad (i_{1*}, i_{2*}) \bar{p}_1 := (\bar{p}_{11}, \bar{p}_{12}), \quad \bar{p}_{1k} \in \mathbb{M}_{m+n}(\Lambda_{k,\mu}), \quad k = 1, 2, \text{ with } j_{1,*}\bar{p}_{11} - j_{2,*}\bar{p}_{12} = 0,$$

$$(130) \quad (i_{1*}, i_{2*}) \bar{p}_2 := (e_{(0,2n)}, e_{(0,2n)}), \quad e_{(0,2n)} \in \mathbb{M}_{2n}(\Lambda_{k,\mu}).$$

By hypothesis we know also that

$$(131) \quad [\tilde{p}_{11}] - [e_{(0,2n)}] = 0 \in K_0(\Lambda_{1,\mu})$$

$$(132) \quad [\tilde{p}_{12}] - [e_{(0,2n)}] = 0 \in K_0(\Lambda_{2,\mu}).$$

Equations (131), (132) tell that after possible stabilisations (let  $\tilde{p}_{11} \in \mathbb{I}demp_{2n+q}(\Lambda_{1,\mu})$ , resp.  $\tilde{p}_{12} \in \mathbb{I}demp_{2n+q}(\Lambda_{2,\mu})$  and  $e_{(q,2n)}$  be the stabilised idempotents (with the same  $q$ , sufficiently large), there exist invertible elements  $u_1 \in \mathbb{G}L_{q+2n}(\Lambda_{1,\mu})$ ,  $u_2 \in \mathbb{G}L_{q+2n}(\Lambda_{2,\mu})$ , such that

$$(133) \quad \tilde{p}_{11} = u_1 \tilde{p}_{21} u_1^{-1} = u_1 e_{(q,2n)} u_1^{-1}$$

$$(134) \quad \tilde{p}_{12} = u_2 \tilde{p}_{22} u_2^{-1} = u_2 e_{(q,2n)} u_2^{-1}.$$

After such stabilisations, the new double matrix  $\tilde{p}_1 := (\tilde{p}_{11}, \tilde{p}_{12})$  is still an idempotent in  $\mathbb{M}_{q+2n}(\Lambda_\mu)$ .

The isomorphisms  $u_1, u_2$  tell that the pair  $\tilde{p}_1 := (\tilde{p}_{11}, \tilde{p}_{12})$  is  $(\Lambda_{1,\mu}, \Lambda_{2,\mu})$ -isomorphic to the pair of trivial idempotents  $(e_{(q,2n)}, e_{(q,2n)})$ . Notice, however, that in general,  $j_{1*}u_1 \neq j_{2*}u_2$ ; this is why this isomorphism is not necessarily a  $\Lambda_\mu$ -isomorphism.

We may further simplify the description of the idempotent  $\tilde{p}_1 \in \mathbb{I}demp(\Lambda_\mu)$ . To do this we replace the idempotent  $\tilde{p}_1$  by the idempotent  $\tilde{\tilde{p}}_1 := (\tilde{\tilde{p}}_{11}, \tilde{\tilde{p}}_{12}) := U\tilde{p}_1U^{-1}$ , where  $U$  is the double matrix

$$(135) \quad \tilde{U} := (u_2^{-1}, u_2^{-1}) \in \mathbb{G}L_{q+2n}(\Lambda_\mu).$$

After this modification

$$(136) \quad \tilde{\tilde{p}}_{11} = u_2^{-1} \tilde{p}_{11} u_2 = (u_2^{-1} u_1) e_{(q,2n)} (u_2^{-1} u_1)^{-1}$$

$$(137) \quad \tilde{\tilde{p}}_{12} = u_2^{-1} \tilde{p}_{12} u_2 = u_2^{-1} (u_2 e_{(q,2n)} u_2^{-1}) u_2 = e_{(q,2n)}.$$

As a result of all these transformations, the element  $[p]$  is stably  $\Lambda_\mu$ -isomorphic to

$$(138) \quad [p] = [p_1] - [p_2] = [(u_2^{-1} u_1) e_{(q,2n)} (u_2^{-1} u_1)^{-1}, e_{(q,2n)}] - [e_{(q,2n)}, e_{(q,2n)}].$$

From the equations (133), (134) we get

$$(139) \quad j_{1,*}\tilde{\tilde{p}}_{11} = j_{1,*}((u_2^{-1} u_1) e_{(q,2n)} (u_2^{-1} u_1)^{-1}) = j_{2,*}\tilde{\tilde{p}}_{12}.$$

This shows that

$$(140) \quad \phi := j_{1*}(u_2^{-1} u_1) \in \mathbb{G}L(\Lambda'_{1,\mu})$$

establishes an isomorphism between  $j_{1*}\tilde{\tilde{p}}_{11}$  and  $j_{1*}e_{(q,2n)} = e_{(q,2n)}$ .

Recall (see §10.2) that the construction of  $\partial(U)$  involves the lifting  $A$ , resp.  $B$ , of  $U$ , resp.  $U^{-1}$ , in  $\mathbb{M}_{2n}(\Lambda_{1,\mu})$ . We look for  $A, B \in \mathbb{M}_{2n}(\Lambda_{1,\mu})$  such that  $j_{1*}A = U$  and  $j_{1*}B = U^{-1}$ .

We may choose the liftings

$$(141) \quad A = u_1 u_2^{-1} \in \mathbb{G}L_{q+2n}(\Lambda_{1,\mu})$$

$$(142) \quad B = u_2 u_1^{-1} \in \mathbb{G}L_{q+2n}(\Lambda_{1,\mu}).$$

From this we get  $S_0 = 0$  and  $S_1 = 0$ .

We get further

$$(143) \quad P = \begin{pmatrix} S_0^2 & S_0(1 + S_0)B \\ S_1A & 1 - S_1^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_{2(m+n+q)}(\Lambda_\mu).$$

This shows that

$$(144) \quad \partial U = [\tilde{p}_{11}] - [\tilde{p}_{22}] = ([\tilde{p}_{11}] - [e_{0,2n}]) - ([\tilde{p}_{22}] - [e_{0,2n}]) = [p_1] - [p_2].$$

These complete the proof of Theorem 51. □

## REFERENCES

- [1] Atiyah M., Singer I. M.: The Index of elliptic operators: I, Ann. of Math. 87 (1968), 484 - 530
- [2] Atiyah M., Singer I. M.: The Index of elliptic operators: IV, Ann. of Math. 92 (1970), 119 - 138
- [3] Bass H.:  $K$ -theory and stable algebra. Publ. Math. I.H.E.S. 22 (1964), pp. 5 - 60.
- [4] Blackadar B.:  $K$ -Theory for Operator Algebras, Second Ed, 1998, Cambridge University Press.
- [5] Connes A.: Connes A.: Noncommutative Geometry, Academic Press, 1994.
- [6] Connes A., Moscovici H.: Cyclic Cohomology, the Novikov Conjecture Vol. 29 and Hyperbolic Groups, Topology Vol. 29, pp.345-388, 1990.
- [7] Connes A., Sullivan D., Teleman N.: Quasiconformal Mappings, Operators on Hilbert Space and Local Formulae for Characteristic Classes, Topology Vol. 33, Nr. 4, pp. 663-681, 1994.
- [8] Cuntz J.: Cyclic Theory, Bivariant  $K$ -theory and the Bivariant Chern-Connes Character. Encyclopedia of Mathematical Sciences. Operator Algebras and Non-commutative Geometry. J. Cuntz, V. F. R. Jones Eds., Springer - Verlag, Berlin, 2004.
- [9] Donaldson S. K., Sullivan D.: Quasi-conformal 4-Manifolds, Acta Mathematica., Vol. 163 (1989), pp. 181252.
- [10] Hirzebruch F.: Neue topologische Methoden in der algebraischen Geometrie. Berlin, Springer, 1956.
- [11] Hirzebruch F.: Elliptische Differentialoperatoren auf Mannigfaltigkeiten, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, 33 (1965), 583 - 608.
- [12] Karoubi M.:  $K$ -THEORY. An elementary introduction. Conference at the Clay Mathematics Research Academy.
- [13] Karoubi M.:  $K$ -Theory. An elementary introduction, Conference at the Clay Mathematics Research Academy.
- [14] Kasparov G. G.: Topological invariants of elliptic operators, I:  $K$ -homology. Math. USSR Izvestija, Vol. 9 (1975), 751 - 792
- [15] Milnor J.: Introduction to Algebraic  $K$ -Theory, Annals of Mathematics Studies Nr. 72, Princeton, 1971.
- [16] Puschnigg M.: Diffeotopy Functors of Ind-Algebras and Local Cyclic Cohomology. Documenta Mathematica, Vol. 8, pp. 143-245, 2003.
- [17] Rordam M., Lansen F., Lautsen N.: An Introduction to  $K$ -Theory for  $C^*$ -Algebras. London Math. Soc. Student Texts Nr. 49, Cambridge, 2000
- [18] Rosenberg J.: Algebraic  $K$ -Theory and its Applications. Graduate Texts Nr. 147, Springer, Berlin, 1994.
- [19] Spanier E. H.: Algebraic Topology, McGraw - Hill Series in Higher Mathematics, New York, 1966.
- [20] Swan R.: Algebraic  $K$ -Theory. Lecture Notes in Mathematics, Vol. 76, Springer, 1968.
- [21] Teleman N.: The Index of Signature Operators on Lipschitz Manifolds. Publ. Math. Paris, IHES, Vol. 58, pp. 251-290, 1983
- [22] Teleman N.: The Index Theorem on Topological Manifolds. Acta Mathematica ???
- [23] Teleman N.:  $Local^3$  Index Theorem. arXiv: 1109.6095v1 [math.KT], 28 Sep. 2011.
- [24] Teleman N.:  $Local$  Hochschild Homology of Hilbert-Schmidt Operators on Simplicial Spaces. arXiv hal-00707040, Version 1, 11 June 2012.
- [25] Teleman N.: The  $Local$  Index Theorem. HAL-00825083, arXiv :1305.5329v2 [math.KT] , 24 May 2013.
- [26] Weibel C. A.: Algebraic  $K$ -Theory, 2012.
- [27] Whitehead J. H. C.: Simple Homotopy Types. Amer. J. Math. (72), 1 - 57, 1950.
- [28] Wegge-Olsen N. E.:  $K$ -Theory and  $C^*$ -Algebras. A Friendly Approach. Oxford University Press, 1994.